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Asymptotic derivation of quasistatic frictional contact models with wear for elastic rods

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ABSTRACT

The aim of this paper is to derive mathematical models for the bending–stretching of an elastic rod in contact with a moving foundation, when the resulting wear is taken into account. The process is assumed to be quasistatic, the contact is modeled with normal compliance and the evolution of the wear function is described with Archard's law. To derive the models we start with the corresponding 3D problem, introduce a change of variable together with the scaling of the unknowns and then we use arguments of asymptotic analysis to obtain error estimates and a convergence result to the limit model.

1. Introduction

In solid mechanics, the obtention of models for rods, beams, bars, plates and shells is based on *a priori* hypotheses on the displacement and/or stress fields which, upon substitution in the equilibrium and constitutive equations of three-dimensional elasticity, lead to useful simplifications. Nevertheless, from both constitutive and geometrical point of views, there is a need to justify the validity of most of the models obtained in this way.

In the past decades many models have been derived and justified by using the asymptotic expansion method, whose foundations can be found in [16]. Earlier works were performed in [3,5] to justify the linearized theory of plate bending, and later in [2] to justify the Bernoulli–Navier model for bending–stretching of elastic thin rods. More recently, the error estimation of higher-order terms in the asymptotic expansion was provided in [12]. The nonlinear case was studied in [4]. The asymptotic method was used in [30] to justify the Saint-Venant, Timoshenko and Vlassov models of beams and in [23,24] to derive models for the stretching–bending of viscoelastic rods.

Contact phenomena involving deformable bodies abound in industry and everyday life. For this reason, the engineering literature concerning this topic is rather extensive. An early attempt to the study of frictional contact problems within the framework of variational inequalities was made in [6]. Comprehensive references on analysis and numerical approximation of variational inequalities arising from contact problems include [8,9,13]. Mathematical, mechanical and numerical state of the art on the Contact Mechanics can be found in the proceedings [17,20], in the special issue [26] and in the monograph [27], as well.

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Wear of mechanical systems is a major factor which affects their proper functioning over time. Most of the wear is generated by the frictional contact between various parts and components of the mechanical systems. A simple and common example is the wear of the car tires resulting from frictional contact with the road. The literature dealing with various aspects of wear includes [1,14,29]. Variational analyses of quasistatic viscoelastic frictional contact problems with wear can be found in [21,22,25], for instance.

Models of beams in frictional contact with wear, both in the dynamic and the quasistatic case, can be found in [7,28]. Nevertheless, these papers focus on the analysis of the models, including existence and uniqueness results for the weak solutions, without providing an explanation on how the corresponding models were derived. Therefore, despite the progress made in the previous two papers, there is a real need to justify such kinds of models of contact with wear involving thin structures.

The aim of the present paper is to contribute to the filling of this gap. More precisely, we derive models for the contact with wear of an elastic rod, by using the asymptotic expansion method. To the best of our knowledge, this is the first time such kinds of models have been rigorously derived. We obtain as a particular case of our main result, Theorem 6.1, the following model:

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 \xi}{\partial x^2}\right) = f + p(\xi - w - s) \quad \text{in } (0, L) \times (0, T), \tag{1.1}$$

$$\dot{w} = -k_w|v^*|p(\xi - w - s) \quad \text{in } (0, L) \times (0, T). \tag{1.2}$$

Here ξ is the flexion of the rod, w denotes the wear function, s is the initial gap between the rod and the obstacle, p is the normal compliance function (which satisfies $p(r) = 0$ if $r \leq 0$), v^* the velocity of the moving foundation, k_w is the wear coefficient, and f represents external loads. Also, E is the Young's modulus of the material, I the inertia moment for the cross section, L is the length of the rod and T the period of observation.

The rest of the paper is organized as follows. In Section 2 we introduce some notations and preliminary material. Then, in Section 3 we show the existence and uniqueness of solution of the three-dimensional elasticity contact problem, formulated in the volume Ω^ε occupied by the rod, ε being the size of the diameter of the transversal section ω^ε. The unknowns are denoted $\boldsymbol{u}^\varepsilon, \boldsymbol{\sigma}^\varepsilon$ and w^ε, which represent the displacement, the stress and the wear function, respectively. The main ingredient in our approach is developed in Section 4, and it consists of introducing a change of variable together with the scaling of the unknowns. In this way, the problem is reduced to an equivalent one, formulated in a reference domain Ω, with contact boundary Γ_C. The unknowns of this new problem are denoted $\boldsymbol{u}(\varepsilon), \boldsymbol{\sigma}(\varepsilon)$ and $w(\varepsilon)$, and represent the scaled displacement, the scaled stress and the scaled wear, respectively, for which we assume asymptotic expansions. The mathematical justification of the model is provided in Section 5, and it is supported by error estimates and a convergence result of the form $\boldsymbol{u}(\varepsilon) \to \boldsymbol{u}^0$ in $[H^1(\Omega)]^3$ and $w(\varepsilon) \to w^0$ in $L^2(\Gamma_C)$, where $\boldsymbol{u}^0 = (u_i^0)$ and w^0 are the first order terms of the respective asymptotic expansions. Finally, in Section 6, after the "descaling" of $\{\boldsymbol{u}(\varepsilon), w(\varepsilon)\}$, which gives an asymptotic expansion of $\{\boldsymbol{u}^\varepsilon, w^\varepsilon\}$, we characterize the zeroth order term of such expansions in terms of the solution of a problem, which describes the axial deformation and bendings of an elastic beam in frictional contact with a moving foundation.

2. Notation and preliminaries

We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$, while "$\cdot$" will represent the inner product on $\mathbb{S}^d$ and $\mathbb{R}^d$ (in practice, $d = 3$). In addition, in what follows, unless the contrary is explicitly written, we use summation convention on repeated indices. Moreover, Latin indices $i, j, k, \ldots$ take values in the set $\{1, 2, 3\}$ whereas Greek indices $\alpha, \beta, \rho, \ldots$ (ε excluded) do it in the set $\{1, 2\}$. Also, an index which follows a comma means a partial derivative with respect to the corresponding spatial variable, while a prime $'$ denotes the derivative of a function with respect to its single spatial variable.

Let $T > 0$. For any real Banach space X we employ the usual notation for the spaces $C([0, T]; X)$, $C^k([0, T]; X)$, $L^p(0, T; X)$, $H^k(0, T; X)$ with $1 \leq p \leq \infty$, $k = 1, 2, \ldots$. Moreover, for a function $u : [0, T] \to X$, we denote by $\dot{u}$ and $\ddot{u}$ the first and second derivatives of u with respect to the time variable, when these derivatives exist.

Let ω be an open, bounded and connected set in $\mathbb{R}^2$ with area $A(\omega)$ which, for simplicity, we assume equal to one. Denote by $\gamma = \partial\omega$ the boundary of ω, which is be supposed to be sufficiently smooth and divided into two disjoint parts: $\gamma = \gamma_C \cup \gamma_N$, $\mathring{\gamma}_C \cap \mathring{\gamma}_N = \emptyset$. The coordinates system Ox_1x_2 will be a principal system of inertia associated with the section ω, which means that

$$\int_\omega x_1 d\omega = \int_\omega x_2 d\omega = \int_\omega x_1 x_2 d\omega = 0.$$

Given $L > 0$, we denote by Ω the reference rod $\Omega = \omega \times (0, L)$. A generic point of Ω is denoted by $\boldsymbol{x} = (x_1, x_2, x_3)$. The boundary $\partial\Omega$ contains the following parts:

$$\Gamma = \gamma \times (0, L) = \Gamma_N \cup \Gamma_C, \quad \Gamma_N = \gamma_N \times (0, L), \Gamma_C = \gamma_C \times (0, L)$$

$$\Gamma_0 = \omega \times \{0\}, \qquad \Gamma_L = \omega \times \{L\}, \qquad \Gamma_D = \Gamma_0 \cup \Gamma_L.$$

Also, $\boldsymbol{n}$ stands for the unitary outer normal vector on Γ. Note that $n_3 = 0$. For $\boldsymbol{v} \in [H^1(\Omega)]^3$ we write $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ on $\partial\Omega$ and denote by $v_n = \boldsymbol{v} \cdot \boldsymbol{n}$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_n\boldsymbol{n}$ the normal and the tangential components of $\boldsymbol{v}$ on $\partial\Omega$. When $\boldsymbol{\sigma} : \Omega \to \mathbb{S}^d$

is a sufficiently smooth function (say C^1), then σ_n and $\boldsymbol{\sigma}_\tau$ denote its normal and the tangential traces, i.e. $\sigma_n = (\boldsymbol{\sigma n}) \cdot \boldsymbol{n}$, $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma n} - \sigma_n \boldsymbol{n}$.

2.1. Three-dimensional contact problem

We now describe the physical setting of the contact problem. Given a real parameter ε such that $0 < \varepsilon \le 1$, we define

$$\omega^\varepsilon = \varepsilon\omega, \qquad \gamma^\varepsilon = \varepsilon\gamma = \partial\omega^\varepsilon, \qquad \gamma_N^\varepsilon = \varepsilon\gamma_N, \qquad \gamma_C^\varepsilon = \varepsilon\gamma_C,$$

and we denote by $\Omega^\varepsilon = \omega^\varepsilon \times (0, L)$ the prismatic set that we will identify as the reference configuration of the beam, with area $A^\varepsilon = \varepsilon^2$. A generic point of Ω^ε is denoted by $\boldsymbol{x}^\varepsilon = (x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon)$. The boundary $\partial\Omega^\varepsilon$ contains the following parts:

$$\begin{aligned}
&\Gamma^\varepsilon = \gamma^\varepsilon \times (0, L) = \Gamma_N^\varepsilon \cup \Gamma_C^\varepsilon, \quad \Gamma_N^\varepsilon = \gamma_N^\varepsilon \times (0, L), \Gamma_C^\varepsilon = \gamma_C^\varepsilon \times (0, L),\\
&\Gamma_0^\varepsilon = \omega^\varepsilon \times \{0\}, \qquad \Gamma_L^\varepsilon = \omega^\varepsilon \times \{L\}, \qquad \Gamma_D^\varepsilon = \Gamma_0^\varepsilon \cup \Gamma_L^\varepsilon.
\end{aligned}$$

Also, $\boldsymbol{n}^\varepsilon$ stands for the unitary outer normal vector on Γ^ε. Note that $n_3^\varepsilon = 0$. We remark that writing superscript 1 is equivalent to dropping it (for example $\Omega^1 = \Omega$).

We suppose that the rod Ω^ε is submitted to the action of body forces of volume density $\boldsymbol{f}^\varepsilon = (f_i^\varepsilon)$ and surface forces acting on Γ_N^ε of density $\boldsymbol{g}^\varepsilon = (g_i^\varepsilon)$. Also, the rod is clamped on both ends Γ_0^ε and Γ_L^ε and may arrive in contact on Γ_C^ε with an obstacle, the so-called foundation. The foundation is sliding with a constant velocity $\boldsymbol{v}^{\star\varepsilon} : \Gamma_C^\varepsilon \to \mathbb{R}^3$. We denote $\alpha^{\star\varepsilon} = \|\boldsymbol{v}^{\star\varepsilon}\|$ and we assume that $\alpha^{\star\varepsilon} > 0$ in Γ_C^ε. We introduce the unitary vector $\boldsymbol{\delta}^\varepsilon = (\delta_i^\varepsilon) : \Gamma_C^\varepsilon \to \mathbb{R}^3$ defined by $\boldsymbol{\delta}^\varepsilon = \boldsymbol{v}^{\star\varepsilon}/\|\boldsymbol{v}^{\star\varepsilon}\|$. In the reference configuration there is a gap between Γ_C^ε and the foundation, measured along the direction of $\boldsymbol{n}^\varepsilon$, denoted s^ε. When the contact arises, some material of the contact surface is worn out and immediately removed from the system. This process is measured by the wear function w^ε.

We are interested in the evolution of the displacement field $\boldsymbol{u}^\varepsilon = (u_i^\varepsilon)$, the stress field $\boldsymbol{\sigma}^\varepsilon = (\sigma_{ij}^\varepsilon)$ and the wear function w^ε in the time interval of interest $(0, T)$ with $T > 0$. Therefore, the classical formulation of the contact problem is the following.

Problem 2.1. Find a displacement field $\boldsymbol{u}^\varepsilon : \Omega^\varepsilon \times [0, T] \to \mathbb{R}^3$, a stress field $\boldsymbol{\sigma}^\varepsilon : \Omega^\varepsilon \times [0, T] \to \mathbb{S}^3$ and a wear function $w^\varepsilon : \Omega^\varepsilon \times [0, T] \to \mathbb{R}$ such that

$$\boldsymbol{\sigma}^\varepsilon = \lambda^\varepsilon tr(\boldsymbol{e}^\varepsilon(\boldsymbol{u}^\varepsilon))\mathcal{I} + 2\mu^\varepsilon \boldsymbol{e}^\varepsilon(\boldsymbol{u}^\varepsilon) \quad \text{in } \Omega^\varepsilon \times (0, T), \tag{2.1}$$

$$\mathrm{Div}\,\boldsymbol{\sigma}^\varepsilon + \boldsymbol{f}^\varepsilon = \boldsymbol{0} \quad \text{in } \Omega^\varepsilon \times (0, T), \tag{2.2}$$

$$\boldsymbol{u}^\varepsilon = \boldsymbol{0} \quad \text{on } \Gamma_D^\varepsilon \times (0, T), \tag{2.3}$$

$$\boldsymbol{\sigma}^\varepsilon \boldsymbol{n}^\varepsilon = \boldsymbol{g}^\varepsilon \quad \text{on } \Gamma_N^\varepsilon \times (0, T), \tag{2.4}$$

$$\sigma_n^\varepsilon = -p_n^\varepsilon(u_n^\varepsilon - w^\varepsilon - s^\varepsilon) \quad \text{on } \Gamma_C^\varepsilon \times (0, T), \tag{2.5}$$

$$\boldsymbol{\sigma}_\tau^\varepsilon = -p_\tau^\varepsilon(u_n^\varepsilon - w^\varepsilon - s^\varepsilon)\frac{\boldsymbol{v}^{\star\varepsilon}}{\|\boldsymbol{v}^{\star\varepsilon}\|} \quad \text{on } \Gamma_C^\varepsilon \times (0, T), \tag{2.6}$$

$$\dot{w}^\varepsilon = -k_w^\varepsilon \alpha^{\star\varepsilon}\sigma_n^\varepsilon = k_w^\varepsilon \alpha^{\star\varepsilon} p_n^\varepsilon(u_n^\varepsilon - w^\varepsilon - s^\varepsilon) \quad \text{on } \Gamma_C^\varepsilon \times (0, T), \tag{2.7}$$

$$w^\varepsilon(0) = w_0^\varepsilon \quad \text{on } \Gamma_C^\varepsilon. \tag{2.8}$$

Here and below, in order to simplify the notation, we voluntary omit the dependence of the various functions on $\boldsymbol{x}^\varepsilon \in \Omega^\varepsilon$ and $t \in (0, T)$. We now proceed to describe the equations and the boundary conditions (2.1)–(2.8).

First, we assume that the rod is made from an elastic material which is homogeneous and isotropic, i.e., it follows a constitutive law of the form (2.1) where $\boldsymbol{e}^\varepsilon(\boldsymbol{u}^\varepsilon) = (e_{ij}^\varepsilon(\boldsymbol{u}^\varepsilon)) = (\frac{1}{2}(\frac{\partial u_j^\varepsilon}{\partial x_i^\varepsilon} + \frac{\partial u_i^\varepsilon}{\partial x_j^\varepsilon}))$ denotes the linearized strain tensor and λ^ε and μ^ε denote the Lamé coefficients. The process is assumed to be quasistatic and, therefore, we use the equilibrium equation (2.2), in which the effects of inertia have been neglected. Eqs. (2.3) and (2.4) are the displacement and traction boundary conditions, respectively. The contact in the normal direction is assumed to satisfy the normal compliance condition (2.5), so the contact pressure $-\sigma_n^\varepsilon$ is related to the interpenetration of surface asperities. The gap at the point $\boldsymbol{x}^\varepsilon$ is given by $s^\varepsilon(\boldsymbol{x}^\varepsilon) + w^\varepsilon(\boldsymbol{x}^\varepsilon) - u_n^\varepsilon(\boldsymbol{x}^\varepsilon)$, when it is nonnegative. The function p_n has to vanish when its argument is negative, since then there is no contact and, therefore, the normal stress vanishes. A power law was used in [15,18] and more general expressions can be found in [1,14]. In this work we consider the following two choices:

$$p_n^\varepsilon(r) = c_n^\varepsilon r_+, \qquad p_n^\varepsilon(r) = \begin{cases} c_n^\varepsilon r_+, & \text{if } r \le \alpha^\varepsilon, \\ c_n^\varepsilon \alpha^\varepsilon, & \text{if } r > \alpha^\varepsilon, \end{cases} \quad \alpha^\varepsilon > 0, \tag{2.9}$$

where $r_+ = \max\{r, 0\}$. The tangential stress $\boldsymbol{\sigma}_\tau^\varepsilon$ is assumed to satisfy condition (2.6) in which p_τ^ε is a given positive function. This means that the shear is opposite to the direction of the velocity of the foundation $\boldsymbol{v}^{\star\varepsilon}$ and depends on the normal pressure. A usual choice for p_τ^ε is

$$p_\tau^\varepsilon(r) = \mu_\tau^\varepsilon p_n^\varepsilon(r). \tag{2.10}$$

The evolution of the wear of the contacting surface depends on the normal pressure and therefore it is governed by the Archard's law (2.7), where $k_w^\varepsilon > 0$ is the wear coefficient. For the sake of simplicity we assume that $\boldsymbol{v}^{\star\varepsilon} = (0, 0, v_3^{\star\varepsilon})$, which implies $\alpha^{*\varepsilon} = |v_3^{*\varepsilon}|$ and $\boldsymbol{\delta}^\varepsilon = \boldsymbol{\delta} = (0, 0, \pm 1)$. This means that the foundation is sliding in the axial direction of the rod. Finally, condition (2.8) represents the initial condition for the wear function in which w_0^ε is the given initial wear.

2.2. Variational formulation

Given a set $\mathcal{O}^\varepsilon$ in $\mathbb{R}^n$ we denote by $|\cdot|_{0,\mathcal{O}^\varepsilon}$ both the usual norm in the Hilbert space $L^2(\mathcal{O}^\varepsilon)$ and the usual product norm in $[L^2(\mathcal{O}^\varepsilon)]^m$, $m = 1, 2, \ldots$. In particular, in the next sections we use the norms $|\cdot|_{0,\Omega^\varepsilon}$ and $|\cdot|_{0,\Gamma_C^\varepsilon}$. For the stress field we define the space

$$\Sigma(\Omega^\varepsilon) = \{\boldsymbol{\tau}^\varepsilon = (\tau_{ij}^\varepsilon) \mid \tau_{ij}^\varepsilon = \tau_{ji}^\varepsilon \in L^2(\Omega^\varepsilon)\}, \tag{2.11}$$

which is a Hilbert space with the inner product

$$(\boldsymbol{\sigma}^\varepsilon, \boldsymbol{\tau}^\varepsilon) = \int_{\Omega^\varepsilon} \boldsymbol{\sigma}^\varepsilon \cdot \boldsymbol{\tau}^\varepsilon \, d\boldsymbol{x}^\varepsilon = \int_{\Omega^\varepsilon} \sigma_{ij}^\varepsilon \tau_{ij}^\varepsilon \, d\boldsymbol{x}^\varepsilon \quad \forall \boldsymbol{\sigma}^\varepsilon, \boldsymbol{\tau}^\varepsilon \in \Sigma(\Omega^\varepsilon).$$

Remark that $\Sigma(\Omega^\varepsilon) = [L^2(\Omega^\varepsilon)]_s^9$ where s stands for symmetry, and the associated norm is $\|\cdot\|_{\Sigma(\Omega^\varepsilon)} := (\cdot,\cdot)^{1/2} = |\cdot|_{0,\Omega^\varepsilon}$. Also, for any element $\boldsymbol{\tau}^\varepsilon \in \Sigma(\Omega^\varepsilon)$ we use the notation $|\tau_{3\beta}^\varepsilon|_{0,\Omega^\varepsilon}$ for the norm of $(\tau_{3\beta}^\varepsilon)$ in $[L^2(\Omega^\varepsilon)]^2$ and $|\tau_{\alpha\beta}^\varepsilon|_{0,\Omega^\varepsilon}$ for the norm of $(\tau_{\alpha\beta}^\varepsilon)$ in $[L^2(\Omega^\varepsilon)]_s^4$.

We denote $H^1(\Omega^\varepsilon)$ the usual Sobolev space endowed with the classical norm $\|\cdot\|_{1,\Omega^\varepsilon}$. We use the same notation for the product norm in $[H^1(\Omega^\varepsilon)]^m$, $m = 1, 2, \ldots$. The space of admissible displacements is defined by

$$V(\Omega^\varepsilon) = \{\boldsymbol{v}^\varepsilon \in [H^1(\Omega^\varepsilon)]^3 \mid \boldsymbol{v}^\varepsilon = \boldsymbol{0} \text{ on } \Gamma_D^\varepsilon\}. \tag{2.12}$$

The space $V(\Omega^\varepsilon)$ endowed with the inner product

$$(\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon) = \int_{\Omega^\varepsilon} \boldsymbol{e}^\varepsilon(\boldsymbol{u}^\varepsilon) \cdot \boldsymbol{e}^\varepsilon(\boldsymbol{v}^\varepsilon) \, d\boldsymbol{x}^\varepsilon = \int_{\Omega^\varepsilon} e_{ij}^\varepsilon(\boldsymbol{u}^\varepsilon) e_{ij}^\varepsilon(\boldsymbol{v}^\varepsilon) \, d\boldsymbol{x}^\varepsilon \quad \forall \boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon \in V(\Omega^\varepsilon),$$

is a real Hilbert space. Since *meas* $(\Gamma_D^\varepsilon) > 0$, it follows from Korn's inequality (see, for instance, [19]) that the associated norm $\|\cdot\|_{V(\Omega^\varepsilon)} = (\cdot,\cdot)^{1/2}$ is equivalent to the usual norm $\|\cdot\|_{1,\Omega^\varepsilon}$. Although we use the same notation for the inner products in $V(\Omega^\varepsilon)$ and $\Sigma(\Omega^\varepsilon)$, no confusion will arise in the following sections. From the Sobolev trace theorem we have

$$|\boldsymbol{v}^\varepsilon|_{0,\Gamma_C^\varepsilon} \le c_0^\varepsilon \, \|\boldsymbol{v}^\varepsilon\|_{1,\Omega^\varepsilon} \quad \forall \boldsymbol{v}^\varepsilon \in V(\Omega^\varepsilon). \tag{2.13}$$

Here and below, c_k^ε $(k = 0, 1, 2, \ldots)$ represent various positive constants which are independent on the time variable.

We turn to the variational formulation of Problem 2.1. First, we assume that the normal and tangential compliance functions p_e^ε $(e = n, \tau)$ satisfy

$$\begin{cases} \text{(a) } p_e^\varepsilon : \Gamma_C^\varepsilon \times \mathbb{R} \to \mathbb{R}_+. \\ \text{(b) There exists } \mathcal{L}_e^\varepsilon > 0 \text{ such that } |p_e^\varepsilon(\boldsymbol{x}^\varepsilon, r_1) - p_e^\varepsilon(\boldsymbol{x}^\varepsilon, r_2)| \le \mathcal{L}_e^\varepsilon |r_1 - r_2|, \quad \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x}^\varepsilon \in \Gamma_C^\varepsilon. \\ \text{(c) The mapping } p_e^\varepsilon(\cdot, r) : \boldsymbol{x}^\varepsilon \mapsto p_e(\boldsymbol{x}^\varepsilon, r) \text{ is measurable on } \Gamma_C, \quad \text{for all } r \in \mathbb{R}. \\ \text{(d) The mapping } p_e^\varepsilon(\cdot, r) : \boldsymbol{x}^\varepsilon \mapsto p_e^\varepsilon(\boldsymbol{x}^\varepsilon, r) \text{ vanishes for all } r \le 0. \end{cases} \tag{2.14}$$

We note that the assumptions on the functions p_n^ε and p_τ^ε are quite general, with the exception of (2.14)(b) which requires the functions to grow asymptotically at most linearly. From the practical point of view this assumption is not restrictive, since the interpenetration is likely to be very small. It is easily seen that the functions defined in (2.9) satisfy the condition (2.14)(b). Also, to conform to the usual practice, we may write $p_\tau^\varepsilon = \mu_\tau^\varepsilon p_n^\varepsilon$, and we notice that if p_n^ε satisfies the condition (2.14)(b), then p_τ^ε also satisfies the condition (2.14)(b) with $\mathcal{L}_\tau^\varepsilon = \mu_\tau^\varepsilon \mathcal{L}_n^\varepsilon$. So the results below are valid for the boundary value problems associated with these choices of the normal compliance functions.

We assume that the force and traction densities satisfy

$$\boldsymbol{f}^\varepsilon \in C([0,T]; L^2(\Omega^\varepsilon)^3), \qquad \boldsymbol{g}^\varepsilon \in C([0,T]; L^2(\Gamma_N^\varepsilon)). \tag{2.15}$$

Also, we assume the following regularity for the initial wear function, the gap and the normal and tangential compliance functions:

$$w_0^\varepsilon \in L^2(\Gamma_C^\varepsilon), \qquad s^\varepsilon \in L^2(\Gamma_C^\varepsilon), \qquad s^\varepsilon \ge 0 \quad \text{a.e. on } \Gamma_C^\varepsilon, \qquad p_e^\varepsilon(\cdot, r) \in L^2(\Gamma_C^\varepsilon), \quad \forall r \in \mathbb{R}. \tag{2.16}$$

We use the Riesz representation theorem to define the function $\boldsymbol{F}^\varepsilon : [0,T] \to V(\Omega^\varepsilon)$ by

$$(\boldsymbol{F}^\varepsilon(t), \boldsymbol{v}^\varepsilon) = \int_{\Omega^\varepsilon} \boldsymbol{f}^\varepsilon(t) \cdot \boldsymbol{v}^\varepsilon \, d\boldsymbol{x}^\varepsilon + \int_{\Gamma_N^\varepsilon} \boldsymbol{g}^\varepsilon(t) \cdot \boldsymbol{v}^\varepsilon \, d\boldsymbol{a}^\varepsilon,$$

and we note that (2.15) implies that $\boldsymbol{F}^\varepsilon \in C([0,T]; V(\Omega^\varepsilon))$. We also consider the functional $j^\varepsilon : V(\Omega^\varepsilon) \times L^2(\Gamma_C^\varepsilon) \times V(\Omega^\varepsilon) \to \mathbb{R}$ given by

$$j^\varepsilon(\boldsymbol{u}^\varepsilon, w^\varepsilon, \boldsymbol{v}^\varepsilon) = \int_{\Gamma_C^\varepsilon} p_n^\varepsilon(u_n^\varepsilon - w^\varepsilon - s^\varepsilon) v_n^\varepsilon \, d\boldsymbol{a}^\varepsilon + \int_{\Gamma_C^\varepsilon} p_\tau^\varepsilon(u_n^\varepsilon - w^\varepsilon - s^\varepsilon) \boldsymbol{\delta}^\varepsilon \cdot \boldsymbol{v}_\tau^\varepsilon \, d\boldsymbol{a}^\varepsilon, \tag{2.17}$$

and we note that assumption (2.14) and inequality (2.13) yield

$$|j^\varepsilon(\boldsymbol{u}_1^\varepsilon, w_1^\varepsilon, \boldsymbol{v}^\varepsilon) - j^\varepsilon(\boldsymbol{u}_2^\varepsilon, w_2^\varepsilon, \boldsymbol{v}^\varepsilon)| \le c_0^\varepsilon(\mathcal{L}_n^\varepsilon + \mathcal{L}_\tau^\varepsilon)(c_0^\varepsilon \|\boldsymbol{u}_1^\varepsilon - \boldsymbol{u}_2^\varepsilon\|_{1,\Omega^\varepsilon} + |w_1^\varepsilon - w_2^\varepsilon|_{0,\Gamma_C^\varepsilon}) \|\boldsymbol{v}^\varepsilon\|_{1,\Omega^\varepsilon}, \tag{2.18}$$

for all $\boldsymbol{u}_1^\varepsilon, \boldsymbol{u}_2^\varepsilon, \boldsymbol{v}^\varepsilon \in V(\Omega^\varepsilon)$, $w_1^\varepsilon, w_2^\varepsilon, \in L^2(\Gamma_C^\varepsilon)$.

Next, we use standard arguments based on Green's formula to obtain the following variational formulation of Problem 2.1.

Problem 2.2. Find a displacement field $\boldsymbol{u}^\varepsilon : [0,T] \to V(\Omega^\varepsilon)$, a stress field $\boldsymbol{\sigma}^\varepsilon : [0,T] \to \Sigma(\Omega^\varepsilon)$, verifying (2.1), and a wear function $w^\varepsilon : [0,T] \to L^2(\Gamma_C^\varepsilon)$ such that

$$(\boldsymbol{\sigma}^\varepsilon, \boldsymbol{e}^\varepsilon(\boldsymbol{v}^\varepsilon)) + j^\varepsilon(\boldsymbol{u}^\varepsilon, w^\varepsilon, \boldsymbol{v}^\varepsilon) = (\boldsymbol{F}^\varepsilon(t), \boldsymbol{v}^\varepsilon) \quad \forall \boldsymbol{v}^\varepsilon \in V(\Omega^\varepsilon), \tag{2.19}$$

$$\dot{w}^\varepsilon = \alpha^{\star\varepsilon} k_w^\varepsilon p_n^\varepsilon(u_n^\varepsilon - w^\varepsilon - s^\varepsilon) \quad \text{on } \Gamma_C^\varepsilon \times (0,T), \qquad w^\varepsilon(0) = w_0^\varepsilon. \tag{2.20}$$

The unique solvability of the Problem 2.2 will be stated and proved in the next section.

3. An existence and uniqueness result

Consider the bilinear form $a^\varepsilon : V(\Omega^\varepsilon) \times V(\Omega^\varepsilon) \to \mathbb{R}$ defined by

$$a^\varepsilon(\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon) = \int_{\Omega^\varepsilon} \lambda^\varepsilon e_{kk}^\varepsilon(\boldsymbol{u}^\varepsilon) e_{ll}^\varepsilon(\boldsymbol{v}^\varepsilon) \, d\boldsymbol{x}^\varepsilon + 2\mu^\varepsilon \int_{\Omega^\varepsilon} e_{ij}^\varepsilon(\boldsymbol{u}^\varepsilon) e_{ij}^\varepsilon(\boldsymbol{v}^\varepsilon) \, d\boldsymbol{x}^\varepsilon \tag{3.1}$$

for all $\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon \in V(\Omega^\varepsilon)$. Since λ^ε and μ^ε are positive, we note that there exist $c_1^\varepsilon > 0$ and $c_2^\varepsilon > 0$ such that

$$|a^\varepsilon(\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon)| \le c_1^\varepsilon \|\boldsymbol{u}^\varepsilon\|_{1,\Omega^\varepsilon} \|\boldsymbol{v}^\varepsilon\|_{1,\Omega^\varepsilon}, \quad a^\varepsilon(\boldsymbol{u}^\varepsilon, \boldsymbol{u}^\varepsilon) \ge c_2^\varepsilon \|\boldsymbol{u}^\varepsilon\|_{1,\Omega^\varepsilon}^2,$$

for all $\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon \in V(\Omega^\epsilon)$. We have the following result.

Theorem 3.1. *Assume that (2.14)–(2.16) hold and, in addition the smallness assumption*

$$\mathcal{L}_n^\varepsilon + \mathcal{L}_\tau^\varepsilon < \mathcal{L}_0^\varepsilon, \tag{3.2}$$

where $\mathcal{L}_0^\varepsilon = c_2^\varepsilon/(c_0^\varepsilon)^2$. Then Problem 2.2 has a unique solution, with regularity

$$\boldsymbol{u}^\varepsilon \in C([0,T]; V(\Omega^\varepsilon)), \qquad \boldsymbol{\sigma}^\varepsilon \in C([0,T]; \Sigma(\Omega^\varepsilon)), \qquad w^\varepsilon \in C([0,T]; L^2(\Gamma_C^\varepsilon)). \tag{3.3}$$

The proof of the theorem is based on arguments similar to those used in [8,21,22,27] and, therefore, we skip the details. The first step consists in the following existence and uniqueness result.

Lemma 3.2. *Given $\eta^\varepsilon \in C([0,T]; L^2(\Gamma_C^\varepsilon))$ there exists a unique function $\boldsymbol{u}_\eta^\varepsilon \in C([0,T]; V(\Omega^\varepsilon))$ which satisfies*

$$a^\varepsilon(\boldsymbol{u}_\eta^\varepsilon(t), \boldsymbol{v}) + j^\varepsilon(\boldsymbol{u}^\varepsilon(t), \eta^\varepsilon(t), \boldsymbol{v}^\varepsilon) = (\boldsymbol{F}^\varepsilon(t), \boldsymbol{v}^\varepsilon) \quad \forall \boldsymbol{v}^\varepsilon \in V(\Omega^\varepsilon), \forall t \in [0,T]. \tag{3.4}$$

We use the function $\boldsymbol{u}_\eta^\varepsilon$ obtained in Lemma 3.2 to obtain the following result.

Lemma 3.3. *For each $\eta^\varepsilon \in C([0,T]; L^2(\Gamma_C^\varepsilon))$ there exists a unique wear function $w_\eta^\varepsilon \in C([0,T]; L^2(\Gamma_C^\varepsilon))$ such that*

$$\dot{w}_\eta^\varepsilon(t) = \alpha^{\star\varepsilon} k_w^\varepsilon p_n^\varepsilon(u_{\eta n}^\varepsilon(t) - w_\eta^\varepsilon(t) - s^\varepsilon) \quad \forall t \in [0,T], \qquad w_\eta^\varepsilon(0) = w_0^\varepsilon. \tag{3.5}$$

Next, we define the operator $\Lambda^\varepsilon : C([0,T]; L^2(\Gamma_C^\varepsilon)) \to C([0,T]; L^2(\Gamma_C^\varepsilon))$ by the equality $\Lambda^\varepsilon \eta^\varepsilon(t) = w_\eta^\varepsilon$. We have the following fixed point result.

Lemma 3.4. *The operator Λ^ε has a unique fixed point $\eta_*^\varepsilon \in C([0,T]; L^2(\Gamma_C^\varepsilon))$.*

We have now all the ingredients to provide the proof of Theorem 3.1.

Proof of Theorem 3.1. *Existence.* Let $\eta^\varepsilon_* \in C([0,T]; L^2(\Gamma^\varepsilon_C))$ be the fixed point of Λ^ε and let $\boldsymbol{u}^\varepsilon$ be the solution of Eq. (3.4) for $\eta^\varepsilon = \eta^\varepsilon_*$, i.e., $\boldsymbol{u}^\varepsilon = \boldsymbol{u}^\varepsilon_{\eta^*}$. We denote by $\boldsymbol{\sigma}^\varepsilon$ the function given by (2.1) and, finally, let w^ε denote the solution of the Cauchy problem (3.5) for $\eta^\varepsilon = \eta^\varepsilon_*$, i.e., $w^\varepsilon = w^\varepsilon_{\eta^\varepsilon_*}$. Clearly, equalities (2.1) and (2.20) hold. Moreover, since $\eta^\varepsilon_* = \Lambda^\varepsilon \eta^\varepsilon_* = w^\varepsilon_{\eta^\varepsilon_*} = w^\varepsilon$, equalities (3.1) and (3.4) imply that (2.19) holds, too. Also, it follows from Lemma 3.2 that $\boldsymbol{u}^\varepsilon \in C([0,T]; V(\Omega^\varepsilon))$ and, therefore, $\boldsymbol{\sigma}^\varepsilon \in C([0,T]; \Sigma(\Omega^\varepsilon))$. In addition, Lemma 3.3 shows that $w^\varepsilon \in C([0,T]; L^2(\Gamma^\varepsilon_C))$. We conclude from above that $(\boldsymbol{u}^\varepsilon, \boldsymbol{\sigma}^\varepsilon, w^\varepsilon)$ is a solution of Problem 2.2 which satisfies (3.3).

Uniqueness. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of operator Λ^ε combined with the unique solvability of problems considered in Lemmas 3.2 and 3.3. □

4. Asymptotic analysis

In order to transport the problem in the reference domain Ω, we associate to the unknowns $(\boldsymbol{u}^\varepsilon, \boldsymbol{\sigma}^\varepsilon) \in V(\Omega^\varepsilon) \times \Sigma(\Omega^\varepsilon)$ the scaled unknowns $(\boldsymbol{u}(\varepsilon), \boldsymbol{\sigma}(\varepsilon)) \in V(\Omega) \times \Sigma(\Omega)$, defined below, where $V(\Omega)$ and $\Sigma(\Omega)$ are defined as in (2.11) and (2.12) for $\varepsilon = 1$. Following [30] we introduce the space of Bernoulli–Navier displacement fields defined by

$$V_{BN}(\Omega) = \{\boldsymbol{v} = (v_i) \in V(\Omega) \mid e_{\alpha\beta}(\boldsymbol{v}) = e_{3\beta}(\boldsymbol{v}) = 0\}.$$

It was proved in [30] that $V_{BN}(\Omega)$ coincides with the space

$$V_{BN}(\Omega) = \{\boldsymbol{v} = (v_i) \in [H^1(\Omega)]^3 \mid v_\alpha(\boldsymbol{x}) = \chi_\alpha(x_3),\ v_3(\boldsymbol{x}) = \chi_3(x_3) - x_\alpha \chi'_\alpha(x_3),\ \chi_\alpha \in H^2_0(0,L),\ \chi_3 \in H^1_0(0,L)\}.$$

We introduce the operator $\Pi^\varepsilon : \bar{\Omega} \to \bar{\Omega}^\varepsilon$ defined by $\Pi^\varepsilon(\boldsymbol{x}) = \boldsymbol{x}^\varepsilon$, where $\boldsymbol{x} = (x_1, x_2, x_3) \in \bar{\Omega}$ and $\boldsymbol{x}^\varepsilon = (x^\varepsilon_1, x^\varepsilon_2, x^\varepsilon_3) = (\varepsilon x_1, \varepsilon x_2, x_3) \in \bar{\Omega}^\varepsilon$. Moreover, for all $\boldsymbol{v}^\varepsilon \in V(\Omega^\varepsilon)$ we define the function $\boldsymbol{v}(\varepsilon) \in V(\Omega)$ by equalities $v_\alpha(\varepsilon)(\boldsymbol{x}) = \varepsilon v^\varepsilon_\alpha(\boldsymbol{x}^\varepsilon)$ and $v_3(\varepsilon)(\boldsymbol{x}) = v^\varepsilon_3(\boldsymbol{x}^\varepsilon)$, where $\boldsymbol{x}^\varepsilon = \Pi^\varepsilon(\boldsymbol{x})$, $\boldsymbol{x} \in \bar{\Omega}$. Particularly, for the displacement field $\boldsymbol{u}^\varepsilon$ the corresponding function $\boldsymbol{u}(\varepsilon) \in V(\Omega)$ is given by

$$u_\alpha(\varepsilon)(\boldsymbol{x}) = \varepsilon u^\varepsilon_\alpha(\boldsymbol{x}^\varepsilon), \qquad u_3(\varepsilon)(\boldsymbol{x}) = u^\varepsilon_3(\boldsymbol{x}^\varepsilon). \tag{4.1}$$

We make the following hypotheses on the data. We assume that the various coefficients and parameters of the material and laws involved in Problem 2.1 have the following orders of magnitude with respect to ε,

$$\begin{aligned} &\lambda^\varepsilon = \lambda, \qquad \mu^\varepsilon = \mu, \qquad c^\varepsilon_n = \varepsilon^3 c_n, \qquad \alpha^\varepsilon = \varepsilon^{-1}\alpha, \qquad \mu^\varepsilon_\tau = \varepsilon^{-1}\mu_\tau, \qquad k^\varepsilon_w = \varepsilon^{-3}k_w, \\ &v^{*\varepsilon} = v^* = (0,0,v^*_3), \qquad \alpha^{\star\varepsilon} = \alpha^\star = |v^*_3|, \end{aligned} \tag{4.2}$$

where $\lambda, \mu, c_n, \alpha, \mu_\tau, k_w, v^*_3$ and $\alpha^\star$ are independent of ε. Also, we define

$$p_n(r) = \varepsilon^{-2} p^\varepsilon_n(\varepsilon^{-1} r), \qquad p_\tau(r) = \varepsilon^{-1} p^\varepsilon_\tau(\varepsilon^{-1} r). \tag{4.3}$$

Note that if p^ε_n is given by (2.9) then, taking into account (4.2), we find that p_n is also of the form (2.9). Moreover, it is easy to check that if p^ε_τ is given by (2.10) then

$$p_\tau(r) = \varepsilon^{-1} p^\varepsilon_\tau(\varepsilon^{-1} r) = \varepsilon^{-1}\mu^\varepsilon_\tau p^\varepsilon_n(\varepsilon^{-1} r) = \varepsilon \mu^\varepsilon_\tau p_n(r) = \mu_\tau p_n(r),$$

which means that p_τ is also of the form (2.10). Next, we assume that there exist functions f_i and g_i independent of ε, such that

$$f^\varepsilon_\alpha(\boldsymbol{x}^\varepsilon) = \varepsilon f_\alpha(\boldsymbol{x}), \qquad f^\varepsilon_3(\boldsymbol{x}^\varepsilon) = f_3(\boldsymbol{x}), \qquad g^\varepsilon_\alpha(\boldsymbol{x}^\varepsilon) = \varepsilon^2 g_\alpha(\boldsymbol{x}), \qquad g^\varepsilon_3(\boldsymbol{x}^\varepsilon) = \varepsilon g_3(\boldsymbol{x}), \tag{4.4}$$

for all $\boldsymbol{x}^\varepsilon = \Pi^\varepsilon(\boldsymbol{x})$ with $\boldsymbol{x} \in \bar{\Omega}$. We also define functions $w(\varepsilon)$, $w_0(\varepsilon)$ and $s(\varepsilon) \in L^2(\Gamma_C)$ by

$$w(\varepsilon)(x) = \varepsilon w^\varepsilon(\boldsymbol{x}^\varepsilon), \qquad w_0(\varepsilon)(x) = \varepsilon w^\varepsilon_0(\boldsymbol{x}^\varepsilon), \qquad s(\varepsilon)(x) = \varepsilon s^\varepsilon(\boldsymbol{x}^\varepsilon). \tag{4.5}$$

Finally, we define the functional $j : V(\Omega) \times L^2(\Gamma_C) \times V(\Omega) \to \mathbb{R}$ as in (2.17) for $\varepsilon = 1$.

By following a similar procedure as those described for other problems in [10–12,30] we obtain the following problem.

Problem 4.1. Find a displacement field $\boldsymbol{u}(\varepsilon) : [0,T] \to V(\Omega)$ and a wear function $w(\varepsilon) : [0,T] \to L^2(\Gamma_C)$ such that

$$a(\varepsilon)(\boldsymbol{u}(\varepsilon), \boldsymbol{v}) + \varepsilon^4 j(\boldsymbol{u}(\varepsilon), w(\varepsilon), \boldsymbol{v}) = \varepsilon^4 (\boldsymbol{F}(t), \boldsymbol{v}), \quad \forall \boldsymbol{v} \in V(\Omega), \tag{4.6}$$

$$\dot{w}(\varepsilon) = \alpha^\star k_w p_n\,(u_n(\varepsilon) - w(\varepsilon) - s(\varepsilon)) \quad \text{a.e. in } (0,T), \qquad w(\varepsilon)(0) = w_0(\varepsilon). \tag{4.7}$$

Here and below we use the bilinear forms on the space $[H^1(\Omega)]^3$ defined by

$$a(\varepsilon)(\boldsymbol{v},\boldsymbol{w}) = a_0(\boldsymbol{v},\boldsymbol{w}) + \varepsilon^2 a_2(\boldsymbol{v},\boldsymbol{w}) + \varepsilon^4 a_4(\boldsymbol{v},\boldsymbol{w}),$$
$$a_0(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \left(\lambda e_{\rho\rho}(\boldsymbol{u})e_{\beta\beta}(\boldsymbol{v}) + 2\mu e_{\alpha\beta}(\boldsymbol{u})e_{\alpha\beta}(\boldsymbol{v})\right) d\boldsymbol{x},$$
$$a_2(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \left(\lambda e_{33}(\boldsymbol{u})e_{\beta\beta}(\boldsymbol{v}) + 2\mu e_{3\beta}(\boldsymbol{u})e_{3\beta}(\boldsymbol{v}) + \lambda e_{\rho\rho}(\boldsymbol{u})e_{33}(\boldsymbol{v})\right) d\boldsymbol{x},$$
$$a_4(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega (\lambda+2\mu)e_{33}(\boldsymbol{u})e_{33}(\boldsymbol{v})\, d\boldsymbol{x},$$

for all $\boldsymbol{u},\boldsymbol{v} \in [H^1(\Omega)]^3$. Using Theorem 3.1, we deduce the following result.

Theorem 4.2. *Assume the hypotheses of Theorem 3.1 and, in addition, assume that (4.2)–(4.5) hold. Let $\boldsymbol{u}(\varepsilon)$ be the displacement field associated to $\boldsymbol{u}^\varepsilon$ by using (4.1) and let $w(\varepsilon)$ be the wear function associated to w^ε by using (4.5). Then the pair $\{\boldsymbol{u}(\varepsilon), w(\varepsilon)\}$ is the unique solution of Problem 4.1 and, moreover, $\boldsymbol{u}(\varepsilon) \in C([0,T];V(\Omega))$ and $w(\varepsilon) \in C([0,T];L^2(\Gamma_C))$.*

Next, we assume that the displacements $\boldsymbol{u}(\varepsilon)$ have an asymptotic expansion of the form

$$\boldsymbol{u}(\varepsilon) = \boldsymbol{u}^0 + \varepsilon^2\boldsymbol{u}^2 + \varepsilon^4\boldsymbol{u}^4 + \mathcal{U}(\varepsilon), \qquad \lim_{\varepsilon\to 0}\varepsilon^{-4}\mathcal{U}(\varepsilon) = \boldsymbol{0}, \tag{4.8}$$

and the scaled wear $w(\varepsilon)$ has an asymptotic expansion of the form

$$w(\varepsilon) = w^0 + \mathcal{W}(\varepsilon), \qquad w_0(\varepsilon) = w_0^0 + \mathcal{W}_0(\varepsilon), \qquad \lim_{\varepsilon\to 0}\mathcal{W}(\varepsilon) = \lim_{\varepsilon\to 0}\mathcal{W}_0(\varepsilon) = 0, \tag{4.9}$$

with the compatibility condition $w^0(0) = w_0^0$, $\mathcal{W}(\varepsilon)(0) = \mathcal{W}_0(\varepsilon)$. We also assume that there exists a function $s : \Gamma_C \to \mathbb{R}$ such that $\lim_{\varepsilon\to 0} s(\varepsilon) = s$. Using (4.8) and (4.9) in Problem 4.1, we find that

$$a_0(\boldsymbol{u}^0,\boldsymbol{v}) + \varepsilon^2\left[a_0(\boldsymbol{u}^2,\boldsymbol{v}) + a_2(\boldsymbol{u}^0,\boldsymbol{v})\right] + \varepsilon^4\left[a_0(\boldsymbol{u}^4,\boldsymbol{v}) + a_2(\boldsymbol{u}^2,\boldsymbol{v}) + a_4(\boldsymbol{u}^0,\boldsymbol{v})\right] + a_0(\mathcal{U}(\varepsilon),\boldsymbol{v}) + \varepsilon^2 a_2(\mathcal{U}(\varepsilon),\boldsymbol{v})$$
$$+\varepsilon^4 a_4(\mathcal{U}(\varepsilon),\boldsymbol{v}) + \varepsilon^4 j(\boldsymbol{u}^0 + \varepsilon^2\boldsymbol{u}^2 + \varepsilon^4\boldsymbol{u}^4 + \mathcal{U}(\varepsilon), w^0 + \mathcal{W}(\varepsilon), \boldsymbol{v}) = \varepsilon^4(\boldsymbol{F}(t),\boldsymbol{v}), \quad \forall \boldsymbol{v} \in V(\Omega),$$
$$\dot{w}^0 + \dot{\mathcal{W}}(\varepsilon) = \alpha^\star k_w p_n\left(u_n^0 + \varepsilon^2 u_n^2 + \varepsilon^4 u_n^4 + \mathcal{U}_n(\varepsilon) - w^0 - \mathcal{W}(\varepsilon) - s(\varepsilon)\right) \quad \text{a.e. in } (0,T),$$
$$w^0(0) + \mathcal{W}(\varepsilon)(0) = w_0^0 + \mathcal{W}_0(\varepsilon).$$

Multiplying the previous equalities by $\varepsilon^0, \varepsilon^{-2}, \varepsilon^{-4}$, respectively, and taking the limit as $\varepsilon \to 0$, we find that

$$a_0(\boldsymbol{u}^0,\boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in V(\Omega), \tag{4.10}$$
$$a_0(\boldsymbol{u}^2,\boldsymbol{v}) + a_2(\boldsymbol{u}^0,\boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in V(\Omega), \tag{4.11}$$
$$a_0(\boldsymbol{u}^4,\boldsymbol{v}) + a_2(\boldsymbol{u}^2,\boldsymbol{v}) + a_4(\boldsymbol{u}^0,\boldsymbol{v}) + j(\boldsymbol{u}^0, w^0, \boldsymbol{v}) = (\boldsymbol{F}(t),\boldsymbol{v}), \quad \forall \boldsymbol{v} \in V(\Omega), \tag{4.12}$$
$$\dot{w}^0 = \alpha^\star k_w p_n\left(u_n^0 - w^0 - s\right) \quad \text{a.e. in } (0,T), \qquad w^0(0) = w_0^0. \tag{4.13}$$

Moreover, taking $\boldsymbol{v} = \boldsymbol{u}^0$ in (4.10)–(4.11) we obtain $e_{\alpha\beta}(\boldsymbol{u}^0) = e_{3\beta}(\boldsymbol{u}^0) = 0$, which shows that $\boldsymbol{u}^0 \in V_{BN}(\Omega)$. This implies that

$$u_\alpha^0(x_1,x_2,x_3) = \xi_\alpha(x_3), \quad \xi_\alpha \in H_0^2(0,L), \qquad u_3^0(x_1,x_2,x_3) = \xi_3(x_3) - x_\alpha\xi_\alpha'(x_3), \quad \xi_3 \in H_0^1(0,L).$$

Next, from (4.12) we deduce that

$$\int_\Omega \left((\lambda+2\mu)e_{33}(\boldsymbol{u}^0) + \lambda e_{\rho\rho}(\boldsymbol{u}^2)\right) e_{33}(\boldsymbol{v})\, d\boldsymbol{x} + j(\boldsymbol{u}^0, w^0, \boldsymbol{v}) = (\boldsymbol{F},\boldsymbol{v}), \quad \forall \boldsymbol{v} \in V_{BN}(\Omega). \tag{4.14}$$

Using (4.11) and arguments similar to those used in [10] we obtain that

$$\int_\Omega e_{\rho\rho}(\boldsymbol{u}^2)e_{33}(\boldsymbol{v})\, d\boldsymbol{x} = -\frac{\lambda}{\lambda+\mu}\int_\Omega e_{33}(\boldsymbol{u}^0)e_{33}(\boldsymbol{v})\, d\boldsymbol{x}, \quad \forall \boldsymbol{v} \in V_{BN}(\Omega),$$

and, therefore, from (4.14) we find that

$$\int_\Omega E e_{33}(\boldsymbol{u}^0)e_{33}(\boldsymbol{v})\, d\boldsymbol{x} + j(\boldsymbol{u}^0, w^0, \boldsymbol{v}) = (\boldsymbol{F},\boldsymbol{v}), \quad \forall \boldsymbol{v} \in V_{BN}(\Omega). \tag{4.15}$$

Moreover, since $\boldsymbol{v} \in V_{BN}(\Omega)$ we deduce that

$$v_\alpha(x_1,x_2,x_3) = \zeta_\alpha(x_3), \quad \zeta_\alpha \in H_0^2(0,L), \qquad v_3(x_1,x_2,x_3) = \zeta_3(x_3) - x_\alpha\zeta_\alpha'(x_3), \quad \zeta_3 \in H_0^1(0,L),$$

which implies that

$$\int_\Omega e_{33}(\boldsymbol{u}^0)e_{33}(\boldsymbol{v})\,d\boldsymbol{x} = \int_0^L A(\omega)\xi_3'\zeta_3'\,dx_3 + \int_0^L I_\alpha \xi_\alpha''\zeta_\alpha''\,dx_3, \tag{4.16}$$

where I_α denote the moments of inertia. Note that since $\boldsymbol{n} = (n_1(x_1,x_2), n_2(x_1,x_2), 0)$, then

$$v_n = \zeta_\alpha n_\alpha, \qquad \boldsymbol{v}_\tau = \boldsymbol{v} - v_n\boldsymbol{n}, \qquad (v_\tau)_3 = v_3 = \zeta_3 - x_\rho\zeta_\rho', \qquad (v_\tau)_\beta = \zeta_\beta - \zeta_\rho n_\rho n_\beta.$$

Therefore,

$$j(\boldsymbol{u}^0, w^0, \boldsymbol{v}) = \int_0^L\int_{\gamma_C} p_n(\xi_\rho n_\rho - w^0 - s)\zeta_\beta n_\beta\,d\gamma_C\,dx_3 + \int_0^L\int_{\gamma_C} p_\tau(\xi_\rho n_\rho - w^0 - s)\left(\delta_3\zeta_3 - \delta_3 x_\rho\zeta_\rho'\right)d\gamma_C\,dx_3. \tag{4.17}$$

Also,

$$(\boldsymbol{F}, \boldsymbol{v}) = \int_0^L F_\alpha\zeta_\alpha\,dx_3 + \int_0^L F_3\zeta_3\,dx_3 - \int_0^L M_\alpha\zeta_\alpha'\,dx_3, \tag{4.18}$$

where

$$F_\alpha = \int_\omega f_\alpha\,d\omega + \int_{\gamma_N} g_\alpha\,d\gamma_N, \qquad F_3 = \int_\omega f_3\,d\omega + \int_{\gamma_N} g_3\,d\gamma_N, \qquad M_\alpha = \int_\omega x_\alpha f_3\,d\omega + \int_{\gamma_N} x_\alpha g_3\,d\gamma_N.$$

On the other hand, using (4.16)–(4.18) we find that (4.15) is equivalent to

$$\begin{aligned} &E\left(\int_0^L A(\omega)\xi_3'\zeta_3'\,dx_3 + \int_0^L I_\alpha\xi_\alpha''\zeta_\alpha''\,dx_3\right) + \int_0^L\int_{\gamma_C} p_n(\xi_\rho n_\rho - w^0 - s)\zeta_\beta n_\beta\,d\gamma_C\,dx_3 \\ &\quad + \int_0^L\int_{\gamma_C} p_\tau(\xi_\rho n_\rho - w^0 - s)\left(\delta_3\zeta_3 - \delta_3 x_\rho\zeta_\rho'\right)d\gamma_C\,dx_3 \\ &= \int_0^L F_\alpha\zeta_\alpha\,dx_3 + \int_0^L F_3\zeta_3\,dx_3 - \int_0^L M_\alpha\zeta_\alpha'\,dx_3. \end{aligned} \tag{4.19}$$

We test in (4.19) with particular functions $\boldsymbol{v} \in V_{BN}(\Omega)$ to deduce the following equations, in which no summation on α is involved:

$$\begin{aligned} &E\int_0^L I_\alpha\xi_\alpha''\zeta_\alpha''\,dx_3 + \int_0^L\int_{\gamma_C} p_n(\xi_\rho n_\rho - w^0 - s)\zeta_\alpha n_\alpha\,d\gamma_C\,dx_3 \\ &\quad - \int_0^L\int_{\gamma_C} p_\tau(\xi_\rho n_\rho - w^0 - s)\left(\delta_3 x_\alpha\zeta_\alpha'\right)d\gamma_C\,dx_3 = \int_0^L F_\alpha\zeta_\alpha\,dx_3 - \int_0^L M_\alpha\zeta_\alpha'\,dx_3, \end{aligned} \tag{4.20}$$

$$E\int_0^L A(\omega)\xi_3'\zeta_3'\,dx_3 + \int_0^L\int_{\gamma_C} p_\tau(\xi_\rho n_\rho - w^0 - s)\delta_3\zeta_3\,d\gamma_C\,dx_3 = \int_0^L F_3\zeta_3\,dx_3. \tag{4.21}$$

5. Strong convergence

We start with the following result.

Lemma 5.1. *There exists a constant $c_2 > 0$ which does not depend on ε such that*

$$a(\varepsilon)(\boldsymbol{v},\boldsymbol{v}) \ge c_2\left(|e_{\alpha\beta}(\boldsymbol{v})|_{0,\Omega}^2 + \varepsilon^2|e_{3\beta}(\boldsymbol{v})|_{0,\Omega}^2 + \varepsilon^4|e_{33}(\boldsymbol{v})|_{0,\Omega}^2\right) \quad \forall \boldsymbol{v} \in [H^1(\Omega)]^3.$$

Moreover, if (3.2) holds, then there exists a constant $c > 0$ independent of ε such that

$$\begin{aligned} &a(\varepsilon)(\boldsymbol{v}-\boldsymbol{w},\boldsymbol{v}-\boldsymbol{w}) + \varepsilon^4\left[j(\boldsymbol{v},\varphi,\boldsymbol{v}-\boldsymbol{w}) - j(\boldsymbol{w},\varphi,\boldsymbol{v}-\boldsymbol{w})\right] \\ &\quad \ge c\left(|e_{\alpha\beta}(\boldsymbol{v}-\boldsymbol{w})|_{0,\Omega}^2 + \varepsilon^2|e_{3\beta}(\boldsymbol{v}-\boldsymbol{w})|_{0,\Omega}^2 + \varepsilon^4|e_{33}(\boldsymbol{v}-\boldsymbol{w})|_{0,\Omega}^2\right) \quad \forall \boldsymbol{v},\boldsymbol{w} \in V(\Omega), \varphi \in L^2(\Gamma_C). \end{aligned} \tag{5.1}$$

Proof. The proof of the first inequality is straightforward. For the second one we use

$$|j(\boldsymbol{v},\varphi,\boldsymbol{v}-\boldsymbol{w}) - j(\boldsymbol{w},\varphi,\boldsymbol{v}-\boldsymbol{w})| \le c_0^2(\mathcal{L}_n + \mathcal{L}_\tau)\|\boldsymbol{v}-\boldsymbol{w}\|_{1,\Omega}^2$$

combined with the smallness assumption (3.2). □

Next, we define the second order approximation of $\boldsymbol{u}(\varepsilon)$ as $\boldsymbol{u}^2(\varepsilon) = \boldsymbol{u}^0 + \varepsilon^2\boldsymbol{u}^2$. From (4.10)–(4.12) we obtain that

$$\begin{aligned} a(\varepsilon)(\boldsymbol{u}^2(\varepsilon),\boldsymbol{v}) + \varepsilon^4 j(\boldsymbol{u}^2(\varepsilon), w(\varepsilon), \boldsymbol{v}) &= \varepsilon^6 a_4(\boldsymbol{u}^2,\boldsymbol{v}) - \varepsilon^4 a_0(\boldsymbol{u}^4,\boldsymbol{v}) + \varepsilon^4\left[j(\boldsymbol{u}^2(\varepsilon), w(\varepsilon), \boldsymbol{v}) - j(\boldsymbol{u}^0, w^0, \boldsymbol{v})\right] \\ &\quad + \varepsilon^4(\boldsymbol{F}(t),\boldsymbol{v}) \quad \forall \boldsymbol{v} \in V(\Omega). \end{aligned} \tag{5.2}$$

Two possibilities arise: $\boldsymbol{u}^2 \in V(\Omega)$ and $\boldsymbol{u}^2 \notin V(\Omega)$. We shall consider them separately.

5.1. No boundary layer case

Assume that $\boldsymbol{u}^2 \in V(\Omega)$ and, therefore, $\boldsymbol{u}^2(\varepsilon) \in V(\Omega)$. Denote $\boldsymbol{r}(\varepsilon) = \boldsymbol{u}(\varepsilon) - \boldsymbol{u}^2(\varepsilon) \in V(\Omega)$. We subtract (5.2) from (4.6), take $\boldsymbol{v} = \boldsymbol{r}(\varepsilon)$ and, after some algebra, we find that

$$\begin{aligned} &a(\varepsilon)(\boldsymbol{r}(\varepsilon), \boldsymbol{r}(\varepsilon)) + \varepsilon^4 \left[j(\boldsymbol{u}(\varepsilon), w(\varepsilon), \boldsymbol{r}(\varepsilon)) - j(\boldsymbol{u}^2(\varepsilon), w(\varepsilon), \boldsymbol{r}(\varepsilon))\right] \\ &\quad = \varepsilon^4 a_0(\boldsymbol{u}^4, \boldsymbol{r}(\varepsilon)) - \varepsilon^6 a_4(\boldsymbol{u}^2, \boldsymbol{r}(\varepsilon)) - \varepsilon^4 \left[j(\boldsymbol{u}^2(\varepsilon), w(\varepsilon), \boldsymbol{r}(\varepsilon)) \pm j(\boldsymbol{u}^2(\varepsilon), w^0, \boldsymbol{r}(\varepsilon)) - j(\boldsymbol{u}^0, w^0, \boldsymbol{r}(\varepsilon))\right]. \end{aligned} \tag{5.3}$$

Also, arguments similar to those used in the proof of (2.18) yield

$$\left|j(\boldsymbol{u}^2(\varepsilon), w(\varepsilon), \boldsymbol{r}(\varepsilon)) - j(\boldsymbol{u}^2(\varepsilon), w^0, \boldsymbol{r}(\varepsilon))\right| \le c|w(\varepsilon) - w^0|_{0,\Gamma_C} \|\boldsymbol{r}(\varepsilon)\|_{1,\Omega}, \tag{5.4}$$

$$\left|j(\boldsymbol{u}^2(\varepsilon), w^0, \boldsymbol{r}(\varepsilon)) - j(\boldsymbol{u}^0, w^0, \boldsymbol{r}(\varepsilon))\right| \le c\varepsilon^2 \|\boldsymbol{u}^2\|_{1,\Omega} \|\boldsymbol{r}(\varepsilon)\|_{1,\Omega}. \tag{5.5}$$

Next, from (4.7) and (4.13) we obtain that

$$\dot{w}(\varepsilon) - \dot{w}^0 = \alpha^\star k_w \left(p_n\left(u_n(\varepsilon) - w(\varepsilon) - s(\varepsilon)\right) - p_n\left(u_n^0 - w^0 - s\right)\right) \quad \text{a.e. in } (0, T).$$

Taking into account (2.14), Fatou's lemma and Cauchy–Schwartz inequality, we find that

$$\begin{aligned} |w(\varepsilon)(t) - w^0(t)|_{0,\Gamma_C} &\le |w_0(\varepsilon) - w_0^0|_{0,\Gamma_C} + c\Bigg(t|s(\varepsilon) - s|_{0,\Gamma_C} \\ &\quad + \int_0^t \left(|u_n(\varepsilon)(s) - u_n^0(s)|_{0,\Gamma_C} + |w(\varepsilon)(s) - w^0(s)|_{0,\Gamma_C}\right) ds\Bigg) \quad \text{a.e. in } (0, T). \end{aligned}$$

Using Gronwall's lemma and adding and subtracting conveniently $\boldsymbol{u}^2(\varepsilon)$ we deduce that

$$\begin{aligned} |w(\varepsilon)(t) - w^0(t)|_{0,\Gamma_C} &\le c\Bigg(|w_0(\varepsilon) - w_0^0|_{0,\Gamma_C} + |s(\varepsilon) - s|_{0,\Gamma_C} + \varepsilon^2 \int_0^t \|\boldsymbol{u}^2(s)\|_{1,\Omega}\, ds \\ &\quad + \int_0^t \|\boldsymbol{r}(\varepsilon)(s)\|_{1,\Omega}\, ds\Bigg) \quad \text{a.e. in } (0, T). \end{aligned} \tag{5.6}$$

Here c is a positive constant which depends on T but is independent of ε.

Theorem 5.2 (*Error Estimate Under No Boundary Layer Phenomenon*)**.** *Assume the hypotheses of Theorem 3.1, and let $\boldsymbol{u}^0$ and $\boldsymbol{u}^2$ be the functions introduced in (4.8). Moreover, assume that the second order term $\boldsymbol{u}^2$ satisfies the boundary condition $\boldsymbol{u}^2 = \boldsymbol{0}$ on $\Gamma_0 \cup \Gamma_L$ and, in addition, the initial gap and wear have second order asymptotic expansions, i.e.,*

$$|w_0(\varepsilon) - w_0^0|_{0,\Gamma_C} \le c\varepsilon^2, \qquad |s(\varepsilon) - s|_{0,\Gamma_C} \le c\varepsilon^2. \tag{5.7}$$

Then the second order approximation of the scaled displacements $\boldsymbol{u}^2(\varepsilon) = \boldsymbol{u}^0 + \varepsilon^2 \boldsymbol{u}^2 \in V(\Omega)$ is such that

$$\|\boldsymbol{u}(\varepsilon) - \boldsymbol{u}^2(\varepsilon)\|_{1,\Omega} \le C\varepsilon^2, \qquad \|\boldsymbol{u}(\varepsilon) - \boldsymbol{u}^0\|_{1,\Omega} \le C\varepsilon^2 \quad \text{a.e. in } (0, T), \tag{5.8}$$

where C is a positive constant which does not depend on ε. Furthermore, the first order approximation of the scaled wear w^0 satisfies

$$|w(\varepsilon) - w^0|_{0,\Gamma_C} \le C\varepsilon^2 \quad \text{a.e. in } (0, T). \tag{5.9}$$

Proof. We use (5.1) and (5.3) with $\boldsymbol{v} = \boldsymbol{u}(\varepsilon)$, $\boldsymbol{w} = \boldsymbol{u}^2(\varepsilon)$ and $\varphi = w(\varepsilon)$, the definitions of a_0 and a_4, (5.4)–(5.6) to find that

$$\begin{aligned} &c\left(|e_{\alpha\beta}(\boldsymbol{r}(\varepsilon))|_{0,\Omega}^2 + \varepsilon^2 |e_{3\beta}(\boldsymbol{r}(\varepsilon))|_{0,\Omega}^2 + \varepsilon^4 |e_{33}(\boldsymbol{r}(\varepsilon))|_{0,\Omega}^2\right) \le \varepsilon^4 |e_{\alpha\beta}(\boldsymbol{r}(\varepsilon))|_{0,\Omega} \\ &\quad + \varepsilon^6 |e_{33}(\boldsymbol{r}(\varepsilon))|_{0,\Omega} + \varepsilon^6 \|\boldsymbol{r}(\varepsilon)\|_{1,\Omega} + \varepsilon^4 \int_0^t \|\boldsymbol{r}(\varepsilon)(s)\|_{1,\Omega}\, ds \|\boldsymbol{r}(\varepsilon)(t)\|_{1,\Omega} \\ &\quad + \varepsilon^4 \left[|w_0(\varepsilon) - w_0^0|_{0,\Gamma_C} + |s(\varepsilon) - s|_{0,\Gamma_C} + \varepsilon^2 \int_0^t \|\boldsymbol{u}^2\|_{1,\Omega}\, ds\right] \|\boldsymbol{r}(\varepsilon)\|_{1,\Omega}, \end{aligned} \tag{5.10}$$

where the constant c depends on $\boldsymbol{u}^2$, $\boldsymbol{u}^4$ and the material coefficients. We define

$$r(\varepsilon) = \|\boldsymbol{r}(\varepsilon)(t)\|_{1,\Omega} = \left(|e_{\alpha\beta}(\boldsymbol{r}(\varepsilon))|_{0,\Omega}^2 + |e_{3\beta}(\boldsymbol{r}(\varepsilon))|_{0,\Omega}^2 + |e_{33}(\boldsymbol{r}(\varepsilon))|_{0,\Omega}^2\right)^{\frac{1}{2}},$$

$$\bar{r}(\varepsilon) = \left(|e_{\alpha\beta}(\boldsymbol{r}(\varepsilon))|_{0,\Omega}^2 + \varepsilon^2 |e_{3\beta}(\boldsymbol{r}(\varepsilon))|_{0,\Omega}^2 + \varepsilon^4 |e_{33}(\boldsymbol{r}(\varepsilon))|_{0,\Omega}^2\right)^{\frac{1}{2}}.$$

Then, it is easy to see that $\varepsilon^2 r(\varepsilon) \le \bar{r}(\varepsilon) \le r(\varepsilon)$. Using these inequalities and (5.10) we find

$$c\,(\bar{r}(\varepsilon)(t))^2 \le \varepsilon^4 \bar{r}(\varepsilon)(t) + \int_0^t \bar{r}(\varepsilon)(s)\,ds\,\bar{r}(\varepsilon)(t) + \varepsilon^2\left[|w_0(\varepsilon) - w_0^0|_{0,\Gamma_C} + |s(\varepsilon) - s|_{0,\Gamma_C} + \varepsilon^2 \int_0^t \|\boldsymbol{u}^2\|_{1,\Omega}\,ds\right]\bar{r}(\varepsilon)(t).$$

Therefore, by using Gronwall's lemma, we deduce that

$$\bar{r}(\varepsilon)(t) \le c\varepsilon^4 + c\varepsilon^2\big(|w_0(\varepsilon) - w_0^0|_{0,\Gamma_C} + |s(\varepsilon) - s|_{0,\Gamma_C}\big),$$

which combined with (5.7) implies (5.8). Finally, (5.9) is a consequence of (5.6). □

5.2. Boundary layer case

We turn now to the general case, in which $\boldsymbol{u}^2$ does not belong to $V(\Omega)$. In this case we need to define a corrector function $\boldsymbol{\Psi}_{u^2}(\varepsilon)$ with the property that $\boldsymbol{u}^* = \boldsymbol{u}^2(\varepsilon) + \boldsymbol{\Psi}_{u^2}(\varepsilon)$ belongs to $V(\Omega)$. More precisely, given an arbitrary displacement $\boldsymbol{v} \in [H^1(0, L; H^1(\omega))]^3$, we define the corrector function $\boldsymbol{\Psi}_v(\varepsilon)$ as in [12] by equality

$$\boldsymbol{\Psi}_v(\varepsilon)(x_1, x_2, x_3) = -\varepsilon^2\left[\boldsymbol{v}(x_1, x_2, 0)\eta_1(\varepsilon)(x_3) + \boldsymbol{v}(x_1, x_2, L)\eta_2(\varepsilon)(x_3)\right],$$

where

$$\eta_1(\varepsilon) = \begin{cases} \dfrac{1}{\varepsilon}(\varepsilon - x_3), & \text{if } 0 \le x_3 \le \varepsilon, \\ 0, & \text{if } \varepsilon \le x_3 \le L, \end{cases} \qquad \eta_2(\varepsilon) = \begin{cases} 0, & \text{if } 0 \le x_3 \le L - \varepsilon, \\ -\dfrac{1}{\varepsilon}(L - \varepsilon - x_3), & \text{if } L - \varepsilon \le x_3 \le L. \end{cases}$$

It is obvious that the corrected functions $\boldsymbol{v} + \varepsilon^{-2}\boldsymbol{\Psi}_v(\varepsilon)$ and $\varepsilon^2\boldsymbol{v} + \boldsymbol{\Psi}_v(\varepsilon)$ belong to $V(\Omega)$ for all $0 < \varepsilon \le 1$. Moreover, it is easy to obtain the following estimates (see [12]):

Proposition 5.3. *For each $\boldsymbol{v} \in [H^1(0, L; H^1(\omega))]^3$, there exists a constant C_v which depends on $\boldsymbol{v}$ but is independent of ε such that for all $0 < \varepsilon \le 1$,*

$$\|\boldsymbol{\Psi}_v(\varepsilon)\|_{1,\Omega} \le C_v\varepsilon^{3/2}, \qquad |\boldsymbol{\Psi}_v(\varepsilon)|_{0,\Omega} \le C_v\varepsilon^3, \qquad |(e_{\alpha\beta}(\boldsymbol{\Psi}_v(\varepsilon)))|_{0,\Omega} \le C_v\varepsilon^{5/2},$$
$$|(e_{3\alpha}(\boldsymbol{\Psi}_v(\varepsilon)))|_{0,\Omega} \le C_v\varepsilon^{3/2}, \qquad |e_{33}(\boldsymbol{\Psi}_v(\varepsilon))|_{0,\Omega} \le C_v\varepsilon^{3/2}.$$

Next, define $\boldsymbol{r}(\varepsilon) = \boldsymbol{u}(\varepsilon) - \boldsymbol{u}^* \in V(\Omega)$. Then, using (5.2) we find that

$$a(\varepsilon)(\boldsymbol{u}^*, \boldsymbol{v}) + \varepsilon^4 j(\boldsymbol{u}^*, w(\varepsilon), \boldsymbol{v}) = \varepsilon^6 a_4(\boldsymbol{u}^2, \boldsymbol{v}) - \varepsilon^4 a_0(\boldsymbol{u}^4, \boldsymbol{v}) + \varepsilon^4\left[j(\boldsymbol{u}^*, w(\varepsilon), \boldsymbol{v}) - j(\boldsymbol{u}^0, w^0, \boldsymbol{v})\right] + \varepsilon^4(\boldsymbol{F}(t), \boldsymbol{v}) + a(\varepsilon)(\boldsymbol{\Psi}_{u^2}(\varepsilon), \boldsymbol{v}) \quad \forall \boldsymbol{v} \in V(\Omega). \tag{5.11}$$

Subtracting (5.11) from (4.6) and taking $\boldsymbol{v} = \boldsymbol{r}(\varepsilon)$ in the resulting equality we find that

$$a(\varepsilon)(\boldsymbol{r}(\varepsilon), \boldsymbol{r}(\varepsilon)) + \varepsilon^4\left[j(\boldsymbol{u}(\varepsilon), w(\varepsilon), \boldsymbol{r}(\varepsilon)) - j(\boldsymbol{u}^*, w(\varepsilon), \boldsymbol{r}(\varepsilon))\right] = \varepsilon^4 a_0(\boldsymbol{u}^4, \boldsymbol{r}(\varepsilon)) - \varepsilon^6 a_4(\boldsymbol{u}^2, \boldsymbol{r}(\varepsilon)) - a(\varepsilon)(\boldsymbol{\Psi}_{u^2}(\varepsilon), \boldsymbol{r}(\varepsilon)) - \varepsilon^4\left[j(\boldsymbol{u}^*, w(\varepsilon), \boldsymbol{r}(\varepsilon)) \pm j(\boldsymbol{u}^*, w^0, \boldsymbol{r}(\varepsilon)) - j(\boldsymbol{u}^0, w^0, \boldsymbol{r}(\varepsilon))\right]. \tag{5.12}$$

By proceeding as in [12] we can proof that $\boldsymbol{u}^2 \in [H^1(0, L; H^1(\omega))]^3$. Therefore, it is easy to see that the following inequalities hold:

$$a(\varepsilon)(\boldsymbol{\Psi}_{u^2}(\varepsilon), \boldsymbol{r}(\varepsilon)) \le c\left(\varepsilon^{\frac{5}{2}}|e_{\alpha\beta}(\boldsymbol{r}(\varepsilon))|_{0,\Omega} + \varepsilon^{\frac{9}{2}}|e_{33}(\boldsymbol{r}(\varepsilon))|_{0,\Omega} + \varepsilon^{\frac{7}{2}}|e_{3\beta}(\boldsymbol{r}(\varepsilon))|_{0,\Omega}\right), \tag{5.13}$$

$$\left|j(\boldsymbol{u}^*, w(\varepsilon), \boldsymbol{r}(\varepsilon)) - j(\boldsymbol{u}^*, w^0, \boldsymbol{r}(\varepsilon))\right| \le c|w(\varepsilon) - w^0|_{0,\Gamma_C}\|\boldsymbol{r}(\varepsilon)\|_{1,\Omega}, \tag{5.14}$$

$$\left|j(\boldsymbol{u}^*, w^0, \boldsymbol{r}(\varepsilon)) - j(\boldsymbol{u}^0, w^0, \boldsymbol{r}(\varepsilon))\right| \le c\left(\varepsilon^2\|\boldsymbol{u}^2\|_{1,\Omega} + \|\boldsymbol{\Psi}_{u^2}(\varepsilon)\|_{1,\Omega}\right)\|\boldsymbol{r}(\varepsilon)\|_{1,\Omega}. \tag{5.15}$$

We have the following error estimate result.

Theorem 5.4 (*General Error Estimate*)**.** *Assume the hypotheses of Theorem 3.1, and let $\boldsymbol{u}^0$ and $\boldsymbol{u}^2$ be the functions introduced in (4.8). Moreover, assume that the initial gap and the initial wear have one half order asymptotic expansions, i.e.,*

$$|w_0(\varepsilon) - w_0^0|_{0,\Gamma_C} \le c\varepsilon^{1/2}, \qquad |s(\varepsilon) - s|_{0,\Gamma_C} \le c\varepsilon^{1/2}. \tag{5.16}$$

Then the corrected second order approximation of the scaled displacements $\boldsymbol{u}^* = \boldsymbol{u}^0 + \varepsilon^2\boldsymbol{u}^2 + \boldsymbol{\Psi}_{u^{(2)}}(\varepsilon) \in V(\Omega)$ is such that

$$\|\boldsymbol{u}(\varepsilon) - \boldsymbol{u}^*\|_{1,\Omega} \le C\varepsilon^{1/2}, \qquad \|\boldsymbol{u}(\varepsilon) - \boldsymbol{u}^0\|_{1,\Omega} \le C\varepsilon^{1/2} \quad \text{a.e. in } (0, T), \tag{5.17}$$

where C is a positive real constant independent of ε. Furthermore, the first order approximation of the scaled wear w^0 satisfies

$$|w(\varepsilon) - w^0|_{0,\Gamma_C} \le C\varepsilon^{1/2} \quad \text{a.e. in } (0, T). \tag{5.18}$$

Proof. We use (5.1) and (5.12) with $\boldsymbol{v} = \boldsymbol{u}^*$, $\boldsymbol{w} = \boldsymbol{u}(\varepsilon)$ and $\varphi = w(\varepsilon)$, the definitions of a_0 and a_4, (5.13)–(5.15) and an inequality similar to (5.6) to find that

$$c\left(|e_{\alpha\beta}(\boldsymbol{r}(\varepsilon))|^2_{0,\Omega} + \varepsilon^2|e_{3\beta}(\boldsymbol{r}(\varepsilon))|^2_{0,\Omega} + \varepsilon^4|e_{33}(\boldsymbol{r}(\varepsilon))|^2_{0,\Omega}\right)$$
$$\leq \varepsilon^{\frac{5}{2}}|e_{\alpha\beta}(\boldsymbol{r}(\varepsilon))|_{0,\Omega} + \varepsilon^{\frac{9}{2}}|e_{33}(\boldsymbol{r}(\varepsilon))|_{0,\Omega} + \varepsilon^{\frac{7}{2}}|e_{3\beta}(\boldsymbol{r}(\varepsilon))|_{0,\Omega} + \varepsilon^{\frac{11}{2}}\|\boldsymbol{r}(\varepsilon)\|_{1,\Omega}$$
$$+\varepsilon^4\left[|w_0(\varepsilon) - w_0^0|_{0,\Gamma_C} + |s(\varepsilon) - s|_{0,\Gamma_C} + \int_0^t (\varepsilon^2\|\boldsymbol{u}^2\|_{1,\Omega} + \|\boldsymbol{\Psi}_{u^2}(\varepsilon)\|_{1,\Omega})\, ds\right]\|\boldsymbol{r}(\varepsilon)\|_{1,\Omega}$$
$$+\varepsilon^4\int_0^t \|\boldsymbol{r}(\varepsilon)(s)\|_{1,\Omega}\, ds\|\boldsymbol{r}(\varepsilon)(t)\|_{1,\Omega}.$$

Then, using the notation introduced in the proof of Theorem 5.2, we see that

$$c\,(\bar{r}(\varepsilon)(t))^2 \leq \varepsilon^{\frac{5}{2}}\bar{r}(\varepsilon)(t) + \int_0^t \bar{r}(\varepsilon)(s)\, ds\, \bar{r}(\varepsilon)(t)$$
$$+\varepsilon^2\left[|w_0(\varepsilon) - w_0^0|_{0,\Gamma_C} + |s(\varepsilon) - s|_{0,\Gamma_C} + \int_0^t (\varepsilon^2\|\boldsymbol{u}^2\|_{1,\Omega} + \|\boldsymbol{\Psi}_{u^2}(\varepsilon)\|_{1,\Omega})\, ds\right]\bar{r}(\varepsilon)(t).$$

Therefore, by using Gronwall's lemma, we deduce that

$$\bar{r}(\varepsilon)(t) \leq c\varepsilon^{\frac{5}{2}} + c\varepsilon^2\left[|w_0(\varepsilon) - w_0^0|_{0,\Gamma_C} + |s(\varepsilon) - s|_{0,\Gamma_C}\right],$$

and, using (5.16) we obtain (5.17). Finally, (5.18) is a consequence of (5.6). □

6. Conclusion

In the previous sections we showed that, if the cross sectional area is small, then the scaled displacement $\boldsymbol{u}(\varepsilon)$ is approximated in Ω by the first term of the asymptotic expansion (4.8). Consequently, the "descaling" of the unknowns and data through (4.1) and (4.4) leads to successive approximations of the true displacement $\boldsymbol{u}^\varepsilon$ in Ω^ε (see [30]). Let us denote

$$F_3^\varepsilon = \int_{\omega^\varepsilon} f_3^\varepsilon d\omega^\varepsilon + \int_{\gamma^\varepsilon} g_3^\varepsilon d\gamma^\varepsilon, \qquad F_\alpha^\varepsilon = \int_{\omega^\varepsilon} f_\alpha^\varepsilon d\omega^\varepsilon + \int_{\gamma^\varepsilon} g_\alpha^\varepsilon d\gamma^\varepsilon, \qquad M_\alpha^\varepsilon = \int_{\omega^\varepsilon} x_\alpha^\varepsilon f_3^\varepsilon d\omega^\varepsilon + \int_{\gamma^\varepsilon} x_\alpha^\varepsilon g_3^\varepsilon d\gamma^\varepsilon.$$

Also, for each positive integer n we define components $\boldsymbol{u}^{n\varepsilon} \in C([0,T]; V(\Omega^\varepsilon))$ of the asymptotic expansion of $\boldsymbol{u}^\varepsilon$ by equalities

$$u_\alpha^{n\varepsilon}(\boldsymbol{x}^\varepsilon) = \varepsilon^{-1+n}u_\alpha^n(\boldsymbol{x}), \qquad u_3^{n\varepsilon}(\boldsymbol{x}^\varepsilon) = \varepsilon^n u_3^n(\boldsymbol{x}) \quad \forall \boldsymbol{x}^\varepsilon = \Pi^\varepsilon(\boldsymbol{x}) \in \bar{\Omega}^\varepsilon. \tag{6.1}$$

We characterize the zeroth order terms of displacements $\boldsymbol{u}^{0\varepsilon}$ and wear $w^{0\varepsilon}$ as follows.

Theorem 6.1. *The first order displacement field $\boldsymbol{u}^{0\varepsilon}$ defined by (6.1) is a Bernoulli–Navier displacement given by*

$$u_\alpha^{0\varepsilon}(x_1^\varepsilon, x_2^\varepsilon, x_3) = \xi_\alpha^\varepsilon(x_3), \qquad u_3^{0\varepsilon}(x_1^\varepsilon, x_2^\varepsilon, x_3) = \xi_3^\varepsilon(x_3) - x_\alpha^\varepsilon(\xi^\varepsilon)'_\alpha(x_3),$$

where the flexions $(\xi_1^\varepsilon, \xi_2^\varepsilon)$ and the stretching ξ_3^ε represent the unique solution of the system

$$\xi_\alpha^\varepsilon \in H_0^2(0,L), \quad E\int_0^L I_\alpha^\varepsilon \xi_\alpha^{\varepsilon\prime\prime}\zeta_\alpha^{\varepsilon\prime\prime}\, dx_3 + \int_0^L\int_{\gamma_C^\varepsilon} p_n^\varepsilon(\xi_\rho^\varepsilon n_\rho - w^{0\varepsilon} - s^\varepsilon)\zeta_\alpha^\varepsilon n_\alpha\, d\gamma_C^\varepsilon\, dx_3$$
$$-\int_0^L\int_{\gamma_C^\varepsilon} p_\tau^\varepsilon(\xi_\rho^\varepsilon n_\rho - w^{0\varepsilon} - s^\varepsilon)\left(\delta_3^\varepsilon x_\alpha^\varepsilon \zeta_\alpha^{\varepsilon\prime}\right)\, d\gamma_C^\varepsilon\, dx_3$$
$$= \int_0^L F_\alpha^\varepsilon \zeta_\alpha^\varepsilon\, dx_3 - \int_0^L M_\alpha^\varepsilon \zeta_\alpha^{\varepsilon\prime}\, dx_3, \quad \forall \zeta_\alpha^\varepsilon \in H_0^1(0,L), \alpha = 1, 2, \tag{6.2}$$

(no summation on α),

$$\xi_3^\varepsilon \in H_0^1(0,L), \quad E\int_0^L A(\omega^\varepsilon)\xi_3^{\varepsilon\prime}\zeta_3^{\varepsilon\prime}\, dx_3 + \int_0^L\int_{\gamma_C^\varepsilon} p_\tau^\varepsilon(\xi_\rho^\varepsilon n_\rho - w^{0\varepsilon} - s^\varepsilon)\delta_3^\varepsilon\zeta_3^\varepsilon\, d\gamma_C^\varepsilon\, dx_3$$
$$= \int_0^L F_3^\varepsilon \zeta_3^\varepsilon\, dx_3 \quad \forall \zeta_3^\varepsilon \in H_0^1(0,L), \tag{6.3}$$

$$\dot{w}^{0\varepsilon} = \alpha^{\star\varepsilon}k_w^\varepsilon p_n^\varepsilon\left(\xi_i^\varepsilon n_i - x_\alpha^\varepsilon \xi_\alpha^{\varepsilon\prime} - w^{0\varepsilon} - s^\varepsilon\right) \quad \text{a.e. in } (0,T), \qquad w^{0\varepsilon}(0) = w_0^{0\varepsilon}. \tag{6.4}$$

The proof of Theorem 6.1 is a direct consequence of equalities (4.20)–(4.21). Note that the model (6.2)–(6.4) obtained above couples Eq. (6.2) (which represents a general Bernoulli–Navier bending model in contact with an elastic foundation) with the Cauchy problem (6.4) (which describes the evolution of the wear of the contact surfaces). We now consider the particular case of a planar contact boundary Γ_C^ε. Without any additional loss of generality, we take $\boldsymbol{n} = (-1, 0, 0)$. Assume that $s^\varepsilon(x_1^\varepsilon, x_2^\varepsilon, x_3) \equiv s^\varepsilon(x_3)$, which implies that the wear function $w^{0\varepsilon}$ depends only on x_3. Therefore, the strong formulation of the characterization of $\boldsymbol{u}^{0\varepsilon}$ and $w^{0\varepsilon}$ above is given by the following system:

$$\begin{cases} E\left(I_1^\varepsilon \xi_1^{\varepsilon\prime\prime}\right)'' - |\gamma_C^\varepsilon| p_n^\varepsilon(-\xi_1^\varepsilon - w^{0\varepsilon} - s^\varepsilon) + \left(\displaystyle\int_{\gamma_C^\varepsilon} x_1^\varepsilon \, d\gamma_C^\varepsilon\right) \delta_3^\varepsilon p_\tau^\varepsilon(-\xi_1^\varepsilon - w^{0\varepsilon} - s^\varepsilon)' = F_1^\varepsilon + M_1^{\varepsilon\prime}, \\ \dot{w}^{0\varepsilon} = \alpha^{\star\varepsilon} k_w^\varepsilon p_n^\varepsilon\left(-\xi_1^\varepsilon - w^{0\varepsilon} - s^\varepsilon\right) \quad \text{a.e. in } (0,T),\ w^{0\varepsilon}(0) = w_0^{0\varepsilon} \\ \xi_1^\varepsilon(0) = \xi_1^{\varepsilon\prime}(0) = \xi_1^\varepsilon(L) = \xi_1^{\varepsilon\prime}(L) = 0. \end{cases} \tag{6.5}$$

$$\begin{cases} E\left(I_2^\varepsilon \xi_2^{\varepsilon\prime\prime}\right)'' + \left(\displaystyle\int_{\gamma_C^\varepsilon} x_2^\varepsilon \, d\gamma_C^\varepsilon\right) \delta_3^\varepsilon p_\tau^\varepsilon(-\xi_1^\varepsilon - w^{0\varepsilon} - s^\varepsilon)' = F_2^\varepsilon + M_2^{\varepsilon\prime}, \\ \xi_2^\varepsilon(0) = \xi_2^{\varepsilon\prime}(0) = \xi_2^\varepsilon(L) = \xi_2^{\varepsilon\prime}(L) = 0. \end{cases} \tag{6.6}$$

$$\begin{cases} -E\left(A(\omega^\varepsilon)\xi_3^{\varepsilon\prime}\right)' + |\gamma_C^\varepsilon| \delta_3^\varepsilon p_\tau^\varepsilon(-\xi_1^\varepsilon - w^{0\varepsilon} - s^\varepsilon) = F_3^\varepsilon, \\ \xi_3^\varepsilon(0) = \xi_3^\varepsilon(L) = 0. \end{cases} \tag{6.7}$$

Consider now the particular case $\gamma_C^\varepsilon = \{c\} \times (a_1, a_2)$, with $c \in \mathbb{R}$ and $a_1 < 0 < a_2$. Then,

$$|\gamma_C^\varepsilon| = |a_2 - a_1|, \qquad \int_{\gamma_C^\varepsilon} x_1^\varepsilon \, d\gamma_C = c|a_2 - a_1|, \qquad \int_{\gamma_C^\varepsilon} x_2^\varepsilon \, d\gamma_C^\varepsilon = \frac{1}{2}(a_2^2 - a_1^2).$$

In addition, take $a_1 = -\frac{1}{2}$, $a_2 = \frac{1}{2}$ and $c = 0$. Then, the system (6.5) becomes

$$\begin{cases} E\left(I_1^\varepsilon \xi_1^{\varepsilon\prime\prime}\right)'' = F_1^\varepsilon + M_1^{\varepsilon\prime} + p_n^\varepsilon(-\xi_1^\varepsilon - w^{0\varepsilon} - s^\varepsilon), \\ \dot{w}^{0\varepsilon} = \alpha^{\star\varepsilon} k_w^\varepsilon p_n^\varepsilon\left(-\xi_1^\varepsilon - w^{0\varepsilon} - s^\varepsilon\right) \quad \text{a.e. in } (0,T),\ w^{0\varepsilon}(0) = w_0^{0\varepsilon}, \\ \xi_1^\varepsilon(0) = \xi_1^{\varepsilon\prime}(0) = \xi_1^\varepsilon(L) = \xi_1^{\varepsilon\prime}(L) = 0. \end{cases} \tag{6.8}$$

Note that, neglecting the derivative of the moment (i.e., assuming that $M_1^{\varepsilon\prime} = 0$), the model (6.8) represents, with a slightly different notation, the model (1.1)–(1.2) described in the Introduction. Moreover, (6.6)–(6.7) yields

$$\begin{cases} E\left(I_2^\varepsilon \xi_2^{\varepsilon\prime\prime}\right)'' = F_2^\varepsilon + M_2^{\varepsilon\prime}, \\ \xi_2^\varepsilon(0) = \xi_2^{\varepsilon\prime}(0) = \xi_2^\varepsilon(L) = \xi_2^{\varepsilon\prime}(L) = 0. \end{cases} \tag{6.9}$$

$$\begin{cases} -E\left(A(\omega^\varepsilon)\xi_3^{\varepsilon\prime}\right)' = F_3^\varepsilon - \delta_3^\varepsilon p_\tau^\varepsilon(-\xi_1^\varepsilon - w^{0\varepsilon} - s^\varepsilon), \\ \xi_3^\varepsilon(0) = \xi_3^\varepsilon(L) = 0. \end{cases} \tag{6.10}$$

To conclude, the main result of this paper consists in obtaining the system of Eqs. (6.5)–(6.7), together with its particular version (6.8)–(6.10). They provide models for the one-dimensional quasistatic problem of frictional contact with wear between an elastic rod and a foundation, when the contact is modeled with normal compliance and the wear is described by the Archard's law. And, at the best of our knowledge, this is the first time that such mathematical models have been provided by using arguments of asymptotic analysis.

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