



Functoriality of Cuntz–Pimsner correspondence maps



S. Kaliszewski^a, John Quigg^a, David Robertson^{b,*}

^a School of Mathematical and Statistical Sciences, Arizona State University, Tempe, AZ 85287, United States

^b School of Mathematics and Applied Statistics, University of Wollongong, NSW 2522, Australia

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ABSTRACT

We show that the passage from a C^* -correspondence to its Cuntz–Pimsner C^* -algebra gives a functor on a category of C^* -correspondences with appropriately defined morphisms. Applications involving topological graph C^* -algebras are discussed, and an application to crossed-product correspondences is presented in detail.

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1. Introduction

Cuntz–Pimsner algebras were first introduced by Pimsner in [14] as C^* -algebras associated to C^* -correspondences with injective left actions; Katsura extended the definition in [12] to include C^* -correspondences with non-injective left actions. The class of Cuntz–Pimsner algebras is very rich, containing all Cuntz–Krieger algebras, crossed products by $\mathbb{Z}$, and topological graph algebras. Accordingly, there is a pressing need to understand how constructions at the level of C^* -correspondences carry over to the Cuntz–Pimsner algebras.

Now, many C^* -correspondence constructions naturally and necessarily involve *multiplier correspondences* $(M(Y), M(B))$. For example, if a group G acts on a correspondence (X, A) , then (X, A) embeds in the multipliers $(M(X \rtimes G), M(A \rtimes G))$ of the crossed-product correspondence, but not in general in $(X \rtimes G, A \rtimes G)$ itself. Our main result (Corollary 3.6) is that a C^* -correspondence homomorphism $(X, A) \rightarrow (M(Y), M(B))$ that is *covariant* in an appropriate sense (Definition 3.1) induces a C^* -algebra homomorphism $\mathcal{O}_X \rightarrow M(\mathcal{O}_Y)$ of the corresponding Cuntz–Pimsner algebras.

Given that the natural embedding of a C^* -correspondence in its Cuntz–Pimsner algebra is often degenerate, there is no reason to expect that a correspondence homomorphism $(X, A) \rightarrow (M(Y), M(B))$ will automatically extend to $(M(X), M(A))$; this makes composing two homomorphisms problematic. To deal with this, our notion of covariance incorporates the assumption that the image of X lies in the so-called *restricted multipliers* $M_B(Y)$ that were introduced in [3, Appendix A]. We then can show (Theorem 3.5) that the C^* -homomorphisms $\mathcal{O}_X \rightarrow M(\mathcal{O}_Y)$ are obtained in a functorial way on an appropriately-defined category of C^* -correspondences. Some earlier work has been done on functoriality of Cuntz–Pimsner algebras [17, 18], but our approach is the first to incorporate multipliers.

In Section 4 we give three applications of our techniques: to topological graph actions, topological graph coactions, and to crossed-product C^* -correspondences. The first two are really just brief indications of applications, the first to a recent result [3] concerning free and proper actions of groups on topological graphs, where the argument was essentially a “bare-hands” special case of our Corollary 3.6, and the second is a foreshadowing of how we plan to use the technique to define

* Corresponding author.

E-mail addresses: kaliszewski@asu.edu (S. Kaliszewski), quigg@asu.edu (J. Quigg), droberts@uow.edu.au (D. Robertson).

coactions on Cuntz–Pimsner algebras from suitable coactions on the correspondences. For the third application, we bring our methods to bear on crossed-product C^* -correspondences $(X \rtimes_{\gamma} G, A \rtimes_{\alpha} G)$. We show that the canonical embedding $(X, A) \rightarrow (M(X \rtimes_{\gamma} G), M(A \rtimes_{\alpha} G))$ induces via functoriality a covariant homomorphism $(\mathcal{O}_X, G) \rightarrow M(\mathcal{O}_{X \rtimes_{\gamma} G})$. As a result we find a surjective homomorphism $\mathcal{O}_X \rtimes_{\beta} G \rightarrow \mathcal{O}_{X \rtimes_{\gamma} G}$, which serves as an alternative approach to the result of Hao and Ng [6, Theorem 2.10] that when the group G is amenable there is an isomorphism $\mathcal{O}_{X \rtimes_{\gamma} G} \cong \mathcal{O}_X \rtimes_{\beta} G$. By showing there is a maximal dual coaction on $\mathcal{O}_{X \rtimes_{\gamma} G}$, we can show that our surjection is actually an isomorphism when the group G is amenable, though this requires background on C^* -correspondence coactions and is saved for a forthcoming paper [10].

2. Preliminaries

Given C^* -algebras A and B , an $A - B$ correspondence (or a C^* -correspondence over A and B) is a Hilbert B -module X equipped with a left A -module action which is implemented by a homomorphism of A into the C^* -algebra $\mathcal{L}(X)$ of adjointable operators on X . (We refer the reader to [4, 13, 15] for background on Hilbert modules and further details on C^* -correspondences.) The homomorphism is generically called φ_A , but we usually suppress this notation and just write $a \cdot \xi$ for $\varphi_A(a)(\xi)$, where $a \in A$ and $\xi \in X$. We say X is full if $\text{span}\langle X, X \rangle \subset B$ is dense. If in addition, the left action is an isomorphism $\varphi_A : A \rightarrow \mathcal{K}(X)$ we will call X an $A - B$ imprimitivity bimodule. We write (A, X, B) to denote an $A - B$ correspondence X ; when $A = B$ we just write (X, A) , and call X an A -correspondence (or a C^* -correspondence over A). In this paper we will make the standing assumption that all C^* -correspondences are nondegenerate in the sense that $A \cdot X = X$.

Given a C^* -correspondence (A, X, B) , the set $\mathcal{K}(X) = \overline{\text{span}}\{\theta_{\xi, \eta} \mid \xi, \eta \in X\}$ is a (closed) two-sided ideal in $\mathcal{L}(X)$ called the compact operators on X , where by definition $\theta_{\xi, \eta}(\zeta) = \xi \cdot \langle \eta, \zeta \rangle$ for $\zeta \in X$. The Banach space $\mathcal{L}(B, X)$ of adjointable operators from B to X (where B is viewed as a Hilbert B -module in the natural way) becomes an $M(A) - M(B)$ correspondence when equipped with the natural left action of $M(A)$, right action of $M(B)$, and $M(B)$ -valued inner product $\langle m, n \rangle = m^*n$ all given by composition of operators. This is called the multiplier correspondence of X , and is denoted by $M(X)$. There is an embedding $\xi \mapsto T_{\xi} : X \rightarrow M(X)$ given by $T_{\xi}(b) = \xi \cdot b$ for $b \in B$. The strict topology on $M(X)$ is generated by the seminorms

$$m \mapsto \|T \cdot m\| \quad \text{and} \quad m \mapsto \|m \cdot b\| \quad \text{for } T \in \mathcal{K}(X), b \in B.$$

A correspondence homomorphism between two C^* -correspondences (A, X, B) and (C, Y, D) is a triple (π, ψ, ρ) consisting of C^* -homomorphisms $\pi : A \rightarrow M(C)$ and $\rho : B \rightarrow M(D)$ and a linear map $\psi : X \rightarrow M(Y)$ satisfying

- (i) $\psi(a \cdot \xi) = \pi(a) \cdot \psi(\xi)$,
- (ii) $\psi(\xi \cdot b) = \psi(\xi) \cdot \rho(b)$, and
- (iii) $\rho(\langle \xi, \eta \rangle) = \langle \psi(\xi), \psi(\eta) \rangle$.

(Note that condition (ii) follows from (iii).)

A correspondence homomorphism (π, ψ, ρ) is nondegenerate when π and ρ are nondegenerate C^* -homomorphisms and $\psi(X) \cdot B = Y$. In this case, by [4, Theorem 1.30] there is a unique strictly continuous extension $\overline{\psi} : M(X) \rightarrow M(Y)$ such that

$$(\overline{\pi}, \overline{\psi}, \overline{\rho}) : (M(A), M(X), M(B)) \rightarrow (M(C), M(Y), M(D))$$

is a correspondence homomorphism, where $\overline{\pi}$ and $\overline{\rho}$ are the usual extensions of nondegenerate C^* -homomorphisms to multiplier algebras.

If (X, A) is an A -correspondence and B is a C^* -algebra, a correspondence homomorphism of the form $(\pi, \psi, \pi) : (X, A) \rightarrow (B, B, B)$ (where B is viewed as a B -correspondence in the natural way) is called a Toeplitz representation of (X, A) in B , and is denoted by (ψ, π) . It is a critical observation, usually attributed to Pimsner [14, Lemma 3.2] (see also [7, Lemma 2.2]), that a Toeplitz representation of a C^* -correspondence determines a homomorphism of the algebra of compact operators. Here we derive this fact (Corollary 2.2) from the more fundamental Proposition 2.1 to emphasize that it is really a property of the underlying Hilbert module structure on X and B .

Proposition 2.1. *Let X and Y be Hilbert modules over C^* -algebras A and B , respectively, and suppose $(\psi, \rho) : (X, A) \rightarrow (M(Y), M(B))$ is a correspondence homomorphism. Then there is a unique homomorphism $\psi^{(1)} : \mathcal{K}(X) \rightarrow M(\mathcal{K}(Y))$ such that*

$$\psi^{(1)}(\theta_{\xi, \eta}) = \psi(\xi)\psi(\eta)^* \quad \text{for } \xi, \eta \in X.$$

If $\psi(X) \subseteq Y$, then $\psi^{(1)}(\mathcal{K}(X)) \subseteq \mathcal{K}(Y)$, with $\psi^{(1)}(\theta_{\xi, \eta}) = \theta_{\psi(\xi), \psi(\eta)}$.

Proof. Obviously there can be at most one such $\psi^{(1)}$. For existence, first suppose $\psi(X) \subseteq Y$. Without loss of generality, we may assume that X is full (otherwise, replace A by the closed span of $\langle X, X \rangle$); similarly, we may assume that Y is full, and that $\psi(X) = Y$ and $\rho(A) = B$.

Next, let $C = \mathcal{K}(X)$, so that X is a $C - A$ imprimitivity bimodule. Let $I = \ker \rho$. The Rieffel correspondence (see [4, Definition 1.7]) induces an ideal $J = X - \text{Ind } I := \{c \in C : cX \subset XI\}$ so that XI and X/XI are $J - I$ and $C/J - A/I$ imprimitivity bimodules respectively. Then the quotient maps (q_C, q_X, q_A) comprise a surjective imprimitivity bimodule homomorphism

of (C, X, A) onto $(C/J, X/XI, A/I)$, and there is an imprimitivity bimodule isomorphism $(\pi, \tilde{\psi}, \tilde{\rho})$ of $(C/J, X/XI, A/I)$ onto $(\mathcal{K}(Y), Y, B)$ such that $\tilde{\psi}(\xi + J) = \psi(\xi)$ for $\xi \in X$. Taking $\psi^{(1)} = \pi \circ q_C$, for $\xi, \eta \in X$ we have

$$\begin{aligned}\psi^{(1)}(\theta_{\xi, \eta}) &= \pi(q_C(\langle \xi, \eta \rangle)) = \pi(\langle \xi, \eta \rangle + J) = \pi(\langle \xi + J, \eta + J \rangle) \\ &= {}_{\mathcal{K}(Y)}\langle \tilde{\psi}(\xi + J), \tilde{\psi}(\eta + J) \rangle = {}_{\mathcal{K}(Y)}\langle \psi(\xi), \psi(\eta) \rangle = \theta_{\psi(\xi), \psi(\eta)}.\end{aligned}$$

For the general case, apply the above to the Hilbert $M(B)$ -module $M(Y)$, and note that by [3, Remark A.10] we have $\mathcal{K}(M(Y)) \subset M(\mathcal{K}(Y))$, with ${}_{\mathcal{K}(M(Y))}\langle m, n \rangle = mn^*$ for $m, n \in M(Y)$. $\square$

Corollary 2.2 ([14,7]). *Let (X, A) be a C^* -correspondence, and let (ψ, π) be a Toeplitz representation of (X, A) in a C^* -algebra B . Then there is a unique homomorphism $\psi^{(1)}: \mathcal{K}(X) \rightarrow B$ such that*

$$\psi^{(1)}(\theta_{\xi, \eta}) = \psi(\xi)\psi(\eta)^*.$$

For a C^* -correspondence (X, A) , we follow Katsura's convention [13] and define an ideal J_X of A by

$$J_X = \{a \in A \mid \varphi_A(a) \in \mathcal{K}(X) \text{ and } ab = 0 \text{ for all } b \in \ker(\varphi_A)\}.$$

A Toeplitz representation (ψ, π) of (X, A) in B is *Cuntz–Pimsner covariant* if

$$\psi^{(1)}(\varphi_A(a)) = \pi(a) \quad \text{for } a \in J_X.$$

We denote the universal Cuntz–Pimsner covariant Toeplitz representation by (k_X, k_A) , and the *Cuntz–Pimsner algebra* by $\mathcal{O}_X$. Note that $k_A: A \rightarrow \mathcal{O}_X$ will always be nondegenerate as a C^* -homomorphism because of our assumption that all correspondences (X, A) are nondegenerate. Again, we refer the reader to [12] for details.

Finally, we will need the theory of “relative multipliers” from [3, Appendix A], which is useful for extending degenerate homomorphisms. If (X, A) is a nondegenerate correspondence and $\kappa: C \rightarrow M(A)$ is a nondegenerate homomorphism, the *C-multipliers* of X are by definition

$$M_C(X) = \{m \in M(X) \mid \kappa(C) \cdot m \cup m \cdot \kappa(C) \subset X\}.$$

The *C-strict* topology on $M_C(X)$ is generated by the seminorms

$$m \mapsto \|\kappa(c) \cdot m\| \quad \text{and} \quad m \mapsto \|m \cdot \kappa(c)\| \quad \text{for } c \in C.$$

When A is viewed as an A -correspondence over itself in the usual way, $M_C(A)$ is a C^* -subalgebra of $M(A)$.

Relative multipliers possess the following elementary properties.

- (i) The *C-strict* topology is stronger than the relative strict topology on $M_C(X)$.
- (ii) $M_C(X)$ is an $M_C(A)$ -correspondence with respect to the restrictions of the operations of the $M(A)$ -correspondence $M(X)$, and the operations are separately *C-strictly* continuous.
- (iii) If $X = A$, then $M_C(A)$ is a C^* -subalgebra of $M(A)$, and the multiplication and involution on $M_C(A)$ are separately *C-strictly* continuous.
- (iv) $\mathcal{K}(M_C(X)) \subset M_C(\mathcal{K}(X))$.
- (v) $M_C(X)$ is the *C-strict* completion of X .
- (vi) $M_C(X)$ is an $M(C)$ -sub-bimodule of $M(X)$.

The main purpose of relative multipliers is the following extension theorem [3, Proposition A.11]: suppose (X, A) and (Y, B) are (nondegenerate) C^* -correspondences and $\kappa: C \rightarrow M(A)$ and $\sigma: D \rightarrow M(B)$ are nondegenerate homomorphisms. For any correspondence homomorphism $(\psi, \pi): (X, A) \rightarrow (M_D(Y), M_D(B))$, if there is a nondegenerate homomorphism $\lambda: C \rightarrow M(\sigma(D))$ such that

$$\pi(\kappa(c)a) = \lambda(c)\pi(a) \quad \text{for all } c \in C \text{ and } a \in A,$$

then there is a unique *C-strict* to *D-strictly* continuous correspondence homomorphism $(\overline{\psi}, \overline{\pi}): (M_C(X), M_C(A)) \rightarrow (M_D(Y), M_D(B))$ that extends (ψ, π) .

A closely related concept is the following, due to Baaj and Skandalis [2].

Definition 2.3. For an ideal I of a C^* -algebra A , let

$$M(A; I) = \{m \in M(A) \mid mA \cup Am \subset I\}.$$

Lemma 2.4. *If I is an ideal of a C^* -algebra A , then:*

- (i) $M(A; I)$ is the strict closure of I in $M(A)$;
- (ii) if $\pi: A \rightarrow M(B)$ is a nondegenerate homomorphism such that $\pi(I) \subset B$, then $\overline{\pi}(M(A; I)) \subset M_A(B)$, and $\overline{\pi}|: M(A; I) \rightarrow M_A(B)$ is strict to *A-strictly* continuous.

Proof. This is elementary, and the techniques are similar to those of [3, Appendix A]. For (i), we first show that I is strictly dense in $M(A; I)$. Let $m \in M(A; I)$, and let $\{e_i\}$ be an approximate identity for A . Then $me_i \in I$ for all i , and $me_i \rightarrow m$ strictly in $M(A)$.

To see that $M(A; I)$ is strictly closed in $M(A)$, let $\{m_i\}$ be a net in $M(A; I)$ converging strictly to m in $M(A)$. We must show that $m \in M(A; I)$. For $a \in A$ we have

$$\|m_i a - ma\| \rightarrow 0 \quad \text{and} \quad \|am_i - am\| \rightarrow 0,$$

so $ma, am \in I$ because $m_i a, am_i \in I$ for all i .

For (ii), note that by (i) it suffices to show that $\pi|_I: I \rightarrow B$ is continuous for the relative strict topology of I in $M(A)$ and the relative A -strict topology of $B \subset M_A(B)$. Let $\{c_i\}$ be a net in I converging strictly to 0 in $M(A)$, and let $a \in A$. Then $\pi(c_i)\pi(a) = \pi(c_i a)$ and $\pi(a)\pi(c_i) = \pi(ac_i)$ converge to 0 in norm, so $\pi(c_i) \rightarrow 0$ A -strictly in B . $\square$

Remark 2.5. Let I be an ideal of a C^* -algebra A , and let $\rho: A \rightarrow M(I)$ be the canonical homomorphism. Then by Lemma 2.4 the canonical extension $\bar{\rho}: M(A) \rightarrow M(I)$ maps $M(A; I)$ into $M_A(I)$. In fact, the restriction $\bar{\rho}|_{M(A; I)}$ is injective, although we will not need this here.

3. Functoriality

Definition 3.1. Let (X, A) and (Y, B) be nondegenerate C^* -correspondences. A correspondence homomorphism $(\psi, \pi): (X, A) \rightarrow (M(Y), M(B))$ is *Cuntz–Pimsner covariant* if

- (i) $\psi(X) \subset M_B(Y)$,
- (ii) $\pi: A \rightarrow M(B)$ is nondegenerate,
- (iii) $\pi(J_X) \subset M(B; J_Y)$, and
- (iv) the diagram

$$\begin{array}{ccc} J_X & \xrightarrow{\pi|} & M(B; J_Y) \\ \varphi_A \downarrow & & \downarrow \overline{\varphi_B} \\ \mathcal{K}(X) & \xrightarrow{\psi^{(1)}} & M_B(\mathcal{K}(Y)) \end{array}$$

commutes, where $\psi^{(1)}$ is the homomorphism provided by Proposition 2.1.

The above definition simplifies when the correspondence homomorphism is nondegenerate:

Lemma 3.2. A nondegenerate correspondence homomorphism $(\psi, \pi): (X, A) \rightarrow (M(Y), M(B))$ is Cuntz–Pimsner covariant if and only if items (i) and (iii) hold in Definition 3.1.

The lemma follows immediately from the following elementary result. Lemma 3.3 is presumably well-known, but we could not find a reference in the literature.

Lemma 3.3. For any nondegenerate correspondence homomorphism $(\pi, \psi, \rho): (A, X, B) \rightarrow (M(C), M(Y), M(D))$, the diagram

$$\begin{array}{ccc} A & \xrightarrow{\pi} & M(C) \\ \varphi_A \downarrow & & \downarrow \overline{\varphi_C} \\ \mathcal{L}(X) & \xrightarrow{\overline{\psi^{(1)}}} & \mathcal{L}(Y) \end{array}$$

commutes.

Proof. Fix $a \in A$; we must show that $\overline{\psi^{(1)}}(\varphi_A(a))\eta = \overline{\varphi_C}(\pi(a))\eta$ for all $\eta \in Y$. By nondegeneracy it suffices to consider elements of the form $\eta = \psi(\xi)d$ with $\xi \in X$ and $d \in D$, in that case we have

$$\begin{aligned} \overline{\psi^{(1)}}(\varphi_A(a))\eta &= \overline{\psi^{(1)}}(\varphi_A(a))\psi(\xi)d = \psi(\varphi_A(a)\xi)d = \psi(a \cdot \xi)d \\ &= \pi(a) \cdot \psi(\xi)d = \overline{\varphi_C}(\pi(a))\psi(\xi)d = \overline{\varphi_C}(\pi(a))\eta. \quad \square \end{aligned}$$

The following lemma addresses the overlap between Cuntz–Pimsner covariant correspondence homomorphisms and Cuntz–Pimsner covariant Toeplitz representations.

Lemma 3.4. Let (X, A) be a nondegenerate C^* -correspondence, and let $(\psi, \pi): (X, A) \rightarrow M(B)$ be a Toeplitz representation of (X, A) in a C^* -algebra B . Then (ψ, π) is Cuntz–Pimsner covariant as a correspondence homomorphism into (B, B) (as

in [Definition 3.1](#)) if and only if $\pi: A \rightarrow M(B)$ is nondegenerate and (ψ, π) is Cuntz–Pimsner covariant as a Toeplitz representation. In particular, $(k_X, k_A): (X, A) \rightarrow (\mathcal{O}_X, \mathcal{O}_X)$ is Cuntz–Pimsner covariant in the sense of [Definition 3.1](#).

Proof. This follows from the identifications $J_B = \mathcal{K}(B) = B$ and $M_B(B) = M(B; B) = M_B(\mathcal{K}(B)) = M(B)$, and the observation that φ_B is the inclusion $B \hookrightarrow M(B)$. $\square$

The Cuntz–Pimsner covariant homomorphisms between correspondences are the morphisms in a suitable category, which we now define.

Theorem 3.5. *There is a category CPCorres that has:*

- nondegenerate C^* -correspondences as objects, and
- Cuntz–Pimsner covariant homomorphisms $(\psi, \pi): (X, A) \rightarrow (M(Y), M(B))$ (as in [Definition 3.1](#)) as morphisms from (X, A) to (Y, B) ;
- and in which the composition of $(\psi, \pi): (X, A) \rightarrow (Y, B)$ and $(\sigma, \tau): (Y, B) \rightarrow (Z, C)$ is $(\bar{\sigma} \circ \psi, \bar{\tau} \circ \pi)$.

Proof. First of all, to see that composition is well-defined, note that since $\tau: B \rightarrow M(C)$ is nondegenerate by definition, it follows from [\[3, Proposition A.11\]](#) that $\sigma: Y \rightarrow M_C(Z)$ extends uniquely to a B -strict to C -strictly continuous homomorphism

$$(\bar{\sigma}, \bar{\tau}): (M_B(Y), M(B)) \rightarrow (M_C(Z), M(C)).$$

Thus we get a correspondence homomorphism

$$(\bar{\sigma} \circ \psi, \bar{\tau} \circ \pi): (X, A) \rightarrow (M_C(Z), M(C)),$$

which we must check is Cuntz–Pimsner covariant in the sense of [Definition 3.1](#). Certainly $\bar{\tau} \circ \pi: A \rightarrow M(C)$ is nondegenerate, so it remains to verify items (iii)–(iv) in [Definition 3.1](#).

For (iii), since τ is nondegenerate we have

$$\begin{aligned} (\bar{\tau} \circ \pi)(J_X)C &= \bar{\tau}(\pi(J_X))\tau(B)C \subset \bar{\tau}(M(B; J_Y))\tau(B)C \\ &= \tau(M(B; J_Y)B)C \subset \tau(J_Y)C \subset J_Z, \end{aligned}$$

and similarly $C(\bar{\tau} \circ \pi)(J_X) \subset J_Z$. Therefore $\bar{\tau} \circ \pi(J_X) \subset M(C; J_Z)$.

For (iv), let $a \in J_X$. We must show that

$$\bar{\varphi}_C \circ (\bar{\tau} \circ \pi)(a) = (\bar{\sigma} \circ \psi)^{(1)} \circ \varphi_A(a)$$

in $\mathcal{L}(Z)$, and by nondegeneracy it suffices to show equality after multiplying on the right by $\varphi_C(c)$ for an arbitrary $c \in C$. Again by nondegeneracy we can factor $c = \tau(b)c'$ for some $b \in B$ and $c' \in C$, and then

$$\begin{aligned} \bar{\varphi}_C \circ (\bar{\tau} \circ \pi)(a)\varphi_C(c) &= \bar{\varphi}_C(\bar{\tau}(\pi(a)))\varphi_C(\tau(b)c') \\ &= \bar{\varphi}_C \circ \tau(\pi(a)b)\varphi_C(c') \\ &= \sigma^{(1)} \circ \varphi_B(\pi(a)b)\varphi_C(c') \\ &= \sigma^{(1)}(\bar{\varphi}_B \circ \pi(a)\varphi_B(b))\varphi_C(c') \\ &= \sigma^{(1)}(\psi^{(1)} \circ \varphi_A(a)\varphi_B(b))\varphi_C(c') \\ &= \bar{\sigma}^{(1)} \circ \psi^{(1)} \circ \varphi_A(a)\sigma^{(1)} \circ \varphi_B(b)\varphi_C(c') \\ &= (\bar{\sigma} \circ \psi)^{(1)} \circ \varphi_A(a)\varphi_C \circ \tau(b)\varphi_C(c') \\ &= (\bar{\sigma} \circ \psi)^{(1)} \circ \varphi_A(a)\varphi_C(\tau(b)c') \\ &= (\bar{\sigma} \circ \psi)^{(1)} \circ \varphi_A(a)\varphi_C(c). \end{aligned}$$

We have thus verified that composition is well-defined in CPCorres.

To see that composition is associative is a routine exercise in the definitions and the properties of “barring” (see, e.g., [\[1, Appendix A\]](#)): if also $(\zeta, \rho): (Z, C) \rightarrow (W, D)$ in CPCorres then

$$\begin{aligned} (\zeta, \rho) \circ ((\sigma, \tau) \circ (\psi, \pi)) &= (\zeta, \rho) \circ (\bar{\sigma} \circ \psi, \bar{\tau} \circ \pi) \\ &= (\bar{\zeta} \circ (\bar{\sigma} \circ \psi), \bar{\rho} \circ (\bar{\tau} \circ \pi)) \\ &= ((\bar{\zeta} \circ \bar{\sigma}) \circ \psi, (\bar{\rho} \circ \bar{\tau}) \circ \pi) \\ &= (\bar{\zeta} \circ \sigma \circ \psi, \bar{\rho} \circ \tau \circ \pi) \\ &= (\bar{\zeta} \circ \sigma, \bar{\rho} \circ \tau) \circ (\psi, \pi) \\ &= ((\zeta, \rho) \circ (\sigma, \tau)) \circ (\psi, \pi). \end{aligned}$$

It is now clear that $(\text{id}_X, \text{id}_A)$ is an identity morphism on each object (X, A) , and therefore CPCorres is a category. $\square$

Corollary 3.6. Let (X, A) and (Y, B) be nondegenerate C^* -correspondences, and let $(\psi, \pi): (X, A) \rightarrow (M(Y), M(B))$ be a Cuntz–Pimsner covariant correspondence homomorphism. Then there is a unique homomorphism $\mathcal{O}_{\psi, \pi}$ making the diagram

$$\begin{array}{ccc} (X, A) & \xrightarrow{(\psi, \pi)} & (M_B(Y), M(B)) \\ (k_X, k_A) \downarrow & & \downarrow (\overline{k_Y}, \overline{k_B}) \\ \mathcal{O}_X & \xrightarrow{\mathcal{O}_{\psi, \pi}} & M_B(\mathcal{O}_Y) \end{array}$$

commute. Moreover, $\mathcal{O}_{\psi, \pi}$ is nondegenerate, and is injective if π is.

Proof. Applying Theorem 3.5 with (Z, C) being the C^* -algebra $\mathcal{O}_Y$ viewed as a correspondence over itself in the canonical way, we see that $(\overline{k_Y} \circ \psi, \overline{k_B} \circ \pi)$ is a Cuntz–Pimsner covariant Toeplitz representation $(\sigma, \nu): X \rightarrow M_B(\mathcal{O}_Y)$. Then the universal property of $\mathcal{O}_X$ gives the unique homomorphism

$$\mathcal{O}_{\psi, \pi} = (\overline{k_Y} \circ \psi) \times (\overline{k_B} \circ \pi).$$

Nondegeneracy of $\mathcal{O}_{\psi, \pi}$ follows from nondegeneracy of π and k_B , since

$$\begin{aligned} \mathcal{O}_Y &= \overline{k_B} \circ \pi(A) \mathcal{O}_Y = \mathcal{O}_{\psi, \pi} \circ k_A(A) \mathcal{O}_Y \\ &= \mathcal{O}_{\psi, \pi}(k_A(A)) \mathcal{O}_Y \subseteq \mathcal{O}_{\psi, \pi}(\mathcal{O}_X) \mathcal{O}_Y \subseteq \mathcal{O}_Y \end{aligned}$$

implies equality throughout.

If π is injective, then so is $\overline{k_B} \circ \pi$, so to show $\mathcal{O}_{\psi, \pi}$ is injective we can apply the Gauge-Invariant Uniqueness Theorem [12, Theorem 6.4]: let $\gamma: \mathbb{T} \rightarrow \text{Aut } \mathcal{O}_Y$ be the gauge action. It suffices to observe that for all $z \in \mathbb{T}$, $\xi \in X$, and $a \in A$ we have

$$\begin{aligned} \overline{\gamma_z} \circ \overline{k_Y} \circ \psi(\xi) &= z \overline{k_Y} \circ \psi(\xi) \\ \overline{\gamma_z} \circ \overline{k_B} \circ \pi(a) &= \overline{k_B} \circ \pi(a). \quad \square \end{aligned}$$

Recall from, e.g., [1], that there is a category $\mathbf{C}_{\text{nd}}^*$ that has:

- C^* -algebras as *objects*, and
- nondegenerate homomorphisms $\pi: A \rightarrow M(B)$ as *morphisms* from A to B ;
- and in which the *composition* of $\pi: A \rightarrow B$ and $\tau: B \rightarrow C$ is $\overline{\tau} \circ \pi$.

Theorem 3.7. The assignments $X \mapsto \mathcal{O}_X$ and $(\psi, \pi) \mapsto \mathcal{O}_{\psi, \pi}$ define a functor from CPCorres to $\mathbf{C}_{\text{nd}}^*$.

Proof. First of all, it follows from Corollary 3.6 that if $(\psi, \pi): (X, A) \rightarrow (Y, B)$ in CPCorres then $\mathcal{O}_{\psi, \pi}$ is a morphism from $\mathcal{O}_X$ to $\mathcal{O}_Y$ in $\mathbf{C}_{\text{nd}}^*$. Moreover, $\mathcal{O}_{\text{id}_X, \text{id}_A} = \text{id}_{\mathcal{O}_X}$ by uniqueness.

To see that compositions are preserved, let $(\psi, \pi): (X, A) \rightarrow (Y, B)$ and $(\sigma, \tau): (Y, B) \rightarrow (Z, C)$ in CPCorres . We have

$$\begin{aligned} \mathcal{O}_{(\sigma, \tau) \circ (\psi, \pi)} \circ k_X &= \overline{k_Z} \circ (\overline{\sigma} \circ \psi) = \overline{k_Z} \circ \overline{\sigma} \circ \psi \\ &= \overline{\mathcal{O}_{\sigma, \tau}} \circ \overline{k_Y} \circ \psi = \overline{\mathcal{O}_{\sigma, \tau}} \circ \mathcal{O}_{\psi, \pi} \circ k_X, \end{aligned}$$

and similarly

$$\mathcal{O}_{(\sigma, \tau) \circ (\psi, \pi)} \circ k_A = \overline{\mathcal{O}_{\sigma, \tau}} \circ \mathcal{O}_{\psi, \pi} \circ k_A,$$

so that $\mathcal{O}_{(\sigma, \tau) \circ (\psi, \pi)} = \overline{\mathcal{O}_{\sigma, \tau}} \circ \mathcal{O}_{\psi, \pi}$ in $\mathbf{C}_{\text{nd}}^*$. $\square$

4. Applications

We give three applications of Corollary 3.6.

Topological graph actions

Our first application is historical; we show that in [3] the germ of the idea of Corollary 3.6 was introduced in an ad-hoc way, in a very special case.

Theorem 5.6 of [3] shows that if a locally compact group G acts freely and properly on a topological graph E , then the quotient E/G is also a topological graph, and $C^*(E) \rtimes_r G$ and $C^*(E/G)$ are Morita equivalent. The strategy for proving Morita equivalence in [3] is to construct an isomorphism of $C^*(E/G)$ with Rieffel's generalized fixed-point algebra $C^*(E)^G$, after showing that the action of G on $C^*(E)$ is saturated and proper in the sense of [16], and then appealing to the imprimitivity theorem $C^*(E) \rtimes_r G \sim_M C^*(E)^G$ of [16].

By definition, $C^*(E)^G$ is a C^* -subalgebra of $M(C^*(E))$, and the isomorphism of $C^*(E/G)$ onto $C^*(E)^G$ is constructed from a Cuntz–Pimsner covariant Toeplitz representation (τ, π) of the topological-graph correspondence $(X(E/G), C_0((E/G)^0))$ in $M(C^*(E))$, which in turn is constructed via a correspondence homomorphism (μ, ν) from $(X(E/G), C_0((E/G)^0))$ to $(M(X(E)), M(C^0(E^0)))$. The proof in [3] that (τ, π) is Cuntz–Pimsner covariant essentially uses a special case of the concept of Cuntz–Pimsner covariant correspondence homomorphisms defined in Definition 3.1.

We will now explain this in more detail. First, recall that the $C_0(E^0)$ -correspondence $X(E)$ is a completion of $C_c(E^1)$, the Katsura ideal $J_{X(E)}$ can be identified with $C_0(E_{\text{rg}}^0)$, where E_{rg}^0 is a certain open subset of E^0 , and the topological-graph algebra $C^*(E)$ is the Cuntz–Pimsner algebra $\mathcal{O}_{X(E)}$. It will help the exposition to introduce the following temporary notation:

- $A = C_0((E/G)^0)$
- $X = X((E/G))$
- $A_{\text{rg}} = J_X$
- $B = C_0(E^0)$
- $Y = X(E)$
- $B_{\text{rg}} = J_Y$
- $q : E \rightarrow E/G$ is the quotient map (both for edges E^1 and vertices E^0).

(Warning: the roles of X, A and Y, B between [3] and here are switched, to allow more convenient reference to the methods of the current paper.) [3] constructed the correspondence homomorphism

$$(\mu, \nu) : (X, A) \rightarrow (M_B(Y), M(B))$$

starting with

- $\nu(f) = f \circ q$
- $(\mu(\xi) \cdot g)(e) = \xi(q(e))g(e)$ for $\xi \in C_c((E/G)^1)$, $g \in C_c(E^0)$, and $e \in E^1$.

Then the pair (X, A) was mapped into $M(\mathcal{O}_Y)$ in [3] by the commutative diagram

$$\begin{array}{ccc} (X, A) & \xrightarrow{(\mu, \nu)} & (M_B(Y), M(B)) \\ & \searrow (\tau, \pi) & \downarrow (\overline{k_Y}, \overline{k_B}) \\ & & M_B(\mathcal{O}_Y) \end{array}$$

In [3] it was then recognized that Cuntz–Pimsner covariance of (τ, π) is expressed by commutativity of the left-hand triangle of the following diagram, which is a version of [3, page 1547, diagram (5)] in which some of the notation has been modified to be consistent with the current paper:

$$\begin{array}{ccc} J_X & \xrightarrow{\nu'} & M_B(J_Y) \\ \varphi_A| \downarrow & \swarrow \pi| \quad \nwarrow \overline{k_B}| & \downarrow \overline{\varphi_B}| \\ & M_B(\mathcal{O}_Y) & \\ \tau^{(1)} \nearrow & & \nwarrow \overline{k_Y^{(1)}} \\ \mathcal{K}(X) & \xrightarrow{\mu^{(1)}} & M_B(\mathcal{K}(Y)), \end{array} \quad (4.1)$$

and the strategy was to verify that the other parts of the diagram commute. Here the homomorphism ν' is constructed from the commutative diagram

$$\begin{array}{ccc} J_X & \xrightarrow{\nu|} & M(B; J_Y) \\ & \searrow \nu' & \swarrow \\ & M_B(J_Y), & \end{array}$$

where the inclusion $M(B : J_Y) \hookrightarrow M_B(J_Y)$ is given by restriction

$$g \mapsto g|_{Y_{\text{rg}}^0}.$$

In [3] it was not recognized that in fact it would be better to do away with the map ν' altogether, so that the outer square of (4.1) is replaced by

$$\begin{array}{ccc} J_X & \xrightarrow{\nu|} & M(B; J_Y) \\ \varphi_A| \downarrow & & \downarrow \overline{\varphi_B}| \\ \mathcal{K}(X) & \xrightarrow{\mu^{(1)}} & M_B(\mathcal{K}(Y)), \end{array}$$

whose commutativity is precisely our definition [Definition 3.1](#) of Cuntz–Pimsner covariance of the correspondence homomorphism (μ, ν) . The computations in [3] were much more painstakingly “bare-hands” than in the current paper, because, again, the techniques were entirely ad-hoc, whereas here we take a more conceptual and systematic approach, developing appropriate machinery along the way.

Topological graph coactions

In a forthcoming paper [8], a continuous map (“cocycle”) κ on a topological graph E with values in a locally compact group G is used to construct a coaction δ of G on $C^*(E)$, with an eye toward proving that the crossed product $C^*(E) \rtimes_{\delta} G$ is isomorphic to the C^* -algebra of the skew-product topological graph $E \rtimes_{\kappa} G$, thereby generalizing [9, Theorem 2.4] from the discrete case.

To describe this application, let

- $A = C_0(E^0)$
- $X = X(E)$,

and recall, e.g., from the discussion at the beginning of this section, that $C^*(E)$ is the Cuntz–Pimsner algebra of the C^* -correspondence (X, A) .

To be a coaction, δ must in particular be a homomorphism from $C^*(E)$ to $M_{1 \otimes C^*(G)}(C^*(E) \otimes C^*(G))$. As usual, δ is constructed from a Cuntz–Pimsner covariant Toeplitz representation (δ_X, δ_A) of (X, A) in the C^* -algebra $M(C^*(E) \otimes C^*(G))$, which in turn is constructed via a correspondence homomorphism $(\sigma, \text{id}_A \otimes 1)$ from (X, A) to $(M(X \otimes C^*(G)), M(A \otimes C^*(G)))$. Techniques based upon [Corollary 3.6](#) will be used to show that the correspondence homomorphism $(\sigma, \text{id}_A \otimes 1)$ gives rise to a coaction δ on $C^*(E)$. Interestingly, however, we will need a slight strengthening of the Cuntz–Pimsner covariance condition of [Definition 3.1](#). The problem is that, due to nonexactness of minimal C^* -tensor products, we have no reason to believe that the minimal tensor product $\mathcal{O}_X \otimes C^*(G)$ coincides with the Cuntz–Pimsner algebra $\mathcal{O}_{X \otimes C^*(G)}$ of the external-tensor-product correspondence (where $C^*(G)$ is regarded as a correspondence over itself in the standard way). In fact, the basic theory of coactions on Cuntz–Pimsner algebras will require a significant amount of work, which we will do in [10].

Anyway, once we have the machinery necessary to construct coactions on Cuntz–Pimsner algebras, our application in [8] will go roughly as follows: first of all, a G -valued cocycle on a topological graph E is just a continuous map $\kappa : E^1 \rightarrow G$. Since the A -correspondence X is a completion of $C_c(E^1)$, and the group G embeds as unitary multipliers of $C^*(G)$, we are led to regard the cocycle κ as an adjointable operator ν on $X \otimes C^*(G)$, and then we are able to define a coaction on X by $\sigma(\xi) = \nu(\xi \otimes 1)$, where $\xi \otimes 1$ is regarded as a multiplier of the correspondence $X \otimes C^*(G)$. This is a continuous version of the coaction $\chi_{\{e\}} \mapsto \chi_{\{e\}} \otimes \kappa(e)$ of [9]. We emphasize that the justification that this actually gives a coaction will depend upon the preparation to come in [10].

C^* -correspondence action crossed products

Let (γ, α) be an action of a locally compact group G on a nondegenerate correspondence (X, A) . The *crossed product* is the completion $(X \rtimes_{\gamma} G, A \rtimes_{\alpha} G)$ of the pre-correspondence $(C_c(G, X), C_c(G, A))$ with operations

$$\begin{aligned} (f \cdot \xi)(s) &= \int_G f(t) \cdot \gamma_t(\xi(t^{-1}s)) dt \\ (\xi \cdot f)(s) &= \int_G \xi(t) \cdot \alpha_t(f(t^{-1}s)) dt \\ \langle \xi, \eta \rangle(s) &= \int_G \alpha_{t^{-1}}(\langle \xi(t), \eta(ts) \rangle) dt \end{aligned}$$

for $f, g \in C_c(G, A)$ and $\xi, \eta \in C_c(G, X)$. (We refer the reader to, e.g., [5,6], [4, Chapters 2 and 3], and [11] for the elementary theory of actions and crossed products for correspondences.)

Since $(\gamma_s, \alpha_s): (X, A) \rightarrow (X, A)$ is a correspondence homomorphism for each $s \in G$, [Proposition 2.1](#) provides homomorphisms $\gamma_s^{(1)}: \mathcal{K}(X) \rightarrow \mathcal{K}(X)$, which by uniqueness give rise to an action $(\gamma^{(1)}, \gamma, \alpha)$ of G on the correspondence $(\mathcal{K}(X), X, A)$ such that $\varphi_A: A \rightarrow M(\mathcal{K}(X))$ is $\alpha - \gamma^{(1)}$ equivariant.

There is an isomorphism (see, e.g., [11, 3.11])

$$\tau: \mathcal{K}(X \rtimes_{\gamma} G) \xrightarrow{\cong} \mathcal{K}(X) \rtimes_{\gamma(1)} G$$

satisfying

$$\tau(\theta_{\xi, \eta})(s) = \int_G \theta_{\xi(t), \gamma_s(\eta(s^{-1}t))} \Delta(s^{-1}t) dt,$$

where $\xi, \eta \in C_c(G, X)$, $s \in G$ and Δ is the modular function of G , and moreover the diagram

$$\begin{array}{ccc} A \rtimes_{\alpha} G & \xrightarrow{\varphi_{A \rtimes_{\alpha} G}} & M(\mathcal{K}(X \rtimes_{\gamma} G)) \\ & \searrow \varphi_A \rtimes G & \downarrow \cong \tau \\ & & M(\mathcal{K}(X) \rtimes_{\gamma(1)} G) \end{array}$$

commutes.

Let $(i_X, i_G): (A, G) \rightarrow M(A \rtimes_{\alpha} G)$ be the canonical covariant homomorphism.

Proposition 4.1. *There is a Cuntz–Pimsner covariant correspondence homomorphism*

$$(i_X, i_A): (X, A) \rightarrow (M(X \rtimes_{\gamma} G), M(A \rtimes_{\alpha} G))$$

such that for $x \in X, f \in C_c(G, A), s \in G$ we have

$$(i_X(x) \cdot f)(s) = x \cdot f(s). \quad (4.2)$$

Proof. We first claim that for fixed $x \in X$, (4.2) uniquely determines an operator $i_X(x): A \rtimes_{\alpha} G \rightarrow X \rtimes_{\gamma} G$ with adjoint given by

$$(i_X(x)^* \xi)(s) = \langle x, \xi(s) \rangle$$

for $\xi \in C_c(G, X)$ and $s \in G$. Indeed, (4.2) certainly defines a right $C_c(G, A)$ -module map $C_c(G, A) \rightarrow C_c(G, X)$, and we can check the adjoint property on generators: for any $\xi \in C_c(G, X), f \in C_c(G, A)$, and $s \in G$ we simply calculate

$$\begin{aligned} \langle i_X(x)f, \xi \rangle(s) &= \int_G \alpha_{t^{-1}}(\langle (i_X(x)f)(t), \xi(ts) \rangle) dt \\ &= \int_G \alpha_{t^{-1}}(f(t)^* \langle x, \xi(ts) \rangle) dt \\ &= \int_G \alpha_t(f(t^{-1}) \Delta(t^{-1}) \langle x, \xi(t^{-1}s) \rangle) dt \\ &= \int_G f^*(t) \alpha_t(\langle x, \xi(t^{-1}s) \rangle) dt, \end{aligned}$$

proving the claim.

It is now straightforward to verify that the pair (i_X, i_A) is a correspondence homomorphism. For example, for $x, y \in X, f \in C_c(G, A)$ and $s \in G$ we calculate

$$\begin{aligned} \langle (i_X(x), i_X(y))f \rangle(s) &= \langle i_X(x)^* i_X(y) \cdot f \rangle(s) = \langle x, i_X(y) \cdot f(s) \rangle \\ &= \langle x, y \cdot f(s) \rangle = \langle x, y \rangle f(s) = \langle i_A(\langle x, y \rangle) f \rangle(s) \end{aligned}$$

as required.

To show that (i_X, i_A) is Cuntz–Pimsner covariant, by Lemma 3.2 it suffices to show that (i_X, i_A) is nondegenerate and satisfies items (i) and (iii) in Definition 3.1.

We already know that the coefficient map i_A is nondegenerate. Then for $x \in X, a \in A, g \in C_c(G)$ we have

$$i_X(x) \cdot (i_A(a) i_G(g)) = i_X(x \cdot a) i_G(g),$$

so nondegeneracy of (i_X, i_A) follows since $X \cdot A = X$ and

$$X \rtimes_{\gamma} G = \overline{i_X(X) \cdot i_G(C_c(G))}.$$

Item (i), that $i_X(X) \subset M_{A \rtimes_{\alpha} G}(X \rtimes_{\gamma} G)$, is clear from the definition. To verify item (iii), that $i_A(J_X) \subset M(A \rtimes_{\alpha} G; J_{X \rtimes_{\gamma} G})$, fix $a \in J_X$. First, [6, Lemma 2.6(a)] says that J_X is α -invariant, and we know from [6, Proposition 2.7] that $J_X \rtimes_{\alpha} G \subset J_{X \rtimes_{\gamma} G}$. For any $f \in C_c(G, A)$ and $s \in G$ we have

$$(i_A(a)f)(s) = af(s) \in J_X$$

so $i_A(a)f \in C_c(G, J_X)$. Hence we get

$$i_A(J_X)(A \rtimes_{\alpha} G) \subset J_X \rtimes_{\alpha} G \subset J_X \rtimes_{\gamma} G$$

as required. $\square$

Proposition 4.2. With i_X as in Proposition 4.1, let

$$\mathcal{O}_{i_X, i_A} : \mathcal{O}_X \rightarrow M(\mathcal{O}_{X \rtimes_{\gamma} G})$$

be the nondegenerate homomorphism vouchsafed by Corollary 3.6. Also define a strictly continuous unitary homomorphism $u : G \rightarrow M(\mathcal{O}_{X \rtimes_{\gamma} G})$ by

$$u = \overline{k_{A \rtimes_{\alpha} G}} \circ i_G.$$

Then the pair $(\mathcal{O}_{i_X, i_A}, u)$ defines a covariant homomorphism of the C^* -dynamical system $(\mathcal{O}_X, G, \beta)$ in the Cuntz–Pimsner algebra $\mathcal{O}_{X \rtimes_{\gamma} G}$.

Proof. We only need to verify the covariance condition, namely that for each $t \in G$ we have

$$\text{Ad } u(t) \circ \mathcal{O}_{i_X, i_A} = \mathcal{O}_{i_X, i_A} \circ \beta_t,$$

and it is enough to check this on the generators from X and A . For $x \in X$ we have

$$\begin{aligned} u(t)\mathcal{O}_{i_X, i_A} \circ k_X(x) &= \overline{k_{A \rtimes_{\alpha} G}}(i_G(t))\overline{k_{X \rtimes_{\gamma} G}}(i_X(x)) \\ &= \overline{k_{X \rtimes_{\gamma} G}}(i_G(t) \cdot i_X(x)) \\ &= \overline{k_{X \rtimes_{\gamma} G}}(i_X(\gamma_t(x)) \cdot i_G(t)) \\ &= \overline{k_{X \rtimes_{\gamma} G}}(i_X(\gamma_t(x)))\overline{k_{A \rtimes_{\alpha} G}}(i_G(t)) \\ &= \mathcal{O}_{i_X, i_A} \circ k_X \circ \gamma_t(x)u(t) \\ &= \mathcal{O}_{i_X, i_A} \circ \beta_t \circ k_X(x)u(t), \end{aligned}$$

and the calculation for generators from A is similar. $\square$

Proposition 4.3. The integrated form

$$\mathcal{O}_{i_X, i_A} \times u : \mathcal{O}_X \rtimes_{\beta} G \rightarrow \mathcal{O}_{X \rtimes_{\gamma} G}$$

of the covariant pair $(\mathcal{O}_{i_X, i_A}, u)$ is surjective.

Proof. Let $f \in C_c(G, A)$. Then $k_A \circ f \in C_c(G, \mathcal{O}_X)$, and since (i_A, i_G) is a covariant pair we have

$$\begin{aligned} (\mathcal{O}_{i_X, i_A} \times u)(k_A \circ f) &= \int_G \mathcal{O}_{i_X, i_A}(k_A(f(t)))u(t) dt \\ &= \overline{k_{A \rtimes_{\alpha} G}} \left(\int_G i_A(f(t))i_G(t) dt \right) \\ &= k_{A \rtimes_{\alpha} G}((i_A \times i_G)(f)) \\ &= k_{A \rtimes_{\alpha} G}(f). \end{aligned}$$

Therefore the image of $\mathcal{O}_{i_X, i_A} \times u$ contains $k_{A \rtimes_{\alpha} G}(A \rtimes_{\alpha} G)$.

Now let $\xi \in C_c(G, X)$. A similar calculation shows that

$$(\mathcal{O}_{i_X, i_A} \times u)(k_X \circ \xi) = \overline{k_{X \rtimes_{\gamma} G}} \left(\int_G i_X(\xi(t)) \cdot i_G(t) dt \right).$$

Now, for any $f \in C_c(G, A)$, $s \in G$ we can calculate

$$\begin{aligned} \left(\int_G i_X(\xi(t)) \cdot i_G(t) dt f \right)(s) &= \left(\int_G i_X(\xi(t)) \cdot i_G(t) f dt \right)(s) \\ &= \int_G (i_X(\xi(t)) \cdot i_G(t) f)(s) dt \\ &= \int_G \xi(t) \cdot (i_G(t) f)(s) dt \\ &= \int_G \xi(t) \cdot \alpha_t(f(t^{-1}s)) dt \\ &= (\xi \cdot f)(s), \end{aligned}$$

and so we have

$$\int_G i_A(\xi(t)) \cdot i_G(t) dt = \xi.$$

Thus the image of $\mathcal{O}_{i_X, i_A} \times u$ also contains $k_{X \rtimes_\gamma G}(X \rtimes_\gamma G)$, and it now follows that $\mathcal{O}_{i_X, i_A}$ is surjective. $\square$

Remark 4.4. Ideally we would like the map in Proposition 4.3 to be injective, which would give $\mathcal{O}_{X \rtimes_\beta G} \cong \mathcal{O}_{X \rtimes_\gamma G}$ for an arbitrary locally compact group G . In a forthcoming paper [10] we will use our techniques to prove this when the group G is amenable, thereby recovering the result previously obtained by Hao and Ng [6, Theorem 2.10] using the theory of correspondence coactions.

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