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Optimal investment, consumption and timing of annuity purchase under a preference change

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ABSTRACT

In this paper, we study the optimal investment and consumption strategies for a retired individual who has the opportunity of choosing a discretionary stopping time to purchase an annuity. We assume that the individual receives a fixed annuity income and changes his/her preference after paying a fixed cost for annuitization. By using the martingale method and the variational inequality method, we tackle this problem and obtain the optimal strategies and the value function explicitly for the case of constant force of mortality and constant relative risk aversion (CRRA) utility function.

1. Introduction

Since the seminal paper of Yaari [27], there have been a number of papers about the optimal annuitization and portfolio selection in the literature, see, for example, [9,22,24–26] and the references therein. Yaari [27] first demonstrated that under some specific assumptions rational individuals with no bequest motives should annuitize all their wealth at retirement. However, the volume of voluntary purchases by retirees is much smaller than predicted by theoretical models, which is the so-called “annuity puzzle”.

Bequest motives play a central role in limiting the demand for annuities. Davidoff et al. [6] showed that if annuities are priced fairly, then people annuitize all of their wealth except what they wish to bequeath. Lockwood [21] argued that bequest motives are strong enough to reduce or eliminate purchases of available annuities. There are other explanations for the annuity puzzle. Inkmann et al. [11] found that the annuity market participation increases with financial wealth, life expectancy and education, while decreases with other pension income and a possible bequest motive for surviving spouses. Benartzi et al. [2] exploited that behavioral and institutional factors are important in explaining why there seems to be so little demand to annuitize wealth at retirement. Wang and Young [25,26] proposed that if life annuities were commutable, namely the individuals can surrender their early purchased life annuities, then retirees would purchase more annuities.

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Milevsky and Young [22] claimed that, as a result of adverse selection, the annuity purchase is an irreversible investment that creates an incentive to delay. Therefore, we consider a model which allows the retirees to prearrange their annuitization using part of their wealth in the future, the amount is fixed as a constant F. In view of significant medical spending, emergent events or bequest motives, full annuitization may not be optimal for the individual. Hence, we suppose that there is a nonnegative constant d as the lowest wealth level around the annuitization time τ, that is, $X_\tau \geqslant F + d$, where $X.$ is the wealth process of the individual.

In this paper, we investigate the optimal investment, consumption strategies and the optimal annuitization time for a retired individual subject to a constant force of mortality. The individual can determine a discretionary stopping time as the annuitization time, at that point, after paying a fixed cost for annuitization, the individual will receive a long-life annuity income. Moreover, we do not allow the individual to borrow his or her future annuity income because this income is contingent on his or her being alive. In contrast to the restrictive all-or-nothing arrangement, explored by [9,22,24], we assume that the individual continues investing and consuming and changes his or her preference after annuitization. Finally, four types of solutions are obtained depending on the free parameters of the problem. We find the optimal annuitization region is a band form. In some special cases, the two annuitization thresholds coincide or degenerate to $+\infty$.

The problem of utility maximization with discretionary stopping was first studied by Karatzas and Wang [16] via the martingale method. They introduced a family of stopping time problems to reduce the original problem into an easy form. Farhi and Panageas [8] applied these techniques to explore an optimal consumption and portfolio choice problem with flexible retirement option. See also [1,3–5,7,17,19,20] for other extensions. In this paper, we use both the martingale method and the variational inequality method to analyze an optimal consumption-portfolio selection problem with discretionary stopping. The wealth process in our model is assumed discontinuous. Another mixed optimal stopping/control problem has been studied by Jeanblanc et al. [12], in which the dynamic programming approach was employed and risk-preference change was not allowed.

The rest of our paper is organized as follows. In Section 2, we present our model and formulate the objective. An auxiliary value function $U(x)$ is introduced to transform the problem into an easily solved form. In Section 3, by using both the martingale method and the variational inequality method, we derive the closed-form solution for the auxiliary problem. In Section 4, the original value function and optimal strategies are provided by employing the same methods. Finally, we give numerical examples to illustrate our results in Section 5. Some of the detailed proofs are deferred to Appendices A and B.

2. Model formulation

2.1. The financial and pension annuity markets

We consider an optimization problem for an individual from the retirement time till the stochastic death time $\tau_d > 0$ in the financial and pension annuity markets. In the financial market, there are one risk-free asset and one risky asset, whose prices evolve according to the following equations:

$$dR_t = rR_t\,dt \quad \text{and} \quad dS_t = \mu S_t\,dt + \sigma S_t\,dW_t,$$

where μ, σ and r are positive constants. W_t is a standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $\{\mathcal{F}_t\}$ is the $\mathbb{P}$-augmentation of the natural filtration generated by W_t. Suppose the death time τ_d is an exponential random variable with parameter h defined on the probability space and is independent of W_t.

Let c_t be the consumption rate process which is nonnegative, progressively measurable with respect to $\mathcal{F}_t$ satisfying $\int_0^t c_s\,ds < \infty$ a.s. for all $t \geqslant 0$, and π_t be the amount invested in the risky asset at time t, which is $\mathcal{F}_t$-progressively measurable satisfying $\int_0^t \pi_s^2\,ds < \infty$ a.s. for all $t \geqslant 0$. We assume that the individual annuitizes a lump sum F at some discretionary stopping time τ, and after that receives a fixed rate $\frac{F}{\bar{a}_\tau^O}$ of annuity income. Here, $\bar{a}_\tau^O$ is the market price of per dollar of annuity income purchased at time τ, which is given by

$$\bar{a}_\tau^O := \mathbb{E}\left[\int_\tau^{\tau_d} e^{-r(t-\tau)}\,dt \;\middle|\; \tau_d > \tau\right] = \int_0^\infty e^{-rt}e^{-\psi t}\,dt = \frac{1}{r+\psi}, \tag{2.1}$$

where $\psi > 0$ is the constant objective hazard rate that is used to price annuities. Then the individual's controlled wealth process $X_\cdot^{x,\pi,c}$ with $X_0 = x \geqslant 0$ satisfies SDE:

$$dX_t^{x,\pi,c} = \left[(\mu - r)\pi_t + rX_t^{x,\pi,c} - c_t\right]dt + \sigma\pi_t\,dW_t, \quad t \leqslant \tau, \tag{2.2}$$

and

$$dX_t^{x,\pi,c} = \left[(\mu - r)\pi_t + rX_t^{x,\pi,c} - c_t + F(\psi + r)\right]dt + \sigma\pi_t\,dW_t, \quad t > \tau, \tag{2.3}$$

with $X_{\tau+}^{x,\pi,c} = X_\tau^{x,\pi,c} - F$.

Define the market-price-of-risk, the discount process, the exponential martingale process, and the state-price-density process, respectively, by

$$\theta \triangleq \frac{\mu - r}{\sigma}, \qquad \zeta_t \triangleq e^{-rt}, \qquad Z_t \triangleq \exp\left\{-\theta W_t - \frac{1}{2}\theta^2 t\right\}, \quad \text{and} \quad H_t \triangleq \zeta_t Z_t.$$

We also define an equivalent martingale measure as

$$\widetilde{\mathbb{P}}^T(A) \triangleq \mathbb{E}[Z_T \mathbf{1}_A],$$

for any fixed $T \in [0,\infty)$ and any $A \in \mathcal{F}_T$. Then the Girsanov theorem implies that $\widetilde{W}_t^T \triangleq W_t + \theta t$, for $0 \leqslant t \leqslant T$, is a standard Brownian motion under the new measure $\widetilde{\mathbb{P}}^T$. There exists a unique probability measure $\widetilde{\mathbb{P}}$ on $\mathcal{F}_\infty$ that agrees with $\widetilde{\mathbb{P}}^T$ on $\mathcal{F}_T$, for any $T \in [0,\infty)$. Furthermore, $\widetilde{W}_t, 0 \leqslant t < \infty$, is a Brownian motion under $\widetilde{\mathbb{P}}$ (see Proposition 7.4 in Section 1.7 of Karatzas and Shreve [15]). So the wealth process before time τ can be rewritten as

$$\zeta_t X_t^{x,\pi,c} + \int_0^t \zeta_s c_s\,ds = x + \int_0^t \zeta_s\sigma\pi_s\,d\widetilde{W}_s, \quad t \leqslant \tau. \tag{2.4}$$

We call a policy $(\tau, \{(\pi_t, c_t),\ t < \infty\})$ admissible if $X_t^{x,\pi,c} \geqslant 0$ and $X_\tau^{x,\pi,c} \geqslant F + d$, for all $0 \leqslant t < \infty$, where $d \geqslant 0$ is a constant, denoting the lowest wealth level after paying a lump annuitization cost F. Here we don't allow the individual to borrow the future annuity income, and we use $\mathcal{A}(x)$ to denote the class of all admissible policies with initial wealth x.

For any triple $(\tau, \{(\pi_t, c_t),\ t < \infty\}) \in \mathcal{A}(x)$, the second term on the right-hand side of Eq. (2.4) is a continuous $\widetilde{\mathbb{P}}$-local martingale bounded below and thus a super-martingale by Fatou's lemma. Then the optional sampling theorem and the Bayes rule lead to

$$\mathbb{E}\left[H_\tau X_\tau^{x,\pi,c} + \int_0^\tau H_s c_s\,ds\right] \leqslant x, \tag{2.5}$$

for every $\tau \in S$, where S denotes the set of all $\mathcal{F}$-stopping times.

2.2. The objective

For an admissible policy $(\tau, \{(\pi_t, c_t),\ t < \infty\})$, the individual's expected utility function $J(c, \pi, \tau; x)$ with initial wealth x is given by

$$J(c,\pi,\tau;x) = \mathbb{E}\left[\int_0^{\tau\wedge\tau_d} e^{-\delta s}\frac{c_s^{1-\gamma_1}}{1-\gamma_1}\,ds + \int_\tau^{\tau_d} e^{-\delta s}\frac{c_s^{1-\gamma_2}}{1-\gamma_2}\,ds\,\mathbf{1}_{\{\tau<\tau_d\}}\right], \tag{2.6}$$

where γ_1 is the individual's coefficient of relative risk aversion before annuitization, and γ_2 is the individual's coefficient of relative risk aversion during the annuity assessment phase. We assume that $0 < \gamma_1 \leqslant \gamma_2 < 1$. The optimal results for the case $\gamma_1 = \gamma_2$ can be easily obtained by taking $\gamma_1 \to \gamma_2$. Hence, we mainly focus on $0 < \gamma_1 < \gamma_2 < 1$. $\delta > 0$ is the constant subjective discount rate.

Under the assumption of the distribution of the death time τ_d, the individual's expected utility function $J(c, \pi, \tau; x)$ can be rewritten as

$$J(c,\pi,\tau;x) = \mathbb{E}\left[\int_0^{\tau} e^{-\beta s}\frac{c_s^{1-\gamma_1}}{1-\gamma_1}\,ds + \int_\tau^{\infty} e^{-\beta s}\frac{c_s^{1-\gamma_2}}{1-\gamma_2}\,ds\right], \tag{2.7}$$

where $\beta \triangleq \delta + h$.

The value function is given by

$$V(x) = \sup_{(c,\pi,\tau)\in\mathcal{A}(x)} J(c,\pi,\tau;x). \tag{2.8}$$

Define

$$K_i \triangleq r + \frac{\beta - r}{\gamma_i} + \frac{\gamma_i - 1}{2\gamma_i^2}\theta^2, \quad i = 1, 2.$$

We make the following assumption that guarantees the finiteness of the value function (see, Karatzas et al. [13, Corollary 14.2]).

Assumption 2.1. $K_i > 0$, $i = 1, 2$.

Definition 2.1. We denote by $\mathcal{A}_1(x) \subset \mathcal{A}(x)$ the class of admissible controls such that

$$\mathbb{E}\left[\int_\tau^\infty e^{-\beta t}\frac{c_t^{1-\gamma_2}}{1-\gamma_2}\,dt\right] = \mathbb{E}\left[e^{-\beta\tau}U\left(X_\tau^{x,\pi,c} - F\right)\mathbf{1}_{\{\tau<\infty\}}\right],$$

where $U(\cdot)$ is given by

$$U(x) = \sup_{(c,\pi)\in\mathcal{A}_2(x)} \mathbb{E}\left[\int_0^\infty e^{-\beta t}\frac{c_t^{1-\gamma_2}}{1-\gamma_2}\,dt\right] \tag{2.9}$$

with

$$dX_t^{x,\pi,c} = \left[(\mu - r)\pi_t + rX_t^{x,\pi,c} - c_t\right]dt + \sigma\pi_t\,dW_t + F(\psi + r)\,dt, \quad t > 0, \tag{2.10}$$

and $\mathcal{A}_2(x)$ denotes the set of policies (c, π) such that $X_t^{x,\pi,c} \geqslant 0$, for all $t > 0$.

According to Proposition 2.4 of Jeanblanc et al. [12], we have the following lemma.

Lemma 2.1. *If $(\tau, \{(\pi_t, c_t),\ t < \infty\}) \in \mathcal{A}(x)$ is an admissible policy and $\{(\pi_t, c_t),\ \tau < t < \infty\} \equiv \{(\tilde{\pi}^*(X_t), \tilde{c}^*(X_t)),\ \tau < t < \infty\}$ (here, $(\tilde{\pi}^*(X_t), \tilde{c}^*(X_t))$ are the corresponding optimal investment/consumption strategies for $U(x)$), then $(\tau, \{(\pi_t, c_t),\ t < \infty\}) \in \mathcal{A}_1(x)$. Additionally, for every $x \in [0, \infty)$,*

$$V(x) = \sup_{(\tau, \{(\pi_t, c_t),\, t<\infty\}) \in \mathcal{A}_1(x)} \mathbb{E}\left[\int_0^\tau e^{-\beta t} \frac{c_t^{1-\gamma_1}}{1-\gamma_1}\, dt + e^{-\beta\tau} U\big(X_\tau^{x,\pi,c} - F\big)\mathbf{1}_{\{\tau<\infty\}}\right]. \tag{2.11}$$

3. The auxiliary problem

In this section we derive the value function $U(x)$ and the corresponding optimal strategies through a duality approach, which is closely related to He and Pagés [10].

With the help of notations defined in Section 2, the wealth process (2.10) can be rewritten as

$$\zeta_t X_t^{x,\pi,c} + \int_0^t \zeta_s c_s\, ds = x + \int_0^t (\psi + r) F \zeta_s\, ds + \int_0^t \zeta_s \sigma \pi_s\, d\widetilde{W}_s. \tag{3.1}$$

Then by Fatou's lemma and the Bayes rule, for sufficiently large T, we obtain

$$\mathbb{E}\left[H_T X_T^{x,\pi,c} + \int_0^T H_s c_s\, ds\right] \leqslant x + \mathbb{E}\left[\int_0^T H_s(\psi + r) F\, ds\right],$$

and also

$$\mathbb{E}\left[\int_0^T H_s c_s\, ds\right] \leqslant x + \mathbb{E}\left[\int_0^T H_s(\psi + r) F\, ds\right].$$

Let $T \to \infty$. The monotone convergence theorem implies

$$\mathbb{E}\left[\int_0^\infty H_s c_s\, ds - \int_0^\infty H_s(\psi + r) F\, ds\right] \leqslant x. \tag{3.2}$$

To guarantee the existence of an optimal portfolio process, the wealth process $X_t^{x,\pi,c}$ should have the form

$$X_t^{x,\pi,c} = \frac{1}{H_t}\mathbb{E}\left[\int_t^\infty H_s c_s\, ds - \int_t^\infty H_s(\psi + r) F\, ds \,\middle|\, \mathcal{F}_t\right].$$

(See Theorem 9.4 in Section 3.9 of Karatzas and Shreve [15].)

Therefore, the nonnegative wealth constraint implies

$$\mathbb{E}\left[\int_t^\infty H_s c_s\, ds - \int_t^\infty H_s(\psi + r) F\, ds \,\middle|\, \mathcal{F}_t\right] \geqslant 0. \tag{3.3}$$

According to He and Pagés [10], the solutions for the following program (P) are the solutions for the problem (2.9) with constraints (3.2) and (3.3). The program (P) is described as follows: minimizing the maximum attainable utility below with any $D \in \mathcal{D}$,

$$U_1(x) = \sup_{(c,\pi)\in\mathcal{A}_2(x)} \mathbb{E}\left[\int_0^\infty e^{-\beta t}\frac{c_t^{1-\gamma_2}}{1-\gamma_2}\,dt\right],$$

with constraint

$$\mathbb{E}\left[\int_0^\infty H_s D_s c_s\,ds - \int_0^\infty H_s D_s(\psi+r)F\,ds\right] \leqslant x, \tag{3.4}$$

where D_t is a positive, decreasing process with $D_0 = 1$, and we use $\mathcal{D}$ to denote the set of all such processes.

For a Lagrange multiplier $\lambda > 0$, let us define a dual value function

$$\begin{aligned}\widetilde{J}(\lambda, D; c, \pi) &\triangleq \sup_{(c,\pi)\in\mathcal{A}_2(x)}\left\{\mathbb{E}\left[\int_0^\infty e^{-\beta t}\frac{c_t^{1-\gamma_2}}{1-\gamma_2}\,dt\right] - \lambda\mathbb{E}\left[\int_0^\infty H_t D_t c_t\,dt - \int_0^\infty H_t D_t(\psi+r)F\,dt\right]\right\}\\ &= \mathbb{E}\left[\int_0^\infty e^{-\beta t}\left\{\tilde{u}_2(Y_t) + (\psi+r)FY_t\right\}dt\right],\end{aligned}$$

where $Y_t \triangleq \lambda e^{\beta t}D_t H_t$, $Y_0 = \lambda$,

$$\tilde{u}_2(y) \triangleq \sup_{c>0}\left\{\frac{c^{1-\gamma_2}}{1-\gamma_2} - cy\right\} = \frac{\gamma_2}{1-\gamma_2}y^{-\frac{1-\gamma_2}{\gamma_2}},$$

and the corresponding optimal consumption is

$$\tilde{c}_t^* = (Y_t)^{-\frac{1}{\gamma_2}}.$$

We can obtain $U(x)$ from $\widetilde{J}(\lambda, D; c, \pi)$ by Legendre transform inverse formula

$$U(x) = \inf_{\{\lambda>0,\ D>0\}}\left\{\widetilde{J}(\lambda, D; c, \pi) + \lambda x\right\} = \inf_{\lambda>0}\left\{\widetilde{U}(\lambda) + \lambda x\right\}, \tag{3.5}$$

where

$$\widetilde{U}(\lambda) \triangleq \inf_{D>0}\widetilde{J}(\lambda, D; c, \pi).$$

In order to obtain $\widetilde{U}(\lambda)$, we define

$$\phi(t, y) \triangleq \inf_{D>0}\mathbb{E}^{Y_t=y}\left[\int_t^\infty e^{-\beta s}\left\{\tilde{u}_2(Y_s) + (\psi+r)FY_s\right\}ds\right], \tag{3.6}$$

with

$$\frac{dY_t}{Y_t} = (\beta - r)\,dt - \theta\,dW_t + \frac{dD_t}{D_t}. \tag{3.7}$$

It is easily checked that ϕ is strictly convex and decreasing in y, since $\tilde{u}_2$ is strictly convex.

Consider the partial differential operator

$$\mathcal{L} \triangleq \frac{\partial}{\partial t} + (\beta - r)y\frac{\partial}{\partial y} + \frac{1}{2}\theta^2 y^2 \frac{\partial^2}{\partial y^2},$$

then a solution to the following free boundary value problem will be a solution to the optimal problem (3.6).

Variational Inequality 3.1. Find a positive number $\hat{y} > 0$ that makes a zero wealth level, and a function $\tilde{\phi}(\cdot,\cdot) \in C^2((0,\infty) \times \mathbb{R}^+)$ satisfying

(1) $\mathcal{L}\tilde{\phi} + e^{-\beta t}\{\tilde{u}_2(y) + (\psi + r)Fy\} = 0$, $0 < y < \hat{y}$,
(2) $\mathcal{L}\tilde{\phi} + e^{-\beta t}\{\tilde{u}_2(y) + (\psi + r)Fy\} \geqslant 0$, $y \geqslant \hat{y}$,
(3) $\frac{\partial \tilde{\phi}}{\partial y} = 0$, $y \geqslant \hat{y}$,
(4) $\frac{\partial \tilde{\phi}}{\partial y} < 0$, $0 < y < \hat{y}$,

for all $t > 0$.

Let $\lambda_1, \lambda_2 (\lambda_1 > \lambda_2)$ be the two roots of the quadratic equation

$$\frac{1}{2}\theta^2\lambda^2 + \left(\beta - r - \frac{1}{2}\theta^2\right)\lambda - \beta = 0.$$

It is easy to check that $\lambda_1 > 1$ and $\lambda_2 < \min(\frac{\gamma_2 - 1}{\gamma_2}, \frac{\gamma_1 - 1}{\gamma_1})$. The proposition below provides a solution for Variational Inequality 3.1.

Proposition 3.1. *Consider the function*

$$v(y) = \begin{cases} C_1 y^{\lambda_1} + \frac{\gamma_2}{K_2(1-\gamma_2)} y^{-\frac{1-\gamma_2}{\gamma_2}} + \frac{(\psi+r)F}{r} y, & 0 < y < \hat{y}, \\ C_1 \hat{y}^{\lambda_1} + \frac{\gamma_2}{K_2(1-\gamma_2)} \hat{y}^{-\frac{1-\gamma_2}{\gamma_2}} + \frac{(\psi+r)F}{r} \hat{y}, & y \geqslant \hat{y}, \end{cases}$$

with

$$\hat{y} = \left(\frac{(\psi + r)FK_2\gamma_2(\lambda_1 - 1)}{r((\lambda_1 - 1)\gamma_2 + 1)}\right)^{-\gamma_2} > 0,$$

and

$$C_1 = -\frac{1}{\gamma_2 K_2 \lambda_1(\lambda_1 - 1)} \hat{y}^{-\frac{(\lambda_1 - 1)\gamma_2 + 1}{\gamma_2}} < 0.$$

Then, $\tilde{\phi}(t, y) = e^{-\beta t} v(y)$ is a solution to Variational Inequality 3.1.

Theorem 3.1. *If the pair $(\hat{y}, \tilde{\phi}(t, y))$ solves Variational Inequality 3.1, and D_t has a continuous sample path such that $\frac{\partial \phi(t,y)}{\partial y} dD_t = 0$, then $\tilde{\phi}(t, y)$ coincides with $\phi(t, y)$ of (3.6) and the optimal process D_t^* is provided by*

$$D_t^* = \min\left\{1, \inf_{0 \leqslant s \leqslant t}\left\{\frac{\hat{y}}{\lambda H_s e^{\beta s}}\right\}\right\}.$$

Proof. The proof is straightforward by applying Theorem 4 of He and Pagés [10]. $\square$

From the results of Theorem 3.1 and Proposition 3.1, we can derive the value function $U(x)$. Note that

$$\widetilde{U}(\lambda) = \phi(0,\lambda) = C_1\lambda^{\lambda_1} + \frac{\gamma_2}{K_2(1-\gamma_2)}\lambda^{-\frac{1-\gamma_2}{\gamma_2}} + \frac{(\psi+r)F}{r}\lambda, \quad 0<\lambda<\hat{y}.$$

$U(x)$ is obtained at $\lambda^*>0$ such that $\widetilde{U}'(\lambda) = -x$.

Theorem 3.2. *The value function $U(x)$ is given by*

$$U(x) = C_1\left(\lambda^*\right)^{\lambda_1} + \frac{\gamma_2}{K_2(1-\gamma_2)}\left(\lambda^*\right)^{-\frac{1-\gamma_2}{\gamma_2}} + \frac{(\psi+r)F}{r}\lambda^* + \lambda^* x, \quad x \geqslant 0, \tag{3.8}$$

and λ^* is the solution to the following algebraic equation

$$C_1\lambda_1\left(\lambda^*\right)^{\lambda_1-1} - \frac{1}{K_2}\left(\lambda^*\right)^{-\frac{1}{\gamma_2}} + \frac{(\psi+r)F}{r} = -x. \tag{3.9}$$

Remark 3.1. It is easy to check the one-to-one correspondence between $\lambda^* \in (0,\hat{y})$ and $x\in(0,\infty)$ in (3.9) using the fact that $\widetilde{U}(\lambda)$ is strictly convex.

Now we derive the optimal strategies. Let Y_t^* be the solution of SDE (3.7) with an initial value $Y_0^* = \lambda^*$ and $D_t = D_t^*$. We obtain the optimal wealth process X_t^* by substituting Y_t^* for λ^* into (3.9) as follows:

$$X_t^* = -C_1\lambda_1\left(Y_t^*\right)^{\lambda_1-1} + \frac{1}{K_2}\left(Y_t^*\right)^{-\frac{1}{\gamma_2}} - \frac{(\psi+r)F}{r}.$$

Theorem 3.3. *The optimal strategies $(\tilde{c}^*, \tilde{\pi}^*)$ are provided by*

$$\tilde{c}_t^* = \left(Y_t^*\right)^{-\frac{1}{\gamma_2}},$$

and

$$\tilde{\pi}_t^* = \frac{\theta}{\sigma}\left[C_1\lambda_1(\lambda_1-1)\left(Y_t^*\right)^{\lambda_1-1} + \frac{1}{K_2\gamma_2}\left(Y_t^*\right)^{-\frac{1}{\gamma_2}}\right].$$

Since $\hat{y}$ makes the zero wealth level, we observe that as the wealth process goes to zero, the individual does not invest in the risky asset, instead, she reduces her consumption to a rate lower than her annuity income in order to accumulate wealth. This can also be seen from the numerical illustrations in Section 5.

4. The main result

In this section we solve the original optimal problem $V(x)$ using the similar method proposed in Section 3, since there is no annuity income before time τ, we need not consider the borrowing constraints.

For any fixed stopping time $\tau \in \mathcal{S}$, let $\Pi_\tau(x)$ be the class of consumption-portfolio plan (c,π) such that $(c,\pi,\tau)\in\mathcal{A}_1(x)$.

For a Lagrange multiplier $\nu > 0$, we define a dual value function

$$\begin{aligned}\widetilde{J}_1(\nu; c, \pi, \tau) &\triangleq \sup_{(c,\pi)\in \Pi_\tau(x)} \left\{ \mathbb{E}\left[\int_0^\tau e^{-\beta t} \frac{c_t^{1-\gamma_1}}{1-\gamma_1}\, dt + e^{-\beta\tau} U\left(X_\tau^{x,\pi,c} - F\right)\right]\right. \\ &\quad \left. - \nu \mathbb{E}\left[\int_0^\tau H_t c_t\, dt + H_\tau\left(X_\tau^{x,\pi,c} - F\right) + H_\tau F\right]\right\} \\ &= \mathbb{E}\left[\int_0^\tau e^{-\beta t} \bar{u}_1\left(Z_t^\nu\right) dt + e^{-\beta\tau}\left(\overline{U}\left(Z_\tau^\nu\right) - Z_\tau^\nu F\right)\right],\end{aligned}$$

where $Z_t^\nu \triangleq \nu e^{\beta t} H_t$, $Z_0^\nu = \nu$,

$$\bar{u}_1(z) \triangleq \sup_{c>0}\left\{\frac{c^{1-\gamma_1}}{1-\gamma_1} - cz\right\} = \frac{\gamma_1}{1-\gamma_1} z^{-\frac{1-\gamma_1}{\gamma_1}},$$

and

$$\overline{U}(z) \triangleq \sup_{X_\tau^{x,\pi,c} - F \geqslant d} \left\{U\left(X_\tau^{x,\pi,c} - F\right) - z\left(X_\tau^{x,\pi,c} - F\right)\right\}.$$

From the derivation of Section 3, we know

$$U\left(X_\tau^{x,\pi,c} - F\right) = C_1(1-\lambda_1)\left(Y_\tau^*\right)^{\lambda_1} + \frac{1}{K_2(1-\gamma_2)}\left(Y_\tau^*\right)^{-\frac{1-\gamma_2}{\gamma_2}},$$

with

$$X_\tau^{x,\pi,c} - F = -C_1\lambda_1\left(Y_\tau^*\right)^{\lambda_1 - 1} + \frac{1}{K_2}\left(Y_\tau^*\right)^{-\frac{1}{\gamma_2}} - \frac{(\psi + r)F}{r}.$$

By direct calculation, we get

$$U'\left(X_\tau^{x,\pi,c} - F\right) = Y_\tau^*.$$

So in order to obtain the maximization, we should have

$$Y_\tau^* = \begin{cases} Z_\tau^\nu, & \text{if } 0 < Z_\tau^\nu \leqslant \ddot{y}, \\ \ddot{y}, & \text{if } Z_\tau^\nu > \ddot{y}, \end{cases}$$

where $\ddot{y}$ satisfies the equation

$$-C_1\lambda_1 \ddot{y}^{\lambda_1 - 1} + \frac{1}{K_2}\ddot{y}^{-\frac{1}{\gamma_2}} - \frac{(\psi + r)F}{r} = d,$$

and it is easy to verify that $\ddot{y} \leqslant \hat{y}$.

Hence,

$$\overline{U}\left(Z_\tau^\nu\right) = \begin{cases} C_1(Z_\tau^\nu)^{\lambda_1} + \frac{\gamma_2}{K_2(1-\gamma_2)}(Z_\tau^\nu)^{-\frac{1-\gamma_2}{\gamma_2}} + \frac{(\psi+r)F}{r} Z_\tau^\nu, & \text{if } 0 < Z_\tau^\nu \leqslant \ddot{y}, \\ U(d) - Z_\tau^\nu d, & \text{if } Z_\tau^\nu > \ddot{y}, \end{cases}$$

and the corresponding optimal strategies are given by

$$c_t^* = \left(Z_t^\nu\right)^{-\frac{1}{\gamma_1}}, \tag{4.1}$$

and

$$X_\tau^* = \begin{cases} -C_1\lambda_1(Z_\tau^\nu)^{\lambda_1-1} + \frac{1}{K_2}(Z_\tau^\nu)^{-\frac{1}{\gamma_2}} - \frac{\psi}{r}F, & \text{if } 0 < Z_\tau^\nu \leqslant \ddot{y}, \\ F + d & \text{if } Z_\tau^\nu > \ddot{y}. \end{cases} \tag{4.2}$$

Therefore, we obtain

$$\begin{aligned} \widetilde{J}_1(\nu; c, \pi, \tau) = \mathbb{E}\Bigg\{ \int_0^\tau e^{-\beta t}\frac{\gamma_1}{1-\gamma_1}\left(Z_t^\nu\right)^{-\frac{1-\gamma_1}{\gamma_1}}dt + e^{-\beta\tau}\left[U(d) - Z_\tau^\nu(F+d)\right]1_{(Z_\tau^\nu > \ddot{y})} \\ + e^{-\beta\tau}\left(C_1\left(Z_\tau^\nu\right)^{\lambda_1} + \frac{\gamma_2}{K_2(1-\gamma_2)}\left(Z_\tau^\nu\right)^{-\frac{1-\gamma_2}{\gamma_2}} + \frac{\psi}{r}FZ_\tau^\nu\right)1_{(0<Z_\tau^\nu\leqslant\ddot{y})}\Bigg\}, \end{aligned}$$

with

$$dZ_t^\nu = Z_t^\nu\left\{(\beta - r)\,dt - \theta\,dW_t\right\}. \tag{4.3}$$

Now we define

$$\widetilde{V}(\nu) \triangleq \sup_{\tau\in\mathcal{S}} \widetilde{J}_1(\nu; c, \pi, \tau), \quad \nu > 0,$$

then the next proposition gives the value function $V(x)$.

Proposition 4.1. *If $\widetilde{V}(\nu)$ exists and is differentiable for $\nu > 0$, then*

$$V(x) = \inf_{\nu>0}\left[\widetilde{V}(\nu) + \nu x\right]$$

holds for every $x \in (0, \infty)$.

Proof. See Theorem 8.5 and Corollary 8.7 of Karatzas and Wang [16]. □

Hence, we first calculate the value of $\widetilde{V}(\nu)$. This is a standard optimal stopping problem and can be solved by HJB variational inequality method, see Øksendal and Sulem [23]. However, the terminal function here is a piecewise function, which will be a little more complex to deal with. For mathematical simplicity, throughout the remainder of the paper we assume d is 0, and consequently $\ddot{y}$ is equal to $\hat{y}$. For the general case, the value function and optimal strategies can be derived similarly.

Consider the partial differential operator

$$\mathcal{L}_1\phi(z) \triangleq (\beta - r)z\phi'(z) + \frac{1}{2}\theta^2z^2\phi''(z) - \beta\phi(z).$$

Then Propositions 4.2–4.5 provide the value of the dual function $\widetilde{V}(\nu)$, the proofs are deferred to Appendix B.

Proposition 4.2.[1] *If* $\bar{z} < \hat{y} \leqslant \tilde{z}_2$, *and* $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)} < U(0)$, *then*

$$\phi(z) = \begin{cases} C_0 z^{\lambda_1} + \frac{\gamma_1}{K_1(1-\gamma_1)} z^{-\frac{1-\gamma_1}{\gamma_1}}, & 0 < z < \bar{z}, \\ C_1 z^{\lambda_1} + \frac{\gamma_2}{K_2(1-\gamma_2)} z^{-\frac{1-\gamma_2}{\gamma_2}} + \frac{\psi}{r} F z, & \bar{z} \leqslant z \leqslant \hat{y}, \\ U(0) - zF, & \hat{y} < z \leqslant \tilde{z}_2, \\ \widetilde{C}_2 z^{\lambda_2} + \frac{\gamma_1}{K_1(1-\gamma_1)} z^{-\frac{1-\gamma_1}{\gamma_1}}, & z > \tilde{z}_2, \end{cases}$$

satisfies:

(1) $\mathcal{L}_1 \phi + \frac{\gamma_1}{1-\gamma_1} z^{-\frac{1-\gamma_1}{\gamma_1}} = 0$, $0 < z < \bar{z}$, $z > \tilde{z}_2$,
(2) $\mathcal{L}_1 \phi + \frac{\gamma_1}{1-\gamma_1} z^{-\frac{1-\gamma_1}{\gamma_1}} \leqslant 0$, $\bar{z} \leqslant z \leqslant \tilde{z}_2$,
(3) $\phi(z) = C_1 z^{\lambda_1} + \frac{\gamma_2}{K_2(1-\gamma_2)} z^{-\frac{1-\gamma_2}{\gamma_2}} + \frac{\psi}{r} F z$, $\bar{z} \leqslant z \leqslant \hat{y}$,
(4) $\phi(z) = U(0) - zF$, $\hat{y} < z \leqslant \tilde{z}_2$,
(5) $\phi(z) > C_1 z^{\lambda_1} + \frac{\gamma_2}{K_2(1-\gamma_2)} z^{-\frac{1-\gamma_2}{\gamma_2}} + \frac{\psi}{r} F z$, $0 < z < \bar{z}$,
(6) $\phi(z) > U(0) - zF$, $z > \tilde{z}_2$.

Moreover $\widetilde{V}(\nu) = \phi(\nu)$, *and the optimal stopping time is provided by*

$$\tau_\nu = \inf\{t > 0 \mid \bar{z} \leqslant Z_t^\nu \leqslant \tilde{z}_2\}.$$

Remark 4.1. Under the assumption of Proposition 4.2, we can derive $C_0 > 0$, see Lemma B.4.

Proposition 4.3.[2] *If* $\bar{z} \geqslant \hat{y}$, *and* $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)} < U(0)$, *then*

$$\phi(z) = \begin{cases} C_2 z^{\lambda_1} + \frac{\gamma_1}{K_1(1-\gamma_1)} z^{-\frac{1-\gamma_1}{\gamma_1}}, & 0 < z < \tilde{z}_1, \\ U(0) - zF, & \tilde{z}_1 \leqslant z \leqslant \tilde{z}_2, \\ \widetilde{C}_2 z^{\lambda_2} + \frac{\gamma_1}{K_1(1-\gamma_1)} z^{-\frac{1-\gamma_1}{\gamma_1}}, & z > \tilde{z}_2, \end{cases}$$

satisfies:

(1) $\mathcal{L}_1 \phi + \frac{\gamma_1}{1-\gamma_1} z^{-\frac{1-\gamma_1}{\gamma_1}} = 0$, $0 < z < \tilde{z}_1$, $z > \tilde{z}_2$,
(2) $\mathcal{L}_1 \phi + \frac{\gamma_1}{1-\gamma_1} z^{-\frac{1-\gamma_1}{\gamma_1}} \leqslant 0$, $\tilde{z}_1 \leqslant z \leqslant \tilde{z}_2$,
(3) $\phi(z) = U(0) - zF$, $\tilde{z}_1 \leqslant z \leqslant \tilde{z}_2$,
(4) $\phi(z) > C_1 z^{\lambda_1} + \frac{\gamma_2}{K_2(1-\gamma_2)} z^{-\frac{1-\gamma_2}{\gamma_2}} + \frac{\psi}{r} F z$, $0 < z \leqslant \hat{y}$,
(5) $\phi(z) > U(0) - zF$, $\hat{y} < z < \tilde{z}_1$, $z > \tilde{z}_2$.

Moreover $\widetilde{V}(\nu) = \phi(\nu)$, *and the optimal stopping time is provided by*

$$\tau_\nu = \inf\{t > 0 \mid \tilde{z}_1 \leqslant Z_t^\nu \leqslant \tilde{z}_2\}.$$

Proposition 4.4. *If* (1) $\bar{z} < \hat{y}$, $C_0 > 0$ *and* $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)} \geqslant U(0)$, *or* (2) $\bar{z} < \hat{y}$, $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)} < U(0)$, *and* $\tilde{z}_2 < \hat{y}$, *then*

[1] The values of $\bar{z}$, $\tilde{z}_2$, C_0, and $\widetilde{C}_2$ are presented in Proposition B.1 and Proposition B.2.
[2] The values of $\tilde{z}_1$, C_2 are presented in Proposition B.2.

$$\phi(z) = \begin{cases} C_0 z^{\lambda_1} + \frac{\gamma_1}{K_1(1-\gamma_1)} z^{-\frac{1-\gamma_1}{\gamma_1}}, & 0 < z < \bar{z}, \\ C_1 z^{\lambda_1} + \frac{\gamma_2}{K_2(1-\gamma_2)} z^{-\frac{1-\gamma_2}{\gamma_2}} + \frac{\psi}{r} F z, & \bar{z} \leqslant z \leqslant \check{y}, \\ \widehat{C}_2 z^{\lambda_2} + \frac{\gamma_1}{K_1(1-\gamma_1)} z^{-\frac{1-\gamma_1}{\gamma_1}}, & z > \check{y}, \end{cases} \tag{4.4}$$

satisfies:

(1) $\mathcal{L}_1 \phi + \frac{\gamma_1}{1-\gamma_1} z^{-\frac{1-\gamma_1}{\gamma_1}} = 0$, $0 < z < \bar{z}$, $z > \check{y}$,
(2) $\mathcal{L}_1 \phi + \frac{\gamma_1}{1-\gamma_1} z^{-\frac{1-\gamma_1}{\gamma_1}} \leqslant 0$, $\bar{z} \leqslant z \leqslant \check{y}$,
(3) $\phi(z) = C_1 z^{\lambda_1} + \frac{\gamma_2}{K_2(1-\gamma_2)} z^{-\frac{1-\gamma_2}{\gamma_2}} + \frac{\psi}{r} F z$, $\bar{z} \leqslant z \leqslant \check{y}$,
(4) $\phi(z) > C_1 z^{\lambda_1} + \frac{\gamma_2}{K_2(1-\gamma_2)} z^{-\frac{1-\gamma_2}{\gamma_2}} + \frac{\psi}{r} F z$, $0 < z < \bar{z}$, $\check{y} < z \leqslant \hat{y}$,
(5) $\phi(z) > U(0) - zF$, $z > \hat{y}$,

where $\check{y}$ is the largest solution of the following equation on $z \in (\bar{z}, \hat{y})$:

$$(\lambda_2 - \lambda_1) C_1 z^{\lambda_1 - 1} + \frac{(\lambda_2 - 1)\gamma_2 + 1}{K_2(1-\gamma_2)} z^{-\frac{1}{\gamma_2}} - \frac{(\lambda_2 - 1)\gamma_1 + 1}{K_1(1-\gamma_1)} z^{-\frac{1}{\gamma_1}} + (\lambda_2 - 1)\frac{\psi}{r} F = 0, \tag{4.5}$$

and

$$\widehat{C}_2 = C_1 \check{y}^{\lambda_1 - \lambda_2} + \frac{\gamma_2}{K_2(1-\gamma_2)} \check{y}^{-\frac{(\lambda_2-1)\gamma_2+1}{\gamma_2}} - \frac{\gamma_1}{K_1(1-\gamma_1)} \check{y}^{-\frac{(\lambda_2-1)\gamma_1+1}{\gamma_1}} + \frac{\psi}{r} F \check{y}^{1-\lambda_2} > 0.$$

Moreover $\widetilde{V}(\nu) = \phi(\nu)$, and the optimal stopping time is provided by

$$\tau_\nu = \inf\{t > 0 \mid \bar{z} \leqslant Z_t^\nu \leqslant \check{y}\}.$$

Proposition 4.5. *If (1) $\bar{z} \geqslant \hat{y}$, $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)} \geqslant U(0)$, or (2) $\bar{z} < \hat{y}$, $C_0 \leqslant 0$, and $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)} \geqslant U(0)$, then $\tau_\nu = \infty$, and $\widetilde{V}(\nu) = \frac{\gamma_1}{K_1(1-\gamma_1)} \nu^{-\frac{1-\gamma_1}{\gamma_1}}$.*

$V(x)$ is obtained at $\nu^* > 0$ such that $\widetilde{V}'(\nu) = -x$. Thus, we get the main results of our paper.

Theorem 4.1. *The value function $V(x)$ is given by*:

1. *If $\bar{z} < \hat{y} \leqslant \tilde{z}_2$, and $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)} < U(0)$, then*

$$V(x) = \begin{cases} C_0 (\nu^*)^{\lambda_1} + \frac{\gamma_1}{K_1(1-\gamma_1)} (\nu^*)^{-\frac{1-\gamma_1}{\gamma_1}} + \nu^* x, & x > \bar{x}, \\ C_1 (\nu^*)^{\lambda_1} + \frac{\gamma_2}{K_2(1-\gamma_2)} (\nu^*)^{-\frac{1-\gamma_2}{\gamma_2}} + \frac{\psi}{r} F \nu^* + \nu^* x, & F \leqslant x \leqslant \bar{x}, \\ \widetilde{C}_2 (\nu^*)^{\lambda_2} + \frac{\gamma_1}{K_1(1-\gamma_1)} (\nu^*)^{-\frac{1-\gamma_1}{\gamma_1}} + \nu^* x, & 0 < x < F, \end{cases} \tag{4.6}$$

with

$$\bar{x} = -C_0 \lambda_1 \bar{z}^{\lambda_1 - 1} + \frac{1}{K_1} \bar{z}^{-\frac{1}{\gamma_1}} = -C_1 \lambda_1 \bar{z}^{\lambda_1 - 1} + \frac{1}{K_2} \bar{z}^{-\frac{1}{\gamma_2}} - \frac{\psi}{r} F,$$

and ν^* satisfies the following algebraic equations

$$-C_0\lambda_1\big(\nu^*\big)^{\lambda_1-1}+\frac{1}{K_1}\big(\nu^*\big)^{-\frac{1}{\gamma_1}}=x,\quad x>\bar{x},\tag{4.7}$$

$$-C_1\lambda_1\big(\nu^*\big)^{\lambda_1-1}+\frac{1}{K_2}\big(\nu^*\big)^{-\frac{1}{\gamma_2}}-\frac{\psi}{r}F=x,\quad F\leqslant x\leqslant\bar{x},\tag{4.8}$$

and

$$-\widetilde{C}_2\lambda_2\big(\nu^*\big)^{\lambda_2-1}+\frac{1}{K_1}\big(\nu^*\big)^{-\frac{1}{\gamma_1}}=x,\quad 0<x<F.\tag{4.9}$$

2. *If* $\bar{z}\geqslant\hat{y}$, *and* $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)}<U(0)$, *then*

$$V(x)=\begin{cases}C_2(\nu^*)^{\lambda_1}+\frac{\gamma_1}{K_1(1-\gamma_1)}(\nu^*)^{-\frac{1-\gamma_1}{\gamma_1}}+\nu^*x, & x>F,\\ U(0), & x=F,\\ \widetilde{C}_2(\nu^*)^{\lambda_2}+\frac{\gamma_1}{K_1(1-\gamma_1)}(\nu^*)^{-\frac{1-\gamma_1}{\gamma_1}}+\nu^*x, & 0<x<F,\end{cases}\tag{4.10}$$

and ν^* *satisfies the following algebraic equations*

$$-C_2\lambda_1\big(\nu^*\big)^{\lambda_1-1}+\frac{1}{K_1}\big(\nu^*\big)^{-\frac{1}{\gamma_1}}=x,\quad x\geqslant F,\tag{4.11}$$

and

$$-\widetilde{C}_2\lambda_2\big(\nu^*\big)^{\lambda_2-1}+\frac{1}{K_1}\big(\nu^*\big)^{-\frac{1}{\gamma_1}}=x,\quad 0<x<F.\tag{4.12}$$

3. *If* (1) $\bar{z}<\hat{y}$, $C_0>0$ *and* $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)}\geqslant U(0)$, *or* (2) $\bar{z}<\hat{y}$, $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)}<U(0)$, *and* $\tilde{z}_2<\hat{y}$, *then*

$$V(x)=\begin{cases}C_0(\nu^*)^{\lambda_1}+\frac{\gamma_1}{K_1(1-\gamma_1)}(\nu^*)^{-\frac{1-\gamma_1}{\gamma_1}}+\nu^*x, & x>\bar{x},\\ C_1(\nu^*)^{\lambda_1}+\frac{\gamma_2}{K_2(1-\gamma_2)}(\nu^*)^{-\frac{1-\gamma_2}{\gamma_2}}+\frac{\psi}{r}F\nu^*+\nu^*x, & \check{x}\leqslant x\leqslant\bar{x},\\ \widehat{C}_2(\nu^*)^{\lambda_2}+\frac{\gamma_1}{K_1(1-\gamma_1)}(\nu^*)^{-\frac{1-\gamma_1}{\gamma_1}}+\nu^*x, & 0<x<\check{x},\end{cases}\tag{4.13}$$

with

$$\check{x}=-\widehat{C}_2\lambda_2\check{y}^{\lambda_2-1}+\frac{1}{K_1}\check{y}^{-\frac{1}{\gamma_1}}=-C_1\lambda_1\check{y}^{\lambda_1-1}+\frac{1}{K_2}\check{y}^{-\frac{1}{\gamma_2}}-\frac{\psi}{r}F,$$

and ν^* *satisfies the following algebraic equations*

$$-C_0\lambda_1\big(\nu^*\big)^{\lambda_1-1}+\frac{1}{K_1}\big(\nu^*\big)^{-\frac{1}{\gamma_1}}=x,\quad x>\bar{x},\tag{4.14}$$

$$-C_1\lambda_1\big(\nu^*\big)^{\lambda_1-1}+\frac{1}{K_2}\big(\nu^*\big)^{-\frac{1}{\gamma_2}}-\frac{\psi}{r}F=x,\quad \check{x}\leqslant x\leqslant\bar{x},\tag{4.15}$$

and

$$-\widehat{C}_2\lambda_2\big(\nu^*\big)^{\lambda_2-1}+\frac{1}{K_1}\big(\nu^*\big)^{-\frac{1}{\gamma_1}}=x,\quad 0<x<\check{x}.\tag{4.16}$$

4. *If* (1) $\bar{z}\geqslant\hat{y}$, $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)}\geqslant U(0)$, *or* (2) $\bar{z}<\hat{y}$, $C_0\leqslant 0$, *and* $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)}\geqslant U(0)$, *then* $V(x)=\frac{x^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)}$.

Proof. For case 1, it is easy to check the one-to-one correspondence between $\nu^* \in (0,\hat{y}] \cup (\tilde{z}_2,\infty)$ and $x \in [F,\infty) \cup (0,F)$ using the fact that $\widetilde{V}(\nu)$ is strictly convex on $(0,\hat{y}] \cup (\tilde{z}_2,\infty)$. By similar arguments, we obtain the results in cases 2, 3 and 4. $\square$

Now we derive the optimal strategies. Let Z_t^* be the solution of SDE (4.3) with an initial value $Z_0^* = \nu^*$. We obtain the optimal wealth process X_t^* by substituting Z_t^* for ν^* into (4.7), (4.9), (4.11), (4.12), (4.14) and (4.16) as follows:

1. If $\bar{z} < \hat{y} \leqslant \tilde{z}_2$, and $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)} < U(0)$,

$$X_t^* = -C_0\lambda_1\left(Z_t^*\right)^{\lambda_1-1} + \frac{1}{K_1}\left(Z_t^*\right)^{-\frac{1}{\gamma_1}}, \quad X_t^* \geqslant \bar{x}, \tag{4.17}$$

and

$$X_t^* = -\widetilde{C}_2\lambda_2\left(Z_t^*\right)^{\lambda_2-1} + \frac{1}{K_1}\left(Z_t^*\right)^{-\frac{1}{\gamma_1}}, \quad 0 < X_t^* \leqslant F. \tag{4.18}$$

2. If $\bar{z} \geqslant \hat{y}$, and $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)} < U(0)$,

$$X_t^* = -C_2\lambda_1\left(Z_t^*\right)^{\lambda_1-1} + \frac{1}{K_1}\left(Z_t^*\right)^{-\frac{1}{\gamma_1}}, \quad X_t^* \geqslant F, \tag{4.19}$$

and

$$X_t^* = -\widetilde{C}_2\lambda_2\left(Z_t^*\right)^{\lambda_2-1} + \frac{1}{K_1}\left(Z_t^*\right)^{-\frac{1}{\gamma_1}}, \quad 0 < X_t^* < F. \tag{4.20}$$

3. If (1) $\bar{z} < \hat{y}$, $C_0 > 0$ and $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)} \geqslant U(0)$, or (2) $\bar{z} < \hat{y}$, $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)} < U(0)$, and $\tilde{z}_2 < \hat{y}$,

$$X_t^* = -C_0\lambda_1\left(Z_t^*\right)^{\lambda_1-1} + \frac{1}{K_1}\left(Z_t^*\right)^{-\frac{1}{\gamma_1}}, \quad X_t^* \geqslant \bar{x}, \tag{4.21}$$

and

$$X_t^* = -\widehat{C}_2\lambda_2\left(Z_t^*\right)^{\lambda_2-1} + \frac{1}{K_1}\left(Z_t^*\right)^{-\frac{1}{\gamma_1}}, \quad 0 < X_t^* \leqslant \check{x}. \tag{4.22}$$

Theorem 4.2. *The optimal strategies (c^*, π^*, τ^*) are provided by:*

1. *If $\bar{z} < \hat{y} \leqslant \tilde{z}_2$, and $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)} < U(0)$, then*

$$c_t^* = \begin{cases} (Z_t^*)^{-\frac{1}{\gamma_1}}, & \text{if } 0 \leqslant t \leqslant \tau^*, \\ (Y_t^*)^{-\frac{1}{\gamma_2}}, & \text{if } t > \tau^*, \end{cases}$$

and

$$\pi_t^* = \begin{cases} \frac{\theta}{\sigma}[C_0\lambda_1(\lambda_1-1)(Z_t^*)^{\lambda_1-1} + \frac{1}{K_1\gamma_1}(Z_t^*)^{-\frac{1}{\gamma_1}}], & \text{if } 0 \leqslant t \leqslant \tau^*,\ X_t^* \geqslant \bar{x}, \\ \frac{\theta}{\sigma}[C_1\lambda_1(\lambda_1-1)(Y_t^*)^{\lambda_1-1} + \frac{1}{K_2\gamma_2}(Y_t^*)^{-\frac{1}{\gamma_2}}], & \text{if } t > \tau^*, \\ \frac{\theta}{\sigma}[\widetilde{C}_2\lambda_2(\lambda_2-1)(Z_t^*)^{\lambda_2-1} + \frac{1}{K_1\gamma_1}(Z_t^*)^{-\frac{1}{\gamma_1}}], & \text{if } 0 \leqslant t \leqslant \tau^*,\ 0 < X_t^* \leqslant F, \end{cases}$$

with

$$\tau^* = \inf\{t > 0 \mid F \leqslant X_t^* \leqslant \bar{x}\},$$

and

$$X_t^* = -C_1\lambda_1 (Y_t^*)^{\lambda_1 - 1} + \frac{1}{K_2}(Y_t^*)^{-\frac{1}{\gamma_2}} - \frac{\psi + r}{r}F, \quad t > \tau^*.$$

2. *If* $\bar{z} \geqslant \hat{y}$*, and* $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)} < U(0)$*, then*

$$c_t^* = \begin{cases} (Z_t^*)^{-\frac{1}{\gamma_1}}, & \text{if } 0 \leqslant t \leqslant \tau^*, \\ (Y_t^*)^{-\frac{1}{\gamma_2}}, & \text{if } t > \tau^*, \end{cases}$$

and

$$\pi_t^* = \begin{cases} \frac{\theta}{\sigma}[C_2\lambda_1(\lambda_1 - 1)(Z_t^*)^{\lambda_1 - 1} + \frac{1}{K_1\gamma_1}(Z_t^*)^{-\frac{1}{\gamma_1}}], & \text{if } 0 \leqslant t \leqslant \tau^*,\ X_t^* \geqslant F, \\ \frac{\theta}{\sigma}[C_1\lambda_1(\lambda_1 - 1)(Y_t^*)^{\lambda_1 - 1} + \frac{1}{K_2\gamma_2}(Y_t^*)^{-\frac{1}{\gamma_2}}], & \text{if } t > \tau^*, \\ \frac{\theta}{\sigma}[\widetilde{C}_2\lambda_2(\lambda_2 - 1)(Z_t^*)^{\lambda_2 - 1} + \frac{1}{K_1\gamma_1}(Z_t^*)^{-\frac{1}{\gamma_1}}], & \text{if } 0 \leqslant t \leqslant \tau^*,\ 0 < X_t^* \leqslant F, \end{cases}$$

with

$$\tau^* = \inf\{t > 0 \mid X_t^* = F\},$$

and

$$X_t^* = -C_1\lambda_1 (Y_t^*)^{\lambda_1 - 1} + \frac{1}{K_2}(Y_t^*)^{-\frac{1}{\gamma_2}} - \frac{\psi + r}{r}F, \quad t > \tau^*.$$

3. *If* (1) $\bar{z} < \hat{y}$, $C_0 > 0$ *and* $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)} \geqslant U(0)$*, or* (2) $\bar{z} < \hat{y}$, $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)} < U(0)$*, and* $\tilde{z}_2 < \hat{y}$*, then*

$$c_t^* = \begin{cases} (Z_t^*)^{-\frac{1}{\gamma_1}}, & \text{if } 0 \leqslant t \leqslant \tau^*, \\ (Y_t^*)^{-\frac{1}{\gamma_2}}, & \text{if } t > \tau^*, \end{cases}$$

and

$$\pi_t^* = \begin{cases} \frac{\theta}{\sigma}[C_0\lambda_1(\lambda_1 - 1)(Z_t^*)^{\lambda_1 - 1} + \frac{1}{K_1\gamma_1}(Z_t^*)^{-\frac{1}{\gamma_1}}], & \text{if } 0 \leqslant t \leqslant \tau^*,\ X_t^* \geqslant \bar{x}, \\ \frac{\theta}{\sigma}[C_1\lambda_1(\lambda_1 - 1)(Y_t^*)^{\lambda_1 - 1} + \frac{1}{K_2\gamma_2}(Y_t^*)^{-\frac{1}{\gamma_2}}], & \text{if } t > \tau^*, \\ \frac{\theta}{\sigma}[\widehat{C}_2\lambda_2(\lambda_2 - 1)(Z_t^*)^{\lambda_2 - 1} + \frac{1}{K_1\gamma_1}(Z_t^*)^{-\frac{1}{\gamma_1}}], & \text{if } 0 \leqslant t \leqslant \tau^*,\ 0 < X_t^* \leqslant \check{x}, \end{cases}$$

with

$$\tau^* = \inf\{t > 0 \mid \check{x} \leqslant X_t^* \leqslant \bar{x}\},$$

and

$$X_t^* = -C_1\lambda_1 (Y_t^*)^{\lambda_1 - 1} + \frac{1}{K_2}(Y_t^*)^{-\frac{1}{\gamma_2}} - \frac{\psi + r}{r}F, \quad t > \tau^*.$$

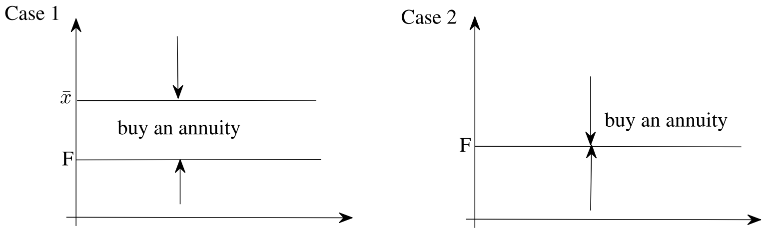

Fig. 1. Case 1: If $\bar{z} < \hat{y} \leqslant \tilde{z}_2$, and $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)} < U(0)$, then it is optimal to buy an annuity when the wealth level lies between F and $\bar{x}$. Case 2: If $\bar{z} \geqslant \hat{y}$, and $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)} < U(0)$, then it is optimal to buy an annuity when the wealth level reaches F.

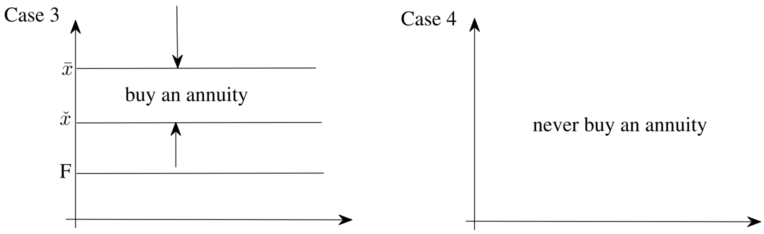

Fig. 2. Case 3: If (1) $\bar{z} < \hat{y}$, $C_0 > 0$ and $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)} \geqslant U(0)$, or (2) $\bar{z} < \hat{y}$, $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)} < U(0)$, and $\tilde{z}_2 < \hat{y}$, then it is optimal to buy an annuity when the wealth level lies between $\check{x}$ and $\bar{x}$. Case 4: If (1) $\bar{z} \geqslant \hat{y}$, $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)} \geqslant U(0)$, or (2) $\bar{z} < \hat{y}$, $C_0 \leqslant 0$, and $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)} \geqslant U(0)$, then it is optimal never to buy an annuity.

4. *If* (1) $\bar{z} \geqslant \hat{y}$, $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)} \geqslant U(0)$, *or* (2) $\bar{z} < \hat{y}$, $C_0 \leqslant 0$, *and* $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)} \geqslant U(0)$, *then* $\tau^* = \infty$, $c_t^* = (Z_t^*)^{-\frac{1}{\gamma_1}}$, $\pi_t^* = \frac{\theta}{\sigma} \frac{1}{K_1\gamma_1}(Z_t^*)^{-\frac{1}{\gamma_1}}$, *and* $X_t^* = \frac{1}{K_1}(Z_t^*)^{-\frac{1}{\gamma_1}}$, *which is obviously the classical Merton solution.*

Proof. The optimal strategies are derived by applying Itô's formula to the wealth process and combining the results of Theorem 3.3. □

Remark 4.2. 1. From case 1 and case 2 in Theorem 4.1, we see that, due to the borrowing constraints the value function is not differentiable at the wealth level F: for case 1 $\lim_{x\uparrow F} V'(x-) = \tilde{z}_2$ and $\lim_{x\downarrow F} V'(x+) = \hat{y}$; for case 2 $\lim_{x\uparrow F} V'(x-) = \tilde{z}_2$ and $\lim_{x\downarrow F} V'(x+) = \tilde{z}_1$.

2. In case 3 of Theorem 4.1, the threshold $\check{x}$ is greater than the fixed annuity amount F, because $C_1\lambda_1 y^{\lambda_1-1} - \frac{1}{K_2} y^{-\frac{1}{\gamma_2}}$ is increasing with y for $y < \hat{y}$.

3. Theorem 4.2 shows that the optimal strategies can be divided into four categories according to the free parameters, see Figs. 1 and 2.

5. Numerical examples

In this section, we provide numerical examples to illustrate the analytical results of Section 4. We focus our attention on the effects of the fixed annuitization cost F, the risk aversion γ, the investment volatility σ and the mortality rate ψ on the optimal strategies.

Figs. 3–6 show the graphical results of the optimal strategies in Theorem 4.2. The dash dot line denotes the classical Merton solution. We observe that when the wealth possessed by the agent is less than 1000 initially, she consumes less and takes risk more before the wealth reaches the level 1000, because she has an incentive to reach the level and annuitize in order to obtain a lifelong annuity income. Figs. 3 and 4 illustrate the optimal consumption and investment strategies with $\gamma_1 = 0.8$ and $\gamma_2 = 0.9$. In this case, the optimal annuitization time is $\tau^* = \inf\{t > 0,\ 1000 \leqslant X_t^* \leqslant 5217.3\}$. Fig. 3 represents the optimal consumption process, which is discontinuous at the annuitization thresholds 1000 and 5217.3. At the wealth level 1000,

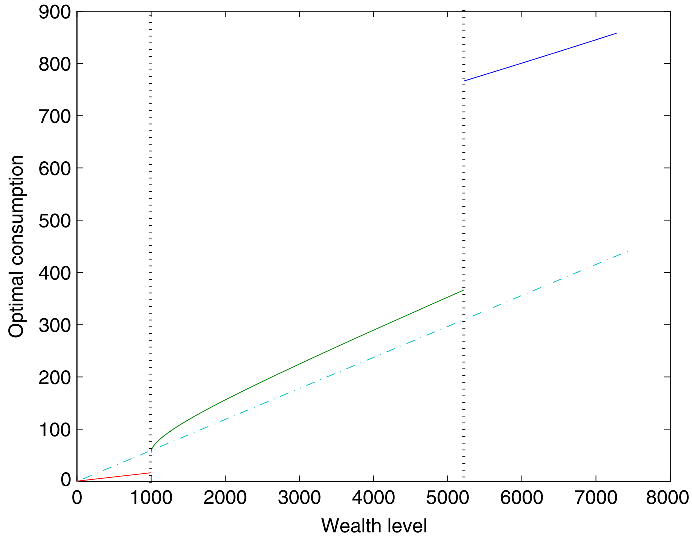

Fig. 3. The optimal consumption rate with $\gamma_1 = 0.8$, $\gamma_2 = 0.9$ ($\beta = 0.06$, $r = 0.04$, $\mu = 0.08$, $\sigma = 0.21$, $\gamma_1 = 0.8$, $\gamma_2 = 0.9$, $F = 1000$ and $\psi = 0.05$).

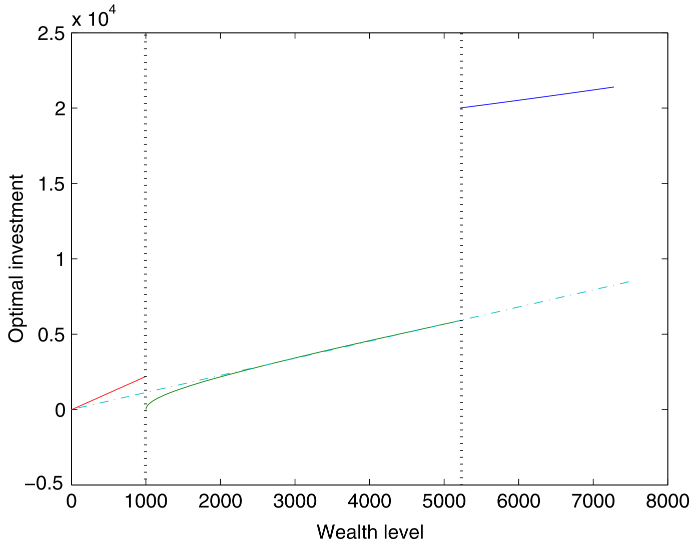

Fig. 4. The optimal investment with $\gamma_1 = 0.8$, $\gamma_2 = 0.9$ ($\beta = 0.06$, $r = 0.04$, $\mu = 0.08$, $\sigma = 0.21$, $\gamma_1 = 0.8$, $\gamma_2 = 0.9$, $F = 1000$ and $\psi = 0.05$).

the agent purchases the annuity immediately then the wealth decreases to zero. The corresponding optimal investment after the annuitization is zero as we see in Fig. 4. While, the corresponding optimal consumption after annuitization is still positive as shown in Fig. 3, since the agent will obtain a continuous annuity income after the annuitization. In Figs. 5 and 6, we assume $\gamma_1 = 0.7$ and $\gamma_2 = 0.8$. In this case it is optimal to annuitize at the wealth level 1000.

In Table 1, for various fixed annuitization cost F and the mortality rate ψ, we give the optimal annuitization region in which the individual will annuitize immediately. Notice that the optimal annuitization region diminishes for a given level of annuitization cost as the individual's expected future lifetime decreases from 33.3 years to 16.7 years. Also, for a given level of mortality rate, the optimal annuitization region becomes smaller as the annuitization cost increases. For the case $F = 30\,000$, the individual will never annuitize, since the annuitization cost is too high for her to afford.

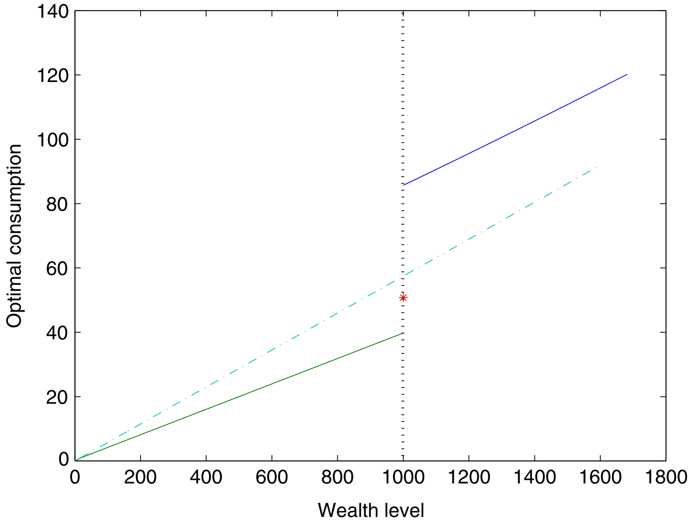

Fig. 5. The optimal consumption rate with $\gamma_1 = 0.7$, $\gamma_2 = 0.8$ ($\beta = 0.06$, $r = 0.04$, $\mu = 0.08$, $\sigma = 0.21$, $\gamma_1 = 0.7$, $\gamma_2 = 0.8$, $F = 1000$ and $\psi = 0.05$).

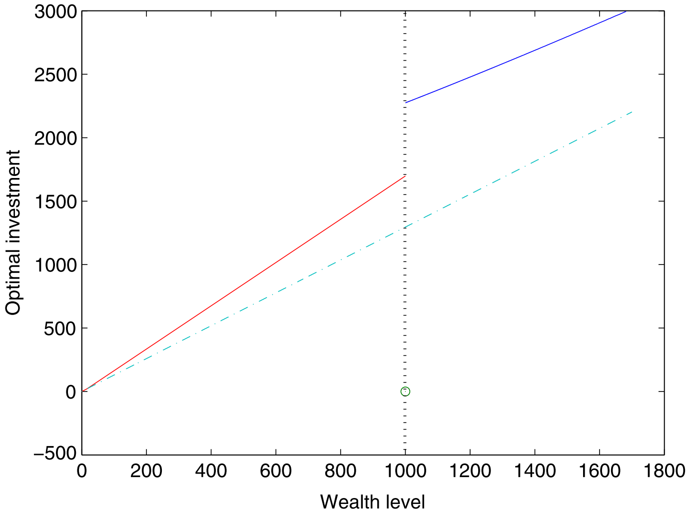

Fig. 6. The optimal investment with $\gamma_1 = 0.7$, $\gamma_2 = 0.8$ ($\beta = 0.06$, $r = 0.04$, $\mu = 0.08$, $\sigma = 0.21$, $\gamma_1 = 0.7$, $\gamma_2 = 0.8$, $F = 1000$ and $\psi = 0.05$).

In Table 2, we investigate the impacts of the investment volatility σ and the risk aversion γ on the annuitization. In Table 2(a), we fix the risk aversion during annuity assessment phase $\gamma_2 = 0.9$ and observe that as the investment volatility σ increases from 0.15 to 0.35, the optimal annuitization region enlarges. This can be seen obviously when $\gamma_1 = 0.7$ and $\gamma_1 = 0.8$. At the same time, as the risk aversion before annuitization goes up, the optimal annuitization region also becomes larger. The economic intuition for these results is quite clear. As the relative risk of investing in the high-return alternative increases or the individual feels more risk averse, it becomes more appealing to annuitize one's wealth. In Table 2(b), we fix the risk aversion $\gamma_1 = 0.55$ and change the value of the investment volatility σ from 0.15 to 0.35 as well as that of the risk aversion γ_2 from 0.55 to 0.90. We find that the higher the investment volatility σ is, the larger the optimal annuitization region becomes as considered in case (a). However, for a fixed volatility rate, as the risk aversion γ_2 increases from 0.55 to 0.70, the optimal annuitization region diminishes; as the risk aversion γ_2 approaches 0.90, the optimal annuitization region seems to be larger. Hence there is no explicit monotonic relationship between the optimal annuitization region and the risk aversion γ_2.

Table 1
How do the fixed cost F and the mortality rate ψ affect annuitization?

Cost F	The optimal annuitization region			
	$\psi = 0.03$	$\psi = 0.04$	$\psi = 0.05$	$\psi = 0.06$
1000	(1000, 5427.4)	(1000, 5320.8)	(1000, 5217.3)	(1000, 5116.6)
2000	(2000, 5442.7)	(2000, 5279.5)	(2000, 5123.3)	(2000, 4973.7)
5000	(5000, 6431.3)	(5000, 6233.2)	(5000, 6052.4)	(5000, 5888.2)
10 000	(10 000, 10 066.4)	(10 000, 10 015.7)	10 000	10 000
15 000	15 000	15 000	15 000	15 000
30 000	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Other parameters $\beta = 0.06$, $r = 0.04$, $\mu = 0.08$, $\sigma = 0.21$, $\gamma_1 = 0.8$, $\gamma_2 = 0.9$.

Table 2
How do the investment volatility σ and the risk aversion γ affect annuitization?

(a) Fix $\gamma_2 = 0.9$, and consider the changes of γ_1 and σ					
Volatility σ	The optimal annuitization region				
	$\gamma_1 = 0.5$	$\gamma_1 = 0.6$	$\gamma_1 = 0.7$	$\gamma_1 = 0.8$	$\gamma_1 = 0.9$
0.15	$\emptyset$	$\emptyset$	1000	$(1000, 3163.3)$	$(1000, +\infty)$
0.20	$\emptyset$	1000	$(1000, 1020.1)$	$(1000, 4937.6)$	$(1000, +\infty)$
0.25	$\emptyset$	1000	$(1000, 1066.1)$	$(1000, 6140.4)$	$(1000, +\infty)$
0.30	1000	1000	$(1000, 1104.3)$	$(1000, 6968.6)$	$(1000, +\infty)$
0.35	1000	1000	$(1000, 1133.0)$	$(1000, 7560.3)$	$(1000, +\infty)$
(b) Fix $\gamma_1 = 0.55$, and consider the changes of γ_2 and σ					
Volatility σ	The optimal annuitization region				
	$\gamma_2 = 0.55$	$\gamma_2 = 0.60$	$\gamma_2 = 0.65$	$\gamma_2 = 0.70$	$\gamma_2 = 0.90$
0.15	$(1069.3, +\infty)$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
0.20	$(1000, +\infty)$	$(1000, 1007.7)$	$\emptyset$	$\emptyset$	$\emptyset$
0.25	$(1000, +\infty)$	$(1000, 1037.7)$	$\emptyset$	$\emptyset$	1000
0.30	$(1000, +\infty)$	$(1000, 1060.9)$	1000	$\emptyset$	1000
0.35	$(1000, +\infty)$	$(1000, 1077.6)$	1000	$\emptyset$	1000

Other parameters $\beta = 0.06$, $r = 0.04$, $\mu = 0.08$, $F = 1000$, $\psi = 0.05$.

Acknowledgments

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Appendix A. Supplementary proofs for Section 3

Proof of Proposition 3.1. First, we consider the partial differential equation (PDE) (1) of Variational Inequality 3.1:

$$\frac{\partial \tilde{\phi}}{\partial t} + (\beta - r)y\frac{\partial \tilde{\phi}}{\partial y} + \frac{1}{2}\theta^2 y^2 \frac{\partial^2 \tilde{\phi}}{\partial y^2} + e^{-\beta t}\left(\frac{\gamma_2}{1-\gamma_2}y^{-\frac{1-\gamma_2}{\gamma_2}} + (\psi + r)Fy\right) = 0, \quad \text{for } 0 < y < \hat{y},$$

with a boundary condition $\frac{\partial \tilde{\phi}}{\partial y}(t, \hat{y}) = 0$.

If we guess a trial solution of the form $\tilde{\phi}(t, y) = e^{-\beta t}v(y)$, then the PDE can be rewritten as an ordinary differential equation (ODE):

$$\frac{1}{2}\theta^2 y^2 v''(y) + (\beta - r)yv'(y) - \beta v(y) + \frac{\gamma_2}{1-\gamma_2}y^{-\frac{1-\gamma_2}{\gamma_2}} + (\psi + r)Fy = 0, \quad \text{for } 0 < y < \hat{y}.$$

The general solution of the above ODE is $A_1 y^{\lambda_1} + A_2 y^{\lambda_2} + \frac{\gamma_2}{K_2(1-\gamma_2)}y^{-\frac{1-\gamma_2}{\gamma_2}} + \frac{(\psi+r)F}{r}y$, where A_1 and A_2 are arbitrary constants. Since the value function is well defined and finite, we guess $v(y)$ has the following form:

$$v(y)=\begin{cases} C_1y^{\lambda_1}+\frac{\gamma_2}{K_2(1-\gamma_2)}y^{-\frac{1-\gamma_2}{\gamma_2}}+\frac{(\psi+r)F}{r}y, & 0<y<\hat{y},\\ C_1\hat{y}^{\lambda_1}+\frac{\gamma_2}{K_2(1-\gamma_2)}\hat{y}^{-\frac{1-\gamma_2}{\gamma_2}}+\frac{(\psi+r)F}{r}\hat{y}, & y\geqslant \hat{y}.\end{cases}$$

By the boundary conditions:

$$\frac{\partial\tilde{\phi}}{\partial y}(t,\hat{y})=0,\qquad \frac{\partial^2\tilde{\phi}}{\partial y^2}(t,\hat{y})=0,$$

we obtain the values of C_1 and $\hat{y}$.

Next we will check the condition (2) of Variational Inequality 3.1. We have for $y\geqslant\hat{y}$,

$$\begin{aligned}\mathcal{L}\tilde{\phi}+e^{-\beta t}\{\tilde{u}_2(y)+(\psi+r)Fy\}&=-\beta e^{-\beta t}\left\{C_1\hat{y}^{\lambda_1}+\frac{\gamma_2}{K_2(1-\gamma_2)}\hat{y}^{-\frac{1-\gamma_2}{\gamma_2}}+\frac{(\psi+r)F}{r}\hat{y}\right\}\\&\quad+e^{-\beta t}\left\{\frac{\gamma_2}{1-\gamma_2}y^{-\frac{1-\gamma_2}{\gamma_2}}+(\psi+r)Fy\right\},\end{aligned}$$

with

$$-\beta e^{-\beta t}\left\{C_1\hat{y}^{\lambda_1}+\frac{\gamma_2}{K_2(1-\gamma_2)}\hat{y}^{-\frac{1-\gamma_2}{\gamma_2}}+\frac{(\psi+r)F}{r}\hat{y}\right\}+e^{-\beta t}\left\{\frac{\gamma_2}{1-\gamma_2}\hat{y}^{-\frac{1-\gamma_2}{\gamma_2}}+(\psi+r)F\hat{y}\right\}=0.$$

Therefore, it only needs to prove

$$\frac{\gamma_2}{1-\gamma_2}y^{-\frac{1-\gamma_2}{\gamma_2}}+(\psi+r)Fy\geqslant\frac{\gamma_2}{1-\gamma_2}\hat{y}^{-\frac{1-\gamma_2}{\gamma_2}}+(\psi+r)F\hat{y},\quad y\geqslant\hat{y}. \tag{A.1}$$

We define a function:

$$f(y)\triangleq\frac{\gamma_2}{1-\gamma_2}y^{-\frac{1-\gamma_2}{\gamma_2}}+(\psi+r)Fy.$$

Then $f'(y)\geqslant 0$, for $y\geqslant\hat{y}$ is equivalent to $y\geqslant((\psi+r)F)^{-\gamma_2}$. We claim that $\hat{y}\geqslant((\psi+r)F)^{-\gamma_2}$, which can be proved by direct calculation and the definition of λ_1 and K_2. Hence (A.1) is proved, and consequently the condition (2) of Variational Inequality 3.1 holds.

In order to verify the condition (4) of Variational Inequality 3.1, we only need to consider the following inequality:

$$C_1\lambda_1y^{\lambda_1-1}-\frac{1}{K_2}y^{-\frac{1}{\gamma_2}}+\frac{(\psi+r)F}{r}<0,\quad 0<y<\hat{y}, \tag{A.2}$$

which becomes an equality at $y=\hat{y}$.

Define a function:

$$g(y)\triangleq C_1\lambda_1y^{\lambda_1-1}-\frac{1}{K_2}y^{-\frac{1}{\gamma_2}}+\frac{(\psi+r)F}{r}.$$

We can verify that $g'(y)>0$, for $0<y<\hat{y}$, is equivalent to

$$y^{1-\lambda_1-\frac{1}{\gamma_2}}>-C_1K_2\gamma_2\lambda_1(\lambda_1-1)=\hat{y}^{1-\lambda_1-\frac{1}{\gamma_2}},\quad 0<y<\hat{y},$$

which is obvious, since $1-\lambda_1-\frac{1}{\gamma_2}<0$. This proves (A.2) and also the condition (4) holds. □

Proof of Theorem 3.3. It is straightforward to see that the optimal consumption process can be written as

$$\tilde{c}_t^* = \left(Y_t^*\right)^{-\frac{1}{\gamma_2}}.$$

In order to find the optimal portfolio process, we apply Itô's formula to X_t^* and obtain

$$\begin{aligned}
dX_t^* &= -\left[C_1\lambda_1(\lambda_1-1)\left(Y_t^*\right)^{\lambda_1-2} + \frac{1}{K_2\gamma_2}\left(Y_t^*\right)^{-\frac{1}{\gamma_2}-1}\right]dY_t^* \\
&\quad - \frac{1}{2}\left[C_1\lambda_1(\lambda_1-1)(\lambda_1-2)\left(Y_t^*\right)^{\lambda_1-3} - \frac{1+\gamma_2}{K_2\gamma_2^2}\left(Y_t^*\right)^{-\frac{1}{\gamma_2}-2}\right]\left(dY_t^*\right)^2 \\
&= r\left[-C_1\lambda_1\left(Y_t^*\right)^{\lambda_1-1} + \frac{1}{K_2}\left(Y_t^*\right)^{-\frac{1}{\gamma_2}} - \frac{(\psi+r)F}{r}\right]dt + (\psi+r)F\,dt \\
&\quad + \left(Y_t^*\right)^{-\frac{1}{\gamma_2}}\left[\frac{r-\beta}{K_2\gamma_2} + \frac{1-\gamma_2}{2K_2\gamma_2^2}\theta^2 - \frac{r}{K_2}\right]dt \\
&\quad + \lambda_1C_1\left(Y_t^*\right)^{\lambda_1-1}\left[-\beta(\lambda_1-1) + r\lambda_1 - \frac{\lambda_1(\lambda_1-1)\theta^2}{2}\right]dt \\
&\quad + \theta^2\left[C_1\lambda_1(\lambda_1-1)\left(Y_t^*\right)^{\lambda_1-1} + \frac{1}{K_2\gamma_2}\left(Y_t^*\right)^{-\frac{1}{\gamma_2}}\right]dt \\
&\quad + \left[-C_1\lambda_1(\lambda_1-1)\left(Y_t^*\right)^{\lambda_1-1} - \frac{1}{K_2\gamma_2}\left(Y_t^*\right)^{-\frac{1}{\gamma_2}}\right]\frac{dD_t^*}{D_t^*} \\
&\quad + \theta\left[C_1\lambda_1(\lambda_1-1)\left(Y_t^*\right)^{\lambda_1-1} + \frac{1}{K_2\gamma_2}\left(Y_t^*\right)^{-\frac{1}{\gamma_2}}\right]dW_t \\
&= r\left[-C_1\lambda_1\left(Y_t^*\right)^{\lambda_1-1} + \frac{1}{K_2}\left(Y_t^*\right)^{-\frac{1}{\gamma_2}} - \frac{(\psi+r)F}{r}\right]dt + (\psi+r)F dt - \left(Y_t^*\right)^{-\frac{1}{\gamma_2}}dt \\
&\quad + \theta\left[C_1\lambda_1(\lambda_1-1)\left(Y_t^*\right)^{\lambda_1-1} + \frac{1}{K_2\gamma_2}\left(Y_t^*\right)^{-\frac{1}{\gamma_2}}\right]dW_t \\
&\quad + \theta^2\left[C_1\lambda_1(\lambda_1-1)\left(Y_t^*\right)^{\lambda_1-1} + \frac{1}{K_2\gamma_2}\left(Y_t^*\right)^{-\frac{1}{\gamma_2}}\right]dt \\
&= rX_t^*\,dt - c_t^*\,dt + (\psi+r)F\,dt + \theta\left[C_1\lambda_1(\lambda_1-1)\left(Y_t^*\right)^{\lambda_1-1} + \frac{1}{K_2\gamma_2}\left(Y_t^*\right)^{-\frac{1}{\gamma_2}}\right]dW_t \\
&\quad + \theta^2\left[C_1\lambda_1(\lambda_1-1)\left(Y_t^*\right)^{\lambda_1-1} + \frac{1}{K_2\gamma_2}\left(Y_t^*\right)^{-\frac{1}{\gamma_2}}\right]dt.
\end{aligned}$$

Here, in deriving the third equality we use the definition of K_2, λ_1 and the fact that $\frac{\partial^2\phi}{\partial y^2}\,dD_t^* = 0$.

Comparing the drift term and the volatility term of the optimal wealth process dX_t^* with Eq. (2.10), we obtain the optimal portfolio $\tilde{\pi}_t^*$ as follows:

$$\tilde{\pi}_t^* = \frac{\theta}{\sigma}\left[C_1\lambda_1(\lambda_1-1)\left(Y_t^*\right)^{\lambda_1-1} + \frac{1}{K_2\gamma_2}\left(Y_t^*\right)^{-\frac{1}{\gamma_2}}\right]. \qquad \square$$

Appendix B. Supplementary proofs for Section 4

In this appendix, we first prove two lemmas for an auxiliary optimal stopping problem, which will be useful for analyzing the value of $\widetilde{V}(\nu)$.

$$G(z) \triangleq \sup_{\tau\in\mathcal{S}} \mathbb{E}^z\left[\int_0^\tau e^{-\beta t}\frac{\gamma_1}{1-\gamma_1}(Z_t)^{-\frac{1-\gamma_1}{\gamma_1}}\,dt + e^{-\beta\tau}\big(U(0)-Z_\tau F\big)\right],$$

with $dZ_t = Z_t\{(\beta-r)\,dt - \theta\,dW_t\}$, $Z_0 = z$, and $\mathbb{E}^z[\cdot] = \mathbb{E}[\cdot \mid Z_0 = z]$.

Lemma B.1. *For the above optimal stopping problem, $\tau^* = \infty$ is equivalent to $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)} \geqslant U(0)$.*

Proof. When $\tau^* = \infty$, we have

$$G(z) = \mathbb{E}^z\left[\int_0^\infty e^{-\beta t}\frac{\gamma_1}{1-\gamma_1}(Z_t)^{-\frac{1-\gamma_1}{\gamma_1}}\,dt\right] \geqslant U(0) - zF, \quad \forall z>0. \tag{B.1}$$

By direct calculation, we see $\mathbb{E}^z[\int_0^\infty e^{-\beta t}\frac{\gamma_1}{1-\gamma_1}(Z_t)^{-\frac{1-\gamma_1}{\gamma_1}}\,dt] = \frac{\gamma_1}{K_1(1-\gamma_1)}z^{-\frac{1-\gamma_1}{\gamma_1}}$.

Define

$$w(z) \triangleq \frac{\gamma_1}{K_1(1-\gamma_1)}z^{-\frac{1-\gamma_1}{\gamma_1}} + zF.$$

Hence (B.1) is equivalent to $w(z) \geqslant U(0)$ for $\forall z>0$. On the other hand, we observe that $\lim_{z\to 0} w(z) = \lim_{z\to\infty} w(z) = \infty$, and $w'(z) = -\frac{1}{K_1}z^{-\frac{1}{\gamma_1}} + F$, so $w(z)$ gets its minimum value $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)}$ at $z = (K_1F)^{-\gamma_1}$. Therefore, $w(z) \geqslant U(0)$ for $\forall z>0$ if and only if $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)} \geqslant U(0)$.

Conversely, when $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)} \geqslant U(0)$, $w(z) \geqslant U(0)$ for $\forall z>0$, i.e., $\widehat{G}(z) \triangleq \frac{\gamma_1}{K_1(1-\gamma_1)}z^{-\frac{1-\gamma_1}{\gamma_1}} \geqslant U(0) - zF$ for $\forall z>0$. Moreover, $\widehat{G}(z)$ also satisfies $\mathcal{L}_1\widehat{G} + \frac{\gamma_1}{1-\gamma_1}z^{-\frac{1-\gamma_1}{\gamma_1}} = 0$. Hence $\tau = \infty$ is optimal. □

Lemma B.2. *If $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)} < U(0)$, then $\beta U(0) > \frac{(rF)^{1-\gamma_1}}{1-\gamma_1}$.*

Proof. First, we denote $S \triangleq \{z\in(0,\infty) \mid G(z) = U(0) - zF\}$, the stopping region of the optimal stopping problem. According to Øksendal and Sulem [23], we know $\{\mathcal{L}_1(U(0)-zF) + \frac{\gamma_1}{1-\gamma_1}z^{-\frac{1-\gamma_1}{\gamma_1}} \leqslant 0\} \supset S$. By careful calculation, $\mathcal{L}_1(U(0)-zF) + \frac{\gamma_1}{1-\gamma_1}z^{-\frac{1-\gamma_1}{\gamma_1}} = -\beta U(0) + rzF + \frac{\gamma_1}{1-\gamma_1}z^{-\frac{1-\gamma_1}{\gamma_1}}$. For the sake of simplicity, define $w_1(z) \triangleq rzF + \frac{\gamma_1}{1-\gamma_1}z^{-\frac{1-\gamma_1}{\gamma_1}}$. Then we obtain $\lim_{z\to 0} w_1(z) = \lim_{z\to\infty} w_1(z) = \infty$, and $w_1'(z) = rF - z^{-\frac{1}{\gamma_1}}$, so $w_1(z)$ gets its minimum value $\frac{(rF)^{1-\gamma_1}}{1-\gamma_1}$ at $z = (rF)^{-\gamma_1}$. Hence, when $\frac{(rF)^{1-\gamma_1}}{1-\gamma_1} > \beta U(0)$, $S = \emptyset$; when $\frac{(rF)^{1-\gamma_1}}{1-\gamma_1} < \beta U(0)$, there exists two different points z_1', z_2', such that, $S \subset \{z_1' \leqslant z \leqslant z_2'\}$; when $\frac{(rF)^{1-\gamma_1}}{1-\gamma_1} = \beta U(0)$, $S = \emptyset$ or S contains only one single point, which is also equivalent to $\tau^* = \infty$. Moreover, from assumption $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)} < U(0)$, the optimal stopping time must be finite. Hence $\beta U(0) > \frac{(rF)^{1-\gamma_1}}{1-\gamma_1}$. □

In order to obtain $\widetilde{V}(\nu)$ we introduce two variational inequalities and derive their corresponding solutions.

Variational Inequality B.1. Find a positive number $0 < \bar{z} \leqslant \hat{y}$ and a function $\phi(\cdot) \in C^1(\mathbb{R}^+) \cap C^2(\mathbb{R}^+\backslash\{\bar{z}\})$ satisfying

(1) $\mathcal{L}_1\phi + \frac{\gamma_1}{1-\gamma_1}z^{-\frac{1-\gamma_1}{\gamma_1}} = 0$, $0 < z < \bar{z}$,
(2) $\mathcal{L}_1\phi + \frac{\gamma_1}{1-\gamma_1}z^{-\frac{1-\gamma_1}{\gamma_1}} \leqslant 0$, $\bar{z} \leqslant z \leqslant \hat{y}$,
(3) $\phi(z) = C_1z^{\lambda_1} + \frac{\gamma_2}{K_2(1-\gamma_2)}z^{-\frac{1-\gamma_2}{\gamma_2}} + \frac{\psi}{r}Fz$, $\bar{z} \leqslant z \leqslant \hat{y}$,
(4) $\phi(z) > C_1z^{\lambda_1} + \frac{\gamma_2}{K_2(1-\gamma_2)}z^{-\frac{1-\gamma_2}{\gamma_2}} + \frac{\psi}{r}Fz$, $0 < z < \bar{z}$.

The above free boundary value problem can be solved by applying the principle of smooth fit, i.e., the C^1-condition.

Proposition B.1. *Assume that $\bar{z}$ is the unique solution of the following algebraic equation*:

$$\frac{(\lambda_1-1)\gamma_1+1}{K_1(1-\gamma_1)}z^{-\frac{1}{\gamma_1}}=\frac{(\lambda_1-1)\gamma_2+1}{K_2(1-\gamma_2)}z^{-\frac{1}{\gamma_2}}+\frac{\psi}{r}F(\lambda_1-1). \tag{B.2}$$

If $0<\bar{z}\leqslant\hat{y}$, and the function

$$\phi(z)=\begin{cases}C_0z^{\lambda_1}+\frac{\gamma_1}{K_1(1-\gamma_1)}z^{-\frac{1-\gamma_1}{\gamma_1}}, & 0<z<\bar{z},\\ C_1z^{\lambda_1}+\frac{\gamma_2}{K_2(1-\gamma_2)}z^{-\frac{1-\gamma_2}{\gamma_2}}+\frac{\psi}{r}Fz, & \bar{z}\leqslant z\leqslant\hat{y},\end{cases}$$

with

$$C_0=C_1-\frac{1}{\lambda_1K_2}\bar{z}^{-\frac{(\lambda_1-1)\gamma_2+1}{\gamma_2}}+\frac{\psi F}{r\lambda_1}\bar{z}^{1-\lambda_1}+\frac{1}{\lambda_1K_1}\bar{z}^{-\frac{(\lambda_1-1)\gamma_1+1}{\gamma_1}}. \tag{B.3}$$

Then, $\phi(z)$ is a solution to Variational Inequality B.1.

Proof. First, we consider the differential equation (DE) (1) of Variational Inequality B.1:

$$\frac{1}{2}\theta^2z^2\phi''(z)+(\beta-r)z\phi'(z)-\beta\phi(z)+\frac{\gamma_1}{1-\gamma_1}z^{-\frac{1-\gamma_1}{\gamma_1}}=0,\quad\text{for } 0<z<\bar{z},$$

with a boundary condition $\phi(\bar{z})=C_1\bar{z}^{\lambda_1}+\frac{\gamma_2}{K_2(1-\gamma_2)}\bar{z}^{-\frac{1-\gamma_2}{\gamma_2}}+\frac{\psi}{r}F\bar{z}$.

The general solution of the above DE is $A_1z^{\lambda_1}+A_2z^{\lambda_2}+\frac{\gamma_1}{K_1(1-\gamma_1)}z^{-\frac{1-\gamma_1}{\gamma_1}}$, where A_1 and A_2 are two arbitrary constants. Since the value function is well defined and finite, we guess $\phi(z)$ has the following form:

$$\phi(z)=\begin{cases}C_0z^{\lambda_1}+\frac{\gamma_1}{K_1(1-\gamma_1)}z^{-\frac{1-\gamma_1}{\gamma_1}}, & 0<z<\bar{z},\\ C_1z^{\lambda_1}+\frac{\gamma_2}{K_2(1-\gamma_2)}z^{-\frac{1-\gamma_2}{\gamma_2}}+\frac{\psi}{r}Fz, & \bar{z}\leqslant z\leqslant\hat{y}.\end{cases}$$

By the boundary condition and the C^1-condition, we have:

$$C_0\bar{z}^{\lambda_1}+\frac{\gamma_1}{K_1(1-\gamma_1)}\bar{z}^{-\frac{1-\gamma_1}{\gamma_1}}=C_1\bar{z}^{\lambda_1}+\frac{\gamma_2}{K_2(1-\gamma_2)}\bar{z}^{-\frac{1-\gamma_2}{\gamma_2}}+\frac{\psi}{r}F\bar{z},$$

and

$$C_0\lambda_1\bar{z}^{\lambda_1-1}-\frac{1}{K_1}\bar{z}^{-\frac{1}{\gamma_1}}=C_1\lambda_1\bar{z}^{\lambda_1-1}-\frac{1}{K_2}\bar{z}^{-\frac{1}{\gamma_2}}+\frac{\psi}{r}F,$$

from which we obtain the values of C_0 and $\bar{z}$.

Next we will prove Eq. (B.2) has a unique solution $\bar{z}$. Define:

$$H(z)\triangleq\frac{(\lambda_1-1)\gamma_1+1}{K_1(1-\gamma_1)}z^{-\frac{1}{\gamma_1}}-\frac{(\lambda_1-1)\gamma_2+1}{K_2(1-\gamma_2)}z^{-\frac{1}{\gamma_2}}, \tag{B.4}$$

and

$$z_0\triangleq\left(\frac{((\lambda_1-1)\gamma_1+1)\gamma_2K_2(1-\gamma_2)}{((\lambda_1-1)\gamma_2+1)\gamma_1K_1(1-\gamma_1)}\right)^{\frac{\gamma_1\gamma_2}{\gamma_2-\gamma_1}},$$

with $0<\gamma_1<\gamma_2<1$.

It is seen that

$$\lim_{z\to 0} H(z) = \lim_{z\to 0} z^{-\frac{1}{\gamma_1}}\left(\frac{(\lambda_1-1)\gamma_1+1}{K_1(1-\gamma_1)} - \frac{(\lambda_1-1)\gamma_2+1}{K_2(1-\gamma_2)} z^{\frac{\gamma_2-\gamma_1}{\gamma_1\gamma_2}}\right) = \infty,$$

and

$$\lim_{z\to \infty} H(z) = \lim_{z\to \infty} z^{-\frac{1}{\gamma_2}}\left(\frac{(\lambda_1-1)\gamma_1+1}{K_1(1-\gamma_1)} z^{\frac{\gamma_1-\gamma_2}{\gamma_1\gamma_2}} - \frac{(\lambda_1-1)\gamma_2+1}{K_2(1-\gamma_2)}\right) = 0.$$

Moreover,

$$H'(z) = -\frac{(\lambda_1-1)\gamma_1+1}{\gamma_1 K_1(1-\gamma_1)} z^{-\frac{1}{\gamma_1}-1} + \frac{(\lambda_1-1)\gamma_2+1}{\gamma_2 K_2(1-\gamma_2)} z^{-\frac{1}{\gamma_2}-1},$$

and $H'(z) > 0$ for $z > z_0$, $H'(z) < 0$ for $0 < z < z_0$, and $H(z_0) = z_0^{-\frac{1}{\gamma_1}} \frac{(\gamma_1-\gamma_2)(\lambda_1\gamma_1+1-\gamma_1)}{K_1\gamma_1(1-\gamma_1)} < 0$. Hence $H(z) = \frac{\psi}{r}F(\lambda_1-1)$ has a unique solution $\bar{z}$.

To check the condition (2) of Variational Inequality B.1, we see for $\bar{z} \leqslant z \leqslant \hat{y}$,

$$\mathcal{L}_1\phi(z) + \frac{\gamma_1}{1-\gamma_1} z^{-\frac{1-\gamma_1}{\gamma_1}} = \frac{\gamma_1}{1-\gamma_1} z^{-\frac{1-\gamma_1}{\gamma_1}} - \frac{\gamma_2}{1-\gamma_2} z^{-\frac{1-\gamma_2}{\gamma_2}} - \psi F z = z\big(h(z) - \psi F\big),$$

where $h(z) \triangleq \frac{\gamma_1}{1-\gamma_1} z^{-\frac{1}{\gamma_1}} - \frac{\gamma_2}{1-\gamma_2} z^{-\frac{1}{\gamma_2}}$, and $h'(z) = -\frac{1}{1-\gamma_1} z^{-\frac{1}{\gamma_1}-1} + \frac{1}{1-\gamma_2} z^{-\frac{1}{\gamma_2}-1}$. It is easily found $h'(z) < 0$, for $0 < z < z_1$, $h'(z) > 0$, for $z > z_1$, with $z_1 \triangleq (\frac{1-\gamma_2}{1-\gamma_1})^{\frac{\gamma_1\gamma_2}{\gamma_2-\gamma_1}}$. Moreover, we observe that $\lim_{z\to 0} h(z) = \infty$, $\lim_{z\to\infty} h(z) = 0$, and $h(z_1) = z_1^{-\frac{1}{\gamma_1}} \frac{\gamma_1-\gamma_2}{1-\gamma_1} < 0$. Thus $h(z) = \psi F$ has a unique solution which we denote by $\tilde{z}$. If we can prove $\bar{z} \geqslant \tilde{z}$, then $h(z) \leqslant \psi F$ holds for $\bar{z} \leqslant z \leqslant \hat{y}$.

In the following we will prove $\bar{z} \geqslant \tilde{z}$. First we know that $H(z)$ is decreasing for $0 < z < \bar{z} < z_0$, so to prove $\bar{z} \geqslant \tilde{z}$ is equivalent to prove

$$H(\tilde{z}) \geqslant H(\bar{z}) = \frac{\psi F}{r}(\lambda_1 - 1). \tag{B.5}$$

Substituting $h(\tilde{z}) = \psi F$, we have

$$\frac{(\lambda_1-1)\gamma_1+1}{K_1(1-\gamma_1)} \tilde{z}^{-\frac{1}{\gamma_1}} - \frac{(\lambda_1-1)\gamma_2+1}{K_2(1-\gamma_2)} \tilde{z}^{-\frac{1}{\gamma_2}} \geqslant \frac{\lambda_1-1}{r}\left(\frac{\gamma_1}{1-\gamma_1}\tilde{z}^{-\frac{1}{\gamma_1}} - \frac{\gamma_2}{1-\gamma_2}\tilde{z}^{-\frac{1}{\gamma_2}}\right),$$

i.e.,

$$\frac{r(\lambda_1\gamma_1+1-\gamma_1) - \gamma_1 K_1(\lambda_1-1)}{K_1(1-\gamma_1)} \tilde{z}^{-\frac{1}{\gamma_1}} \geqslant \frac{r(\lambda_1\gamma_2+1-\gamma_2) - \gamma_2 K_2(\lambda_1-1)}{K_2(1-\gamma_2)} \tilde{z}^{-\frac{1}{\gamma_2}}. \tag{B.6}$$

By the definition of K_1 and λ_1, we get

$$\begin{aligned} r(\lambda_1\gamma_1+1-\gamma_1) - \gamma_1 K_1(\lambda_1-1) &= -\beta(\lambda_1-1) + r\lambda_1 - \frac{\gamma_1-1}{2\gamma_1}(\lambda_1-1)\theta^2 \\ &= \frac{1}{2}\theta^2\lambda_1(\lambda_1-1) - \frac{\gamma_1-1}{2\gamma_1}(\lambda_1-1)\theta^2 \\ &= \frac{1}{2\gamma_1}\theta^2(\lambda_1-1)(\lambda_1\gamma_1+1-\gamma_1). \end{aligned} \tag{B.7}$$

Similarly,

$$r(\lambda_1\gamma_2 + 1 - \gamma_2) - \gamma_2 K_2(\lambda_1 - 1) = \frac{1}{2\gamma_2}\theta^2(\lambda_1 - 1)(\lambda_1\gamma_2 + 1 - \gamma_2). \tag{B.8}$$

Substituting (B.7) and (B.8) into (B.6), we derive the inequality:

$$\tilde{z} \leqslant \left(\frac{((\lambda_1 - 1)\gamma_1 + 1)K_2\gamma_2(1 - \gamma_2)}{((\lambda_1 - 1)\gamma_2 + 1)K_1\gamma_1(1 - \gamma_1)}\right)^{\frac{\gamma_1\gamma_2}{\gamma_2 - \gamma_1}}. \tag{B.9}$$

If we denote the right hand side of (B.9) by z_2, we have

$$h(z_2) = z_2^{-\frac{1}{\gamma_1}}\left(\frac{\gamma_1}{1 - \gamma_1} - \frac{K_2\gamma_2^2((\lambda_1 - 1)\gamma_1 + 1)}{\gamma_1(1 - \gamma_1)K_1((\lambda_1 - 1)\gamma_2 + 1)}\right). \tag{B.10}$$

We claim

$$\frac{K_2\gamma_2((\lambda_1 - 1)\gamma_1 + 1)}{K_1\gamma_1((\lambda_1 - 1)\gamma_2 + 1)} > 1, \quad \text{for } 0 < \gamma_1 < \gamma_2 < 1. \tag{B.11}$$

Then Eq. (B.10) can be calculated as

$$h(z_2) < z_2^{-\frac{1}{\gamma_1}}\frac{\gamma_1 - \gamma_2}{1 - \gamma_1} < 0,$$

which implies that $\tilde{z} < z_2$.

In order to prove (B.11), let us define

$$q(x) \triangleq \frac{x(r + \frac{\beta - r}{x} + \frac{x-1}{2x^2}\theta^2)}{1 - x + \lambda_1 x}.$$

By direct calculation and the definition of λ_1, we have $q'(x) > 0$. Hence $q(\gamma_1) < q(\gamma_2)$, i.e.,

$$\frac{K_1\gamma_1}{(\lambda_1 - 1)\gamma_1 + 1} < \frac{K_2\gamma_2}{(\lambda_1 - 1)\gamma_2 + 1},$$

since $K_i = r + \frac{\beta - r}{\gamma_i} + \frac{\gamma_i - 1}{2\gamma_i^2}\theta^2$, and this proves (B.11).

Now we will check the condition (4), which is equivalent to:

$$m(z) \triangleq C_0 z^{\lambda_1} + \frac{\gamma_1}{K_1(1 - \gamma_1)}z^{-\frac{1-\gamma_1}{\gamma_1}} - C_1 z^{\lambda_1} - \frac{\gamma_2}{K_2(1 - \gamma_2)}z^{-\frac{1-\gamma_2}{\gamma_2}} - \frac{\psi}{r}Fz > 0, \quad 0 < z < \bar{z},$$

the above inequality becomes an equality when $z = \bar{z}$.

We observe that

$$\frac{1}{K_1}\bar{z}^{-\frac{1}{\gamma_1}} > \frac{1}{K_2}\bar{z}^{-\frac{1}{\gamma_2}}, \tag{B.12}$$

which needs to check that

$$\bar{z} < \left(\frac{K_2}{K_1}\right)^{\frac{\gamma_1\gamma_2}{\gamma_2 - \gamma_1}}. \tag{B.13}$$

Denoting the right hand side of (B.13) by z_3, we find

$$H(z_3) = z_3^{-\frac{1}{\gamma_1}} \frac{\lambda_1(\gamma_1 - \gamma_2)}{K_1(1-\gamma_1)(1-\gamma_2)} < 0.$$

Therefore, (B.13) is valid.

Now we have

$$\begin{aligned} m'(z) &= C_0\lambda_1 z^{\lambda_1 - 1} - \frac{1}{K_1} z^{-\frac{1}{\gamma_1}} - C_1\lambda_1 z^{\lambda_1-1} + \frac{1}{K_2} z^{-\frac{1}{\gamma_2}} - \frac{\psi}{r}F \\ &= \left(\frac{1}{K_1}\bar{z}^{-\frac{1}{\gamma_1}} - \frac{1}{K_2}\bar{z}^{-\frac{1}{\gamma_2}}\right)\left(\frac{z}{\bar{z}}\right)^{\lambda_1-1} + \frac{\psi F}{r}\left(\left(\frac{z}{\bar{z}}\right)^{\lambda_1-1} - 1\right) \\ &\quad - \left(\frac{z}{\bar{z}}\right)^{-\frac{1}{\gamma_1}}\left(\frac{1}{K_1}\bar{z}^{-\frac{1}{\gamma_1}} - \frac{1}{K_2}\bar{z}^{-\frac{1}{\gamma_2}}\left(\frac{z}{\bar{z}}\right)^{\frac{\gamma_2-\gamma_1}{\gamma_1\gamma_2}}\right) \\ &< \left(\frac{1}{K_1}\bar{z}^{-\frac{1}{\gamma_1}} - \frac{1}{K_2}\bar{z}^{-\frac{1}{\gamma_2}}\right)\left(\left(\frac{z}{\bar{z}}\right)^{\lambda_1-1} - \left(\frac{z}{\bar{z}}\right)^{-\frac{1}{\gamma_1}}\right) + \frac{\psi F}{r}\left(\left(\frac{z}{\bar{z}}\right)^{\lambda_1-1} - 1\right) \\ &< 0. \end{aligned}$$

The second equality holds by substituting Eq. (B.3), the third inequality follows by $0 < \frac{z}{\bar{z}} < 1$, and the last inequality holds by (B.12), $0 < \frac{z}{\bar{z}} < 1$ and $\lambda_1 > 1$. This proves the condition (4) of Variational Inequality B.1. □

Variational Inequality B.2. Find two positive numbers $0 < \tilde{z}_1 < \tilde{z}_2 < \infty$ and a function $\phi(\cdot) \in C^1(\mathbb{R}^+) \cap C^2(\mathbb{R}^+ \backslash \{\tilde{z}_1, \tilde{z}_2\})$ satisfying

(1) $\mathcal{L}_1\phi + \frac{\gamma_1}{1-\gamma_1} z^{-\frac{1-\gamma_1}{\gamma_1}} = 0$, $0 < z < \tilde{z}_1$, $z > \tilde{z}_2$,
(2) $\mathcal{L}_1\phi + \frac{\gamma_1}{1-\gamma_1} z^{-\frac{1-\gamma_1}{\gamma_1}} \leqslant 0$, $\tilde{z}_1 \leqslant z \leqslant \tilde{z}_2$,
(3) $\phi(z) = U(0) - zF$, $\tilde{z}_1 \leqslant z \leqslant \tilde{z}_2$,
(4) $\phi(z) > U(0) - zF$, $0 < z < \tilde{z}_1$, $z > \tilde{z}_2$.

Proposition B.2. *If $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)} < U(0)$, then we can find $\tilde{z}_1$, $\tilde{z}_2$ such that $0 < \tilde{z}_1 < (K_1F)^{-\gamma_1} < \tilde{z}_2 < \infty$ and*

$$\phi(z) = \begin{cases} C_2 z^{\lambda_1} + \frac{\gamma_1}{K_1(1-\gamma_1)} z^{-\frac{1-\gamma_1}{\gamma_1}}, & 0 < z < \tilde{z}_1, \\ U(0) - zF, & \tilde{z}_1 \leqslant z \leqslant \tilde{z}_2, \\ \widetilde{C}_2 z^{\lambda_2} + \frac{\gamma_1}{K_1(1-\gamma_1)} z^{-\frac{1-\gamma_1}{\gamma_1}}, & z > \tilde{z}_2, \end{cases}$$

where

$$C_2 = \frac{1}{\lambda_1 K_1} \tilde{z}_1^{-\frac{(\lambda_1-1)\gamma_1+1}{\gamma_1}} - \frac{F}{\lambda_1}\tilde{z}_1^{1-\lambda_1} > 0,$$

and

$$\widetilde{C}_2 = \frac{1}{\lambda_2 K_1} \tilde{z}_2^{-\frac{(\lambda_2-1)\gamma_1+1}{\gamma_1}} - \frac{F}{\lambda_2}\tilde{z}_2^{1-\lambda_2} > 0.$$

Then, $\phi(z)$ is a solution to Variational Inequality B.2.

Proof. First, we consider the differential equation (DE) (1) of Variational Inequality B.2:

$$\frac{1}{2}\theta^2 z^2\phi''(z)+(\beta-r)z\phi'(z)-\beta\phi(z)+\frac{\gamma_1}{1-\gamma_1}z^{-\frac{1-\gamma_1}{\gamma_1}}=0.$$

Similarly as in Proposition B.1, we guess $\phi(z)$ has the following form:

$$\phi(z)=\begin{cases} C_2 z^{\lambda_1}+\frac{\gamma_1}{K_1(1-\gamma_1)}z^{-\frac{1-\gamma_1}{\gamma_1}}, & 0<z<\tilde{z}_1,\\ \widetilde{C}_2 z^{\lambda_2}+\frac{\gamma_1}{K_1(1-\gamma_1)}z^{-\frac{1-\gamma_1}{\gamma_1}}, & z>\tilde{z}_2.\end{cases}$$

By the boundary conditions and the C^1-condition, we obtain the values of C_2, $\widetilde{C}_2$ and the equations satisfied by $\tilde{z}_1$ and $\tilde{z}_2$ as follows:

$$\frac{(\lambda_1-1)\gamma_1+1}{K_1(1-\gamma_1)}\tilde{z}_1^{-\frac{1-\gamma_1}{\gamma_1}}+(\lambda_1-1)\tilde{z}_1F=\lambda_1U(0), \tag{B.14}$$

and

$$\frac{(\lambda_2-1)\gamma_1+1}{K_1(1-\gamma_1)}\tilde{z}_2^{-\frac{1-\gamma_1}{\gamma_1}}+(\lambda_2-1)\tilde{z}_2F=\lambda_2U(0). \tag{B.15}$$

Now we find the values of $\tilde{z}_1$ and $\tilde{z}_2$. Define

$$g_1(z)\triangleq\frac{(\lambda_1-1)\gamma_1+1}{K_1(1-\gamma_1)}z^{-\frac{1-\gamma_1}{\gamma_1}}+(\lambda_1-1)zF.$$

It is easily seen $\lim_{z\to\infty}g_1(z)=\lim_{z\to 0}g_1(z)=\infty$ and $g_1'(z)=-\frac{(\lambda_1-1)\gamma_1+1}{K_1\gamma_1}z^{-\frac{1}{\gamma_1}}+(\lambda_1-1)F$. Hence $g_1(z)$ gets its minimum at $z^*=(\frac{K_1\gamma_1F(\lambda_1-1)}{(\lambda_1-1)\gamma_1+1})^{-\gamma_1}$ and $g_1(z^*)=\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)}(\frac{(\lambda_1-1)\gamma_1+1}{(\lambda_1-1)\gamma_1})^{\gamma_1}(\lambda_1-1)$. From the assumption $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)}<U(0)$ and the fact that $(\frac{(\lambda_1-1)\gamma_1+1}{(\lambda_1-1)\gamma_1})^{\gamma_1}<\frac{\lambda_1}{\lambda_1-1}$, we obtain $g_1(z^*)<\lambda_1U(0)$. Therefore, the equation $g_1(z)=\lambda_1U(0)$ has two roots. We claim $\tilde{z}_1$ is the smaller one, i.e., $\tilde{z}_1<z^*$. Simple algebraic calculation yields $g_1((K_1F)^{-\gamma_1})<\lambda_1U(0)$, hence, $\tilde{z}_1<(K_1F)^{-\gamma_1}$, so $C_2>0$.

Similarly, define

$$g_2(z)\triangleq\frac{(\lambda_2-1)\gamma_1+1}{K_1(1-\gamma_1)}z^{-\frac{1-\gamma_1}{\gamma_1}}+(\lambda_2-1)zF.$$

Since $(\lambda_2-1)\gamma_1+1<0$, $\lim_{z\to 0}g_2(z)=\lim_{z\to\infty}g_2(z)=-\infty$, $g_2(z)$ gets its maximum at $z^{**}=(\frac{K_1\gamma_1F(\lambda_2-1)}{(\lambda_2-1)\gamma_1+1})^{-\gamma_1}$ and $g_2(z^{**})=\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)}(\frac{(\lambda_2-1)\gamma_1+1}{(\lambda_2-1)\gamma_1})^{\gamma_1}(\lambda_2-1)$. From the assumption $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)}<U(0)$ and the fact that $(\frac{(\lambda_2-1)\gamma_1+1}{(\lambda_2-1)\gamma_1})^{\gamma_1}<\frac{\lambda_2}{\lambda_2-1}$, we obtain $g_2(z^{**})>\lambda_2U(0)$. Therefore, the equation $g_2(z)=\lambda_2U(0)$ has two roots. We claim $\tilde{z}_2$ is the larger one, i.e., $\tilde{z}_2>z^{**}$. Moreover, we find $g_2((K_1F)^{-\gamma_1})>\lambda_2U(0)$, hence, $\tilde{z}_2>(K_1F)^{-\gamma_1}$, so $\widetilde{C}_2>0$.

To check the condition (2) of Variational Inequality B.2, we see for $\tilde{z}_1\leqslant z\leqslant\tilde{z}_2$,

$$\mathcal{L}_1\phi(z)+\frac{\gamma_1}{1-\gamma_1}z^{-\frac{1-\gamma_1}{\gamma_1}}=rzF+\frac{\gamma_1}{1-\gamma_1}z^{-\frac{1-\gamma_1}{\gamma_1}}-\beta U(0)=\omega_1(z)-\beta U(0),$$

where $\omega_1(z)=rzF+\frac{\gamma_1}{1-\gamma_1}z^{-\frac{1-\gamma_1}{\gamma_1}}$.

From the proof of Lemma B.2, we know if $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)} < U(0)$, then the equation $\omega_1(z) = \beta U(0)$ has two different roots, z_1' and z_2'. So, in order to prove Variational Inequality B.2(2), we only need to prove $\tilde{z}_1 \geqslant z_1'$ and $\tilde{z}_2 \leqslant z_2'$, which are equivalent to $\omega_1(\tilde{z}_1) \leqslant \beta U(0)$ and $\omega_1(\tilde{z}_2) \leqslant \beta U(0)$. First, we prove

$$\omega_1(\tilde{z}_1) \leqslant \beta U(0). \tag{B.16}$$

Substituting Eq. (B.14) into the above inequality yields

$$\left(r - \frac{\beta(\lambda_1 - 1)}{\lambda_1}\right)\tilde{z}_1 F + \left(\frac{\gamma_1}{1-\gamma_1} - \frac{\beta(\lambda_1 - 1)\gamma_1 + \beta}{\lambda_1 K_1(1-\gamma_1)}\right)\tilde{z}_1^{-\frac{1-\gamma_1}{\gamma_1}} \leqslant 0. \tag{B.17}$$

With the definition of λ_1 and K_1, we derive

$$r - \frac{\beta(\lambda_1 - 1)}{\lambda_1} = \frac{1}{2}\theta^2(\lambda_1 - 1),$$

and

$$\frac{\gamma_1}{1-\gamma_1} - \frac{\beta(\lambda_1 - 1)\gamma_1 + \beta}{\lambda_1 K_1(1-\gamma_1)} = -\frac{\theta^2((\lambda_1 - 1)\gamma_1 + 1)}{2\gamma_1 K_1}.$$

Substituting the above results into (B.17), we obtain $\tilde{z}_1 \leqslant (\frac{K_1\gamma_1 F(\lambda_1-1)}{(\lambda_1-1)\gamma_1+1})^{-\gamma_1}$, which is obviously valid from the selection of $\tilde{z}_1$. Hence, (B.16), and $\tilde{z}_1 \geqslant z_1'$. The proof for $\tilde{z}_2 \leqslant z_2'$ is similar, so we omit it.

Next we check the condition (4) for $0 < z < \tilde{z}_1$, which is equivalent to

$$C_2 + \frac{\gamma_1}{K_1(1-\gamma_1)} z^{-\frac{(\lambda_1-1)\gamma_1+1}{\gamma_1}} - U(0)z^{-\lambda_1} + z^{1-\lambda_1}F > 0,$$

with $C_2 = U(0)\tilde{z}_1^{-\lambda_1} - \tilde{z}_1^{1-\lambda_1}F - \frac{\gamma_1}{K_1(1-\gamma_1)}\tilde{z}_1^{-\frac{(\lambda_1-1)\gamma_1+1}{\gamma_1}}$. Therefore, we only need to prove $h_1(z)$ is increasing for $0 < z < \tilde{z}_1$, where $h_1(z) \triangleq U(0)z^{-\lambda_1} - z^{1-\lambda_1}F - \frac{\gamma_1}{K_1(1-\gamma_1)}z^{-\frac{(\lambda_1-1)\gamma_1+1}{\gamma_1}}$. By direct calculation, we have

$$h_1'(z) = z^{-1-\lambda_1}\left(-\lambda_1 U(0) + (\lambda_1 - 1)zF + \frac{(\lambda_1 - 1)\gamma_1 + 1}{K_1(1-\gamma_1)} z^{-\frac{1-\gamma_1}{\gamma_1}}\right).$$

From the definition of $\tilde{z}_1$ we know that $h_1'(z) > 0$, for $0 < z < \tilde{z}_1$. For $z > \tilde{z}_2$, the condition (4) of Variational Inequality B.2 can be verified by using the same method. □

Remark B.1. From the derivation in Section 3, we find $U(0) = \frac{(\lambda_1-1)\gamma_2+1}{K_2\lambda_1\gamma_2(1-\gamma_2)}\hat{y}^{-\frac{1-\gamma_2}{\gamma_2}}$. Moreover by careful computation, we get

$$g_1(\hat{y}) - \lambda_1 U(0) = \hat{y}\left(H(\hat{y}) - \frac{\psi}{r}F(\lambda_1 - 1)\right). \tag{B.18}$$

Hence if $\bar{z} < \hat{y}$, i.e., $H(\hat{y}) - \frac{\psi}{r}F(\lambda_1 - 1) < 0$, then, $g_1(\hat{y}) - \lambda_1 U(0) < 0$, which implies $\tilde{z}_1 < \hat{y}$. If $\bar{z} \geqslant \hat{y}$, then $g_1(\hat{y}) - \lambda_1 U(0) \geqslant 0$. In addition, from $H(\hat{y}) - \frac{\psi}{r}F(\lambda_1 - 1) \geqslant 0$, we obtain $\hat{y}^{-\frac{1}{\gamma_1}} \geqslant \frac{(r\gamma_2+\psi)FK_1(\lambda_1-1)(1-\gamma_1)}{r(1-\gamma_2)((\lambda_1-1)\gamma_1+1)} > \frac{(\lambda_1-1)FK_1\gamma_1}{(\lambda_1-1)\gamma_1+1}$, i.e., $\hat{y} < z^*$. Therefore, we have $\hat{y} \leqslant \tilde{z}_1$.

In order to prove Proposition 4.2, we first prove a lemma.

Lemma B.3. *$\mathbb{E}[\sup_{0<t<\infty} e^{-\beta t}(Z_t^\nu)^{-\frac{1-\gamma_i}{\gamma_i}}] < \infty$, $i = 1, 2$ and $\mathbb{E}[\sup_{0<t<\infty} e^{-\beta t} Z_t^\nu] < \infty$, with $Z_t^\nu = \nu \exp\{(\beta - r)t - \theta W_t - \frac{1}{2}\theta^2 t\}$.*

Proof. First, we define $Y_t \triangleq (-r - \frac{\beta-r}{\gamma_1} + \frac{1-\gamma_1}{2\gamma_1}\theta^2)t + \frac{1-\gamma_1}{\gamma_1}\theta W_t$, then $e^{-\beta t}(Z_t^\nu)^{-\frac{1-\gamma_1}{\gamma_1}} = \nu^{-\frac{1-\gamma_1}{\gamma_1}} e^{Y_t}$. Denote $\overline{Y}_\infty = \sup_{0<t<\infty} Y_t$. From Chapter 3 of [18], $\overline{Y}_\infty$ is exponentially distributed with parameter $\Phi(0)$, where $\Phi(q) = \sup\{\delta \geqslant 0\colon\ \psi(\delta) = q\}$ and $\mathbb{E}[e^{\delta Y_t}] = e^{\psi(\delta)t}$. In our case, $\psi(\delta) = (-r - \frac{\beta-r}{\gamma_1} + \frac{1-\gamma_1}{2\gamma_1}\theta^2)\delta + \frac{(1-\gamma_1)^2}{2\gamma_1^2}\theta^2\delta^2$. Hence, $\Phi(0) = \frac{r+\frac{\beta-r}{\gamma_1} - \frac{1-\gamma_1}{2\gamma_1}\theta^2}{\frac{(1-\gamma_1)^2}{2\gamma_1^2}\theta^2} > 1$, since $K_1 > 0$. Therefore, $\mathbb{E}[\sup_{0<t<\infty} e^{-\beta t}(Z_t^\nu)^{-\frac{1-\gamma_1}{\gamma_1}}] = \nu^{-\frac{1-\gamma_1}{\gamma_1}}\mathbb{E}[e^{\overline{Y}_\infty}] < \infty$. For other cases, the proof is similar, hence we omit it. □

Proof of Proposition 4.2. From Proposition B.1 and Proposition B.2, it is easy to check that $\phi(z)$ satisfies the above conditions. Here, in the following we only verify that $\widetilde{V}(\nu) = \phi(\nu)$.

First, we know that $\phi(\cdot) \in C^1(\mathbb{R}^+) \cap C^2(\mathbb{R}^+ \setminus \{\bar{z}, \tilde{z}_2\})$. $\forall \tau \in \mathcal{S}$, define $\tau_R \triangleq \tau \wedge R$, R is a fixed constant. By applying the generalized Itô rule (see [14, Problem 3.7.3]) to $e^{-\beta\tau_R}\phi(Z_{\tau_R}^\nu)$, we have

$$e^{-\beta\tau_R}\phi\big(Z_{\tau_R}^\nu\big) = \phi(\nu) + \int_0^{\tau_R} e^{-\beta t}\mathcal{L}_1\phi\big(Z_t^\nu\big)\,dt - \int_0^{\tau_R} e^{-\beta t}\theta Z_t^\nu\phi'\big(Z_t^\nu\big)\,dW_t.$$

Taking expectations and noting the above conditions (1)–(6), we obtain

$$\begin{aligned}\phi(\nu) \geqslant{}& \mathbb{E}\Bigg[\int_0^{\tau_R} e^{-\beta t}\frac{\gamma_1}{1-\gamma_1}\big(Z_t^\nu\big)^{-\frac{1-\gamma_1}{\gamma_1}}\,dt\Bigg] + \mathbb{E}\Big[e^{-\beta\tau_R}\big(U(0) - Z_{\tau_R}^\nu F\big)1_{(Z_{\tau_R}^\nu > \hat{y})}\\ &+ e^{-\beta\tau_R}\bigg(C_1\big(Z_{\tau_R}^\nu\big)^{\lambda_1} + \frac{\gamma_2}{K_2(1-\gamma_2)}\big(Z_{\tau_R}^\nu\big)^{-\frac{1-\gamma_2}{\gamma_2}} + \frac{\psi}{r}FZ_{\tau_R}^\nu\bigg)1_{(0<Z_{\tau_R}^\nu \leqslant \hat{y})}\Big].\end{aligned}$$

Since $\mathbb{E}[\sup_{0<t<\infty} e^{-\beta t}(Z_t^\nu)^{-\frac{1-\gamma_i}{\gamma_i}}] < \infty$, $i = 1, 2$ and $\mathbb{E}[\sup_{0<t<\infty} e^{-\beta t}Z_t^\nu] < \infty$ (see Lemma B.3), by the monotone convergence theorem and the dominated convergence theorem, taking $R \to \infty$, we have

$$\begin{aligned}\phi(\nu) \geqslant{}& \mathbb{E}\Bigg[\int_0^{\tau} e^{-\beta t}\frac{\gamma_1}{1-\gamma_1}\big(Z_t^\nu\big)^{-\frac{1-\gamma_1}{\gamma_1}}\,dt\Bigg] + \mathbb{E}\Big[e^{-\beta\tau}\big(U(0) - Z_{\tau}^\nu F\big)1_{(Z_{\tau}^\nu > \hat{y})}\\ &+ e^{-\beta\tau}\bigg(C_1\big(Z_{\tau}^\nu\big)^{\lambda_1} + \frac{\gamma_2}{K_2(1-\gamma_2)}\big(Z_{\tau}^\nu\big)^{-\frac{1-\gamma_2}{\gamma_2}} + \frac{\psi}{r}FZ_{\tau}^\nu\bigg)1_{(0<Z_{\tau}^\nu \leqslant \hat{y})}\Big].\end{aligned}$$

Since τ is arbitrary, $\phi(\nu) \geqslant \widetilde{V}(\nu)$.

Moreover, if we apply the above arguments to $\tau = \tau_\nu$, by the definition of τ_ν, we see

$$\phi(\nu) = \mathbb{E}\Bigg[\int_0^{\tau_R} e^{-\beta t}\frac{\gamma_1}{1-\gamma_1}\big(Z_t^\nu\big)^{-\frac{1-\gamma_1}{\gamma_1}}\,dt\Bigg] + \mathbb{E}\big[e^{-\beta\tau_R}\phi\big(Z_{\tau_R}^\nu\big)\big].$$

Taking $R \to \infty$, we obtain

$$\begin{aligned}\phi(\nu) &= \mathbb{E}\Bigg[\int_0^{\tau_\nu} e^{-\beta t}\frac{\gamma_1}{1-\gamma_1}\big(Z_t^\nu\big)^{-\frac{1-\gamma_1}{\gamma_1}}\,dt\Bigg] + \mathbb{E}\big[e^{-\beta\tau_\nu}\phi\big(Z_{\tau_\nu}^\nu\big)\big]\\ &= \mathbb{E}\Bigg[\int_0^{\tau_\nu} e^{-\beta t}\frac{\gamma_1}{1-\gamma_1}\big(Z_t^\nu\big)^{-\frac{1-\gamma_1}{\gamma_1}}\,dt\Bigg] + \mathbb{E}\Big[e^{-\beta\tau_\nu}\big(U(0) - Z_{\tau_\nu}^\nu F\big)1_{(Z_{\tau_\nu}^\nu > \hat{y})}\\ &\quad+ e^{-\beta\tau_\nu}\bigg(C_1\big(Z_{\tau_\nu}^\nu\big)^{\lambda_1} + \frac{\gamma_2}{K_2(1-\gamma_2)}\big(Z_{\tau_\nu}^\nu\big)^{-\frac{1-\gamma_2}{\gamma_2}} + \frac{\psi}{r}FZ_{\tau_\nu}^\nu\bigg)1_{(0<Z_{\tau_\nu}^\nu \leqslant \hat{y})}\Big].\end{aligned}$$

Hence, we conclude $\phi(\nu) = \widetilde{V}(\nu)$ and τ_ν is optimal. □

Proof of Proposition 4.3. From Proposition B.2, we see that $\phi(z)$ satisfies the above conditions (1), (2), (3), and (5). In the following we verify the correctness of the condition (4). From Remark B.1, we know $\tilde{z}_1 \geqslant \hat{y}$, so $\phi(z) = C_2 z^{\lambda_1} + \frac{\gamma_1}{K_1(1-\gamma_1)} z^{-\frac{1-\gamma_1}{\gamma_1}}$, for $0 < z \leqslant \hat{y}$. Therefore to prove the condition (4) is equivalent to prove

$$C_2 z^{\lambda_1} + \frac{\gamma_1}{K_1(1-\gamma_1)} z^{-\frac{1-\gamma_1}{\gamma_1}} > C_1 z^{\lambda_1} + \frac{\gamma_2}{K_2(1-\gamma_2)} z^{-\frac{1-\gamma_2}{\gamma_2}} + \frac{\psi}{r} F z, \quad \text{for } 0 < z \leqslant \hat{y}. \tag{B.19}$$

Define

$$f_1(z) \triangleq C_2 - C_1 + \frac{\gamma_1}{K_1(1-\gamma_1)} z^{-\frac{(\lambda_1-1)\gamma_1+1}{\gamma_1}} - \frac{\gamma_2}{K_2(1-\gamma_2)} z^{-\frac{(\lambda_1-1)\gamma_2+1}{\gamma_2}} - \frac{\psi}{r} F z^{1-\lambda_1}.$$

Then (B.19) is valid if and only if $f_1(z) > 0$, $0 < z \leqslant \hat{y}$. From the proof of Proposition B.2, we have $f_1(\hat{y}) > 0$. Moreover,

$$f_1'(z) = z^{-\lambda_1}\left(-\frac{(\lambda_1-1)\gamma_1+1}{K_1(1-\gamma_1)} z^{-\frac{1}{\gamma_1}} + \frac{(\lambda_1-1)\gamma_2+1}{K_2(1-\gamma_2)} z^{-\frac{1}{\gamma_2}} + \frac{\psi}{r} F(\lambda_1 - 1)\right) \leqslant 0,$$

for $0 < z \leqslant \hat{y} \leqslant \bar{z}$, by the definition of $\bar{z}$. Hence, (B.19) is proved. The verification for $\widetilde{V}(\nu) = \phi(\nu)$ is similar to Proposition 4.2, so here we omit it. □

Proof of Proposition 4.4. Taking the similar method as in Proposition B.2, we can easily guess $\phi(z)$ has the form of (4.4). Then by the boundary conditions and the C^1-condition, we obtain the values of $\widehat{C}_2$, C_0 (which is the same as in Proposition B.1) and the equation satisfied by $\check{y}$.

Next, we prove Eq. (4.5) do has a solution on $z \in (\bar{z}, \hat{y})$. Define

$$p(z) \triangleq (\lambda_2 - \lambda_1) C_1 z^{\lambda_1 - 1} - \frac{(\lambda_2-1)\gamma_1+1}{K_1(1-\gamma_1)} z^{-\frac{1}{\gamma_1}} + \frac{(\lambda_2-1)\gamma_2+1}{K_2(1-\gamma_2)} z^{-\frac{1}{\gamma_2}} + (\lambda_2 - 1)\frac{\psi}{r} F,$$

and

$$\begin{aligned} p(\bar{z}) &= \frac{(\lambda_1-1)\gamma_1+1}{K_1(1-\gamma_1)} \bar{z}^{-\frac{1}{\gamma_1}} - \frac{(\lambda_1-1)\gamma_2+1}{K_2(1-\gamma_2)} \bar{z}^{-\frac{1}{\gamma_2}} - \frac{\psi}{r} F(\lambda_1 - 1) + p(\bar{z}) \\ &= (\lambda_2 - \lambda_1)\left[C_1 \bar{z}^{\lambda_1 - 1} + \frac{\gamma_2}{K_2(1-\gamma_2)} \bar{z}^{-\frac{1}{\gamma_2}} - \frac{\gamma_1}{K_1(1-\gamma_1)} \bar{z}^{-\frac{1}{\gamma_1}} + \frac{\psi}{r} F\right] \\ &= (\lambda_2 - \lambda_1) C_0 \bar{z}^{\lambda_1 - 1} < 0, \end{aligned}$$

since C_0 is positive in both cases (for case (2) see Lemma B.4).

For the case (1) $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)} \geqslant U(0)$, which is equivalent to $U(0) - zF - \frac{\gamma_1}{K_1(1-\gamma_1)} z^{-\frac{1-\gamma_1}{\gamma_1}} \leqslant 0$, $\forall z > 0$, we obtain

$$\begin{aligned} p(\hat{y}) &> \frac{(\lambda_1-1)\gamma_1+1}{K_1(1-\gamma_1)} \hat{y}^{-\frac{1}{\gamma_1}} - \frac{(\lambda_1-1)\gamma_2+1}{K_2(1-\gamma_2)} \hat{y}^{-\frac{1}{\gamma_2}} - \frac{\psi}{r} F(\lambda_1 - 1) + p(\hat{y}) \\ &= (\lambda_2 - \lambda_1)\hat{y}^{-1}\left[C_1 \hat{y}^{\lambda_1} + \frac{\gamma_2}{K_2(1-\gamma_2)} \hat{y}^{-\frac{1-\gamma_2}{\gamma_2}} - \frac{\gamma_1}{K_1(1-\gamma_1)} \hat{y}^{-\frac{1-\gamma_1}{\gamma_1}} + \frac{\psi}{r} F \hat{y}\right] \\ &= (\lambda_2 - \lambda_1)\hat{y}^{-1}\left[U(0) - \hat{y}F - \frac{\gamma_1}{K_1(1-\gamma_1)} \hat{y}^{-\frac{1-\gamma_1}{\gamma_1}}\right] \geqslant 0, \end{aligned}$$

the inequality holds by the assumption that $\bar{z} < \hat{y}$.

For the case (2) $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)} < U(0), \tilde{z}_2 < \hat{y}$, we get

$$\begin{aligned}
p(\hat{y}) &= (\lambda_2-\lambda_1)\hat{y}^{-1}\left[U(0) - \frac{\gamma_2}{K_2(1-\gamma_2)}\hat{y}^{-\frac{1-\gamma_2}{\gamma_2}} - \frac{\psi+r}{r}F\hat{y}\right] \\
&\quad - \frac{(\lambda_2-1)\gamma_1+1}{K_1(1-\gamma_1)}\hat{y}^{-\frac{1}{\gamma_1}} + \frac{(\lambda_2-1)\gamma_2+1}{K_2(1-\gamma_2)}\hat{y}^{-\frac{1}{\gamma_2}} + \frac{\psi}{r}F(\lambda_2-1) \\
&= \hat{y}^{-1}\left[\lambda_2 U(0) - (\lambda_2-1)\hat{y}F - \frac{(\lambda_2-1)\gamma_1+1}{K_1(1-\gamma_1)}\hat{y}^{-\frac{1-\gamma_1}{\gamma_1}}\right] \\
&\quad - \lambda_1 U(0)\hat{y}^{-1} + \frac{(\lambda_1-1)(\psi+r)F}{r} + \frac{(\lambda_1-1)\gamma_2+1}{K_2(1-\gamma_2)}\hat{y}^{-\frac{1}{\gamma_2}} \\
&> -\lambda_1 U(0)\hat{y}^{-1} + \frac{(\lambda_1-1)(\psi+r)F}{r} + \frac{(\lambda_1-1)\gamma_2+1}{K_2(1-\gamma_2)}\hat{y}^{-\frac{1}{\gamma_2}} \\
&= 0.
\end{aligned}$$

Here, we derive the first equality by substituting the value of $U(0)$, the third inequality by the fact that $\tilde{z}_2 < \hat{y}$, and the last equality by the values of $U(0)$ and $\hat{y}$.

Hence, Eq. (4.5) has a solution on $z \in (\bar{z}, \hat{y})$, we denote $\check{y}$ is the largest one. Moreover, we claim that $\widehat{C}_2 > 0$, because

$$\begin{aligned}
0 = p(\check{y}) &> \frac{(\lambda_1-1)\gamma_1+1}{K_1(1-\gamma_1)}\check{y}^{-\frac{1}{\gamma_1}} - \frac{(\lambda_1-1)\gamma_2+1}{K_2(1-\gamma_2)}\check{y}^{-\frac{1}{\gamma_2}} - \frac{\psi}{r}F(\lambda_1-1) + p(\check{y}) \\
&= (\lambda_2-\lambda_1)\left[C_1\check{y}^{\lambda_1-1} + \frac{\gamma_2}{K_2(1-\gamma_2)}\check{y}^{-\frac{1}{\gamma_2}} - \frac{\gamma_1}{K_1(1-\gamma_1)}\check{y}^{-\frac{1}{\gamma_1}} + \frac{\psi}{r}F\right] \\
&= (\lambda_2-\lambda_1)\widehat{C}_2\check{y}^{\lambda_2-1},
\end{aligned}$$

if $\widehat{C}_2 \leqslant 0$, then there is a contradiction.

From Proposition B.1, we see conditions (1)–(3) are valid, we only need to verify the conditions (4) and (5). Next, we prove the condition (4) (for the case $0 < z < \bar{z}$, see the proof of Proposition B.1), i.e.,

$$\phi(z) > C_1 z^{\lambda_1} + \frac{\gamma_2}{K_2(1-\gamma_2)}z^{-\frac{1-\gamma_2}{\gamma_2}} + \frac{\psi}{r}Fz, \quad \text{for } \check{y} < z \leqslant \hat{y}, \tag{B.20}$$

the inequality becomes an equality at $z = \check{y}$. Moreover, (B.20) can be simplified as

$$\widehat{C}_2 > C_1 z^{\lambda_1-\lambda_2} + \frac{\gamma_2}{K_2(1-\gamma_2)}z^{-\frac{(\lambda_2-1)\gamma_2+1}{\gamma_2}} - \frac{\gamma_1}{K_1(1-\gamma_1)}z^{-\frac{(\lambda_2-1)\gamma_1+1}{\gamma_1}} + \frac{\psi}{r}Fz^{1-\lambda_2},$$

i.e.,

$$\begin{aligned}
&C_1\check{y}^{\lambda_1-\lambda_2} + \frac{\gamma_2}{K_2(1-\gamma_2)}\check{y}^{-\frac{(\lambda_2-1)\gamma_2+1}{\gamma_2}} - \frac{\gamma_1}{K_1(1-\gamma_1)}\check{y}^{-\frac{(\lambda_2-1)\gamma_1+1}{\gamma_1}} + \frac{\psi}{r}F\check{y}^{1-\lambda_2} \\
&\quad > C_1 z^{\lambda_1-\lambda_2} + \frac{\gamma_2}{K_2(1-\gamma_2)}z^{-\frac{(\lambda_2-1)\gamma_2+1}{\gamma_2}} - \frac{\gamma_1}{K_1(1-\gamma_1)}z^{-\frac{(\lambda_2-1)\gamma_1+1}{\gamma_1}} + \frac{\psi}{r}Fz^{1-\lambda_2},
\end{aligned} \tag{B.21}$$

by substituting the value of $\widehat{C}_2$.

Let us denote the right hand side of (B.21) by $q_1(z)$, then

$$q_1'(z) = z^{-\lambda_2}\left[(\lambda_1-\lambda_2)C_1 z^{\lambda_1-1} + \frac{(\lambda_2-1)\gamma_1+1}{K_1(1-\gamma_1)}z^{-\frac{1}{\gamma_1}} - \frac{(\lambda_2-1)\gamma_2+1}{K_2(1-\gamma_2)}z^{-\frac{1}{\gamma_2}} - (\lambda_2-1)\frac{\psi}{r}F\right]$$
$$= -p(z)z^{-\lambda_2}.$$

We see $p(z)>0$ on $z\in(\check{y},\hat{y}]$ from the selection of $\check{y}$. Hence, $q_1(z)$ is decreasing on $z\in(\check{y},\hat{y}]$, and (B.21) is proved.

In the following we will check the condition (5). For case (1), it is easy to verify the correctness since $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)} \geqslant U(0)$ and $\widehat{C}_2 > 0$. For case (2), we need to verify

$$\widehat{C}_2 z^{\lambda_2} + \frac{\gamma_1}{K_1(1-\gamma_1)}z^{-\frac{1-\gamma_1}{\gamma_1}} > U(0) - zF, \tag{B.22}$$

which is valid when $z=\hat{y}$.

Define

$$q_2(z) \triangleq \widehat{C}_2 + \frac{\gamma_1}{K_1(1-\gamma_1)}z^{-\frac{(\lambda_2-1)\gamma_1+1}{\gamma_1}} - U(0)z^{-\lambda_2} + z^{1-\lambda_2}F,$$

and

$$q_2'(z) = z^{-\lambda_2-1}\left[\lambda_2 U(0) - \frac{(\lambda_2-1)\gamma_1+1}{K_1(1-\gamma_1)}z^{-\frac{1-\gamma_1}{\gamma_1}} - (\lambda_2-1)Fz\right].$$

Since $\tilde{z}_2 < \hat{y}$, we have $q_2'(z) > 0$, $\forall z > \hat{y}$ and $q_2(z) > q_2(\hat{y}) > 0$. Hence (B.22) is verified. The verification for $\widetilde{V}(\nu) = \phi(\nu)$ is similar to Proposition 4.2, so here we omit it. □

Proof of Proposition 4.5. When $\tau=\infty$, we have

$$\phi(z) \triangleq \mathbb{E}\left[\int_0^\infty e^{-\beta t}\frac{\gamma_1}{1-\gamma_1}(Z_t)^{-\frac{1-\gamma_1}{\gamma_1}}\,dt\right] = \frac{\gamma_1}{K_1(1-\gamma_1)}z^{-\frac{1-\gamma_1}{\gamma_1}},$$

and $\mathcal{L}_1\phi(z) + \frac{\gamma_1}{1-\gamma_1}z^{-\frac{1-\gamma_1}{\gamma_1}} = 0$.

According to Lemma B.1, we see if $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)} \geqslant U(0)$, then

$$\frac{\gamma_1}{K_1(1-\gamma_1)}z^{-\frac{1-\gamma_1}{\gamma_1}} \geqslant U(0) - zF, \quad \text{for } \hat{y} \leqslant z < \infty. \tag{B.23}$$

Next, we prove

$$\frac{\gamma_1}{K_1(1-\gamma_1)}z^{-\frac{1-\gamma_1}{\gamma_1}} \geqslant C_1 z^{\lambda_1} + \frac{\gamma_2}{K_2(1-\gamma_2)}z^{-\frac{1-\gamma_2}{\gamma_2}} + \frac{\psi}{r}Fz, \quad \text{for } 0 < z \leqslant \hat{y}. \tag{B.24}$$

It is equivalent to prove

$$l(z) \triangleq C_1 + \frac{\gamma_2}{K_2(1-\gamma_2)}z^{-\frac{(\lambda_1-1)\gamma_2+1}{\gamma_2}} - \frac{\gamma_1}{K_1(1-\gamma_1)}z^{-\frac{(\lambda_1-1)\gamma_1+1}{\gamma_1}} + \frac{\psi}{r}Fz^{1-\lambda_1} \leqslant 0, \quad \text{for } 0 < z \leqslant \hat{y},$$

which is satisfied at $z=\hat{y}$, since $C_1\hat{y}^{\lambda_1} + \frac{\gamma_2}{K_2(1-\gamma_2)}\hat{y}^{-\frac{1-\gamma_2}{\gamma_2}} + \frac{\psi}{r}F\hat{y} = U(0) - \hat{y}F$.

Moreover,

$$l'(z) = z^{-\lambda_1}\left(\frac{(\lambda_1-1)\gamma_1+1}{K_1(1-\gamma_1)}z^{-\frac{1}{\gamma_1}} - \frac{(\lambda_1-1)\gamma_2+1}{K_2(1-\gamma_2)}z^{-\frac{1}{\gamma_2}} - \frac{\psi}{r}F(\lambda_1-1)\right).$$

If $0<z<\bar{z}$, then $l'(z)>0$; if $z>\bar{z}$, then $l'(z)<0$, so $l(z)$ gets its maximum at $z=\bar{z}$. Therefore, if $\hat{y}\leqslant\bar{z}$, then $l(z)<l(\hat{y})\leqslant 0$, for $0<z<\hat{y}$; if $\bar{z}<\hat{y}$, and $C_0\leqslant 0$, which implies that $l(\bar{z})\leqslant 0$, then we also have $l(z)\leqslant 0$ for $0<z\leqslant\hat{y}$. Hence, the optimal stopping time is $\tau_\nu=\infty$ and the proof is completed. □

Lemma B.4. *If $\bar{z}<\hat{y}$, and $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)}<U(0)$, then $C_0>0$.*

Proof. (1) We first consider the case when $\hat{y}\leqslant\tilde{z}_2$. Under the assumption $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)}<U(0)$, we see $w(z)=U(0)$ has two roots, denoted by z_1^*, z_2^*, respectively, where $w(z)$ is defined in Lemma B.1. From Remark B.1 and the assumption, we get $\tilde{z}_1<\hat{y}\leqslant\tilde{z}_2$. We claim $\tilde{z}_1>z_1^*$, and $\tilde{z}_2<z_2^*$. Because they are equivalent to $w(\tilde{z}_i)<U(0)$, $i=1,2$, which can be easily verified by the definition of $\tilde{z}_1$, $\tilde{z}_2$ and the fact that $0<\tilde{z}_1<(K_1F)^{-\gamma_1}<\tilde{z}_2<\infty$. Hence, we obtain $U(0)-\hat{y}F>\frac{\gamma_1}{K_1(1-\gamma_1)}\hat{y}^{-\frac{1-\gamma_1}{\gamma_1}}$, i.e., $C_1\hat{y}^{\lambda_1}+\frac{\gamma_2}{K_2(1-\gamma_2)}\hat{y}^{-\frac{1-\gamma_2}{\gamma_2}}+\frac{\psi}{r}F\hat{y}>\frac{\gamma_1}{K_1(1-\gamma_1)}\hat{y}^{-\frac{1-\gamma_1}{\gamma_1}}$. Moreover, from the proof of Proposition 4.5, we have $l'(z)<0$, when $z>\bar{z}$, so $l(\bar{z})>l(\hat{y})>0$. Therefore, $C_0>0$.

(2) For $\hat{y}>\tilde{z}_2$, if we assume $C_0\leqslant 0$, then following the proof of Proposition 4.5, we see

$$\frac{\gamma_1}{K_1(1-\gamma_1)}z^{-\frac{1-\gamma_1}{\gamma_1}} \geqslant C_1z^{\lambda_1}+\frac{\gamma_2}{K_2(1-\gamma_2)}z^{-\frac{1-\gamma_2}{\gamma_2}}+\frac{\psi}{r}Fz, \quad \text{for } 0<z\leqslant\hat{y}.$$

Next, by using the notations in Section 3, we have $\widetilde{U}(z)=C_1z^{\lambda_1}+\frac{\gamma_2}{K_2(1-\gamma_2)}z^{-\frac{1-\gamma_2}{\gamma_2}}+\frac{(\psi+r)F}{r}z$, $0<z<\hat{y}$, which is decreasing and convex in z. So we get $\widetilde{U}(z)>\widetilde{U}(\hat{y})=U(0)$, i.e.,

$$C_1z^{\lambda_1}+\frac{\gamma_2}{K_2(1-\gamma_2)}z^{-\frac{1-\gamma_2}{\gamma_2}}+\frac{\psi}{r}Fz\geqslant U(0)-zF, \quad 0<z\leqslant\hat{y}.$$

On the other hand, combining the assumption $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)}<U(0)$, $(K_1F)^{-\gamma_1}<\tilde{z}_2<\hat{y}$, and the fact $\frac{\gamma_1}{K_1(1-\gamma_1)}\hat{y}^{-\frac{1-\gamma_1}{\gamma_1}}>U(0)-\hat{y}F$, we obtain

$$\frac{\gamma_1}{K_1(1-\gamma_1)}z^{-\frac{1-\gamma_1}{\gamma_1}}>U(0)-zF, \quad z\geqslant\hat{y}.$$

Therefore, $\forall z>0$, $\frac{\gamma_1}{K_1(1-\gamma_1)}z^{-\frac{1-\gamma_1}{\gamma_1}}>U(0)-zF$, which contradicts with the assumption that $\frac{F^{1-\gamma_1}}{K_1^{\gamma_1}(1-\gamma_1)}<U(0)$. Hence $C_0>0$. □

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