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# $L^p$-strong solutions of stochastic partial differential equations with monotonic drifts ☆

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ABSTRACT

In this paper we prove the existence and uniqueness of $L^p$-strong solutions for SPDEs with polynomial growth drifts in a general framework. As applications, we also study the existence of smooth solutions for SPDEs in abstract Wiener space, complete Riemannian manifold as well as the whole Euclidean space.



## 1. Introduction

Consider the following stochastic partial differential equation in $\mathbb{R}^d$

$$\begin{cases} \mathrm{d}u(t,x) = \left[\Delta u(t,x) - u^3(t,x)\right]\mathrm{d}t + g\big(u(t,x)\big)\,\mathrm{d}W(t), \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where $\Delta$ is the Laplace operator in $\mathbb{R}^d$ and $W(t)$ is a one dimensional standard Brownian motion. In the theory of classical partial differential equations (PDE), it is well-known that when $g = 0$ and $u_0$ is smooth enough, there exists a unique smooth solution for Eq. (1.1). In this case, the solution is also smooth with respect to $t$, and the proof of the smoothness of $u(t,x)$ with respect to the spatial variable $x$ is based on the smoothness of $u(t,x)$ with respect to the time variable $t$ by a bootstrap method.

However, when $g \in C_b^\infty(\mathbb{R})$ does not vanish, Eq. (1.1) becomes a stochastic partial differential equation (SPDE). A quite natural question is: Is there a smooth or classical solution for Eq. (1.1) when $u_0$ is smooth

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enough? In view of the non-differentiability of Brownian motion with respect to $t$, it seems that one cannot directly obtain the smoothness of $u(t,x)$ with respect to $x$ by the classical method in PDE. It should be noticed that when $g$ is linear in $u$ and the drift is also linear, the smooth solution for SPDE is well known (cf. [27,4,15,16,13], etc.). With respect to the existence of smooth solutions of semilinear stochastic partial differential equations, Zhang [36] proved that when the diffusion and drift coefficients are smooth and have all bounded derivatives, there exists a unique smooth solution for the Dirichlet boundary problem (some forced compatibility conditions are required). Therein, a Tartar's non-linear interpolation theorem (cf. [31]) plays a key role. Moreover, when $d = 1$, the existence of smooth solutions for Eq. (1.1) was proved in [37] by using the energy method combined with the mollifying argument.

In the past three decades, the theory of SPDEs has been developed extensively by many authors based mainly on two different approaches: the semigroup method (cf. [33,9,4–7,35], etc.) and Galerkin's finite dimensional approximation method or variational method (cf. [22,23,18,27,15–17,26,8], etc.). To our knowledge, most of these well known results are concentrated on the mild or weak solutions. Obviously, to establish a regularity theory for SPDEs, by using Sobolev's embedding theorem, it is natural to consider the $L^p$-solution of SPDEs. Such an $L^p$-theory for SPDE has been established in [4–6,15,11–13,21,17,35], etc. But, it seems that no results can deal with the problem of smooth solution for Eq. (1.1). A very close result related to Eq. (1.1) is given in [6, Theorem 1.3]. Therein, the authors proved the existence of space–time continuous solutions for SPDEs driven by spatial homogeneous Wiener processes. However, none of further regularity about the spatial variable was obtained. We also emphasize that the first order term was not contained in the equation considered in [6].

In the present work, we shall study the existence of $L^p$-strong solutions for SPDEs with monotone drifts in a general framework by using Yosida's approximation. Here, the word "strong" has two aspect meanings: in the sense of SDE as well as in the sense of PDE. Our proof of $L^p$-strong solutions can be considered as a combination of the semigroup method and the variational method mentioned above. The main result is given in Section 2, and will be proved in Section 3. In order to cover more general stochastic equations, we use the notion of *carré du champ* operator so that our equation can admit a first order term. In Section 4, we apply our general result to the abstract Wiener space case. In Section 5, the SPDEs on complete Riemannian manifolds are considered. In order to obtain the existence of smooth solutions, we shall work in weighted Sobolev spaces by using a non-linear interpolation theorem due to Tartar [31] (cf. [36]). In the case of Euclidean space, we also consider the SPDEs driven by spatially homogeneous Wiener processes (cf. [25,6]). Lastly, in Section 6, we deal with Eq. (1.1). There, a sharp result of Littlewood–Paley's inequality due to Krylov [14] will play a crucial role.

## 2. Framework and main result

Let $(E,\mathcal{B},\mu)$ be a $\sigma$-finite measure space. For any $1 \leqslant p \leqslant +\infty$, we denote by $L^p(E,\mu)$ the corresponding real $L^p$-space equipped with the usual norm $\|\cdot\|_{L^p}$. Let $(\mathfrak{T}_t)_{t\geqslant 0}$ be a family of symmetric strongly continuous semigroup on $L^2(E,\mu)$ with $\mathfrak{T}_0 = g$ the identity operator. Suppose also throughout this paper that:

**(A)** $\mathfrak{T}_t$ is contracted on $L^p(E,\mu)$ for each $t > 0$, i.e., $\|\mathfrak{T}_t f\|_p \leqslant \|f\|_p$, $1 \leqslant p \leqslant \infty$;
**(B)** for each $t > 0$, $\mathfrak{T}_t$ is self-adjoint on $L^2(E,\mu)$.

Under **(A)** and **(B)**, it is well known that $(\mathfrak{T}_t)_{t\geqslant 0}$ forms an analytic semigroup on $L^p(E,\mu)$ for each $p \in (1,\infty)$ (cf. Stein [28, p. 67, Theorem 1]). Let $(\mathfrak{L},\mathcal{D}_p(\mathfrak{L}))$ be the generator of $(\mathfrak{T}_t)_{t\geqslant 0}$ in $L^p(E,\mu)$, where

$$\mathcal{D}_p(\mathfrak{L}) := \left\{ u \in L^p(E,\mu)\colon\ \mathfrak{L}u := \lim_{t\to 0} \frac{\mathfrak{T}_t u - u}{t} \text{ exists in } L^p(E,\mu) \right\}.$$

Obviously, $(\mathfrak{L},\mathcal{D}_2(\mathfrak{L}))$ is a negative self-adjoint operator on $L^2(E,\mu)$.

By [24, Section 2.6], for $\alpha \in \mathbb{R}_+$ the fractional power $(g-\mathfrak{L})^\alpha$ of $g-\mathfrak{L}$ is defined as the inverse of the bounded linear operator:

$$(g-\mathfrak{L})^{-\alpha} := c_\alpha^{-1} \int_0^\infty t^{\alpha-1} e^{-t} \mathfrak{T}_t \, \mathrm{d}t,$$

where $c_\alpha \equiv \int_0^\infty t^{\alpha-1}e^{-t}\,\mathrm{d}t$ is the Gamma constant.

For $p>1$ and $\alpha>0$, the Sobolev type spaces are defined by

$$H^p_\alpha := (g-\mathfrak{L})^{-\alpha/2}\big(L^p(E,\mu)\big)$$

with the norm:

$$\|f\|_{H^p_\alpha} := \big\|(g-\mathfrak{L})^{\alpha/2} f\big\|_{L^p}.$$

They are separable and reflexive Banach spaces. Clearly, $H^p_0 = L^p(E,\mu)$, $H^p_2 = \mathcal{D}_p(\mathfrak{L})$ and

$$H^p_{\alpha'} \subset H^p_\alpha \quad \text{if } \alpha' \geqslant \alpha \geqslant 0.$$

Let $l^2$ be the usual Hilbert space of the sequence of square summable real numbers. Similarly, we may define the $l^2$-valued Sobolev type spaces $H^p_\alpha(l^2)$.

For $p>1$ and $\alpha<0$, one defines $H^p_\alpha$ as the dual space of $H^{p^*}_{-\alpha}$, where $\frac{1}{p}+\frac{1}{p^*}=1$. In the sequel, $(p,p^*)$ will always be used to denote a couple of conjugated indexes. It is not hard to see that for any $f\in H^p_\alpha$, there exists a unique element $h\in L^p(E,\mu)$ such that

$${}_{H^p_\alpha}\langle f,g\rangle_{H^{p^*}_{-\alpha}} = {}_{L^p}\big\langle h,(g-\mathfrak{L})^{-\alpha/2}g\big\rangle_{L^{p^*}}, \quad \forall g\in H^{p^*}_{-\alpha}.$$

In this sense, $H^p_\alpha$ may also be regarded as $(g-\mathfrak{L})^{-\alpha/2}(L^p(E,\mu))$ for $\alpha<0$. We remark that for any $\alpha,\beta\in\mathbb{R}$, $(g-\mathfrak{L})^{\alpha/2}$ is an isomorphic mapping from $H^p_\beta$ to $H^p_{\beta-\alpha}$.

Moreover, we also assume that

- **(C)** There exists a set $\mathcal{D}$ such that $\mathcal{D}$ is an algebra and dense in $H^p_\alpha$ for all $p>1$ and $\alpha\in\mathbb{R}_+$.
- **(D)** $(\mathfrak{T}_t)_{t\geqslant 0}$ is sub-Markovian, and associates with a unique local Dirichlet form $(\mathscr{E},\mathcal{D}(\mathscr{E}))$ (see [3, Definitions 2.1.1 and 5.1.2]).

Under **(C)** and **(D)**, the Dirichlet form $(\mathscr{E},\mathcal{D}(\mathscr{E}))$ admits a *carré du champ* operator (see [3, Corollary 4.2.3]), denoted by $\varGamma$, which is the unique positive symmetric continuous form from $\mathcal{D}(\mathscr{E})\times\mathcal{D}(\mathscr{E})$ into $L^1(E,\mu)$, such that for all $u,v,h\in\mathcal{D}(\mathscr{E})\cap L^\infty(E,\mu)$,

$$\mathscr{E}(uh,v)+\mathscr{E}(hv,u)-\mathscr{E}(uv,h) = 2\int_E h\varGamma(u,v)\,\mathrm{d}\mu.$$

We have (cf. [3, Proposition 6.1.1])

$$\mathscr{E}(u,v) = \int_E \varGamma(u,v)\,\mathrm{d}\mu = -{}_{H^2_1}\langle u,\mathfrak{L}v\rangle_{H^2_{-1}}, \quad u,v\in\mathcal{D}(\mathscr{E}). \tag{2.1}$$

Later, we shall simply write $\varGamma(u):=\varGamma(u,u)$, and suppose that the following Meyer type inequality holds:

**(E)** For any $p > 1$, there exists a constant $c_p > 0$ such that for any $u \in \mathcal{D}$,

$$c_p^{-1}\|u\|_{H_1^p} \leqslant \left\|\Gamma^{1/2}(u)\right\|_{L^p} + \|u\|_{L^p} \leqslant c_p\|u\|_{H_1^p}.$$

We need the following function class on $E \times \mathbb{R}$:

$$\mathscr{G} := \left\{ \begin{array}{l} \text{for } \mu\text{-a.a. } x \in E, \text{ the map } z \mapsto f(x,z) \in C^1(\mathbb{R}), \\ \text{and for each } z \in \mathbb{R}, \ f(\cdot,z), \ \partial_z f(\cdot,z) \in H_1^2, \text{ and} \\ \text{for } \mu\text{-a.a. } x \in E, \text{ the map } z \mapsto \Gamma(\partial_z f(\cdot,z))(x) \in C(\mathbb{R}) \end{array} \right\}. \tag{2.2}$$

In the sequel, although it is not true in general that $1 \in \mathcal{D}(\mathfrak{L})$ and $\mathfrak{L}1 = 0$, we still use the following convention: for any constant $c \in \mathbb{R}$,

$$\Gamma(c) = 0.$$

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, P)$ be a complete and right-continuous filtration probability space. Let $\mathcal{P}$ be the predictable $\sigma$-field associated with $(\mathcal{F}_t)_{t\geqslant 0}$. A family of independent one dimensional $(\mathcal{F}_t)$-adapted Brownian motions $\{W_t^k;\ t \geqslant 0,\ k = 1, 2, \ldots\}$ on $(\Omega, \mathcal{F}, P)$ is given. In the following, the letters $c$ or $C$ with or without subscripts will denote different positive constants on different occasions.

In the present paper, we consider the following type SPDE:

$$\begin{cases} \mathrm{d}u(t,x) = \left[\mathfrak{L}u(t,x) + \Gamma\left(f_1\left(t,\cdot,u(t)\right),\varphi\right)(x) + f_0\left(t,x,u(t,x)\right)\right]\mathrm{d}t + \sum_k g_k\left(t,x,u(t,x)\right)\mathrm{d}W_t^k, \\ u(0,x) = u_0(x), \end{cases} \tag{2.3}$$

where $\varphi \in \mathcal{D}(\mathfrak{L})$ satisfies $|\mathfrak{L}\varphi| + \Gamma^{1/2}(\varphi) \leqslant c_\varphi$, and

$$f_1 : \Omega \times \mathbb{R}_+ \times E \times \mathbb{R} \to \mathbb{R},$$
$$f_0 : \Omega \times \mathbb{R}_+ \times E \times \mathbb{R} \to \mathbb{R},$$
$$g : \Omega \times \mathbb{R}_+ \times E \times \mathbb{R} \to l^2$$

are measurable functions.

We impose the following conditions on $f_1$, $f_0$ and $g$:

**(H1)** For each $(x,z) \in E \times \mathbb{R}$, the following maps

$$(\omega,t) \mapsto f_1(\omega,t,x,z),\ f_0(\omega,t,x,z),\ g_k(\omega,t,x,z), \quad k \in \mathbb{N}$$

are $\mathcal{P}$-measurable, and for each $(\omega,t) \in \Omega \times \mathbb{R}_+$ and $k \in \mathbb{N}$,

$$(x,z) \mapsto f_1(\omega,t,x,z),\ g_k(\omega,t,x,z) \in \mathscr{G},$$

and for each $(\omega,t,x) \in \Omega \times \mathbb{R}_+ \times E$,

$$z \mapsto f_0(\omega,t,x,z) \text{ is continuous.}$$

**(H2)** There exist a constant $c_{f_1} > 0$ and $\lambda_{f_1}(x) \in \bigcap_{p\geqslant 2} L^p(E,\mu) =: L^{\infty-}(E,\mu)$ such that for all $(\omega,t,x,z) \in \Omega \times \mathbb{R}_+ \times E \times \mathbb{R}$,

$$\begin{aligned}\Gamma^{1/2}\big(\partial_z f_1(\omega,t,\cdot,z)\big)(x)+\big|\partial_z f_1(\omega,t,x,z)\big| &\leqslant c_{f_1},\\ \Gamma^{1/2}\big(f_1(\omega,t,\cdot,0)\big)(x)+\big|f_1(\omega,t,x,0)\big| &\leqslant \lambda_{f_1}(x).\end{aligned}$$

**(H3)** There exist constants $c_{f_0}, c'_{f_0} > 0$, $n \in \mathbb{N}$ and $\lambda_{f_0}(x) \in L^{\infty-}(E,\mu)$ such that for all $(\omega,t,x,y,z) \in \Omega \times \mathbb{R}_+ \times E \times \mathbb{R}^2$,

$$\begin{aligned}(y-z)\cdot\big(f_0(\omega,t,x,y)-f_0(\omega,t,x,z)\big) &\leqslant c_{f_0}\cdot|y-z|^2,\\ \big|f_0(\omega,t,x,z)-f_0(\omega,t,x,0)\big| &\leqslant c'_{f_0}\cdot\big(|z|^n+z\big),\\ \big|f_0(\omega,t,x,0)\big| &\leqslant \lambda_{f_0}(x).\end{aligned}$$

**(H4)** There exist a constant $c_g > 0$ and $\lambda_g(x) \in L^{\infty-}(E,\mu)$ such that for all $(\omega,t,x,z) \in \Omega \times \mathbb{R}_+ \times E \times \mathbb{R}$,

$$\begin{aligned}\sum_k\Big(\Gamma\big(\partial_z g_k(\omega,t,\cdot,z)\big)(x)+\big|\partial_z g_k(\omega,t,x,z)\big|^2\Big) &\leqslant c_g,\\ \sum_k\Big(\Gamma\big(g_k(\omega,t,\cdot,0)\big)(x)+\big|g_k(\omega,t,x,0)\big|^2\Big) &\leqslant \lambda_g(x).\end{aligned}$$

**Remark 2.1.** Fix an odd number $n$. Let $f_0$ be an $n$-order polynomial with the form

$$f_0(z)=\sum_{j=1}^n c_j z^j, \quad \text{where } c_1,\dots,c_{n-1}\in\mathbb{R} \text{ and } c_n<0.$$

Then $f_0$ satisfies **(H3)**.

For $q \geqslant 2$, we define the Banach space

$$\mathbb{B}^q := H_0^q \cap H_1^2$$

with the norm

$$\|\cdot\|_{\mathbb{B}^q} := \|\cdot\|_{H_0^q} + \|\cdot\|_{H_1^2}.$$

We can now state our main result as follows:

**Theorem 2.2.** *Assume that* **(H1)**–**(H4)** *hold. Then, for any $q \geqslant 2n$, where $n$ satisfy* **(H3)** *and $u_0 \in \mathbb{B}^q$, there exists a unique $H_1^2$-valued continuous and $(\mathcal{F}_t)$-adapted process $u(t)$ (called a solution of Eq. (2.3)) such that for any $T > 0$,*

$$\mathbb{E}\Big(\sup_{t\in[0,T]}\big\|u(t)\big\|_{L^q}^q\Big)+\mathbb{E}\Big(\sup_{t\in[0,T]}\big\|u(t)\big\|_{H_1^2}^2\Big)+\int_0^T\mathbb{E}\big\|u(t)\big\|_{H_2^2}^2\,\mathrm{d}t \leqslant C_T\big(\|u_0\|_{\mathbb{B}^q}^q+1\big), \tag{2.4}$$

*and the following equation holds in $L^2(E,\mu)$,*

$$\begin{aligned}u(t)=u_0&+\int_0^t\big[\mathfrak{L}u(s)+\Gamma\big(f_1\big(s,\cdot,u(s)\big),\varphi\big)+f_0\big(s,\cdot,u(s)\big)\big]\,\mathrm{d}s\\ &+\sum_k\int_0^t g_k\big(s,\cdot,u(s)\big)\,\mathrm{d}W_s^k \quad \text{for any } t\geqslant 0,\ P\text{-a.e.}\end{aligned} \tag{2.5}$$

*Moreover, let $u(t)$ and $\tilde{u}(t)$ be two solutions of Eq.* (2.3) *corresponding to the initial values $u_0, \tilde{u}_0 \in \mathbb{B}^q$. Then*

$$\mathbb{E}\Big(\sup_{t\in[0,T]}\big\|u(t)-\tilde{u}(t)\big\|_{L^q}^q\Big) \leqslant C_T\|u_0-\tilde{u}_0\|_{L^q}^q. \tag{2.6}$$

**Remark 2.3.** If $q>2$, by (2.4) and the interpolation inequality between $L^2(E,\mu)$ and $L^q(E,\mu)$ (cf. [2]), we in fact have the continuity of $t\mapsto u(t)$ in $L^p(E,\mu)$ for any $2<p<q$.

**Remark 2.4.** In terms of $\mathfrak{T}_t$, the solution $u$ can be written as the following mild form

$$u(t)=\mathfrak{T}_t u_0+\int_0^t \mathfrak{T}_{t-s}\big[\Gamma\big(f_1\big(s,u(s)\big),\varphi\big)+f_0\big(s,u(s)\big)\big]\,\mathrm{d}s+\sum_k\int_0^t \mathfrak{T}_{t-s}g_k\big(s,u(s)\big)\,\mathrm{d}W_s^k. \tag{2.7}$$

This expression will be used to improve the regularity of the solution in Sections 5 and 6.

## 3. Proof of main result

We first prepare two lemmas about the *carré du champ* operator for later use. The following lemma is as the chain rule.

**Lemma 3.1.** *Let $f\in\mathscr{G}$. Assume that for some $c_0>0$ and $p\geqslant 0$,*

$$\Gamma^{1/2}\big(\partial_z f(\cdot,z)\big)(x)+\big|\partial_z f(x,z)\big| \leqslant c_0\big(|z|^p+1\big). \tag{3.1}$$

*Then for any $u,v\in H_1^2\cap H_1^{2(p+1)}$, we have*

$$\Gamma\big(f(\cdot,u)\big)(x)=\big(\partial_u f\big(x,u(x)\big)\big)^2\cdot\Gamma(u)(x)+\Gamma\big(f\big(\cdot,u(x)\big)\big)(x)+2\partial_u f\big(x,u(x)\big)\cdot\Gamma\big(u,f\big(\cdot,u(x)\big)\big)(x) \tag{3.2}$$

*and*

$$\Gamma\big(f(\cdot,u),v\big)(x)=\partial_u f\big(x,u(x)\big)\cdot\Gamma(u,v)(x)+\Gamma\big(f\big(\cdot,u(x)\big),v\big)(x). \tag{3.3}$$

*Here and below, $\Gamma(f(\cdot,u(x)))(x):=\Gamma(f(\cdot,z))(x)|_{z=u(x)}$ and similarly for others.*

**Proof.** We only prove the first one, the second is analogous. Denote $\tilde{f}(x,z)=f(x,z)-f(x,0)$ and assume $f(x,0)=0$ for simplicity. For $n\in\mathbb{N}$, let $\chi_n\in C^1(\mathbb{R})$ be a cutoff function with $|\chi_n|\leqslant 1$ and

$$\chi_n(z)=\begin{cases}1, & \text{for } |z|<n,\\ 0, & \text{for } |z|\geqslant n+1.\end{cases}$$

Define

$$g_n(x,z):=\sum_{k=-\infty}^{\infty}\partial_z f\big(x,k2^{-n}\big)\cdot 1_{[k2^{-n},(k+1)2^{-n})}(z)$$

and

$$f_n(x,z):=\int_0^z \chi_n(y)g_n(x,y)\,\mathrm{d}y=\sum_{k=-\infty}^{\infty}\partial_z f\big(x,k2^{-n}\big)\cdot\int_0^z \chi_n(y)1_{[k2^{-n},(k+1)2^{-n})}(y)\,\mathrm{d}y,$$

where the sum only has finite terms. Since $f \in \mathscr{G}$ (see (2.2) for the definition), it is easy to see that for $\mu$-a.a. $x \in E$ and all $z \in \mathbb{R}$,

$$\lim_{n\to\infty} \partial_z f_n(x,z) = \partial_z f(x,z), \tag{3.4}$$

and

$$\lim_{n\to\infty} \Gamma\big(\partial_z f_n(\cdot,z) - \partial_z f(\cdot,z)\big)(x) = 0. \tag{3.5}$$

Moreover, by (3.1) we also have

$$\big|\Gamma^{1/2}\big(\partial_z f_n(\cdot,z)\big)(x)\big| + \big|\partial_z f_n(x,z)\big| \leqslant C \cdot \big(|z|^p + 1\big). \tag{3.6}$$

Now, by $u \in H_1^2 \cap H_1^{2(p+1)}$ and [3, Corollary 6.1.3], we know

$$\begin{aligned}\Gamma\big(f_n(\cdot,u)\big)(x) &= \big(\partial_u f_n\big(x,u(x)\big)\big)^2 \Gamma(u)(x) + \Gamma\big(f_n\big(\cdot,u(x)\big)\big)(x) + 2\partial_u f_n\big(x,u(x)\big)\Gamma\big(u, f_n\big(\cdot,u(x)\big)\big)(x) \\ &=: I_1^n(x) + I_2^n(x) + I_3^n(x).\end{aligned} \tag{3.7}$$

Let $I_i(x)$, $i = 1,2,3$, be the corresponding terms in (3.2). By the dominated convergence theorem and (3.4), we have

$$\lim_{n\to\infty} \big\|I_1^n(\cdot) - I_1(\cdot)\big\|_{L^1} = 0.$$

By a direct calculation, one finds that

$$\Gamma\big(f_n(\cdot,z)\big)(x) = \int_0^z \int_0^z \Gamma\big(\partial_y f_n(\cdot,y), \partial_{y'} f_n\big(\cdot,y'\big)\big)(x)\,\mathrm{d}y\,\mathrm{d}y',$$

which together with (3.5) and (3.6) yields

$$\lim_{n\to\infty} \big\|I_2^n(\cdot) - I_2(\cdot)\big\|_{L^1} = 0.$$

Similarly, we have

$$\lim_{n\to\infty} \big\|I_3^n(\cdot) - I_3(\cdot)\big\|_{L^1} = 0.$$

Hence,

$$\lim_{n,m\to\infty} \big\|\Gamma\big(f_n(\cdot,u)\big) - \Gamma\big(f_m(\cdot,u)\big)\big\|_{L^1} = 0.$$

Noting that

$$\lim_{n\to\infty} \big\|f_n(\cdot,u) - f(\cdot,u)\big\|_{L^2} = 0,$$

we have $f(\cdot,u) \in H_1^2$ and

$$\lim_{n\to\infty} \big\|\Gamma\big(f_n(\cdot,u)\big) - \Gamma\big(f(\cdot,u)\big)\big\|_{L^1} = 0,$$

which gives (3.2) by taking limits for (3.7). $\square$

**Lemma 3.2.** *Assume that* $\varphi \in \mathcal{D}(\mathfrak{L})$ *satisfies*

$$|\mathfrak{L}\varphi| + \Gamma^{1/2}(\varphi) \leqslant c_\varphi.$$

*Then for any* $p > 1$*, there is a constant* $C_{p,\varphi} > 0$ *such that for any* $u \in L^p(E,\mu)$*,*

$$\big\|\Gamma(u,\varphi)\big\|_{H^p_{-1}} \leqslant C_{p,\varphi}\|u\|_{L^p}.$$

**Proof.** Suppose $u \in \mathcal{D}$ and let $p^* = p/(p-1)$. We have

$$\begin{aligned}\big\|\Gamma(u,\varphi)\big\|_{H^p_{-1}} &= \sup_{v\in\mathcal{D};\,\|v\|_{H_1^{p^*}}\leqslant 1}\left|\int_E \Gamma(u,\varphi)\cdot v\,\mathrm{d}\mu\right| = \sup_{v\in\mathcal{D};\,\|v\|_{H_1^{p^*}}\leqslant 1}\left|\int_E \big(\Gamma(uv,\varphi) - \Gamma(v,\varphi)\cdot u\big)\,\mathrm{d}\mu\right| \\ &\leqslant 2\sup_{v\in\mathcal{D};\,\|v\|_{H_1^{p^*}}\leqslant 1}\left|\int_E (uv)\cdot\mathfrak{L}\varphi\,\mathrm{d}\mu\right| + \sup_{v\in\mathcal{D};\,\|v\|_{H_1^{p^*}}\leqslant 1}\left|\int_E \Gamma^{1/2}(v)\cdot\Gamma^{1/2}(\varphi)\cdot u\,\mathrm{d}\mu\right| \\ &\leqslant 2c_\varphi\cdot\|u\|_{L^p} + c_\varphi\cdot\sup_{v\in\mathcal{D};\,\|v\|_{H_1^{p^*}}\leqslant 1}\big\|\Gamma^{1/2}(v)\big\|_{L^{p^*}}\cdot\|u\|_{L^p},\end{aligned}$$

which gives the desired result by **(C)** and **(E)**. □

### *3.1. Lipschitz continuous* $f_0$

Without loss of generality, we assume in what follows that

$$c_{f_0} = 0.$$

Otherwise, we may use the transformation $\tilde{u}(t,x) = u(t,x)e^{-c_{f_0}t}$ to reduce the coefficients to this case.

First of all, we consider the case that $u_0 \in \mathcal{D}$ and $f_0$ is Lipschitz continuous with respect to $z$, i.e., for some $\kappa > 0$

$$\big|f_0(\omega,t,x,y) - f_0(\omega,t,x,z)\big| \leqslant \kappa\cdot|y-z|. \tag{3.8}$$

Under **(H1)**–**(H4)** and (3.8), using Lemmas 3.1 and 3.2, it is easy to see by Corollary 5.4 of [36] and Theorem 2.2 of [37] that there exists a unique strong solution $u$ to Eq. (2.3) such that for any $p \geqslant 2$ and $T > 0$,

$$\mathbb{E}\Big(\sup_{t\in[0,T]}\big\|u(t)\big\|^p_{H^p_1}\Big) + \int_0^T \mathbb{E}\big\|u(t)\big\|^2_{H^2_2}\,\mathrm{d}t \leqslant C_{p,T,\kappa}.$$

In the following, we shall prove three uniform estimates which are independent of the Lipschitz constant $\kappa$ of $f_0$.

**Lemma 3.3.** *For any* $p \geqslant 2$ *and* $T > 0$*, there exists a constant* $C_{p,T} > 0$ *independent of* $\kappa$ *such that*

$$\mathbb{E}\Big(\sup_{t\in[0,T]}\big\|u(t)\big\|^p_{L^p}\Big) \leqslant C_{p,T}\cdot\big(\|u_0\|^p_{L^p} + 1\big). \tag{3.9}$$

**Proof.** By the usual Itô's formula (cf. [6, Theorem A.2]), we have

$$\big\|u(t)\big\|_{L^p}^p = \|u_0\|_{L^p}^p + I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t),$$

where

$$I_1(t) := p\int_0^t \big\langle \big|u(s)\big|^{p-2}u(s), \mathfrak{L}u(s)\big\rangle_{L^2}\,\mathrm{d}s,$$

$$I_2(t) := p\int_0^t \big\langle \big|u(s)\big|^{p-2}u(s), \Gamma\big(f_1\big(s,\cdot,u(s)\big),\varphi\big)\big\rangle_{L^2}\,\mathrm{d}s,$$

$$I_3(t) := p\int_0^t \big\langle \big|u(s)\big|^{p-2}u(s), f_0\big(s,\cdot,u(s)\big)\big\rangle_{L^2}\,\mathrm{d}s,$$

$$I_4(t) := p\sum_k \int_0^t \big\langle \big|u(s)\big|^{p-2}u(s), g_k\big(s,\cdot,u(s)\big)\big\rangle_{L^2}\,\mathrm{d}W_s^k,$$

$$I_5(t) := \frac{p(p-1)}{2}\sum_k \int_0^t \big\langle \big|u(s)\big|^{p-2}, \big|g_k\big(s,\cdot,u(s)\big)\big|^2\big\rangle_{L^2}\,\mathrm{d}s.$$

First of all, we have

$$\begin{aligned} I_1(t) &\overset{(2.1)}{=} -p\int_0^t\int_E \Gamma\big(\big|u(s)\big|^{p-2}u(s), u(s)\big)\,\mathrm{d}\mu\,\mathrm{d}s \\ &\overset{(3.3)}{=} -p(p-1)\int_0^t\int_E \big|u(s)\big|^{p-2}\Gamma\big(u(s)\big)\,\mathrm{d}\mu\,\mathrm{d}s, \end{aligned}$$

and by **(H2)** and Young's inequality,

$$\begin{aligned} I_2(t) &\overset{(3.3)}{=} p\int_0^t\int_E \Gamma\big(\big|u(s)\big|^{p-2}u(s)\cdot f_1\big(s,\cdot,u(s)\big),\varphi\big)\,\mathrm{d}\mu\,\mathrm{d}s - p\int_0^t\int_E f_1\big(s,\cdot,u(s)\big)\cdot\Gamma\big(\big|u(s)\big|^{p-2}u(s),\varphi\big)\,\mathrm{d}\mu\,\mathrm{d}s \\ &\overset{(2.1)}{=} -p\int_0^t\int_E \mathfrak{L}\varphi\cdot\big|u(s)\big|^{p-2}u(s)\cdot f_1\big(s,\cdot,u(s)\big)\,\mathrm{d}\mu\,\mathrm{d}s - p\int_0^t\int_E f_1\big(s,\cdot,u(s)\big)\cdot\big|u(s)\big|^{p-2}\cdot\Gamma\big(u(s),\varphi\big)\,\mathrm{d}\mu\,\mathrm{d}s \\ &\leqslant p\int_0^t\int_E |\mathfrak{L}\varphi|\cdot\big|u(s)\big|^{p-1}\cdot\big(c_{f_1}\big|u(s)\big|+\lambda_{f_1}\big)\,\mathrm{d}\mu\,\mathrm{d}s \\ &\quad + p\int_0^t\int_E \big(c_{f_1}\big|u(s)\big|+\lambda_{f_1}\big)\cdot\big|u(s)\big|^{p-2}\cdot\Gamma^{1/2}\big(u(s)\big)\cdot\Gamma^{1/2}(\varphi)\,\mathrm{d}\mu\,\mathrm{d}s \\ &\leqslant p\int_0^t\int_E \big|u(s)\big|^{p-2}\cdot\Gamma\big(u(s)\big)\,\mathrm{d}\mu\,\mathrm{d}s + C\int_0^t\big(\big\|u(s)\big\|_{L^p}^p+1\big)\,\mathrm{d}s, \end{aligned}$$

where we have used that $|\mathfrak{L}\varphi| + \Gamma^{1/2}(\varphi) \leqslant c_\varphi$. Here and after, the constant $C$ is independent of $\kappa$.

For $I_3(t)$, since we have assumed $c_{f_0} = 0$, by **(H3)** we have

$$I_3(t) = p\int_0^t\int_E \left|u(s)\right|^{p-2}u(s)\cdot f_0\bigl(s,\cdot,u(s)\bigr)\,\mathrm{d}\mu\,\mathrm{d}s$$
$$\leqslant p\int_0^t\int_E \lambda_{f_0}\cdot\left|u(s)\right|^{p-1}\,\mathrm{d}\mu\,\mathrm{d}s \leqslant C\int_0^t\bigl(\left\|u(s)\right\|_{L^p}^p+1\bigr)\,\mathrm{d}s.$$

Moreover, it is clear that by **(H4)**

$$I_5(t)\leqslant C\int_0^t\bigl(\left\|u(s)\right\|_{L^p}^p+1\bigr)\,\mathrm{d}s.$$

Hence

$$\left\|u(t)\right\|_{L^p}^p \leqslant \|u_0\|_{L^p}^p + C\int_0^t\bigl(\left\|u(s)\right\|_{L^p}^p+1\bigr)\,\mathrm{d}s + I_4(t). \tag{3.10}$$

Taking expectations for (3.10) and using Gronwall's inequality, we get that for any $T>0$,

$$\sup_{t\in[0,T]}\mathbb{E}\left\|u(t)\right\|_{L^p}^p \leqslant C\bigl(\|u_0\|_{L^p}^p+1\bigr). \tag{3.11}$$

On the other hand, by BDG's inequality and **(H4)** we have

$$\mathbb{E}\Bigl(\sup_{t\in[0,T]}\left|I_4(t)\right|\Bigr)\leqslant C\mathbb{E}\left(\int_0^T\left(\int_E\left|u(s)\right|^{p-1}\cdot\left\|g\bigl(s,\cdot,u(s)\bigr)\right\|_{l^2}\,\mathrm{d}\mu\right)^2\mathrm{d}s\right)^{1/2}$$
$$\leqslant C\mathbb{E}\left(\int_0^T\bigl(\left\|u(s)\right\|_{L^p}^{2p}+1\bigr)\,\mathrm{d}s\right)^{1/2}$$
$$\leqslant C\mathbb{E}\left(\sup_{s\in[0,T]}\left\|u(s)\right\|_{L^p}^p\int_0^T\left\|u(s)\right\|_{L^p}^p\,\mathrm{d}s\right)^{1/2}+C$$
$$\leqslant \frac{1}{2}\mathbb{E}\Bigl(\sup_{t\in[0,T]}\left\|u(t)\right\|_{L^p}^p\Bigr)+C\int_0^T\mathbb{E}\bigl(\left\|u(s)\right\|_{L^p}^p+1\bigr)\,\mathrm{d}s,$$

which together with (3.10) and (3.11) yields the desired estimate. □

Similarly, we can prove that

**Lemma 3.4.** *Let $u(t)$ and $\tilde u(t)$ be two solutions of Eq. (2.3) corresponding to the initial values $u_0,\tilde u_0\in\mathcal{D}$. For any $p\geqslant 2$ and $T>0$, there exists a constant $C_{p,T}>0$ independent of $\kappa$ such that*

$$\mathbb{E}\Bigl(\sup_{t\in[0,T]}\left\|u(t)-\tilde u(t)\right\|_{L^p}^p\Bigr)\leqslant C_{p,T}\cdot\|u_0-\tilde u_0\|_{L^p}^p.$$

We also have

**Lemma 3.5.** *For any $T > 0$, there exists constant $C_T > 0$ independent of $\kappa$ such that*

$$\mathbb{E}\Big(\sup_{t\in[0,T]}\big\|u(t)\big\|^2_{H^2_1}\Big) + \int_0^T \mathbb{E}\big\|u(t)\big\|^2_{H^2_2}\,\mathrm{d}t \leqslant C_T\big(\|u_0\|^2_{H^2_1} + \|u_0\|^{2n}_{L^{2n}} + 1\big).$$

**Proof.** Consider the following evolution triple:

$$H^2_2 \subset H^2_1 \subset H^2_0$$

with

$${}_{H^2_2}\langle u, v\rangle_{H^2_0} = \big\langle (g-\mathfrak{L})u, v\big\rangle_{H^2_0}, \quad u \in H^2_2,\ v \in H^2_0.$$

By Itô's formula (cf. [27,26]), we have

$$\big\|u(t)\big\|^2_{H^2_1} = \|u_0\|^2_{H^2_1} + I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t),$$

where

$$I_1(t) := 2\int_0^t {}_{H^2_2}\big\langle u(s), \mathfrak{L}u(s)\big\rangle_{H^2_0}\,\mathrm{d}s,$$

$$I_2(t) := 2\int_0^t {}_{H^2_2}\big\langle u(s), \Gamma\big(f_1\big(s,\cdot,u(s)\big),\varphi\big)\big\rangle_{H^2_0}\,\mathrm{d}s,$$

$$I_3(t) := 2\int_0^t {}_{H^2_2}\big\langle u(s), f_0\big(s,\cdot,u(s)\big)\big\rangle_{H^2_0}\,\mathrm{d}s,$$

$$I_4(t) := 2\sum_k\int_0^t \big\langle u(s), g_k\big(s,\cdot,u(s)\big)\big\rangle_{H^2_1}\,\mathrm{d}W^k(s),$$

$$I_5(t) := \sum_k\int_0^t \big\|g_k\big(s,\cdot,u(s)\big)\big\|^2_{H^2_1}\,\mathrm{d}s.$$

First of all, we have

$$I_1(t) = -2\int_0^t \big\|u(s)\big\|^2_{H^2_2}\,\mathrm{d}s + 2\int_0^t \big\|u(s)\big\|^2_{H^2_1}\,\mathrm{d}s$$

and by Young's inequality, Lemma 3.1 and **(H2)**,

$$I_2(t) \leqslant \frac{1}{2}\int_0^t \big\|u(s)\big\|^2_{H^2_2}\,\mathrm{d}s + C\int_0^t \big(\big\|u(s)\big\|^2_{H^2_1} + 1\big)\,\mathrm{d}s.$$

Similarly, by **(H3)** we have

$$\begin{aligned} I_3(t) &\leqslant \frac{1}{2}\int_0^t \|u(s)\|_{H_2^2}^2\,\mathrm{d}s + C\int_0^t \|f_0(s,\cdot,u(s))\|_{L^2}^2\,\mathrm{d}s \\ &\leqslant \frac{1}{2}\int_0^t \|u(s)\|_{H_2^2}^2\,\mathrm{d}s + C\int_0^t \left(\|u(s)\|_{L^{2n}}^{2n} + \|u(s)\|_{L^2}^2 + 1\right)\mathrm{d}s. \end{aligned}$$

Moreover, it is obvious by **(H4)** and Lemma 3.1 that

$$I_5(t) \leqslant C\int_0^t \left(\|u(s)\|_{H_1^2}^2 + 1\right)\mathrm{d}s.$$

Combining the above calculations gives that

$$\|u(t)\|_{H_1^2}^2 \leqslant \|u_0\|_{H_1^2}^2 - \int_0^t \|u(s)\|_{H_2^2}^2\,\mathrm{d}s + C\int_0^t \left(\|u(s)\|_{H_1^2}^2 + 1\right)\mathrm{d}s + C\int_0^t \|u(s)\|_{L^{2n}}^{2n}\,\mathrm{d}s + I_4(t).$$

As in the proof of Lemma 3.3, we deduce that

$$\mathbb{E}\Big(\sup_{t\in[0,T]}\|u(t)\|_{H_1^2}^2\Big) + \int_0^T \mathbb{E}\|u(t)\|_{H_2^2}^2\,\mathrm{d}t \leqslant C_T\left(\|u_0\|_{H_1^2}^2 + \int_0^T \mathbb{E}\|u(s)\|_{L^{2n}}^{2n}\,\mathrm{d}s + 1\right),$$

which gives the desired estimate by (3.9). □

### *3.2. Proof of Theorem 2.2*

Let $f_0^\varepsilon$ be the Yosida approximation of $f_0$ defined as follows:

$$f_0^\varepsilon(\omega,t,x,z) := f_0\big(\omega,t,x,J_\varepsilon(\omega,t,x,z)\big),$$

where $\varepsilon > 0$ and

$$\begin{aligned} J_\varepsilon(\omega,t,x,z) &:= A_\varepsilon(\omega,t,x,\cdot)^{-1}(z), \\ A_\varepsilon(\omega,t,x,z) &:= z - \varepsilon f_0(\omega,t,x,z). \end{aligned}$$

The following lemma is well known (cf. [10, p. 74]).

**Lemma 3.6.** *For all $(\omega,t,x,y,z) \in \Omega\times\mathbb{R}_+\times E\times\mathbb{R}^2$ and $\varepsilon,\varepsilon' > 0$,*

(i)
$$\begin{aligned} &(y-z)\cdot\big(f_0^\varepsilon(\omega,t,x,y) - f_0^\varepsilon(\omega,t,x,z)\big) \leqslant 0, \\ &\big|f_0^\varepsilon(\omega,t,x,y) - f_0^\varepsilon(\omega,t,x,z)\big| \leqslant 1/\varepsilon\cdot|y-z|; \end{aligned}$$

(ii)
$$\big|f_0^\varepsilon(\omega,t,x,z)\big| \leqslant \big|f_0(\omega,t,x,z)\big|;$$

(iii) $$\lim_{\varepsilon\to 0} f_0^\varepsilon(\omega,t,x,z) = f_0(\omega,t,x,z);$$

(iv) $$(y-z)\cdot\big(f_0^\varepsilon(\omega,t,x,y) - f_0^{\varepsilon'}(\omega,t,x,z)\big) \leqslant \big(\varepsilon+\varepsilon'\big)\big(\big|f_0^\varepsilon(\omega,t,x,y)\big|^2 + \big|f_0^{\varepsilon'}(\omega,t,x,z)\big|^2\big).$$

In the following, we shall fix $q \geqslant 2n$ and $u_0, \tilde{u}_0 \in \mathbb{B}^q$. Let $u_\varepsilon$ (resp. $\tilde{u}_\varepsilon$) solve the following equation

$$u_\varepsilon(t) = u_0^\varepsilon + \int_0^t \big[\mathfrak{L}u_\varepsilon(s) + \Gamma\big(f_1\big(s,u_\varepsilon(s)\big),\varphi\big) + f_0^\varepsilon\big(s,u_\varepsilon(s)\big)\big]\,\mathrm{d}s + \sum_k \int_0^t g^k\big(s,u_\varepsilon(s)\big)\,\mathrm{d}W_s^k, \tag{3.12}$$

where $u_0^\varepsilon \in \mathcal{D}$ (resp. $\tilde{u}_0^\varepsilon \in \mathcal{D}$) satisfies

$$\lim_{\varepsilon\downarrow 0}\big\|u_0^\varepsilon - u_0\big\|_{\mathbb{B}^q} = 0, \qquad \lim_{\varepsilon\downarrow 0}\big\|\tilde{u}_0^\varepsilon - \tilde{u}_0\big\|_{\mathbb{B}^q} = 0.$$

Then by Lemmas 3.3, 3.4 and 3.5, for some $\varepsilon_0 > 0$ we have

$$\sup_{\varepsilon\in(0,\varepsilon_0)} \mathbb{E}\Big(\sup_{t\in[0,T]}\big\|u_\varepsilon(t)\big\|_{L^q}^q\Big) \leqslant C\big(\|u_0\|_{L^q}^q + 1\big), \tag{3.13}$$

$$\sup_{\varepsilon\in(0,\varepsilon_0)} \mathbb{E}\Big(\sup_{t\in[0,T]}\big\|u_\varepsilon(t) - \tilde{u}_\varepsilon(t)\big\|_{L^q}^q\Big) \leqslant C\|u_0 - \tilde{u}_0\|_{L^q}^q, \tag{3.14}$$

and

$$\sup_{\varepsilon\in(0,\varepsilon_0)} \mathbb{E}\Big(\sup_{t\in[0,T]}\big\|u_\varepsilon(t)\big\|_{H_1^2}^2\Big) + \sup_{\varepsilon\in(0,\varepsilon_0)} \int_0^T \mathbb{E}\big\|u_\varepsilon(t)\big\|_{H_2^2}^2\,\mathrm{d}t \leqslant C\big(\|u_0\|_{H_1^2}^2 + \|u_0\|_{L^{2n}}^{2n} + 1\big). \tag{3.15}$$

Set

$$v_{\varepsilon,\varepsilon'}(t,x) := u_\varepsilon(t,x) - u_{\varepsilon'}(t,x).$$

By Itô's formula we have

$$\big\|v_{\varepsilon,\varepsilon'}(t)\big\|_{L^2} = \big\|u_0^\varepsilon - u_0^{\varepsilon'}\big\|_{L^2}^2 + I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t), \tag{3.16}$$

where

$$I_1(t) := 2\int_0^t \big\langle v_{\varepsilon,\varepsilon'}(s), \mathfrak{L}v_{\varepsilon,\varepsilon'}(s)\big\rangle_{L^2}\,\mathrm{d}s,$$

$$I_2(t) := 2\int_0^t \big\langle v_{\varepsilon,\varepsilon'}(s), \Gamma\big(f_1\big(s,\cdot,u_\varepsilon(s)\big) - f_1\big(s,\cdot,u_{\varepsilon'}(s)\big),\varphi\big)\big\rangle_{L^2}\,\mathrm{d}s,$$

$$I_3(t) := 2\int_0^t \big\langle v_{\varepsilon,\varepsilon'}(s), f_0^\varepsilon\big(s,\cdot,u_\varepsilon(s)\big) - f_0^{\varepsilon'}\big(s,\cdot,u_{\varepsilon'}(s)\big)\big\rangle_{L^2}\,\mathrm{d}s,$$

$$I_4(t) := 2\sum_k \int_0^t \big\langle v_{\varepsilon,\varepsilon'}(s), g_k\big(s,\cdot,u_\varepsilon(s)\big) - g_k\big(s,\cdot,u_{\varepsilon'}(s)\big)\big\rangle_{L^2}\,\mathrm{d}W_s^k,$$

$$I_5(t) := \sum_k \int_0^t \big\|g_k\big(s,\cdot,u_\varepsilon(s)\big) - g_k\big(s,\cdot,u_{\varepsilon'}(s)\big)\big\|_{L^2}^2\,\mathrm{d}s.$$

First of all, we have

$$I_1(t) \overset{(2.1)}{=} -2\int_0^t \int_E \Gamma\big(v_{\varepsilon,\varepsilon'}(s)\big)\,\mathrm{d}\mu\,\mathrm{d}s.$$

For $I_2(t)$, by **(H2)** and Young's inequality we have

$$\begin{aligned} I_2(t) &\overset{(3.3)}{=} 2\int_0^t \int_E \Gamma\big(v_{\varepsilon,\varepsilon'}(s)\cdot\big(f_1\big(s,\cdot,u_\varepsilon(s)\big)-f_1\big(s,\cdot,u_{\varepsilon'}(s)\big)\big),\varphi\big)\,\mathrm{d}\mu\,\mathrm{d}s \\ &\quad -2\int_0^t \int_E \big(f_1\big(s,\cdot,u_\varepsilon(s)\big)-f_1\big(s,\cdot,u_{\varepsilon'}(s)\big)\big)\cdot\Gamma\big(v_{\varepsilon,\varepsilon'}(s),\varphi\big)\,\mathrm{d}\mu\,\mathrm{d}s \\ &\leqslant 2c_{f_1}\int_0^t \int_E \big|v_{\varepsilon,\varepsilon'}(s)\big|^2\cdot|\mathfrak{L}\varphi|\,\mathrm{d}\mu\,\mathrm{d}s + 2c_{f_1}\int_0^t \int_E \big|v_{\varepsilon,\varepsilon'}(s)\big|\cdot\Gamma^{1/2}\big(v_{\varepsilon,\varepsilon'}(s)\big)\cdot\Gamma^{1/2}(\varphi)\,\mathrm{d}\mu\,\mathrm{d}s \\ &\leqslant C\int_0^t \big\|v_{\varepsilon,\varepsilon'}(s)\big\|_{L^2}^2\,\mathrm{d}s + \int_0^t \int_E \Gamma\big(v_{\varepsilon,\varepsilon'}(s)\big)\,\mathrm{d}\mu\,\mathrm{d}s. \end{aligned}$$

Here and after, the constant $C$ is independent of $\varepsilon$ and $\varepsilon'$.

For $I_3(t)$, we have by (iv) and (ii) of Lemma 3.6 and **(H3)** with $f_0(\omega,t,x,0)=0$,

$$\begin{aligned} I_3(t) &\leqslant 2\big(\varepsilon+\varepsilon'\big)\int_0^t \int_E \big(\big|f_0\big(s,\cdot,u_\varepsilon(s)\big)\big|+\big|f_0\big(s,\cdot,u_{\varepsilon'}(s)\big)\big|\big)^2\,\mathrm{d}\mu\,\mathrm{d}s \\ &\leqslant C\big(\varepsilon+\varepsilon'\big)\int_0^t \big(\big\|u_\varepsilon(s)\big\|_{L^{2n}}^{2n}+\big\|u_\varepsilon(s)\big\|_{L^2}^2+\big\|u_{\varepsilon'}(s)\big\|_{L^{2n}}^{2n}+\big\|u_{\varepsilon'}(s)\big\|_{L^2}^2+1\big)\,\mathrm{d}s. \end{aligned}$$

For $I_5(t)$, by **(H4)** we have

$$I_5(t) \leqslant c_g^2\int_0^t \big\|v_{\varepsilon,\varepsilon'}(s)\big\|_{L^2}^2\,\mathrm{d}s.$$

Hence, by taking expectations for (3.16) and (3.13) we find

$$\mathbb{E}\big\|v_{\varepsilon,\varepsilon'}(t)\big\|_{L^2}^2 \leqslant \big\|u_0^\varepsilon-u_0^{\varepsilon'}\big\|_{L^2}^2 + C\big(\varepsilon+\varepsilon'\big) + C\int_0^t \mathbb{E}\big\|v_{\varepsilon,\varepsilon'}(s)\big\|_{L^2}^2\,\mathrm{d}s,$$

which yields by Gronwall's inequality that

$$\mathbb{E}\big\|u_\varepsilon(t)-u_{\varepsilon'}(t)\big\|_{L^2}^2 = \mathbb{E}\big\|v_{\varepsilon,\varepsilon'}(t)\big\|_{L^2}^2 \leqslant C\big\|u_0^\varepsilon-u_0^{\varepsilon'}\big\|_{L^2}^2 + C\big(\varepsilon+\varepsilon'\big).$$

As in the proof of Lemma 3.3, we further have for any $T>0$,

$$\mathbb{E}\Big(\sup_{t\in[0,T]}\big\|u_\varepsilon(t)-u_{\varepsilon'}(t)\big\|_{L^2}^2\Big) \leqslant C\big\|u_0^\varepsilon-u_0^{\varepsilon'}\big\|_{L^2}^2 + C\big(\varepsilon+\varepsilon'\big).$$

Therefore, there exists an $L^2$-valued continuous $(\mathcal{F}_t)$-adapted process $u$ such that

$$\lim_{\varepsilon\downarrow 0}\mathbb{E}\Big(\sup_{t\in[0,T]}\big\|u_\varepsilon(t)-u(t)\big\|_{L^2}^2\Big)=0. \tag{3.17}$$

Moreover, by (3.13) and (3.15), we also have

$$\mathbb{E}\Big(\sup_{t\in[0,T]}\big\|u(t)\big\|_{L^q}^q\Big)\leqslant C\big(\|u_0\|_{L^q}^q+1\big) \tag{3.18}$$

and

$$\mathbb{E}\Big(\sup_{t\in[0,T]}\big\|u(t)\big\|_{H_1^2}^2\Big)+\int_0^T\mathbb{E}\big\|u(t)\big\|_{H_2^2}^2\,\mathrm{d}t\leqslant C\big(\|u_0\|_{H_1^2}^2+\|u_0\|_{L^{2n}}^{2n}+1\big), \tag{3.19}$$

as well as by the interpolation inequality (cf. [2]),

$$\begin{aligned}\mathbb{E}\int_0^T\big\|u_\varepsilon(s)-u(s)\big\|_{H_1^2}^2\,\mathrm{d}s&\leqslant\mathbb{E}\int_0^T\big\|u_\varepsilon(s)-u(s)\big\|_{H_2^2}\cdot\big\|u_\varepsilon(s)-u(s)\big\|_{H_0^2}\,\mathrm{d}s\\&\leqslant\left(\mathbb{E}\int_0^T\big\|u_\varepsilon(s)-u(s)\big\|_{H_2^2}^2\,\mathrm{d}s\right)^{1/2}\\&\quad\times\left(\mathbb{E}\int_0^T\big\|u_\varepsilon(s)-u(s)\big\|_{H_0^2}^2\,\mathrm{d}s\right)^{1/2}\overset{(3.17)}{\to}0,\quad\text{as }\varepsilon\downarrow 0.\end{aligned} \tag{3.20}$$

It must be noticed that one cannot directly take limits for (3.12) in order to show that $u$ solves Eq. (2.3). Instead of that, we shall use the standard weak convergence method (cf. [26]) to derive that $u(t)$ solves the original equation.

In the following, we fix $T>0$ and set

$$F_\varepsilon(\omega,s,x):=f_0^\varepsilon\big(s,x,u_\varepsilon(s,x)\big).$$

By (ii) of Lemma 3.6, **(H3)** and (3.13), (3.15), one knows that

$$\sup_{\varepsilon\in(0,\varepsilon_0)}\mathbb{E}\left(\int_0^T\big\|F_\varepsilon(s)\big\|_{L^2}^2\,\mathrm{d}s\right)\leqslant C.$$

Therefore, there exists an $F\in L^2(\Omega\times[0,T];L^2(E,\mu))$ and a subsequence $\varepsilon_k$ (still denoted by $\varepsilon$) such that

$$F_\varepsilon\to F\quad\text{weakly in }L^2\big(\Omega\times[0,T];L^2(E,\mu)\big).$$

Now, define

$$\tilde{u}(t):=u_0+\int_0^t\big[\mathfrak{L}u(s)+\Gamma\big(f_1\big(s,u(s)\big),\varphi\big)+F(s)\big]\,\mathrm{d}s+\sum_k\int_0^t g_k\big(s,u(s)\big)\,\mathrm{d}W_t^k. \tag{3.21}$$

By taking weak limits for (3.12), we find that

$$u(t) = \tilde{u}(t), \quad \text{a.s. } \forall t \in [0, T], \tag{3.22}$$

which combining with Itô's formula gives

$$\mathbb{E}\big\|u(T)\big\|_{L^2}^2 = \|u_0\|_{L^2}^2 + J_1 + J_2 + J_3 + J_4,$$

where

$$J_1 := 2\int_0^T \mathbb{E}\big\langle u(s), \mathfrak{L}u(s)\big\rangle_{L^2}\,\mathrm{d}s,$$

$$J_2 := 2\int_0^T \mathbb{E}\big\langle u(s), \Gamma\big(f_1\big(s,\cdot,u(s)\big),\varphi\big)\big\rangle_{L^2}\,\mathrm{d}s,$$

$$J_3 := 2\int_0^T \mathbb{E}\big\langle u(s), F(s)\big\rangle_{L^2}\,\mathrm{d}s,$$

$$J_4 := \sum_k \int_0^T \mathbb{E}\big\|g_k\big(s,\cdot,u(s)\big)\big\|_{L^2}^2\,\mathrm{d}s.$$

On the other hand, from (3.12) we also have

$$\mathbb{E}\big\|u_\varepsilon(T)\big\|_{L^2}^2 = \|u_0\|_{L^2}^2 + J_1^\varepsilon + J_2^\varepsilon + J_3^\varepsilon + J_4^\varepsilon,$$

where

$$J_1^\varepsilon := 2\int_0^T \mathbb{E}\big\langle u_\varepsilon(s), \mathfrak{L}u_\varepsilon(s)\big\rangle_{L^2}\,\mathrm{d}s,$$

$$J_2^\varepsilon := 2\int_0^T \mathbb{E}\big\langle u_\varepsilon(s), \Gamma\big(f_1\big(s,\cdot,u_\varepsilon(s)\big),\varphi\big)\big\rangle_{L^2}\,\mathrm{d}s,$$

$$J_3^\varepsilon := 2\int_0^T \mathbb{E}\big\langle u_\varepsilon(s), f_0^\varepsilon\big(s,\cdot,u_\varepsilon(s)\big)\big\rangle_{L^2}\,\mathrm{d}s,$$

$$J_4^\varepsilon := \sum_k \int_0^T \mathbb{E}\big\|g_k\big(s,\cdot,u_\varepsilon(s)\big)\big\|_{L^2}^2\,\mathrm{d}s.$$

By (3.17) and (3.20), we easily have

$$\lim_{\varepsilon\downarrow 0}\mathbb{E}\big\|u_\varepsilon(T)\big\|_{L^2}^2 = \mathbb{E}\big\|u(T)\big\|_{L^2}^2$$

and

$$\lim_{\varepsilon\downarrow 0} J_1^\varepsilon = J_1.$$

For $J_2^\varepsilon$, by Lemma 3.2 and **(H2)** we have

$$\begin{aligned}\left|J_2^\varepsilon - J_2\right| &\leqslant 2\int_0^T \mathbb{E}\left(\left\|u_\varepsilon(s)-u(s)\right\|_{L^2}\cdot\left\|\Gamma\left(f_1\left(s,\cdot,u_\varepsilon(s)\right),\varphi\right)\right\|_{L_2}\right)\mathrm{d}s\\ &\quad+2\int_0^T \mathbb{E}\left(\left\|u(s)\right\|_{H_1^2}\cdot\left\|\Gamma\left(f_1\left(s,\cdot,u_\varepsilon(s)\right)-f_1\left(s,\cdot,u(s)\right),\varphi\right)\right\|_{H_{-1}^2}\right)\mathrm{d}s\\ &\leqslant C\left(\int_0^T \mathbb{E}\left\|u_\varepsilon(s)-u(s)\right\|_{L^2}^2\,\mathrm{d}s\right)^{1/2}\cdot\left(\int_0^T \mathbb{E}\left\|u_\varepsilon(s)\right\|_{H_1^2}^2\,\mathrm{d}s\right)^{1/2}\\ &\quad+C\left(\int_0^T \mathbb{E}\left\|u(s)\right\|_{H_1^2}^2\,\mathrm{d}s\right)^{1/2}\cdot\left(\int_0^T \mathbb{E}\left\|u_\varepsilon(s)-u(s)\right\|_{L^2}^2\,\mathrm{d}s\right)^{1/2}\end{aligned}$$

which gives by (3.15), (3.17) and (3.19) the limit:

$$\lim_{\varepsilon\downarrow 0}\left|J_2^\varepsilon - J_2\right| = 0.$$

Moreover, by **(H4)** and (3.17) we also have

$$\lim_{\varepsilon\downarrow 0}\left|J_4^\varepsilon - J_4\right| = 0.$$

Combining the above calculations, we obtain

$$\lim_{\varepsilon\downarrow 0} J_3^\varepsilon = J_3. \tag{3.23}$$

Now, for any $\Phi\in(L^2\cap L^{2n})(\Omega\times[0,T]\times E)$, by (ii) and (iii) of Lemma 3.6 and the dominated convergence theorem we have

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0}\left|\mathbb{E}\int_0^T\int_E\left(u_\varepsilon(s)-\Phi(s)\right)\cdot\left[f_0^\varepsilon\left(s,\cdot,\Phi(s)\right)-f_0\left(s,\cdot,\Phi(s)\right)\right]\mathrm{d}\mu\,\mathrm{d}s\right|\\ &\quad\leqslant\lim_{\varepsilon\downarrow 0}\left[\left(\int_0^T\mathbb{E}\left\|f_0^\varepsilon\left(s,\cdot,\Phi(s)\right)-f_0\left(s,\cdot,\Phi(s)\right)\right\|_{L^2}^2\,\mathrm{d}s\right)^{1/2}\left(\int_0^T\mathbb{E}\left\|u_\varepsilon(s)-\Phi(s)\right\|_{L^2}^2\,\mathrm{d}s\right)^{1/2}\right]=0.\end{aligned} \tag{3.24}$$

In virtue of (i) in Lemma 3.6, we have

$$\mathbb{E}\left(\int_0^T\int_E\left(u_\varepsilon(s)-\Phi(s)\right)\cdot\left[f_0^\varepsilon\left(s,\cdot,u_\varepsilon(s)\right)-f_0^\varepsilon\left(s,\cdot,\Phi(s)\right)\right]\mathrm{d}\mu\,\mathrm{d}s\right)\leqslant 0.$$

Thus, by (3.23) and (3.24) we finally arrive at

$$\mathbb{E}\left(\int_0^T\int_E\left(u(s)-\Phi(s)\right)\cdot\left[F(s)-f_0\left(s,\cdot,\Phi(s)\right)\right]\mathrm{d}\mu\,\mathrm{d}s\right)\leqslant 0 \tag{3.25}$$

for any $\Phi\in(L^2\cap L^{2n})(\Omega\times[0,T]\times E)$.

Replacing $\Phi$ in (3.25) by $u - \delta\Phi$, where $\delta > 0$, we have

$$\mathbb{E}\left(\int_0^T \int_E \Phi(s) \cdot \left[F(s) - f_0\left(s, \cdot, u(s) - \delta\Phi(s)\right)\right] \mathrm{d}\mu \, \mathrm{d}s\right) \leqslant 0.$$

Letting $\delta \downarrow 0$ and using the dominated convergence theorem, one finds that

$$\mathbb{E}\left(\int_0^T \int_E \Phi(s) \cdot \left[F(s) - f_0\left(s, \cdot, u(s)\right)\right] \mathrm{d}\mu \, \mathrm{d}s\right) \leqslant 0.$$

By changing $\Phi$ to $-\Phi$, we conclude that by the arbitrariness of $\Phi$,

$$F(s, x) = f_0\left(s, x, u(s, x)\right),$$

which together with (3.21) and (3.22) produces (2.5).

Lastly, the continuity of $t \mapsto u(t)$ in $H_1^2$ follows from the Itô formula in [27], and the estimate (2.6) follows from (3.14) and (3.17).

## 4. Application to SPDEs on abstract Wiener space

Let $(\mathbb{X}, \mathbb{H}, \mu)$ be an abstract Wiener space. Namely, $\mathbb{H}$ is a real and separable Hilbert space, and it is continuously and densely embedded into Banach space $\mathbb{X}$. Therefore, by transposition, the dual space $\mathbb{X}^*$ of $\mathbb{X}$ could be injected in $\mathbb{H}$ and we have the triplet $\mathbb{X}^* \hookrightarrow \mathbb{H} \hookrightarrow \mathbb{X}$. The measure $\mu$ is the Gaussian measure on $\mathcal{B}(\mathbb{X})$.

Let $\mathcal{C}$ be the set of smooth cylindrical functions on $(\mathbb{X}, \mu)$. The Ornstein–Uhlenbeck semigroup is defined by Mehler's formula: for every $f \in \mathcal{C}$,

$$\mathfrak{T}_t f(x) := \int_{\mathbb{X}} f\left(xe^{-t} + y\sqrt{1 - e^{-2t}}\right)\mu(\mathrm{d}y).$$

For any $p > 1$, $\mathfrak{T}_t$ can be extended to a strongly continuous $C_0$-semigroup of contraction on $L^p(\mathbb{X}, \mu)$. The generator $\mathfrak{L}$ of semigroup $\mathfrak{T}_t$ is a non-positive self-adjoint operator on $L^2(\mathbb{X}, \mu)$. For $f \in \mathcal{C}$ with the form

$$f(x) = F\left(\langle x, h_1\rangle, \ldots, \langle x, h_k\rangle\right), \quad F \in C_0^\infty\left(\mathbb{R}^k\right), \ h_i \in \mathbb{H},$$

the Malliavin derivative of $f$ is defined by

$$Df(x) := \sum_i \partial_i F\left(\langle x, h_1\rangle, \ldots, \langle x, h_k\rangle\right) h_i, \quad h_i \in \mathbb{H}.$$

The *carré du champ* operator is then given by

$$\Gamma(f, g) := \langle Df, Dg\rangle_{\mathbb{H}}, \quad f, g \in \mathcal{C}.$$

Note that the following Meyer inequality holds (cf. [20]): for any $p > 1$ and some $c_p > 0$,

$$c_p^{-1}\|f\|_{H_1^p} \leqslant \|f\|_{L^p} + \|Df\|_{L^p(\mathbb{H})} \leqslant c_p\|f\|_{H_1^p}.$$

We can choose $\mathcal{D} := \bigcap_{p>1,\, n>1} H_n^p$ as our test functions space in **(C)**.

Consider the following SPDE:

$$\begin{cases} \mathrm{d}u(t,x) = \big[\mathfrak{L}u(t,x) + D_h f_1\big(t,\cdot,u(t)\big)(x) + f_0\big(t,x,u(t,x)\big)\big]\,\mathrm{d}t + \sum_k g_k\big(t,x,u(t,x)\big)\,\mathrm{d}W_t^k, \\ u(0,x) = u_0(x), \end{cases} \tag{4.1}$$

where $h \in \mathbb{H}$ and $f_1$, $f_0$ and $g$ satisfy **(H1)–(H4)**. Then applying Theorem 2.2 to this case, we can get the same conclusions as Theorem 2.2.

## 5. Application to SPDEs on complete Riemannian manifold

Let $(M,\mathfrak{g})$ be a $d$-dimensional and complete Riemannian manifold with Riemannian metric $\mathfrak{g}$. The Riemannian volume is denoted by $\mathrm{d}x$. Let $\nabla$ denote the gradient or covariant derivatives without confusion, $\Delta$ the Laplace–Beltrami operator, $T(M)$ the tangent bundle. The curvature tensor of $(M,\mathfrak{g})$ is denoted by $\mathcal{R}$, and the Ricci curvature is denoted by $\mathrm{Ric}_{\mathfrak{g}}$.

Given $\rho(x) \in C^\infty(M)$, let $\mu(\mathrm{d}x) := e^{-\rho(x)}\,\mathrm{d}x$. Assume that

$$\mu(M) := \int_M e^{-\rho(x)}\,\mathrm{d}x < +\infty.$$

Let $\mathcal{D} := C_0^\infty(M)$ be the smooth functions on $M$ with compact support. We consider the following distorted Laplace–Beltrami operator:

$$\mathfrak{L}u := \Delta u - \mathfrak{g}(\nabla\rho, \nabla u), \quad u \in \mathcal{D}.$$

It is well known that $(\mathfrak{L},\mathcal{D})$ is an essentially self-adjoint operator on $L^2(M,\mu)$ (cf. [34]), whose closure is denoted by $(\mathfrak{L},\mathcal{D}(\mathfrak{L}))$. Let $(\mathfrak{T}_t)_{t\geqslant 0}$ be the symmetric heat semigroup on $L^2(M,\mu)$ associated to $(\mathfrak{L},\mathcal{D}(\mathfrak{L}))$. Then, $(\mathfrak{T}_t)_{t\geqslant 0}$ can be extended to a strongly continuous contraction semigroup on $L^p(M,\mu)$ for $1 \leqslant p < +\infty$, which is also contracted on $L^\infty(M,\mu)$ (cf. [30,34]). Therefore, for each $1 < p < +\infty$, $(\mathfrak{T}_t)_{t\geqslant 0}$ forms an analytic semigroup on Banach space $L^p(M,\mu)$. The Sobolev spaces are defined by

$$H_n^p := (g - \mathfrak{L})^{-n/2}\big(L^p(M,\mu)\big).$$

The *carré du champ* operator is given by

$$\Gamma(f,g)(x) := \mathfrak{g}_x(\nabla f, \nabla g), \quad f,g \in \mathcal{D}.$$

In this section, we make the following geometric assumptions:

($\boldsymbol{\mathcal{M}_n}$) The tensors $\nabla^2\rho + \mathrm{Ric}_{\mathfrak{g}}$ and $\mathcal{R}$ together with their covariant derivatives up to $n$-th order are bounded. The *trace*$(\mathcal{R} \otimes \nabla\rho)$ together with its covariant derivatives up to $n-1$-th order are bounded.

Under ($\boldsymbol{\mathcal{M}_n}$), an equivalent norm of $H_n^p$ is given by the covariant derivatives up to $n$-th order, i.e., for $p > 1$, there exists a positive constant $c_{n,p}$ such that for any $f \in \mathcal{D}$,

$$c_{n,p}^{-1}\|f\|_{H_n^p} \leqslant \sum_{j=0}^n \big\|\nabla^j f\big\|_{L^p} \leqslant c_{n,p}\|f\|_{H_n^p}. \tag{5.1}$$

In the case of $n = 1$, this equivalence was first proved by Bakry in [1] under the assumption of $\nabla^2\rho + \mathrm{Ric}_{\mathfrak{g}}$ bounded from below. The higher order derivative case was proved by Yoshida in [34].

Consider the following SPDE:

$$\begin{cases} \mathrm{d}u(t,x) = \left[\mathfrak{L}u(t,x) + \mathfrak{g}_x\left(\nabla f_1\left(t,\cdot,u(t)\right),\nabla\varphi\right) + f_0\left(t,x,u(t,x)\right)\right]\mathrm{d}t + \sum_k g_k\left(t,x,u(t,x)\right)\mathrm{d}W_t^k, \\ u(0,x) = u_0(x), \end{cases} \tag{5.2}$$

where $\varphi \in \mathcal{D}$, and in addition to **(H1)–(H4)**, we also assume that

**(H5)** For every $n, m = 0, 1, 2, \ldots$, there exist $C_{nm} > 0$, $l_{nm} \in \mathbb{N}$ and $h_{nm} \in L^{\infty-}(M,\mu)$ such that for all $z \in \mathbb{R}$, $x \in M$ and $(\omega, t) \in \Omega \times \mathbb{R}_+$,

$$\begin{aligned} \left|\left(\nabla_x^n \partial_z^m f_0\right)(\omega,t,x,z)\right| &\leqslant C_{nm}|z|^{l_{nm}} + h_{nm}(x), \\ \left|\left(\nabla_x^n \partial_z^m f_1\right)(\omega,t,x,z)\right| &\leqslant C_{nm}|z|^{l_{nm}} + h_{nm}(x), \\ \left\|\left(\nabla_x^n \partial_z^m g\right)(\omega,t,x,z)\right\|_{l^2} &\leqslant C_{nm}|z|^{l_{nm}} + h_{nm}(x). \end{aligned}$$

In this and next sections, we use the following Banach spaces as in [35]. For $T > 0$, $p \geqslant 2$ and $\alpha \in \mathbb{R}$, define:

$$\mathbb{H}_\alpha^p(T) := L^p\left(\Omega \times [0,T], \mathcal{P}, \mathrm{d}P \times \mathrm{d}t; H_\alpha^p\right), \tag{5.3}$$

$$\mathbb{H}_\alpha^p\left(T; l^2\right) := L^p\left(\Omega \times [0,T], \mathcal{P}, \mathrm{d}P \times \mathrm{d}t; H_\alpha^p\left(l^2\right)\right). \tag{5.4}$$

We now prove the following result.

**Theorem 5.1.** *Assume that* $(\boldsymbol{\mathcal{M}_n})$ *holds for any* $n \in \mathbb{N}$ *and* **(H1)–(H5)** *hold. Then, for any* $u_0 \in \bigcap_{p\geqslant 2,\, n\in\mathbb{N}} H_n^p =: H^\infty$, *there exists a unique solution* $u(t) \in H^\infty \subset C^\infty(M)$ *to Eq.* (5.2) *so that*

$$\begin{aligned} u(t,x) = u_0(x) &+ \int_0^t \left[\mathfrak{L}u(s,x) + \mathfrak{g}_x\left(\nabla f_1\left(s,\cdot,u(s)\right),\nabla\varphi\right) + f_0\left(s,x,u(s,x)\right)\right]\mathrm{d}s \\ &+ \sum_k \int_0^t g_k\left(s,x,u(s,x)\right)\mathrm{d}W_s^k, \quad \forall (t,x) \in \mathbb{R}_+ \times M,\ P\text{-a.s.} \end{aligned}$$

*Moreover, there exists a version* $\tilde{u}$ *such that for any* $j = 1, 2, \ldots$, *the mapping* $(t,x) \mapsto \nabla^j \tilde{u}(t,x)$ *is continuous a.s.*

**Proof.** For $p \geqslant 2$ and $u, v \in L^p(E,\mu)$, by **(H2)**, **(H4)** and Lemma 3.2 we have

$$\begin{aligned} \left\|\mathfrak{g}\left(\nabla\left(f_1(s,\cdot,u) - f_1(s,\cdot,v)\right),\nabla\varphi\right)\right\|_{H_{-1}^p} &\leqslant C\|u - v\|_{L^p}, \\ \left\|g(s,\cdot,u) - g(s,\cdot,v)\right\|_{L^p(l^2)} &\leqslant C\|u - v\|_{L^p} \end{aligned}$$

and by **(H5)**, Lemma 3.1 and (5.1), for any $n \in \mathbb{N}$ and some $k_n \in \mathbb{N}$

$$\begin{aligned} \left\|\mathfrak{g}\left(\nabla\left(f_1(s,\cdot,u)\right),\nabla\varphi\right)\right\|_{H_{n-1}^p} &\leqslant C\left(1 + \|u\|_{H_n^{k_n\cdot p}}^{k_n}\right), \\ \left\|g(s,\cdot,u)\right\|_{H_n^p(l^2)} &\leqslant C\left(1 + \|u\|_{H_n^{k_n\cdot p}}^{k_n}\right). \end{aligned}$$

Thus, by [36, Corollary 3.2] we have, for any $0 < \delta < \frac{1}{n+1}$,

$$\big\|\mathfrak{g}\big(\nabla\big(f_1(s,\cdot,u)\big),\nabla\varphi\big)\big\|_{H^p_{n-1-(n+1)\delta}} \leqslant C\big(1+\|u\|^{k_n}_{H^{k_n\cdot p}_{n-\delta}}\big), \tag{5.5}$$

$$\big\|g(s,\cdot,u)\big\|_{H^p_{n-(n+1)\delta}(l^2)} \leqslant C\big(1+\|u\|^{k_n}_{H^{k_n\cdot p}_{n-\delta}}\big). \tag{5.6}$$

Moreover, by **(H5)**, Lemma 3.1 and (5.1) again we have, for any $\delta \in (0,1)$, $n \in \mathbb{N}$ and some $m_n \in \mathbb{N}$,

$$\big\|f_0(s,\cdot,u)\big\|_{H^p_{n-1-(n+1)\delta}} \leqslant C\big\|f_0(s,\cdot,u)\big\|_{H^p_{n-1}} \leqslant C\big(1+\|u\|^{m_n}_{H^{m_n\cdot p}_{n-1}}\big) \leqslant C\big(1+\|u\|^{m_n}_{H^{m_n\cdot p}_{n-\delta}}\big). \tag{5.7}$$

Fix $T > 0$. By (2.4) and [35, Theorem 3.2], for any $\delta \in (0,1)$ we have

$$u(\cdot) \in \mathbb{H}^p_{1-\delta}(T), \quad \forall p \geqslant 2.$$

Using the representation (2.7), [35, Theorem 3.2], (5.5)–(5.7) and induction method, as in the proof of [36, Theorem 2.5], we may prove that for any $n \in \mathbb{N}$ and $\delta \in (0,1)$,

$$u(\cdot) \in \mathbb{H}^p_{n-\delta}(T), \quad \forall p \geqslant 2.$$

The proof is thus completed by [35, Theorem 3.4] (see also [4]). $\square$

We now consider a special case of $M = \mathbb{R}^d$ endowed with the Euclidean metric. Let $\rho \in C^\infty(\mathbb{R}^d)$ satisfy

$$\int_{\mathbb{R}^d} e^{-\rho(x)}\,\mathrm{d}x < +\infty \quad \text{and} \quad \big|\nabla^n \rho\big| \leqslant c_n \quad \text{for any } n = 1,2,\ldots.$$

For example, $\rho(x) = (1+|x|^2)^{1/2}$ and $\rho(x) = \alpha\log(1+|x|^2)$, $\alpha > d/2$, satisfy the above conditions. In this case, $(\boldsymbol{\mathcal{M}_n})$ clearly holds for any $n \in \mathbb{N}$.

Below, we recall the definition of spatially homogeneous Wiener process (random fields case) (cf. [25, p. 190]):

**Definition 5.2 (and Theorem).** Let $\mu$ be a symmetric and finite measure on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$, and $\Gamma$ the Fourier transform of $\mu$. Then there exists a Gaussian random field $W$ on $\mathbb{R}_+ \times \mathbb{R}^d$ defined on some probability space $(\Omega, \mathcal{F}, P)$ such that

(i) For $P$-almost all $\omega$, $(t,x) \mapsto W(\omega,t,x)$ is measurable.
(ii) For each $x \in \mathbb{R}^d$, $(\omega,t) \mapsto W(\omega,t,x)$ is a one-dimensional Wiener process.
(iii) For $t,s \in \mathbb{R}_+$ and $x,y \in \mathbb{R}^d$, $\mathbb{E}(W(t,x)W(s,y)) = (t\wedge s)\cdot\Gamma(x-y)$.

The $\mu$ is called the spectral measure of $W$, and $\Gamma$ the spatial correlation function of $W$.

In the following, we assume that the spectral measure of $W$ satisfies

$$\int_{\mathbb{R}^d} \big(1+|x|^2\big)^n \mu(\mathrm{d}x) < +\infty, \quad \forall n \in \mathbb{N}, \tag{5.8}$$

and consider the following SPDE in $\mathbb{R}^d$ driven by $W(t,x)$:

$$\begin{cases} \mathrm{d}u(t,x) = \big[\Delta u(t,x) + \big\langle\nabla f_1\big(t,\cdot,u(t)\big),\nabla\rho\big\rangle_{\mathbb{R}^d} + f_0\big(t,x,u(t,x)\big)\big]\,\mathrm{d}t + g\big(t,x,u(t,x)\big)\,\mathrm{d}W(t,x), \\ u(0,x) = u_0(x), \end{cases} \tag{5.9}$$

where $f_1$, $f_0$ and $g$ satisfy **(H1)–(H5)**.

In order to use Theorem 5.1, we need a representation of $W(t,x)$ in terms of $W_t^k$. Let $L^2_{(s)}(\mathbb{R}^d,\mu)$ be the subspace of $L^2(\mathbb{R}^d,\mu;\mathbb{C})$ consisting of all $u$ such that $\overline{u(-x)} = u(x)$, which is a separable Hilbert space. Define

$$H_\mu := \{\widehat{u\mu}:\ u \in L^2_{(s)}(\mathbb{R}^d,\mu)\},$$

and

$$\langle \widehat{u\mu}, \widehat{v\mu}\rangle_{H_\mu} := \langle u, v\rangle_{L^2_{(s)}(\mathbb{R}^d,\mu)},$$

where $\widehat{u\mu}(x) := \int_{\mathbb{R}^d} e^{-i\langle x,y\rangle_{\mathbb{R}^d}} u(y)\mu(\mathrm{d}y)$ denotes the Fourier transform of $u\mu$.

It is well known that $t \mapsto W(t,\cdot)$ is a cylindrical Brownian motion in $H_\mu$, and $H_\mu$ is called the reproducing kernel Hilbert space of $W(t,x)$. Let $\{e_k:\ k\in\mathbb{N}\}$ be an orthogonormal basis of $L^2_{(s)}(\mathbb{R}^d,\mu)$. Then $\{h_k := \widehat{e_k\mu},\ k \in \mathbb{N}\}$ is an orthogonormal basis of $H_\mu$, and

$$W(t,x) = \sum_k h_k(x) W_t^k,$$

where $\{W_t^k := \int_{\mathbb{R}^d} W(t,x)h_k(x)\,\mathrm{d}x,\ k\in\mathbb{N}\}$ is a sequence of independent standard one dimensional Brownian motions (cf. [25, p. 191]).

We need the following simple lemma.

**Lemma 5.3.** *Assume that (5.8) holds. Then $\{h_k, k\in\mathbb{N}\} \in C_b^\infty(\mathbb{R}^d; l^2)$, i.e.,*

$$\sum_k \bigl|\nabla^n h_k(x)\bigr|^2 < +\infty, \quad \forall n = 0,1,2,\ldots.$$

**Proof.** It follows from

$$\sum_k \bigl|\nabla^n h_k(x)\bigr|^2 = \sum_k \Biggl|\int_{\mathbb{R}^d} \bigl(\nabla_x^n e^{-i\langle x,y\rangle_{\mathbb{R}^d}}\bigr)\cdot e_k(y)\mu(\mathrm{d}y)\Biggr|^2$$
$$= \int_{\mathbb{R}^d} \bigl|\nabla_x^n e^{-i\langle x,y\rangle_{\mathbb{R}^d}}\bigr|^2 \mu(\mathrm{d}y) \leqslant C\int_{\mathbb{R}^d} \bigl(1+|y|^2\bigr)^n \mu(\mathrm{d}y) < +\infty. \qquad \square$$

Now we rewrite Eq. (5.9) as follows:

$$\begin{cases} \mathrm{d}u(t,x) = \bigl[\mathfrak{L}u(t,x) + \langle \nabla \tilde f_1(t,\cdot,u(t)), \nabla\rho\rangle_{\mathbb{R}^d} + f_0(t,x,u(t,x))\bigr]\,\mathrm{d}t + \tilde g_k(t,x,u(t,x))\,\mathrm{d}W_t^k, \\ u(0,x) = u_0(x), \end{cases}$$

where $\mathfrak{L}u := \Delta u - \langle \nabla u, \nabla\rho\rangle_{\mathbb{R}^d}$, $\tilde f_1(t,x,u) = f_1(t,x,u) + u$ and $\tilde g_k(t,x,u) = g(t,x,u)\cdot h_k(x)$.

Clearly, by Lemma 5.3, one knows that $\tilde f_1$ and $\tilde g$ satisfy **(H1)**–**(H5)**. Thus, we may use Theorem 5.1 to get

**Theorem 5.4.** *Assume that (5.8) and* **(H1)**–**(H5)** *hold. For any $u_0 \in H^\infty$, there exists a unique solution $u(t) \in H^\infty \subset C^\infty(\mathbb{R}^d)$ of Eq. (5.9) such that*

$$u(t,x) = u_0(x) + \int_0^t \bigl[\Delta u(s,x) + \langle \nabla f_1(s,\cdot,u(s)), \nabla\rho\rangle_{\mathbb{R}^d} + f_0(s,x,u(s,x))\bigr]\,\mathrm{d}s$$
$$+ \int_0^t g(s,x,u(s,x))\,\mathrm{d}W(s,x), \quad \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d.$$

**Remark 5.5.** Since we are working in the weighted space, the initial value $u_0(x)$ can be constant. Obviously, one may also consider the more general uniformly second order symmetric elliptic operator having smooth coefficients with all orders of bounded derivatives.

## 6. Application to stochastic reaction diffusion equations in $\mathbb{R}^d$

For $p > 1$ and $n \in \mathbb{N}$, let $H_n^p$ be the Bessel space in $\mathbb{R}^d$ given by

$$H_n^p := (g - \Delta)^{-n/2}\big(L^p\big(\mathbb{R}^d, \mathrm{d}x\big)\big).$$

It is well known that for any $p > 1$ and $f \in C_0^\infty(\mathbb{R}^d)$ (cf. [29,32]):

$$c_p^{-1}\|f\|_{H_1^p} \leqslant \|f\|_{L^p} + \|\nabla f\|_{L^p} \leqslant c_p\|f\|_{H_1^p}, \tag{6.1}$$

where $\nabla$ is the usual gradient operator.

Consider the following stochastic heat equation in $\mathbb{R}^d$:

$$\begin{cases} \mathrm{d}u(t,x) = \big[\Delta u(t,x) + \big\langle \nabla f_1\big(t,\cdot,u(t)\big), \nabla\varphi\big\rangle_{\mathbb{R}^d}(x) + f_0\big(t,x,u(t,x)\big)\big]\,\mathrm{d}t + \sum_k g_k\big(t,x,u(t,x)\big)\,\mathrm{d}W_t^k, \\ u(0,x) = u_0(x), \end{cases} \tag{6.2}$$

where $\varphi \in C_0^\infty(\mathbb{R}^d)$, and $f_0$, $f_1$, $g$ satisfy **(H1)–(H5)**. In this case, we remark that the result in Section 5 is not available since we are working in the whole space $\mathbb{R}^d$ and without weights.

For $p \geqslant 2$ and $T > 0$, we shall use the following deep result (cf. [19]):

$$\int_0^T \bigg\| \int_0^t \mathfrak{T}_{t-s} f(s)\,\mathrm{d}s \bigg\|_{H_2^p}^p \mathrm{d}t \leqslant C_{d,p} \int_0^T \int_{\mathbb{R}^d} \big|f(s,x)\big|^p\,\mathrm{d}x\,\mathrm{d}s \tag{6.3}$$

as well as a generalization of Littlewood–Paley's inequality due to Krylov [14]

$$\int_0^T \int_{\mathbb{R}^d} \bigg( \int_0^t \big|\nabla \mathfrak{T}_{t-s} g(s,\cdot)(x)\big|_G^2\,\mathrm{d}s \bigg)^{p/2} \mathrm{d}x\,\mathrm{d}t \leqslant C_{d,p} \int_0^T \int_{\mathbb{R}^d} \big|g(t,x)\big|_G^p\,\mathrm{d}x\,\mathrm{d}t, \tag{6.4}$$

where $f \in L^p((0,T) \times \mathbb{R}^d)$ and

$$\mathfrak{T}_t f(x) = \frac{1}{(2\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-|x-y|^2/2t} f(y)\,\mathrm{d}y,$$

$G$ is a Hilbert space, and $g \in L^p((0,T) \times \mathbb{R}^d; G)$. Here, the constant $C_{d,p}$ only depends on $d$ and $p$.

We now prove the following result.

**Theorem 6.1.** *Assume that* **(H1)–(H5)** *hold. Then, for any $u_0 \in \bigcap_{p\geqslant 2,\, n\in\mathbb{N}} H_n^p =: H^\infty$, there exists a unique solution $u(t) \in H^\infty \subset C^\infty(\mathbb{R}^d)$ to Eq. (6.2) so that*

$$\begin{aligned} u(t,x) = u_0(x) &+ \int_0^t \big[\Delta u(s,x) + \big\langle \nabla f_1\big(s,\cdot,u(s)\big), \nabla\varphi\big\rangle_{\mathbb{R}^d} + f_0\big(s,x,u(s,x)\big)\big]\,\mathrm{d}s \\ &+ \sum_k \int_0^t g_k\big(s,x,u(s,x)\big)\,\mathrm{d}W_s^k, \quad \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d,\ P\text{-a.s.} \end{aligned}$$

*Moreover, there exists a version $\tilde{u}$ such that for any $j = 1, 2, \dots,$ the mapping $(t, x) \mapsto \nabla^j \tilde{u}(t, x)$ is continuous a.s.*

**Proof.** Fix $T > 0$. Recall the spaces $\mathbb{H}_0^p(T; l^2)$ and $\mathbb{H}_0^p(T)$ given in (5.3) and (5.4). For any $p \geqslant 2$, by (6.1), (6.4) and Minkowski's inequality we have

$$\begin{aligned}
&\int_0^T \mathbb{E}\left\| \sum_k \int_0^t \mathfrak{T}_{t-s} g_k\big(s, \cdot, u(s)\big)\, \mathrm{d}W_s^k \right\|_{H_1^p}^p \mathrm{d}t \\
&\quad \leqslant C \int_0^T \mathbb{E}\left\| \sum_k \int_0^t \mathfrak{T}_{t-s} g_k\big(s, \cdot, u(s)\big)\, \mathrm{d}W_s^k \right\|_{L^p}^p \mathrm{d}t + C \int_0^T \mathbb{E}\left\| \sum_k \int_0^t \nabla \mathfrak{T}_{t-s} g_k\big(s, \cdot, u(s)\big)\, \mathrm{d}W_s^k \right\|_{L^p}^p \mathrm{d}t \\
&\quad \leqslant C \int_0^T \int_{\mathbb{R}^d} \mathbb{E}\left( \int_0^t \big\| \mathfrak{T}_{t-s} g\big(s, \cdot, u(s)\big)(x) \big\|_{l^2}^2 \,\mathrm{d}s \right)^{p/2} \mathrm{d}x\, \mathrm{d}t \\
&\qquad + C \int_0^T \int_{\mathbb{R}^d} \mathbb{E}\left( \int_0^t \big\| \nabla \mathfrak{T}_{t-s} g\big(s, \cdot, u(s)\big)(x) \big\|_{l^2}^2 \,\mathrm{d}s \right)^{p/2} \mathrm{d}x\, \mathrm{d}t \\
&\quad \leqslant C \int_0^T \mathbb{E}\left( \int_0^t \big\| g\big(s, \cdot, u(s)\big) \big\|_{L^p(l^2)}^2 \,\mathrm{d}s \right)^{p/2} \mathrm{d}t + C \int_0^T \mathbb{E}\big\| g\big(t, \cdot, u(t)\big) \big\|_{L^p(l^2)}^p \,\mathrm{d}t \\
&\quad \leqslant C \int_0^T \mathbb{E}\big\| g\big(t, \cdot, u(t)\big) \big\|_{L^p(l^2)}^p \,\mathrm{d}t = C \big\| g\big(\cdot, \cdot, u(\cdot)\big) \big\|_{\mathbb{H}_0^p(T; l^2)}^p,
\end{aligned}$$

and by (6.3) and Lemma 3.2,

$$\begin{aligned}
&\int_0^T \mathbb{E}\left\| \int_0^t \mathfrak{T}_{t-s} \big\langle \nabla f_1\big(s, \cdot, u(s)\big), \nabla\varphi \big\rangle_{\mathbb{R}^d} \,\mathrm{d}s \right\|_{H_1^p}^p \mathrm{d}t \\
&\quad = \int_0^T \mathbb{E}\left\| \int_0^t \mathfrak{T}_{t-s} (g - \Delta)^{-1/2} \big\langle \nabla f_1\big(s, \cdot, u(s)\big), \nabla\varphi \big\rangle_{\mathbb{R}^d} \,\mathrm{d}s \right\|_{H_2^p}^p \mathrm{d}t \\
&\quad \leqslant C \int_0^T \mathbb{E}\big\| (g - \Delta)^{-1/2} \big\langle \nabla f_1\big(t, \cdot, u(t)\big), \nabla\varphi \big\rangle_{\mathbb{R}^d} \big\|_{L^p}^p \,\mathrm{d}t \\
&\quad \leqslant C \int_0^T \mathbb{E}\big\| f_1\big(t, \cdot, u(t)\big) \big\|_{L^p}^p \,\mathrm{d}t = C \big\| f_1\big(\cdot, \cdot, u(\cdot)\big) \big\|_{\mathbb{H}_0^p(T)}^p.
\end{aligned}$$

Moreover, noting that (cf. [19]),

$$\| \nabla \mathfrak{T}_t f \|_{L^p} \leqslant \frac{C}{t^{1/2}} \| f \|_{L^p},$$

by Minkowski's inequality and Hölder's inequality, we have for $p > 2$,

$$\int_0^T \mathbb{E}\left\|\int_0^t \mathfrak{T}_{t-s} f_0\big(s,\cdot,u(s)\big)\,\mathrm{d}s\right\|_{H_1^p}^p \mathrm{d}t \leqslant \int_0^T \mathbb{E}\left(\int_0^t \frac{1}{(t-s)^{1/2}}\big\|f_0\big(s,\cdot,u(s)\big)\big\|_{L^p}\,\mathrm{d}s\right)^p \mathrm{d}t$$
$$\leqslant \int_0^T \left[\int_0^t \frac{1}{(t-s)^{\frac{p^*}{2}}}\,\mathrm{d}s\right]^{\frac{p}{p^*}} \mathrm{d}t \cdot \int_0^T \mathbb{E}\big\|f_0\big(s,\cdot,u(s)\big)\big\|_{L^p}^p\,\mathrm{d}s$$
$$\leqslant C\big\|f_0\big(\cdot,\cdot,u(\cdot)\big)\big\|_{\mathbb{H}_0^p(T)}^p.$$

By (2.7) and combining the above calculations, we find that for any $p>2$,

$$\|u\|_{\mathbb{H}_1^p(T)}^p \leqslant C\int_0^T \|\mathfrak{T}_t u_0\|_{L^p}^p\,\mathrm{d}t + C\big\|f_1\big(\cdot,u(\cdot)\big)\big\|_{\mathbb{H}_0^p(T)}^p + C\big\|f_0\big(\cdot,u(\cdot)\big)\big\|_{\mathbb{H}_0^p(T)}^p + C\big\|g\big(\cdot,u(\cdot)\big)\big\|_{\mathbb{H}_0^p(T;l^2)}^p$$
$$\leqslant C\|u_0\|_{L^p}^p + C\|u\|_{\mathbb{H}_0^p(T)}^p + C\|u\|_{\mathbb{H}_0^{np}(T)}^{np} < +\infty,$$

where we have used **(H2)**–**(H4)** and (2.4).

By similar calculations as above, we also have, for any $m\in\mathbb{N}$ and $p>2$,

$$\|u\|_{\mathbb{H}_{m+1}^p(T)}^p \leqslant C\int_0^T \|\mathfrak{T}_t u_0\|_{H_{m+1}^p}^p\,\mathrm{d}t + C\big\|f_1\big(\cdot,u(\cdot)\big)\big\|_{\mathbb{H}_m^p(T)}^p + C\big\|f_0\big(\cdot,u(\cdot)\big)\big\|_{\mathbb{H}_m^p(T)}^p + C\big\|g\big(\cdot,u(\cdot)\big)\big\|_{\mathbb{H}_m^p(T;l^2)}^p.$$

Now, by **(H5)** and using induction method, we finally have that for any $m\in\mathbb{N}$ and $p>2$,

$$\|u\|_{\mathbb{H}_{m+1}^p(T)}^p < +\infty.$$

The proof is thus completed by [35, Theorem 3.4] (see also [4]). □

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