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Lifespan of classical discontinuous solutions to general quasilinear hyperbolic systems of conservation laws with small BV initial data: Rarefaction waves ☆

Zhi-Qiang Shao

Department of Mathematics, Fuzhou University, Fuzhou 350002, China

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ABSTRACT

In the present paper the author investigates the global structure stability of Riemann solutions for general quasilinear hyperbolic systems of conservation laws under small BV perturbations of the initial data, where the Riemann solution contains rarefaction waves, while the perturbations are in BV but they are assumed to be C^1-smooth, with bounded and possibly large C^1-norms. Combining the techniques employed by Li–Kong with the modified Glimm's functional, the author obtains a lower bound of the lifespan of the piecewise C^1 solution to a class of generalized Riemann problems, which can be regarded as a small BV perturbation of the corresponding Riemann problem. This result is also applied to the system of traffic flow on a road network using the Aw–Rascle model.

1. Introduction and the main result

Consider the following quasilinear hyperbolic system of conservation laws:

$$\partial_t u + \partial_x f(u) = 0, \quad x \in \mathbf{R},\ t > 0, \tag{1.1}$$

where $u = (u_1, \dots, u_n)^T$ is the unknown vector-valued function of (t, x), $f : \mathbf{R}^n \to \mathbf{R}^n$ is a given C^3 vector function of u.

It is assumed that system (1.1) is strictly hyperbolic, i.e., for any given u on the domain under consideration, the Jacobian $A(u) = \nabla f(u)$ has n real distinct eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u). \tag{1.2}$$

Let $l_i(u) = (l_{i1}(u), \dots, l_{in}(u))$ (resp. $r_i(u) = (r_{i1}(u), \dots, r_{in}(u))^T$) be a left (resp. right) eigenvector corresponding to $\lambda_i(u)$ $(i = 1, \dots, n)$:

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \quad (\text{resp.}\, A(u)r_i(u) = \lambda_i(u)r_i(u)).$$

We have

$$\det|l_{ij}(u)| \neq 0 \quad (\text{equivalently},\ \det|r_{ij}(u)| \neq 0).$$

Without loss of generality, we may assume that on the domain under consideration

$$l_i(u)r_j(u) \equiv \delta_{ij} \quad (i, j = 1, \dots, n) \tag{1.3}$$

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E-mail address: zqshao_fzu@yahoo.com.cn.

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and

$$r_i^T(u)r_i(u) \equiv 1 \quad (i = 1, \dots, n),$$

where δ_{ij} stands for Kronecker's symbol.

Clearly, all $\lambda_i(u)$, $l_{ij}(u)$ and $r_{ij}(u)$ $(i, j = 1, \dots, n)$ have the same regularity as $A(u)$, i.e., C^2 regularity.

We also assume that on the domain under consideration, each characteristic field is either genuinely nonlinear in the sense of Lax (cf. [14]):

$$\nabla\lambda_i(u)r_i(u) \neq 0 \tag{1.4}$$

or linearly degenerate in the sense of Lax:

$$\nabla\lambda_i(u)r_i(u) \equiv 0. \tag{1.5}$$

We are interested in the generalized Riemann problem for the system (1.1) with the following piecewise C^1 initial data:

$$t = 0 : u = \begin{cases} \widehat{u}_- + \varepsilon u_-(x), & x \leq 0, \\ \widehat{u}_+ + \varepsilon u_+(x), & x \geq 0, \end{cases} \tag{1.6}$$

where $\widehat{u}_\pm$ are two constant vectors satisfying

$$\widehat{u}_- \neq \widehat{u}_+,$$

while ε $(0 < \varepsilon \ll |\ \widehat{u}_+ - \widehat{u}_-\ |)$ is a small parameter, $u_-(x)$ and $u_+(x)$ are C^1 vector functions defined on $x \leq 0$ and $x \geq 0$ respectively, which satisfy

$$\|u_-(x)\|_{C^1}, \qquad \|u_+(x)\|_{C^1} \leq K_1 \tag{1.7}$$

and

$$\int_0^{+\infty} |u'_+(x)|dx, \qquad \int_{-\infty}^0 |u'_-(x)|dx \leq K_2, \tag{1.8}$$

where K_1 and K_2 are positive constants independent of ε.

Problem (1.1) and (1.6) can be regarded as a small BV perturbation of the corresponding Riemann problem (1.1) and

$$t = 0 : u = \begin{cases} \widehat{u}_-, & x \leq 0, \\ \widehat{u}_+, & x \geq 0. \end{cases} \tag{1.9}$$

Let

$$\theta = |\widehat{u}_- - \widehat{u}_+|.$$

When $\theta > 0$ is suitably small, by Lax [14], the Riemann problem (1.1) and (1.9) admits a unique self-similar solution $u = U(\frac{x}{t})$ composed of $n+1$ constant states $\widehat{u}^{(0)} = \widehat{u}_-, \widehat{u}^{(1)}, \dots, \widehat{u}^{(n-1)}, \widehat{u}^{(n)} = \widehat{u}_+$ separated by shocks, centered rarefaction waves (corresponding characteristics are genuinely nonlinear) or contact discontinuities (corresponding characteristics are linearly degenerate). As in Kong [11], this kind of solution is simply called Lax's Riemann solution of the system (1.1).

For the self-similar solution of the Riemann problem of general quasilinear hyperbolic systems of conservation laws, the local nonlinear structure stability has been proved by Li and Yu [16] for the one-dimensional case, and by Majda [18] for the multidimensional case. If system (1.1) is strictly hyperbolic and linearly degenerate, Li and Kong [15] proved the global structure stability of the self-similar solution with small amplitude under perturbation (1.6) satisfying certain reasonable hypotheses. In this case the self-similar solution contains only n contact discontinuities. If system (1.1) is strictly hyperbolic and genuinely nonlinear, Li and Zhao [17] proved the global structure stability of the self-similar solution containing only n shocks under perturbation (1.6) satisfying certain reasonable hypotheses. In their work they do not require that the amplitude of the self-similar solution is small, although the existence of the self-similar solution with non small amplitude still remains open. Since many physical systems (for example, the one-dimensional compressible Euler equations of gas dynamics, the system of traffic flow on a road network using the Aw–Rascle model, etc.) do not belong to these two cases, a general consideration is needed for general hyperbolic systems of conservation laws whose characteristic families might be either genuinely nonlinear or linearly degenerate. Recently, for the case that the perturbation (1.6) satisfying the following decay property: there exists a constant $\mu > 0$ such that

$$\varrho \overset{\triangle}{=} \sup_{x\leq 0}\{(1+|x|)^{1+\mu}(|\varepsilon u_-(x)| + |\varepsilon u'_-(x)|)\} + \sup_{x\geq 0}\{(1+|x|)^{1+\mu}(|\varepsilon u_+(x)| + |\varepsilon u'_+(x)|)\} < +\infty$$

is small enough, Kong [11,12] proved that Lax's Riemann solution of general $n \times n$ quasilinear hyperbolic system of conservation laws is globally structurally stable if and only if it contains only non-degenerate shocks and contact discontinuities, but no rarefaction waves and other weak discontinuities.

However, it is well known that the BV space is a suitable framework for the one-dimensional Cauchy problem for the hyperbolic systems of conservation laws (see Bressan [2], Glimm [9]), the result in Bressan [3] suggests that one may achieve global smoothness even if the C^1 norm of the initial data is large. So the following question arises naturally: can we obtain the global structure stability of Riemann solutions for general quasilinear hyperbolic systems of conservation laws under small BV perturbations of the initial data (the perturbations are in BV but they are assumed to be C^1-smooth, with bounded and possibly large C^1-norms), where the Riemann solution contains rarefaction waves. Here, it is important to mention that the global existence of weak solutions to a strictly hyperbolic system of conservation laws in one space dimension when the initial data is a small BV perturbation of a solvable Riemann problem has been proved by Schochet [21], unfortunately his method is not useful to show that the solutions are still either contact discontinuities or shocks or rarefaction waves. An analogous result on stability of a strong shock wave under perturbations of small bounded variation is stated by Corli and Sable-Tougeron [7]. In this paper we exploit to some extent the ideas of Bressan [3], and we will develop the method of using continuous Glimm's functional to provide a new, nontrivial proof of an estimate on the lifespan of the piecewise C^1 solution to the generalized Riemann problem under consideration mentioned above. The basic idea we will use here is to combine the techniques employed by Li–Kong [15], especially both the decomposition of waves and the global behavior of waves on the discontinuity curves, with the method of using continuous Glimm's functional. However, we must modify Glimm's functional in order to take care of the presence of rarefaction waves. This makes our new analysis more complicated than those for the C^1 solutions of the Cauchy problem for linearly degenerate quasilinear hyperbolic systems in Bressan [3], Dai and Kong [8] and Zhou [24].

As in [22], the aim of this paper is to study the global structure stability of Lax's Riemann solution containing shocks, contact discontinuities, particularly centered rarefaction waves. In this case, we shall first get a lower bound of the lifespan of the piecewise C^1 solution containing at least a rarefaction wave to the generalized Riemann problem. To do so, we introduce

$$J_R \overset{\triangle}{=} \left\{ j \mid j \in \{1, \dots, n\},\ j\text{-wave in } u = U\left(\frac{x}{t}\right) \text{ is a centered rarefaction wave} \right\},$$

$$J_S \overset{\triangle}{=} \left\{ j \mid j \in \{1, \dots, n\},\ j\text{-wave in } u = U\left(\frac{x}{t}\right) \text{ is a shock wave} \right\},$$

$$J \overset{\triangle}{=} \{j | j \in \{1, \dots, n\},\ \lambda_j(u) \text{is genuinely nonlinear}\}$$

and

$$I \overset{\triangle}{=} \{i | i \in \{1, \dots, n\},\ \lambda_i(u) \text{ is linearly degenerate}\}.$$

Then, the assumption that each characteristic field is either genuinely nonlinear or linearly degenerate gives

$$I \cup J = \{1, \dots, n\}.$$

To state our result precisely, we now introduce the concept of the lifespan of the piecewise C^1 solution to the generalized Riemann problem (1.1) and (1.6) as follows.

Definition 1.1. The existence of piecewise C^1 local solutions to the generalized Riemann problem (1.1) and (1.6) is guaranteed by the monograph Li–Yu [16]. The life span is defined to be the supremum of the time T such that a Li–Yu solution exists for $0 < t \le T$. This definition will coincide with the usual definition of the life span of the C^1 solution (without shocks).

Our main results can be summarized as follows.

Theorem 1.1. *Suppose that system (1.1) is strictly hyperbolic and each characteristic field is either genuinely nonlinear or linearly degenerate. Suppose furthermore that $u_-(x)$ and $u_+(x)$ are all C^1 vector functions on $x \le 0$ and on $x \ge 0$ respectively satisfying (1.7) and (1.8) as well as*

$$u_-(0) = u_+(0) = 0, \tag{1.10}$$

and

$$\theta = |\widehat{u}_+ - \widehat{u}_-| = |u_0^+(0) - u_0^-(0)| > 0$$

is suitably small. Suppose finally that the self-similar solution $u = U(\frac{x}{t})$ of the Riemann problem (1.1) and (1.9) contains at least either centered rarefaction waves or shocks, i.e., $J_R \cup J_S \neq \emptyset$. Then for small $\theta > 0$, there exists a constant $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$, the lifespan $\widetilde{T}(\varepsilon)$ of the piecewise C^1 solution to the generalized Riemann problem (1.1) and (1.6) satisfies

$$\widetilde{T}(\varepsilon) \ge K_3 \varepsilon^{-1},$$

where K_3 is a positive constant independent of ε. Moreover, when $u = u(t, x)$ blows up in a finite time, $u = u(t, x)$ itself is bounded on the domain $[0, \widetilde{T}(\varepsilon)) \times \mathbf{R}$, while the first-order derivatives of $u = u(t, x)$ tend to be unbounded as $t \nearrow \widetilde{T}(\varepsilon)$.

Remark 1.1. In Theorem 1.1, we say the solution blows up, if the piecewise C^1 solution in the sense of Li–Yu $u = u(t, x)$ to the generalized Riemann problem (1.1) and (1.6) ceases to exist for some $x \in \mathbf{R}$, $t > 0$. After this blow-up time, new waves (particularly, new shocks) will appear (cf. [6,13]).

Remark 1.2. Suppose that (1.1) is a non-strictly hyperbolic system with characteristics with constant multiplicity, say, on the domain under consideration,

$$\lambda_1(u) \equiv \cdots \equiv \lambda_p(u) < \lambda_{p+1}(u) < \cdots < \lambda_n(u) \quad (1 \le p \le n).$$

Then the conclusion of Theorem 1.1 still holds (cf. [8]).

Remark 1.3. Our result implies that classical discontinuous solutions to the generalized Riemann problem under consideration exists almost globally in time, i.e., we prove almost global well-posedness of the generalized Riemann problem for hyperbolic systems of conservation laws.

The rest of this paper is organized as follows. For the sake of completeness, in Section 2, we briefly recall John's formula on the decomposition of waves with some supplements and give a generalized Hörmander Lemma. In Section 3, we first review the definitions of shock, contact discontinuity and centered rarefaction wave, and then analyze some properties of waves on discontinuous curves, which will play an important role in our proof. The main result, Theorem 1.1 is proved in Section 4. Some applications with physical interest will be given in Section 5.

2. John's formula, generalized Hörmander Lemma

For the sake of completeness, in this section we briefly recall John's formula on the decomposition of waves with some supplements, which will play an important role in our proof.

Let

$$v_i = l_i(u)u \quad (i = 1, \dots, n) \tag{2.1}$$

and

$$w_i = l_i(u)u_x \quad (i = 1, \dots, n), \tag{2.2}$$

where $l_i(u) = (l_{i1}(u), \dots, l_{in}(u))$ denotes the ith left eigenvector.

By (1.3), it is easy to see that

$$u = \sum_{k=1}^{n} v_k r_k(u) \tag{2.3}$$

and

$$u_x = \sum_{k=1}^{n} w_k r_k(u). \tag{2.4}$$

Let

$$\frac{d}{d_i t} = \frac{\partial}{\partial t} + \lambda_i(u)\frac{\partial}{\partial x} \tag{2.5}$$

be the directional derivative along the ith characteristic. We have (cf. [10,11,15])

$$\frac{dv_i}{d_i t} = \sum_{j,k=1}^{n} \beta_{ijk}(u) v_j w_k \quad (i = 1, \dots, n), \tag{2.6}$$

where

$$\beta_{ijk}(u) = (\lambda_k(u) - \lambda_i(u)) l_i(u) \nabla r_j(u) r_k(u). \tag{2.7}$$

Hence, we have

$$\beta_{iji}(u) \equiv 0, \quad \forall i, j. \tag{2.8}$$

On the other hand, we have (cf. [10,11,15])

$$\frac{dw_i}{d_i t} = \sum_{j,k=1}^{n} \gamma_{ijk}(u) w_j w_k \quad (i = 1, \dots, n), \tag{2.9}$$

where

$$\gamma_{ijk}(u) = \frac{1}{2}\{(\lambda_j(u) - \lambda_k(u)) l_i(u) \nabla r_k(u) r_j(u) - \nabla\lambda_i(u) r_j(u) \delta_{ik} + (j|k)\}, \tag{2.10}$$

in which $(j|k)$ denotes all the terms obtained by changing j and k in the previous terms. We have

$$\gamma_{ijj}(u) \equiv 0, \quad \forall j \neq i \ (i, j = 1, \ldots, n) \tag{2.11}$$

and

$$\gamma_{iii}(u) \equiv -\nabla\lambda_i(u) r_i(u) \quad (i = 1, \ldots, n). \tag{2.12}$$

Noting (2.4), by (2.9) we have (cf. [8])

$$\frac{\partial w_i}{\partial t} + \frac{\partial(\lambda_i(u)w_i)}{\partial x} = \sum_{j,k=1}^{n} \Gamma_{ijk}(u) w_j w_k \stackrel{def}{=} G_i(t, x), \tag{2.13}$$

equivalently,

$$d[w_i(dx - \lambda_i(u)dt)] = \sum_{j,k=1}^{n} \Gamma_{ijk}(u) w_j w_k dt \wedge dx = G_i(t, x) dt \wedge dx, \tag{2.14}$$

where

$$\Gamma_{ijk}(u) = \frac{1}{2}(\lambda_j(u) - \lambda_k(u)) l_i(u) [\nabla r_k(u) r_j(u) - \nabla r_j(u) r_k(u)]. \tag{2.15}$$

Hence, we have

$$\Gamma_{ijj}(u) \equiv 0, \quad \forall i, j. \tag{2.16}$$

Lemma 2.1 (*Generalized Hörmander Lemma*). *Suppose that $u = u(t, x)$ is a piecewise C^1 solution to system (1.1), τ_1 and τ_2 are two C^1 arcs which are never tangent to the ith characteristic direction, and $\mathcal{D}$ is the domain bounded by τ_1, τ_2 and two ith characteristic curves L_i^- and L_i^+. Suppose furthermore that the domain $\mathcal{D}$ contains mC^1 curves of discontinuity of u, denoted by $\widehat{C}_j : x = x_j(t)$ $(j = 1, \ldots, m)$, which are never tangent to the ith characteristic direction. Then we have*

$$\begin{aligned}\int_{\tau_1} |w_i(dx - \lambda_i(u)dt)| \le \int_{\tau_2} |w_i(dx - \lambda_i(u)dt)| + \sum_{j=1}^{m} \int_{\widehat{C}_j} |[w_i]dx - [w_i\lambda_i(u)]dt| \\ + \int\int_{\mathcal{D}} \left| \sum_{j,k=1}^{n} \Gamma_{ijk}(u) w_j w_k \right| dt dx,\end{aligned} \tag{2.17}$$

where $\Gamma_{ijk}(u)$ is given by (2.15) and $[w_i] = w_i^+ - w_i^-$ denotes the jump of w_i over the curve of discontinuity $\widehat{C}_j$ $(j = 1, \ldots, m)$, etc.

The proof can be found in Li and Kong [15].

3. Shock, contact discontinuity and centered rarefaction wave

In this section, we first review the definitions of shock, contact discontinuity and centered rarefaction wave, and then analyze some properties of waves on the discontinuous curves, which will play an important role in our proof.

Definition 3.1. A piecewise C^1 vector function $u = u(t, x)$ is called a piecewise C^1 solution containing a kth shock $x = x_k(t)(x_k(0) = 0)$ for system (1.1), if $u = u(t, x)$ satisfies system (1.1) away from $x = x_k(t)$ in the classical sense and satisfies on $x = x_k(t)$ the following Rankine–Hugoniot condition:

$$f(u^+) - f(u^-) = s(u^+ - u^-) \tag{3.1}$$

and the Lax entropy condition:

$$\lambda_k(u^+) < s < \lambda_k(u^-), \quad \lambda_{k+1}(u^+) > s > \lambda_{k-1}(u^-), \tag{3.2}$$

where $u^{\pm} = u^{\pm}(t, x_k(t)) \stackrel{\triangle}{=} u(t, x_k(t) \pm 0)$ and $s = \frac{dx_k(t)}{dt}$ (when $k = 1$ (resp. $k = n$), the term $\lambda_{k-1}(u^-)$ (resp. $\lambda_{k+1}(u^+)$) disappears in (3.2)).

Definition 3.2. A piecewise C^1 vector function $u = u(t, x)$ is called a piecewise C^1 solution containing a kth contact discontinuity $x = x_k(t)$ $(x_k(0) = 0)$ for system (1.1), if $u = u(t, x)$ satisfies system (1.1) away from $x = x_k(t)$ in the classical sense and satisfies on $x = x_k(t)$ the Rankine–Hugoniot condition (3.1) and

$$s = \lambda_k(u^+) = \lambda_k(u^-), \tag{3.3}$$

where $u^{\pm} = u^{\pm}(t, x_k(t)) \stackrel{\triangle}{=} u(t, x_k(t) \pm 0)$ and $s = \frac{dx_k(t)}{dt}$.

Definition 3.3. Let

$$\nabla = \{(t, x) \mid t \geq 0,\ \xi_L t \leq x \leq \xi_R t\}$$

be an angular domain, where $\xi_L,\ \xi_R$ are two constants with $\xi_L < \xi_R$. If $u_0(\xi)$ is a C^1 function of $\xi \in [\xi_L, \xi_R]$ with the following properties:

$$\lambda_k(u_0(\xi)) = \xi \quad \text{and} \quad \frac{du_0(\xi)}{d\xi} // r_k(u_0(\xi)),$$

then $u = u_0(\frac{x}{t})$ defined on ∇ is called a kth standard centered rarefaction wave with the center point $(0, 0)$.

Definition 3.4. Let

$$\widetilde{\nabla} = \{(t, x) \mid t \geq 0,\ x_L(t) \leq x \leq x_R(t)\}$$

be an angular domain, where $x_L(t),\ x_R(t)$ are two C^1 functions of t with the following properties:

$$x_L(0) = x_R(0) = 0 \quad \text{and} \quad \xi_L \triangleq \frac{dx_L}{dt}(0) < \frac{dx_R}{dt}(0) \triangleq \xi_R.$$

A vector function $u = u(t, x)$ defined on $\widetilde{\nabla}$ is called a kth centered rarefaction wave for system (1.1) with the center point $(0, 0)$, if the following conditions are satisfied:

(i) let $\xi = \frac{x}{t}$ and

$$v(t, \xi) = \begin{cases} u(t, t\xi), & \text{in } t > 0, \\ \lim\limits_{\tau \to 0+} u(\tau, \tau\xi), & \text{on } t = 0, \end{cases}$$

we have

$$v(t, \xi) \in C^1[\widetilde{\widetilde{\nabla}}] \quad \text{and} \quad \frac{\partial v}{\partial \xi}(0, \xi) \neq 0, \quad \forall \xi \in [\xi_L, \xi_R],$$

where

$$\widetilde{\widetilde{\nabla}} = \left\{ (t, \xi) \middle| t \geq 0, \begin{array}{ll} \frac{x_L(t)}{t} \leq \xi \leq \frac{x_R(t)}{t}, & \text{in } t > 0 \\ \xi_L \leq \xi \leq \xi_R, & \text{on } t = 0 \end{array} \right\};$$

(ii) $u(t, x)$, i.e., $v(t, \frac{x}{t})$ satisfies system (1.1) on $\widetilde{\nabla} \setminus \{(0, 0)\}$ in the classical sense;

(iii) both boundaries $x = x_H(t) (H = L, R)$ of $\widetilde{\nabla}$ are the kth characteristic curves passing through $(0, 0)$, i.e.,

$$\frac{dx_H(t)}{dt} = \lambda_k(u(t, x_H(t)))(H = L, R), \quad \forall t > 0. \tag{3.4}$$

A continuous vector function $u = u(t, x)$ defined on $\mathbf{R}^+ \times \mathbf{R} \setminus \{(0, 0)\}$ is called a piecewise C^1 solution with a kth centered rarefaction wave on $\widetilde{\nabla}$ for system (1.1), if $u = u(t, x)$ is a kth centered rarefaction wave on $\widetilde{\nabla}$ and satisfies system (1.1) out of $\widetilde{\nabla}$ in the classical sense.

Definitions 3.1–3.4 can be found in [14,16].

Definition 3.5. We call the piecewise C^1 solution containing a finite number of shocks, contact discontinuities or centered rarefaction waves as a classical discontinuous solution.

The following lemmas give some properties of waves on shock, contact discontinuity or centered rarefaction wave.

Lemma 3.1. *Let $u = u(t, x)$ be a piecewise C^1 solution with a kth centered rarefaction wave on $\widetilde{\nabla}$ for system (1.1). Then on $x = x_H(t)(H = L, R)$ it holds that*

$$v_i^+ = v_i^- \quad (i = 1, \ldots, n) \tag{3.5}$$

and

$$w_i^+ = w_i^- \quad (i = 1, \ldots, k-1, k+1, \ldots, n), \tag{3.6}$$

provided that $|u^\pm|$ is suitably small, where v_i, w_i are defined by (2.1) and (2.2), respectively, $v_i^\pm = v_i^\pm(t, x_H(t)) \triangleq v_i(t, x_H(t) \pm 0)$, etc.

Lemma 3.2. *On the kth shock or contact discontinuity $x = x_k(t)$, it holds that*

$$v_i^+ = v_i^- + O(|v^\pm|^2) \quad (i = 1, \ldots, k-1, k+1, \ldots, n), \tag{3.7}$$

provided that $|u^\pm|$ is suitably small, where v_i is defined by (2.1) and $v_i^\pm \triangleq v_i(t, x_k(t) \pm 0)$.

Lemma 3.3. *On the kth contact discontinuity $x = x_k(t)$, it holds that*

$$w_i^- = w_i^+ + O\left(|u^+ - u^-| \cdot \sum_{j\neq k} |w_j^\pm|\right) \quad (i = 1, \dots, k-1, k+1, \dots, n), \tag{3.8}$$

provided that $|u^\pm|$ is suitably small, where w_i are defined by (2.2) and $w_i^\pm \triangleq w_i(t, x_k(t) \pm 0)$.

Lemma 3.4. *On the kth shock $x = x_k(t)$, it holds that*

$$\begin{aligned} w_i^- &= w_i^+ + O\left(|u^+ - u^-| \cdot \sum_{j\neq k} |w_j^\pm|\right) + O\Big(|u^+ - u^-| \cdot |(\lambda_k(u^-, u^+) - \lambda_k(u^+))w_k^+|\Big) \\ &\quad + O\Big(|u^+ - u^-| \cdot |(\lambda_k(u^-, u^+) - \lambda_k(u^-))w_k^-|\Big) \quad (i = 1, \dots, k-1, k+1, \dots, n), \end{aligned} \tag{3.9}$$

provided that $|u^\pm|$ is suitably small, where $\lambda_k(u^-, u^+)$ is the kth eigenvalue of the matrix

$$A(u^-, u^+) \triangleq \int_0^1 \nabla f(u^- + \varsigma(u^+ - u^-))d\varsigma.$$

Remark 3.1. By (1.2), if $|u^+ - u^-|$ is suitably small, then the matrix $A(u^-, u^+)$ has n distinct real eigenvalues:

$$\lambda_1(u^-, u^+) < \lambda_2(u^-, u^+) < \cdots < \lambda_n(u^-, u^+).$$

The proofs of Lemmas 3.1–3.4 can be found in Kong [11,12].

Corollary 3.1. *On the kth contact discontinuity $x = x_k(t)$, it holds that*

$$(w_i\lambda_i(u))^+ = (w_i\lambda_i(u))^- + O\left(|u^+ - u^-| \cdot \sum_{j\neq k} |w_j^\pm|\right) \quad (i = 1, \dots, k-1, k+1, \dots, n), \tag{3.10}$$

provided that $|u^\pm|$ is small.

Proof. Noting

$$(w_i\lambda_i(u))^+ - (w_i\lambda_i(u))^- = [w_i^+ - w_i^-](\lambda_i(u))^+ + w_i^-[(\lambda_i(u))^+ - (\lambda_i(u))^-],$$

from (3.8), we immediately get (3.10). □

4. Proof of Theorem 1.1

For the sake of simplicity and without loss of generality, we may suppose that

$$0 < \lambda_1(0) < \lambda_2(0) < \cdots < \lambda_n(0) \tag{4.1}$$

and

$$|\widehat{u}_\pm| \leq \theta. \tag{4.2}$$

By the existence and uniqueness of local classical discontinuous solutions of quasilinear hyperbolic systems of conservation laws (see [16]), when $\theta > 0$ is suitably small, the generalized Riemann problem (1.1) and (1.6) admits a unique piecewise C^1 solution $u = u(t, x)$ containing only n shocks, centered rarefaction waves (corresponding characteristics are genuinely nonlinear) or contact discontinuities (corresponding characteristics are linearly degenerate) $x = x_k(t)$ $(x_k(0) = 0)$ $(k = 1, \dots, n)$ on the strip $[0, h] \times \mathbf{R}$, where $h > 0$ is a small number; moreover, this solution has a local structure similar to the one of the self-similar solution to the corresponding Riemann problem. In order to prove Theorem 1.1, we first establish some uniform a priori estimates on u and u_x on the domain of existence of the piecewise C^1 solution $u = u(t, x)$.

By (4.1), there exist sufficiently small positive constants δ and δ_0 such that

$$\lambda_{i+1}(u) - \lambda_i(v) \geq \delta_0, \quad \forall |u|, |v| \leq \delta \; (i = 1, \dots, n-1). \tag{4.3}$$

For the time being it is supposed that on the domain of existence of the piecewise C^1 solution $u = u(t, x)$ to the generalized Riemann problem (1.1) and (1.6), we have

$$|u(t, x)| \leq \delta. \tag{4.4}$$

At the end of the proof of Lemma 4.3, we will explain that this hypothesis is reasonable.

For any fixed $T > 0$, let

$$U_\infty(T) = \sup_{0\le t\le T}\sup_{x\in\mathbf{R}} |u(t,x)|, \tag{4.5}$$

$$V_\infty(T) = \sup_{0\le t\le T}\sup_{x\in\mathbf{R}} |v(t,x)|, \tag{4.6}$$

$$W_\infty(T) = \sup_{0\le t\le T}\sup_{x\in\mathbf{R}} |w(t,x)|, \tag{4.7}$$

$$\widetilde{W}_1(T) = \max_{i=1,\dots,n}\max_{j\neq i}\sup_{\widetilde{C}_j}\int_{\widetilde{C}_j} |w_i(t,x)|dt, \tag{4.8}$$

$$W_1(T) = \max_{j\in J_S}\int_0^T |(x_j'(t) - \lambda_j(u(t,x_j(t)\pm 0)))w_j(t,x_j(t)\pm 0)|dt, \tag{4.9}$$

where $|\cdot|$ stands for the Euclidean norm in $\mathbf{R}^n$, $v = (v_1,\dots,v_n)^T$ and $w = (w_1,\dots,w_n)^T$ in which v_i and w_i are defined by (2.1) and (2.2) respectively, while $\widetilde{C}_j$ stands for any given jth characteristic on the domain $[0,T]\times\mathbf{R}$. In (4.4)–(4.7), on any contact discontinuity or shock $x = x_k(t)$ the values of $u(t,x)$, $v(t,x)$ and $w(t,x)$ are taken to be $u^\pm(t,x) = u(t,x_k(t)\pm 0)$, $v^\pm(t,x) = v(t,x_k(t)\pm 0)$ and $w^\pm(t,x) = w(t,x_k(t)\pm 0)$. Clearly, $V_\infty(T)$ is equivalent to $U_\infty(T)$.

In the present situation, similar to some basic L^1 estimates that are essentially due to Schatzman [19,20] and Zhou [24], we have the following careful L^1 estimates of the piecewise C^1 solution.

Lemma 4.1. *Under the assumptions of Theorem 1.1, on any given domain of existence $[0,T]\times\mathbf{R}$ of the piecewise C^1 solution $u = u(t,x)$ to the generalized Riemann problem (1.1) and (1.6), there exists a positive constant k_1 independent of θ, ε and T such that*

$$\int_{-\infty}^{+\infty} |w_i(t,x)|dx \le k_1\Big\{\varepsilon + W_1(T) + V_\infty(T)(\widetilde{W}_1(T) + W_1(T)) + \int_0^T\int_{-\infty}^{+\infty} |G_i(t,x)|dxdt\Big\} \quad (i = 1,\dots,n),\ \forall t\le T, \tag{4.10}$$

provided that the right-hand side of the inequality is bounded.

Proof. To estimate $\int_{-\infty}^{+\infty} |w_i(t,x)|dx$, we need only to estimate

$$\int_{-a}^{a} |w_i(t,x)|dx$$

for any given $a > 0$ and then let $a\to +\infty$.

For $i = 1,\dots,n$, for any given t with $0\le t\le T$, passing through point $A(t,a)$ $(a > x_n(t))$ (resp. $B(t,-a)$), we draw the ith backward characteristic which intersects the x-axis at a point $D(0,x_D)$ (resp. $C(0,x_C)$). In what follows, we assume that the ith wave (shock or contact discontinuity or rarefaction wave) passing through $O(0,0)$ is a centered rarefaction wave, while other cases can be dealt with in a manner similar to [23]. Let $E(t,x_E)$ (*resp.* $F(t,x_F)$) be the intersection point of the straight line $t = t$ with the left (resp. right) boundary $x = x_L(t)$ (*resp.* $x = x_R(t)$) of the rarefaction wave. Then, applying (2.17) on the domain $BCOE$ bounded by the left boundary of the rarefaction wave, the straight line $t = t$, the ith characteristic passing through B and the x-axis, we get

$$\int_B^E |w_i(t,x)|dx \le \int_{x_C}^0 |w_i(0,x)|dx + \sum_{k\in S_1}\int_{\widehat{C}_k} |([w_i]x_k'(t) - [w_i\lambda_i(u)])dt| + \int\int_{BCOE} |G_i|dxdt, \tag{4.11}$$

where S_1 stands for the set of all indices k such that the kth discontinuous curve (shock or contact discontinuity) $\widehat{C}_k : x = x_k(t)$ is partly contained in the region $BCOE$, and

$$x_k'(t) = \frac{dx_k(t)}{dt} = \begin{cases}\lambda_k(u^\pm), & \text{as } k\in I,\\ \lambda_k(u^-,u^+), & \text{as } k\in J_S.\end{cases} \tag{4.12}$$

Noting (3.6), in (4.11) we need not consider the case that the kth wave is a rarefaction wave. Using (1.8), (3.8)–(3.10), (4.4) and (4.12), and noting the fact that $i\notin S_1$, we have

$$\begin{aligned}\int_{-a}^{x_E} |w_i(t,x)|dx &\le \int_{-\infty}^0 |w_i(0,x)|dx + c_1V_\infty(T)(\widetilde{W}_1(T) + W_1(T)) + \int_0^T\int_{-\infty}^{+\infty} |G_i|dxdt\\ &\le c_2\Big\{\varepsilon + V_\infty(T)(\widetilde{W}_1(T) + W_1(T)) + \int_0^T\int_{-\infty}^{+\infty} |G_i|dxdt\Big\},\end{aligned} \tag{4.13}$$

where here and henceforth, c_i $(i = 1, 2, \ldots)$ will denote positive constants independent of θ, ε and T. Similarly, we have

$$\int_{x_F}^{a} |w_i(t,x)|dx \le c_3\left\{\varepsilon + V_\infty(T)(\widetilde{W}_1(T) + W_1(T)) + \int_0^T \int_{-\infty}^{+\infty} |G_i|dxdt\right\}. \tag{4.14}$$

$$\int_{x_E}^{x_F} |w_i(t,x)|dx \le \int_0^T \int_{-\infty}^{+\infty} |G_i|dxdt. \tag{4.15}$$

Combining (4.17) in [23] and (4.13)–(4.15) all together, we finally get

$$\int_{-a}^{a} |w_i(t,x)|dx \le c_4\left\{\varepsilon + W_1(T) + V_\infty(T)(\widetilde{W}_1(T) + W_1(T)) + \int_0^T \int_{-\infty}^{+\infty} |G_i|dxdt\right\}.$$

Letting $a \to +\infty$, we immediately get the assertion in (4.10). The proof of Lemma 4.1 is finished. □

Lemma 4.2. *Under the assumptions of Theorem 1.1, on any given domain of existence $[0, T] \times \mathbf{R}$ of the piecewise C^1 solution $u = u(t, x)$ to the generalized Riemann problem (1.1) and (1.6), there exists a positive constant k_2 independent of θ, ε and T such that*

$$\begin{aligned}\int_0^T \int_{-\infty}^{+\infty} |w_i(t,x)||w_j(t,x)|dxdt &\le k_2\left(\varepsilon + W_1(T) + V_\infty(T)(\widetilde{W}_1(T) + W_1(T)) + \int_0^T \int_{-\infty}^{+\infty} |G_i(t,x)|dxdt\right)\\ &\quad\times\left(\varepsilon + W_1(T) + V_\infty(T)(\widetilde{W}_1(T) + W_1(T)) + \int_0^T \int_{-\infty}^{+\infty} |G_j(t,x)|dxdt\right), \quad \forall i \ne j\ (i, j = 1, \ldots, n),\end{aligned} \tag{4.16}$$

provided that the right-hand side of the inequality is bounded.

Proof. The proof of Lemma 4.2 is essentially similar to Lemma 3.2 proved by Yi Zhou in [24]. The author points out that here the line of discontinuity (center rarefaction wave) is considered; and some differences will emerge. Such points are similar to the cases considered by the author in [23,25]. As in [24], to estimate

$$\int_0^T \int_{-\infty}^{+\infty} |w_i(t,x)||w_j(t,x)|dxdt,$$

it is enough to estimate

$$\int_0^T \int_{-L}^{L} |w_i(t,x)||w_j(t,x)|dxdt$$

for any given $L > 0$ and then let $L \to +\infty$.

For $i, j \in \{1, \ldots, n\}$ and $i \ne j$, without loss of generality, we suppose that $i < j$. Let $x = x_i(t, L)$ $(0 \le t \le T)$ be the ith forward characteristic passing through point $(0, L)$ $(L > x_n(T))$. Then, we draw the ith backward characteristic $x = s_i(t)$ $(0 \le t \le T)$ passing through point (T, a) $(a > x_i(T, L))$. In the meantime, passing through the point $(T, -L)$, we draw the jth backward characteristic $x = s_j(t)$ $(0 \le t \le T)$ which intersects the x-axis at a point.

We introduce the "continuous Glimm's functional" (cf. [3,4,24])

$$Q(t) = \int\int_{s_j(t)<x<y<s_i(t)} |w_j(t,x)||w_i(t,y)|dxdy.$$

Because of the piecewise C^1 solution $u = u(t, x)$ containing only n shocks, centered rarefaction waves or contact discontinuities $x = x_k(t)$ $(x_k(0) = 0)$ $(k = 1, \ldots, n)$, without loss of generality, we may suppose that $u = u(t, x)$ containing only a centered rarefaction wave, while other cases can be dealt with in a manner similar to [23]. Without loss of generality, we may suppose that the 1st wave is a centered rarefaction wave. Let $x = x_L(t)$(resp. $x = x_R(t)$) be the left (resp. right) boundary of the rarefaction wave, we divide the bounded domain $\widetilde{\Omega} \triangleq \{(x, y)|s_j(t) < x < y < s_i(t)\}$ by the straight lines $y = x_H(t)$ $(H = L, R)$ and $y = x_k(t)$ $(k = 2, \ldots, n)$ into some parts. Then, the straightforward calculations on all parts of the domain $\widetilde{\Omega}$ reveal that

$$\begin{aligned}\frac{dQ(t)}{dt} &= s_i'(t)|w_i(t, s_i(t))|\int_{s_j(t)}^{s_i(t)} |w_j(t,x)|dx - s_j'(t)|w_j(t, s_j(t))|\int_{s_j(t)}^{s_i(t)} |w_i(t,x)|dx\\ &\quad + x_L'(t)\{|w_i(t, x_L(t) - 0)| - |w_i(t, x_L(t) + 0)|\}\int_{s_j(t)}^{x_L(t)} |w_j(t,x)|dx + x_R'(t)\{|w_i(t, x_R(t) - 0)|\\ &\quad - |w_i(t, x_R(t) + 0)|\}\int_{s_j(t)}^{x_R(t)} |w_j(t,x)|dx + \sum_{k=2}^{n} x_k'(t)\{|w_i(t, x_k(t) - 0)|\end{aligned}$$

$$
\begin{aligned}
&- |w_i(t, x_k(t)+0)|\} \int_{s_j(t)}^{x_k(t)} |w_j(t,x)|dx + \int\int_{s_j(t)<x<y<s_i(t)} \frac{\partial}{\partial t}(|w_j(t,x)|)|w_i(t,y)|dxdy \\
&+ \int\int_{s_j(t)<x<y<s_i(t)} |w_j(t,x)|\frac{\partial}{\partial t}(|w_i(t,y)|)dxdy \\
=\ & s_i'(t)|w_i(t,s_i(t))| \int_{s_j(t)}^{s_i(t)} |w_j(t,x)|dx - s_j'(t)|w_j(t,s_j(t))| \int_{s_j(t)}^{s_i(t)} |w_i(t,x)|dx \\
&+ x_L'(t)\{|w_i(t,x_L(t)-0)| - |w_i(t,x_L(t)+0)|\} \int_{s_j(t)}^{x_L(t)} |w_j(t,x)|dx + x_R'(t) \\
&\times \{|w_i(t,x_R(t)-0)| - |w_i(t,x_R(t)+0)|\} \int_{s_j(t)}^{x_R(t)} |w_j(t,x)|dx + \sum_{k=2}^{n} x_k'(t)\{|w_i(t,x_k(t)-0)| \\
&- |w_i(t,x_k(t)+0)|\} \int_{s_j(t)}^{x_k(t)} |w_j(t,x)|dx - \int\int_{s_j(t)<x<y<s_i(t)} \frac{\partial}{\partial x}(\lambda_j(u)|w_j(t,x)|)|w_i(t,y)|dxdy \\
&- \int\int_{s_j(t)<x<y<s_i(t)} |w_j(t,x)|\frac{\partial}{\partial y}(\lambda_i(u)|w_i(t,y)|)dxdy + \int\int_{s_j(t)<x<y<s_i(t)} \mathrm{sgn}(w_j) \\
&\times G_j(t,x)|w_i(t,y)|dxdy + \int\int_{s_j(t)<x<y<s_i(t)} |w_j(t,x)|\mathrm{sgn}(w_i)G_i(t,y)dxdy \\
=\ & -\int_{s_j(t)}^{s_i(t)} (\lambda_j(u(t,x)) - \lambda_i(u(t,x)))|w_i(t,x)||w_j(t,x)|dx + (s_i'(t) - \lambda_i(u(t,s_i(t))))|w_i(t,s_i(t))| \\
&\times \int_{s_j(t)}^{s_i(t)} |w_j(t,x)|dx + (\lambda_j(u(t,s_j(t))) - s_j'(t))|w_j(t,s_j(t))| \\
&\times \int_{s_j(t)}^{s_i(t)} |w_i(t,x)|dx + x_L'(t)\{|w_i(t,x_L(t)-0)| - |w_i(t,x_L(t)+0)|\} \\
&\times \int_{s_j(t)}^{x_L(t)} |w_j(t,x)|dx + x_R'(t)\{|w_i(t,x_R(t)-0)| - |w_i(t,x_R(t)+0)|\} \\
&\times \int_{s_j(t)}^{x_R(t)} |w_j(t,x)|dx + \sum_{k=2}^{n} x_k'(t)\{|w_i(t,x_k(t)-0)| - |w_i(t,x_k(t)+0)|\} \int_{s_j(t)}^{x_k(t)} |w_j(t,x)|dx \\
&+ \{\lambda_i(u(t,x_L(t)))|w_i(t,x_L(t)+0)| - \lambda_i(u(t,x_L(t)))|w_i(t,x_L(t)-0)|\} \\
&\times \int_{s_j(t)}^{x_L(t)} |w_j(t,x)|dx + \{\lambda_i(u(t,x_R(t)))|w_i(t,x_R(t)+0)| - \lambda_i(u(t,x_R(t)))|w_i(t,x_R(t)-0)|\} \\
&\times \int_{s_j(t)}^{x_R(t)} |w_j(t,x)|dx + \sum_{k=2}^{n} \{\lambda_i(u(t,x_k(t)+0))|w_i(t,x_k(t)+0)| - \lambda_i(u(t,x_k(t)-0)) \\
&\times |w_i(t,x_k(t)-0)|\} \int_{s_j(t)}^{x_k(t)} |w_j(t,x)|dx + \int\int_{s_j(t)<x<y<s_i(t)} \mathrm{sgn}(w_j)G_j(t,x)|w_i(t,y)|dxdy \\
&+ \int\int_{s_j(t)<x<y<s_i(t)} |w_j(t,x)|\mathrm{sgn}(w_i)G_i(t,y)dxdy.
\end{aligned} \tag{4.17}
$$

(i) For $i = 2, \ldots, n-1$, noting (3.6), we get from (4.17) that

$$
\begin{aligned}
\frac{dQ(t)}{dt} =\ & -\int_{s_j(t)}^{s_i(t)} (\lambda_j(u(t,x)) - \lambda_i(u(t,x)))|w_i(t,x)||w_j(t,x)|dx + (s_i'(t) - \lambda_i(u(t,s_i(t))))|w_i(t,s_i(t))| \\
&\times \int_{s_j(t)}^{s_i(t)} |w_j(t,x)|dx + (\lambda_j(u(t,s_j(t))) - s_j'(t))|w_j(t,s_j(t))| \\
&\times \int_{s_j(t)}^{s_i(t)} |w_i(t,x)|dx + (x_i'(t) - \lambda_i(u(t,x_i(t)-0)))|w_i(t,x_i(t)-0)|
\end{aligned}
$$

$$\times \int_{s_j(t)}^{x_i(t)} |w_j(t,x)|dx + (\lambda_i(u(t,x_i(t)+0)) - x_i'(t))|w_i(t,x_i(t)+0)| \int_{s_j(t)}^{x_i(t)} |w_j(t,x)|dx$$
$$+ \sum_{k=2,k\neq i}^{n} x_k'(t)\{|w_i(t,x_k(t)-0)| - |w_i(t,x_k(t)+0)|\} \int_{s_j(t)}^{x_k(t)} |w_j(t,x)|dx + \sum_{k=2,k\neq i}^{n} \{\lambda_i(u(t,x_k(t)+0))$$
$$\times |w_i(t,x_k(t)+0)| - \lambda_i(u(t,x_k(t)-0))|w_i(t,x_k(t)-0)|\} \int_{s_j(t)}^{x_k(t)} |w_j(t,x)|dx$$
$$+ \int\int_{s_j(t)<x<y<s_i(t)} \mathrm{sgn}(w_j)G_j(t,x)|w_i(t,y)|dxdy + \int\int_{s_j(t)<x<y<s_i(t)} |w_j(t,x)|\mathrm{sgn}(w_i)G_i(t,y)dxdy. \quad (4.18)$$

Using (3.2), (3.3) and (4.3), we get

$$\frac{dQ(t)}{dt} \le -\delta_0 \int_{s_j(t)}^{s_i(t)} |w_i(t,x)||w_j(t,x)|dx + |(x_i'(t) - \lambda_i(u(t,x_i(t)\pm 0)))w_i(t,x_i(t)\pm 0)|$$
$$\times \int_{s_j(t)}^{x_i(t)} |w_j(t,x)|dx + \sum_{k=2,k\neq i}^{n} x_k'(t)\{|w_i(t,x_k(t)-0) - w_i(t,x_k(t)+0)|\} \int_{s_j(t)}^{x_k(t)} |w_j(t,x)|dx$$
$$+ \sum_{k=2,k\neq i}^{n} \{|\lambda_i(u(t,x_k(t)+0))w_i(t,x_k(t)+0) - \lambda_i(u(t,x_k(t)-0))w_i(t,x_k(t)-0)|\}$$
$$\times \int_{s_j(t)}^{x_k(t)} |w_j(t,x)|dx + \int_{s_j(t)}^{s_i(t)} |G_j(t,x)|dx \int_{s_j(t)}^{s_i(t)} |w_i(t,x)|dx + \int_{s_j(t)}^{s_i(t)} |G_i(t,x)|dx \int_{s_j(t)}^{s_i(t)} |w_j(t,x)|dx$$
$$\le -\delta_0 \int_{s_j(t)}^{s_i(t)} |w_i(t,x)||w_j(t,x)|dx + |(x_i'(t) - \lambda_i(u(t,x_i(t)\pm 0)))w_i(t,x_i(t)\pm 0)| \int_{-\infty}^{+\infty} |w_j(t,x)|dx$$
$$+ \sum_{k=2,k\neq i}^{n} x_k'(t)\{|w_i(t,x_k(t)-0) - w_i(t,x_k(t)+0)|\} \int_{-\infty}^{+\infty} |w_j(t,x)|dx + \sum_{k=2,k\neq i}^{n} \{|\lambda_i(u(t,x_k(t)+0))$$
$$\times w_i(t,x_k(t)+0) - \lambda_i(u(t,x_k(t)-0))w_i(t,x_k(t)-0)|\} \int_{-\infty}^{+\infty} |w_j(t,x)|dx$$
$$+ \int_{-\infty}^{+\infty} |G_j(t,x)|dx \int_{-\infty}^{+\infty} |w_i(t,x)|dx + \int_{-\infty}^{+\infty} |G_i(t,x)|dx \int_{-\infty}^{+\infty} |w_j(t,x)|dx.$$

It then follows from Lemma 4.1 that

$$\frac{dQ(t)}{dt} + \delta_0 \int_{s_j(t)}^{s_i(t)} |w_i(t,x)||w_j(t,x)|dx$$
$$\le k_1 \int_{-\infty}^{+\infty} |G_j(t,x)|dx \left(\varepsilon + W_1(T) + V_\infty(T)(\widetilde{W}_1(T) + W_1(T)) + \int_0^T \int_{-\infty}^{+\infty} |G_i(t,x)|dxdt\right)$$
$$+ k_1\left(|(x_i'(t) - \lambda_i(u(t,x_i(t)\pm 0)))w_i(t,x_i(t)\pm 0)| + \sum_{k=2,k\neq i}^{n} x_k'(t)\{|w_i(t,x_k(t)-0) - w_i(t,x_k(t)+0)|\}\right.$$
$$\left.+ \sum_{k=2,k\neq i}^{n} \{|\lambda_i(u(t,x_k(t)+0))w_i(t,x_k(t)+0) - \lambda_i(u(t,x_k(t)-0))w_i(t,x_k(t)-0)|\} + \int_{-\infty}^{+\infty} |G_i(t,x)|dx\right)$$
$$\times \left(\varepsilon + W_1(T) + V_\infty(T)(\widetilde{W}_1(T) + W_1(T)) + \int_0^T \int_{-\infty}^{+\infty} |G_j(t,x)|dxdt\right).$$

Therefore

$$\delta_0 \int_0^T \int_{s_j(t)}^{s_i(t)} |w_i(t,x)||w_j(t,x)|dxdt \le Q(0) + k_1 \int_0^T \int_{-\infty}^{+\infty} |G_j(t,x)|dxdt \left(\varepsilon + W_1(T) + V_\infty(T)(\widetilde{W}_1(T) + W_1(T))\right.$$
$$\left. + \int_0^T \int_{-\infty}^{+\infty} |G_i(t,x)|dxdt\right) + k_1\left(\int_0^T |(x_i'(t) - \lambda_i(u(t,x_i(t)\pm 0)))\right.$$

$$\times w_i(t,x_i(t)\pm 0)|dt+\sum_{k=2,k\neq i}^{n}\int_{\widehat{C}_k}|[w_i]|\lambda_k(u^{\pm})dt$$
$$+\sum_{k=2,k\neq i}^{n}\int_{\widehat{C}_k}|[w_i\lambda_i(u)]|dt+\int_0^T\int_{-\infty}^{+\infty}|G_i(t,x)|dxdt\Big)$$
$$\times\Big(\varepsilon+W_1(T)+V_\infty(T)(\widetilde{W}_1(T)+W_1(T))+\int_0^T\int_{-\infty}^{+\infty}|G_j(t,x)|dxdt\Big).$$

Using (3.8)–(3.10) and noting (4.4), we obtain

$$\delta_0\int_0^T\int_{s_j(t)}^{s_i(t)}|w_i(t,x)||w_j(t,x)|dxdt\le Q(0)+c_5\Big(\varepsilon+W_1(T)+V_\infty(T)(\widetilde{W}_1(T)+W_1(T))$$
$$+\int_0^T\int_{-\infty}^{+\infty}|G_i(t,x)|dxdt\Big)\Big(\varepsilon+W_1(T)+V_\infty(T)(\widetilde{W}_1(T)+W_1(T))+\int_0^T\int_{-\infty}^{+\infty}|G_j(t,x)|dxdt\Big).$$

Noting

$$Q(0)\le\int_{-\infty}^{+\infty}|w_i(0,x)|dx\int_{-\infty}^{+\infty}|w_j(0,x)|dx,$$

we get

$$\delta_0\int_0^T\int_{s_j(t)}^{s_i(t)}|w_i(t,x)||w_j(t,x)|dxdt\le c_6\Big(\varepsilon+W_1(T)+V_\infty(T)(\widetilde{W}_1(T)+W_1(T))+\int_0^T\int_{-\infty}^{+\infty}|G_i(t,x)|dxdt\Big)$$
$$\times\Big(\varepsilon+W_1(T)+V_\infty(T)(\widetilde{W}_1(T)+W_1(T))+\int_0^T\int_{-\infty}^{+\infty}|G_j(t,x)|dxdt\Big).$$

It then follows

$$\int_0^T\int_{s_j(t)}^{s_i(t)}|w_i(t,x)||w_j(t,x)|dxdt\le\frac{c_6}{\delta_0}\Big(\varepsilon+W_1(T)+V_\infty(T)(\widetilde{W}_1(T)+W_1(T))+\int_0^T\int_{-\infty}^{+\infty}|G_i(t,x)|dxdt\Big)$$
$$\times\Big(\varepsilon+W_1(T)+V_\infty(T)(\widetilde{W}_1(T)+W_1(T))+\int_0^T\int_{-\infty}^{+\infty}|G_j(t,x)|dxdt\Big).$$

Therefore

$$\int_0^T\int_{-L}^{L}|w_i(t,x)||w_j(t,x)|dxdt\le c_7\Big(\varepsilon+W_1(T)+V_\infty(T)(\widetilde{W}_1(T)+W_1(T))+\int_0^T\int_{-\infty}^{+\infty}|G_i(t,x)|dxdt\Big)$$
$$\times\Big(\varepsilon+W_1(T)+V_\infty(T)(\widetilde{W}_1(T)+W_1(T))+\int_0^T\int_{-\infty}^{+\infty}|G_j(t,x)|dxdt\Big)$$

and the desired conclusion follows by taking $L\to+\infty$.

(ii) For $i=1$, the argument in step (ii) is similar to the one in step (i), so instead of giving all the details one can just refer to (i) and briefly describe the changes one needs to introduce. Instead of formula (4.18) we have

$$\frac{dQ(t)}{dt}=-\int_{s_j(t)}^{s_i(t)}(\lambda_j(u(t,x))-\lambda_i(u(t,x)))|w_i(t,x)||w_j(t,x)|dx+(s_i'(t)-\lambda_i(u(t,s_i(t))))|w_i(t,s_i(t))|$$
$$\times\int_{s_j(t)}^{s_i(t)}|w_j(t,x)|dx+(\lambda_j(u(t,s_j(t)))-s_j'(t))|w_j(t,s_j(t))|\int_{s_j(t)}^{s_i(t)}|w_i(t,x)|dx$$
$$+(x_L'(t)-\lambda_i(u(t,x_L(t))))|w_i(t,x_L(t)-0)|\int_{s_j(t)}^{x_L(t)}|w_j(t,x)|dx+(\lambda_i(u(t,x_L(t)))-x_L'(t))$$
$$\times|w_i(t,x_L(t)+0)|\int_{s_j(t)}^{x_L(t)}|w_j(t,x)|dx+(x_R'(t)-\lambda_i(u(t,x_R(t))))|w_i(t,x_R(t)-0)|$$
$$\times\int_{s_j(t)}^{x_R(t)}|w_j(t,x)|dx+(\lambda_i(u(t,x_R(t)))-x_R'(t))|w_i(t,x_R(t)+0)|\int_{s_j(t)}^{x_R(t)}|w_j(t,x)|dx$$
$$+\sum_{k=2}^{n}x_k'(t)\{|w_i(t,x_k(t)-0)|-|w_i(t,x_k(t)+0)|\}\int_{s_j(t)}^{x_k(t)}|w_j(t,x)|dx+\sum_{k=2}^{n}\{\lambda_i(u(t,x_k(t)+0))$$

$$\times|w_i(t,x_k(t)+0)|-\lambda_i(u(t,x_k(t)-0))|w_i(t,x_k(t)-0)|\}\int_{s_j(t)}^{x_k(t)}|w_j(t,x)|dx$$
$$+\int\int_{s_j(t)<x<y<s_i(t)}\mathrm{sgn}(w_j)G_j(t,x)|w_i(t,y)|dxdy+\int\int_{s_j(t)<x<y<s_i(t)}|w_j(t,x)|\mathrm{sgn}(w_i)G_i(t,y)dxdy.$$

Using (3.4) and (4.3), we get

$$\begin{aligned}\frac{dQ(t)}{dt}\le{}&-\delta_0\int_{s_j(t)}^{s_i(t)}|w_i(t,x)||w_j(t,x)|dx+\sum_{k=2}^{n}x_k'(t)\{|w_i(t,x_k(t)-0)-w_i(t,x_k(t)+0)|\}\\&\times\int_{s_j(t)}^{x_k(t)}|w_j(t,x)|dx+\sum_{k=2}^{n}\{|\lambda_i(u(t,x_k(t)+0))w_i(t,x_k(t)+0)-\lambda_i(u(t,x_k(t)-0))w_i(t,x_k(t)-0)|\}\\&\times\int_{s_j(t)}^{x_k(t)}|w_j(t,x)|dx+\int_{s_j(t)}^{s_i(t)}|G_j(t,x)|dx\int_{s_j(t)}^{s_i(t)}|w_i(t,x)|dx+\int_{s_j(t)}^{s_i(t)}|G_i(t,x)|dx\int_{s_j(t)}^{s_i(t)}|w_j(t,x)|dx\\\le{}&-\delta_0\int_{s_j(t)}^{s_i(t)}|w_i(t,x)||w_j(t,x)|dx+\sum_{k=2}^{n}x_k'(t)\{|w_i(t,x_k(t)-0)-w_i(t,x_k(t)+0)|\}\\&\times\int_{-\infty}^{+\infty}|w_j(t,x)|dx+\sum_{k=2}^{n}\{|\lambda_i(u(t,x_k(t)+0))w_i(t,x_k(t)+0)\\&-\lambda_i(u(t,x_k(t)-0))w_i(t,x_k(t)-0)|\}\int_{-\infty}^{+\infty}|w_j(t,x)|dx\\&+\int_{-\infty}^{+\infty}|G_j(t,x)|dx\int_{-\infty}^{+\infty}|w_i(t,x)|dx+\int_{-\infty}^{+\infty}|G_i(t,x)|dx\int_{-\infty}^{+\infty}|w_j(t,x)|dx.\end{aligned}\tag{4.19}$$

By exploiting the same arguments as in (i), and noting (3.8)–(3.10), we can deduce from (4.19), formula (4.16). The proof of Lemma 4.2 is finished. □

Lemma 4.3. *Under the assumptions of Theorem 1.1, for small $\theta>0$ there exists a constant $\varepsilon>0$ so small that on any given domain of existence $[0,T]\times\mathbf{R}$ of the piecewise C^1 solution $u=u(t,x)$ to the generalized Riemann problem (1.1) and (1.6), there exist positive constants k_i $(i=3,\dots,7)$ independent of θ, ε and T, such that the following uniform a priori estimates hold:*

$$W_1(T)\le k_3\varepsilon,\tag{4.20}$$
$$\widetilde{W}_1(T)\le k_4\varepsilon,\tag{4.21}$$
$$U_\infty(T),V_\infty(T)\le k_5\theta\tag{4.22}$$

and

$$W_\infty(T)\le k_6\varepsilon,\tag{4.23}$$

where T satisfies

$$T\varepsilon\le k_7.\tag{4.24}$$

Proof. The proof of Lemma 4.3 is similar to that in Yi Zhou's paper [24]. The only differences between them are emerged in Lemma 4.2 caused by the line of discontinuity (center rarefaction wave). We sketch the proof and details for the readers' convenience. We introduce

$$Q_W(T)=\sum_{j=1}^{n}\sum_{i\ne j}\int_0^T\int_{-\infty}^{+\infty}|w_i(t,x)||w_j(t,x)|dxdt.$$

By (2.13), it follows from Lemma 4.2 that

$$Q_W(T)\le c_8\left(\varepsilon+W_1(T)+V_\infty(T)(\widetilde{W}_1(T)+W_1(T))+\int_0^T\int_{-\infty}^{+\infty}|G(t,x)|dxdt\right)^2,\tag{4.25}$$

where $G=(G_1,G_2,\dots,G_n)$.

Noting (2.16), we have

$$\int_0^T\int_{-\infty}^{+\infty}|G(t,x)|dxdt\le c_9Q_W(T).\tag{4.26}$$

Substituting (4.26) into (4.25), we obtain

$$Q_W(T) \le c_{10}\left(\varepsilon + W_1(T) + V_\infty(T)(\widetilde{W}_1(T) + W_1(T)) + Q_W(T)\right)^2. \tag{4.27}$$

We next estimate $\widetilde{W}_1(T)$.

Let

$$\widetilde{C}_j : x = x_j(t) \quad (0 \le t_1 \le t \le t_2 \le T)$$

be any given jth characteristic on the domain $[0, T] \times \mathbf{R}$. Then, passing through the point $P_1(t_1, x_j(t_1))$ (resp. $P_2(t_2, x_j(t_2))$) we draw the ith characteristic which intersects the x-axis at a point $A_1(0, y_1)$ (resp. $A_2(0, y_2)$). Without loss of generality, we assume that the whole ith wave (shock or contact discontinuity or rarefaction wave) passing through $O(0, 0)$ is contained in the domain $P_1A_1A_2P_2$. In what follows, we only consider the case of rarefaction wave, while other cases can be dealt with in a manner similar to [23]. Let $x = x_L(t)$(resp. $x = x_R(t)$) be the left (resp. right) boundary of the rarefaction wave. Then, applying (2.17) on the domain $P_1A_1A_2P_2$ and noting (2.16), it is easy to see that

$$\begin{aligned}
&\int_{t_1}^{t_2} |w_i(t, x_j(t))||\lambda_j(u(t, x_j(t))) - \lambda_i(u(t, x_j(t)))|dt \\
&\le \int_{y_1}^{y_2} |w_i(0, x)|dx + \int_0^T |(x_L'(t) - \lambda_i(u(t, x_L(t))))w_i(t, x_L(t) \pm 0)|dt \\
&\quad + \int_0^T |(x_R'(t) - \lambda_i(u(t, x_R(t))))w_i(t, x_R(t) \pm 0)|dt + \sum_{k\in S_2} \int_{\widehat{C}_k} |([w_i]x_k'(t) - [w_i\lambda_i(u)])dt| \\
&\quad + \int\int_{P_1A_1A_2P_2} \sum_{j\ne k} |\Gamma_{ijk}(u)w_jw_k|dtdx,
\end{aligned} \tag{4.28}$$

where S_2 stands for the set of all indices k such that the kth discontinuous curve (shock or contact discontinuity) $\widehat{C}_k : x = x_k(t)$ is partly contained in the domain $P_1A_1A_2P_2$, and

$$x_k'(t) = \frac{dx_k(t)}{dt} = \begin{cases} \lambda_k(u^\pm), & \text{as } k \in I, \\ \lambda_k(u^-, u^+), & \text{as } k \in J_S. \end{cases} \tag{4.29}$$

Noting (3.6), in (4.28) we need not consider the case that the kth wave is a rarefaction wave. Using (1.8), (3.4), (3.8)–(3.10), (4.3), (4.4) and (4.29), we have

$$\int_{t_1}^{t_2} |w_i(t, x_j(t))|dt \le c_{11}\left\{\varepsilon + V_\infty(T)(\widetilde{W}_1(T) + W_1(T)) + Q_W(T)\right\}.$$

Thus, noting (4.42) in [23], we get immediately

$$\widetilde{W}_1(T) \le c_{12}\left\{\varepsilon + W_1(T) + V_\infty(T)(\widetilde{W}_1(T) + W_1(T)) + Q_W(T)\right\}. \tag{4.30}$$

We next estimate $W_1(T)$.

(i) For $i = n$, passing through any fixed point $A(T, a)$ ($a > x_n(T)$), we draw the nth backward characteristic which intersects the x-axis at a point $B(0, x_B)$.

We rewrite (2.14) as

$$d(|w_i(t, x)|(dx - \lambda_i(u)dt)) = \operatorname{sgn}(w_i)G_i dxdt. \tag{4.31}$$

Without loss of generality, we assume that the nth wave (shock or contact discontinuity or rarefaction wave) $x = x_n(t)$ passing through $O(0, 0)$ is the shock curve. Let D denotes the point $(T, x_n(T))$. Then, integrating (4.31) (in which we take $i = n$) on the domain $ABOD$ gives

$$\int_0^T (x_n'(t) - \lambda_n(u(t, x_n(t) + 0)))|w_n(t, x_n(t) + 0)|dt + \int_D^A |w_n(T, x)|dx \le \int_0^{x_B} |w_n(0, x)|dx + \int\int_{ABOD} |G_n|dxdt.$$

Hence, noting (1.8) and (4.26), we get

$$\int_0^T (x_n'(t) - \lambda_n(u(t, x_n(t) + 0)))|w_n(t, x_n(t) + 0)|dt \le c_{13}\{\varepsilon + Q_W(T)\}.$$

In view of (3.2), this implies

$$\int_0^T |(x_n'(t) - \lambda_n(u(t, x_n(t) + 0)))w_n(t, x_n(t) + 0)|dt \le c_{13}\{\varepsilon + Q_W(T)\}. \tag{4.32}$$

(ii) For $i = 1, \ldots, n-1$, passing through point $A(T, a)$ $(a > x_n(T))$, we draw the ith backward characteristic which intersects the x-axis at a point $B(0, x_B)$. Without loss of generality, we assume that the ith discontinuous curve $x = x_i(t)$ passing through the origin is the shock curve.

Let D denotes the point $(T, x_i(T))$. Thanks to the piecewise C^1 solution $u = u(t, x)$ containing only n shocks, centered rarefaction waves or contact discontinuities $x = x_k(t)(x_k(0) = 0)$ $(k = 1, \ldots, n)$, without loss of generality, we may suppose that $u = u(t, x)$ containing only a centered rarefaction wave, while other cases can be dealt with in a manner similar to [23]. Without loss of generality, we may suppose that the $i+1$th wave is a centered rarefaction wave. Let $x = x_L(t)(\text{resp.}\, x = x_R(t))$ be the left (resp. right) boundary of the rarefaction wave, we divide the bounded domain $ABOD$ by both boundaries of the rarefaction wave $x = x_H(t)$ $(H = L, R)$ and the discontinuous curves (shocks or contact discontinuities) $x = x_k(t)$ $(x_k(0) = 0)$ $(k = i+2, \ldots, n)$ into some parts. Then, integrating (4.31) on all parts of the domain $ABOD$ gives

$$\begin{aligned}
&\int_0^T (x_i'(t) - \lambda_i(u(t, x_i(t) + 0)))|w_i(t, x_i(t) + 0)|dt + \int_D^A |w_i(T, x)|dx \\
&\quad\le \int_0^{x_B} |w_i(0, x)|dx + \int_0^T x_L'(t)(|w_i(t, x_L(t) - 0)| - |w_i(t, x_L(t) + 0)|)dt \\
&\qquad + \int_0^T \lambda_i(u(t, x_L(t)))(|w_i(t, x_L(t) + 0)| - |w_i(t, x_L(t) - 0)|)dt \\
&\qquad + \int_0^T x_R'(t)(|w_i(t, x_R(t) - 0)| - |w_i(t, x_R(t) + 0)|)dt \\
&\qquad + \int_0^T \lambda_i(u(t, x_R(t)))(|w_i(t, x_R(t) + 0)| - |w_i(t, x_R(t) - 0)|)dt \\
&\qquad + \sum_{k=i+2}^{n} \int_{\widehat{C}_k} |[w_i]x_k'(t) - [w_i\lambda_i(u)]|dt + \int\int_{ABOD} |G_i|dxdt,
\end{aligned}$$

where $\widehat{C}_k : x = x_k(t)$ stands for the kth discontinuous curve (shock or contact discontinuity) passing through the origin, which is contained in the domain $ABOD$. Thus, using (1.8), (3.6), (3.8)–(3.10), (4.4) and (4.29), we obtain

$$\int_0^T (x_i'(t) - \lambda_i(u(t, x_i(t) + 0)))|w_i(t, x_i(t) + 0)|dt \le c_{14}\{\varepsilon + V_\infty(T)(\widetilde{W}_1(T) + W_1(T)) + Q_W(T)\}.$$

In view of (3.2), this implies

$$\int_0^T |(x_i'(t) - \lambda_i(u(t, x_i(t) + 0)))w_i(t, x_i(t) + 0)|dt \le c_{14}\{\varepsilon + V_\infty(T)(\widetilde{W}_1(T) + W_1(T)) + Q_W(T)\}. \tag{4.33}$$

(iii) For $i = 1$, passing through any fixed point $A(T, a)$ $(a < x_1(T))$, we draw the 1st backward characteristic which intersects the x-axis at a point $B(0, x_B)$. Without loss of generality, we assume that the 1st discontinuous curve $x = x_1(t)$ passing through the origin is the shock curve.

Let D denotes the point $(T, x_1(T))$. Then, integrating (4.31) (in which we take $i = 1$) on the domain $ABOD$ gives

$$\int_A^D |w_1(T, x)|dx + \int_T^0 (x_1'(t) - \lambda_1(u(t, x_1(t) - 0)))|w_1(t, x_1(t) - 0)|dt \le \int_{x_B}^0 |w_1(0, x)|dx + \int\int_{ABOD} |G_1|dxdt.$$

Hence, noting (1.8) and (4.26), we get

$$\int_0^T (\lambda_1(u(t, x_1(t) - 0)) - x_1'(t))|w_1(t, x_1(t) - 0)|dt \le c_{15}\{\varepsilon + Q_W(T)\}.$$

In view of (3.2), this implies

$$\int_0^T |(x_1'(t) - \lambda_1(u(t, x_1(t) - 0)))w_1(t, x_1(t) - 0)|dt \le c_{15}\{\varepsilon + Q_W(T)\}. \tag{4.34}$$

(iv) For $i = 2, \ldots, n$, passing through point $A(T, a)$ $(a < x_1(T))$, we draw the ith backward characteristic which intersects the x-axis at a point $B(0, x_B)$. Without loss of generality, we assume that the ith discontinuous curve $x = x_i(t)$ passing through the origin is the shock curve.

Let D denotes the point $(T, x_i(T))$, without loss of generality, we may suppose that the piecewise C^1 solution $u = u(t, x)$ containing only a centered rarefaction wave, while other cases can be dealt with in a manner similar to [23]. Without loss of generality, we may suppose that the 1st wave is a centered rarefaction wave. Let $x = x_L(t)$(resp. $x = x_R(t)$) be the left (resp. right) boundary of the rarefaction wave, we divide the bounded domain $ABOD$ by both boundaries of the rarefaction wave $x = x_H(t)$ $(H = L, R)$ and the discontinuous curves (shocks or contact discontinuities) $x = x_k(t)$ $(x_k(0) = 0)$ $(k = 2, \dots, i-1)$ into some parts. Thus, integrating (4.31) on all parts of the domain $ABOD$ gives

$$\begin{aligned}
&\int_0^T (\lambda_i(u(t, x_i(t)-0)) - x_i'(t))|w_i(t, x_i(t)-0)|dt + \int_A^D |w_i(T, x)|dx \\
&\quad \le \int_{x_B}^0 |w_i(0, x)|dx + \int_0^T x_L'(t)(|w_i(t, x_L(t)-0)| - |w_i(t, x_L(t)+0)|)dt \\
&\qquad + \int_0^T \lambda_i(u(t, x_L(t)))(|w_i(t, x_L(t)+0)| - |w_i(t, x_L(t)-0)|)dt \\
&\qquad + \int_0^T x_R'(t)(|w_i(t, x_R(t)-0)| - |w_i(t, x_R(t)+0)|)dt \\
&\qquad + \int_0^T \lambda_i(u(t, x_R(t)))(|w_i(t, x_R(t)+0)| - |w_i(t, x_R(t)-0)|)dt \\
&\qquad + \sum_{k=2}^{i-1} \int_{\widehat{C}_k} |[w_i]x_k'(t) - [w_i\lambda_i(u)]|dt + \int\int_{ABOD} |G_i|dxdt
\end{aligned}$$

where $\widehat{C}_k : x = x_k(t)$ stands for the kth discontinuous curve (shock or contact discontinuity) passing through the origin, which is contained in the domain $ABOD$. Then, using (1.8), (3.6), (3.8)–(3.10), (4.4) and (4.29), we obtain

$$\int_0^T (\lambda_i(u(t, x_i(t)-0)) - x_i'(t))|w_i(t, x_i(t)-0)|dt \le c_{16}\big\{\varepsilon + V_\infty(T)(\widetilde{W}_1(T) + W_1(T)) + Q_W(T)\big\}.$$

In view of (3.2), this implies

$$\int_0^T |(x_i'(t) - \lambda_i(u(t, x_i(t)-0)))w_i(t, x_i(t)-0)|dt \le c_{16}\big\{\varepsilon + V_\infty(T)(\widetilde{W}_1(T) + W_1(T)) + Q_W(T)\big\}. \tag{4.35}$$

Combining (4.32)–(4.34) and (4.35) all together, we have

$$W_1(T) \le c_{17}\big\{\varepsilon + V_\infty(T)(\widetilde{W}_1(T) + W_1(T)) + Q_W(T)\big\}. \tag{4.36}$$

We next estimate $U_\infty(T)$ and $V_\infty(T)$.

Passing through any fixed point $(t, x) \in [0, T] \times \mathbf{R}$, we draw the ith backward characteristic C_i which intersects the x-axis at a point $(0, y)$. Integrating (2.6) along this characteristic C_i and noting (2.8) yield

$$v_i(t, x) = v_i(0, y) + \sum_{k\in S_3}[v_i]_k + \int_{C_i} \sum_{j,k=1,k\ne i}^n \beta_{ijk}(u)v_jw_kdt, \tag{4.37}$$

where S_3 denotes the set of all indices k such that this characteristic C_i intersects the kth discontinuous curve (shock or contact discontinuity) $x = x_k(t)$ at a point $(t_k, x_k(t_k))$, and $[v_i]_k = v_i(t_k, x_k(t_k)+0) - v_i(t_k, x_k(t_k)-0)$. Noting (1.8) and using (1.10), we have

$$|u_+(x)| \le \int_0^{+\infty} |u_+'(x)|dx \le K_2, \quad \forall x \in \mathbf{R}^+ \tag{4.38}$$

and

$$|u_-(x)| \le \int_{-\infty}^0 |u_-'(x)|dx \le K_2, \quad \forall x \in \mathbf{R}^-. \tag{4.39}$$

Therefore, noting the fact that $i \notin S_3$, and using (1.6), (2.1), (3.7), (4.2) and (4.4), we get from (4.37)–(4.39) that

$$V_\infty(T) \le c_{18}\{\theta + \varepsilon + V_\infty(T)(V_\infty(T) + \widetilde{W}_1(T))\}. \tag{4.40}$$

Combining (4.27), (4.30), (4.36) and (4.40) all together, and noting (4.2), (4.38) and (4.39), we can prove (4.20)–(4.22) and

$$Q_W(T) \le c_{19}\varepsilon^2.$$

We finally estimate $W_\infty(T)$.

For any fixed point $(t,x) \in [0,T] \times \mathbf{R}$, we draw the ith backward characteristic C_i passing through the point (t,x), which intersects the x-axis at a point $(0,y)$. Integrating (2.9) along this characteristic C_i and noting (2.11) yield

$$w_i(t,x) = w_i(0,y) + \sum_{k \in S_4} [w_i]_k + \int_{C_i} \left[\sum_{j,k=1, j\neq k}^{n} \gamma_{ijk}(u) w_j w_k + \gamma_{iii}(u) w_i^2 \right] dt,$$

where S_4 denotes the set of all indices k such that this characteristic C_i intersects the kth discontinuous curve $x = x_k(t)$ at a point $(t_k, x_k(t_k))$, and $[w_i]_k = w_i(t_k, x_k(t_k)+0) - w_i(t_k, x_k(t_k)-0)$. Using (3.6), (3.8), (3.9) and (4.4) and noting the fact that $i \notin S_4$, we have

$$W_\infty(T) \le c_{20}\{\varepsilon + V_\infty(T) W_\infty(T) + W_\infty(T)\widetilde{W}_1(T) + T(W_\infty(T))^2\}. \tag{4.41}$$

Noting (1.7), by continuity there exists a positive constant k_6 independent of θ, ε and T such that (4.23) holds at least for $T > 0$ suitably small. Thus, in order to prove (4.23) it suffices to show that we can choose k_6 and k_7 in such a way that for any fixed T_0 $(0 < T_0 \le T)$ with $T_0 \varepsilon \le k_7$ such that

$$W_\infty(T_0) \le 2k_6 \varepsilon, \tag{4.42}$$

we have

$$W_\infty(T_0) \le k_6 \varepsilon. \tag{4.43}$$

Substituting (4.42) into the right-hand side of (4.41) (in which we take $T = T_0$), and noting (4.21)–(4.22) and (4.24), it is easy to see that, when $\theta > 0$ is suitably small, we have

$$W_\infty(T_0) \le 2c_{20}(1 + 2k_6^2 k_7)\varepsilon.$$

Hence, if $k_6 \ge 6c_{20}$ and $k_6^2 k_7 = 1$, then we have (4.43), provided that θ is suitably small. Therefore (4.23) is proved.

Finally, we observe that when $\theta > 0$ is suitably small, by (4.22) we have

$$U_\infty(T) \le k_5 \theta \le \frac{1}{2}\delta.$$

This implies the validity of hypothesis (4.4). The proof of Lemma 4.3 is finished. □

Proof of Theorem 1.1. By (4.22)–(4.23), we know that for small $\theta > 0$ there exists $\varepsilon > 0$ suitably small such that the generalized Riemann problem (1.1) and (1.6) admits a unique piecewise C^1 solution $u = u(t,x)$ containing shocks, contact discontinuities and rarefaction waves on the strip $[0,T] \times \mathbf{R}$, where T satisfies (4.24). Therefore, the lifespan $\widetilde{T}(\varepsilon)$ of the piecewise C^1 solution satisfies

$$\widetilde{T}(\varepsilon) \ge K_3 \varepsilon^{-1},$$

where $K_3 (= k_7)$ is a positive constant independent of ε. Moreover, by Lemma 4.3, when the piecewise C^1 solution $u = u(t,x)$ blows up in a finite time, $u = u(t,x)$ itself must be bounded on the domain $[0, \widetilde{T}(\varepsilon)) \times \mathbf{R}$. Hence, the first-order derivative u_x of $u = u(t,x)$ should tend to be unbounded as $t \nearrow \widetilde{T}(\varepsilon)$. The proof of Theorem 1.1 is finished. □

Remark 4.1. In this paper, the author considered the below bound of the life span of the classical piecewise C^1 solution. However, how to search the above bound of the life span is a more important and more interesting problem, this problem is worth studying in the future.

5. Applications

In this section, two concrete examples are given to show some applications of our main results, they are both the one-dimensional compressible Euler equations in Eulerian coordinates and the system of traffic flow on a road network using the Aw–Rascle model. Here, we only give the latter for which our work is of importance, for the basic model on the former we refer the reader to Chen and Frid [5] and the references therein. The system of traffic flow on a road network using the Aw–Rascle model reads (cf. [1]):

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho(v + p(\rho))) + \partial_x(\rho v(v + p(\rho))) = 0, \end{cases} \tag{5.1}$$

where $\rho > 0$ and v are, respectively, the density and the velocity of the cars at point x and time t, $y = \rho v + \rho p(\rho)$ is the momentum, the "pressure" $p = p(\rho)$ is a suitably smooth function of ρ and satisfies

$$p'(\rho) > 0 \quad \text{and} \quad p''(\rho) > 0, \quad \forall \rho > 0.$$

Set

$$u = \begin{pmatrix} \rho \\ v \end{pmatrix}.$$

Then, we rewrite system (5.1) as

$$u_t + A(u)u_x = 0,$$

where the Jacobian matrix is

$$A(u) = \begin{pmatrix} v & \rho \\ 0 & v - \rho p'(\rho) \end{pmatrix}.$$

It is known that (5.1) is strictly hyperbolic for $\rho > 0$, the first characteristic field is genuinely nonlinear, while the second characteristic field is linearly degenerate. Therefore, Theorem 1.1 is obviously applicable to the system of traffic flow on a road network using the Aw–Rascle model.

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