



# On split common fixed point problems



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## ABSTRACT

Based on the convergence theorem recently proved by the second author, we modify the iterative scheme studied by Moudafi for quasi-nonexpansive operators to obtain strong convergence to a solution of the split common fixed point problem. It is noted that Moudafi's original scheme can conclude only weak convergence. As a consequence, we obtain strong convergence theorems for split variational inequality problems for Lipschitz continuous and monotone operators, split common null point problems for maximal monotone operators, and Moudafi's split feasibility problem.

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## 1. Introduction

Let  $C$  and  $Q$  be closed convex subsets of Hilbert spaces  $\mathcal{H}_1$  and  $\mathcal{H}_2$ , respectively and  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  be a bounded linear operator. The *split feasibility problem* (SFP) which was first introduced by Censor and Elfving [4] is to find

$$\hat{x} \in C \quad \text{such that} \quad A\hat{x} \in Q. \quad (1)$$

Suppose that  $P_C$  and  $P_Q$  are the (orthogonal) projections onto the sets  $C$  and  $Q$ , respectively. Assuming that SFP is consistent (i.e., (1) has a solution), it is not difficult to see that  $\hat{x} \in \mathcal{H}_1$  solves (1) if and only if it solves the fixed-point equation

$$\hat{x} = P_C(I + \gamma A^*(P_Q - I)A)\hat{x},$$

where  $\gamma > 0$  is any positive constant,  $I$  is the identity operator and  $A^*$  denotes the adjoint of  $A$ . To solve (1), in the setting of the finite dimensional case, Byrne [2] proposed the following so-called  $CQ$  algorithm:

$$x_{n+1} = P_C(x_n + \gamma A^t(P_Q - I)Ax_n), \quad n \in \mathbb{N},$$

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where  $\gamma \in ]0, \frac{2}{L}[$ , with  $L$  being the largest eigenvalue of the matrix  $A^t A$  ( $t$  stands for matrix transposition). SFP is important and has been widely studied because it plays a prominent role in the signal processing and image reconstruction problem. Initiated by SFP, several “split type” problems have been investigated and studied, for example, the split variational inequality problem (SVIP) and the split null point problem (SCNP). We will consolidate these problems. Let  $U : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  and  $T : \mathcal{H}_2 \rightarrow \mathcal{H}_2$  be two operators with nonempty fixed point sets  $\text{Fix}(U) := \{x \in \mathcal{H}_1 : x = Ux\}$  and  $\text{Fix}(T)$ , respectively. The *split common fixed point problem* (SCFP) is to find

$$\hat{x} \in \text{Fix}(U) \quad \text{such that} \quad A\hat{x} \in \text{Fix}(T).$$

If  $U := P_C$  and  $T := P_Q$ , then  $\text{Fix}(U) = C$  and  $\text{Fix}(T) = Q$  and hence SCFP immediately reduces to SFP. In the case that  $U$  and  $T$  are directed operators, Censor and Segal [5] proposed and proved, still in finite-dimensional spaces, the convergence of the following algorithm:

$$x_{n+1} = U(x_n + \gamma A^t(T - I)Ax_n), \quad n \in \mathbb{N},$$

where  $\gamma$  and  $L$  are as mentioned before. Note that a class of directed operators includes the metric projections. Hence the result of Censor et al. recovers Byrne’s  $CQ$  algorithm.

Moudafi [9] recently studied the convergence properties of a relaxed algorithm for SCFP for a class of quasi-nonexpansive operators  $T$  such that  $I - T$  is demiclosed at zero. He also proved a weak convergence theorem as shown below.

**Theorem 1.1.** *Given a bounded linear operator  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ , let  $U : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  and  $T : \mathcal{H}_2 \rightarrow \mathcal{H}_2$  be two quasi-nonexpansive operators with nonempty sets  $\text{Fix}(U) = C$  and  $\text{Fix}(T) = Q$ . Assume that  $I - U$  and  $I - T$  are demiclosed at zero. Suppose  $\Gamma := \{x \in C : Ax \in Q\} \neq \emptyset$  and define an iterative sequence  $\{x_n\}$  by*

$$\begin{cases} x_0 \in \mathcal{H}_1, \\ u_n = x_n + \gamma\beta A^*(T - I)Ax_n, \\ x_{n+1} = (1 - \alpha_n)u_n + \alpha_n U(u_n), \end{cases}$$

where  $\beta \in ]0, 1[$ ,  $\alpha_n \in ]0, 1[$  and  $\gamma \in ]0, \frac{1}{\lambda\beta}[$  with  $\lambda = \|A^*A\|$ . Then  $\{x_n\}$  converges weakly to  $\hat{x} \in \Gamma$  provided that  $\alpha_n \in ]\delta, 1 - \delta[$  for a small enough  $\delta > 0$ .

Note that, in the setting of finite dimensional spaces, weak and strong convergences are equivalent. Differently, in infinite dimensional cases, they are not the same. Furthermore, Moudafi’s result [9] can guarantee only weak convergence. In most cases, strong convergence is more desirable than weak convergence. In this paper, we slightly modify the algorithm to obtain a strong convergence.

## 2. Definitions and preliminaries

Throughout, let  $\mathcal{H}$  be a real Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and the induced norm  $\|\cdot\|$ . We denote the strong and weak convergence of a sequence  $\{x_n\}$  in  $\mathcal{H}$  to an element  $x \in \mathcal{H}$  by  $x_n \rightarrow x$  and  $x_n \rightharpoonup x$ , respectively. For a closed convex subset  $C$  of  $\mathcal{H}$ , the (metric) projection  $P_C : \mathcal{H} \rightarrow C$  is defined for each  $x \in \mathcal{H}$  as the unique element  $P_C x \in C$  such that

$$\|x - P_C x\| = \inf\{\|x - z\| : z \in C\}.$$

For  $x \in \mathcal{H}$  and  $y \in C$ , it is known that

$$y = P_C x \iff \langle y - x, z - y \rangle \geq 0 \quad \text{for all } z \in C.$$

In this paper, the fixed-point set of an operator  $T : \mathcal{H} \rightarrow \mathcal{H}$  is denoted by  $\text{Fix}(T)$ , that is,  $\text{Fix}(T) = \{x \in \mathcal{H} : x = Tx\}$ .

Let us recall some definitions of operators involved in our study.

**Definition 2.1.** An operator  $T : \mathcal{H} \rightarrow \mathcal{H}$  is called:

- *L-Lipschitzian* if

$$\|Tx - Ty\| \leq L\|x - y\| \quad \text{for all } x, y \in \mathcal{H};$$

- *a contraction* if it is  $\alpha$ -Lipschitzian with  $\alpha \in [0, 1[$ , and in this case, we also say that  $T$  is a contraction with the coefficient  $\alpha$ ;
- *nonexpansive* if  $T$  is 1-Lipschitzian;
- *quasi-nonexpansive* if  $\text{Fix}(T) \neq \emptyset$  and

$$\|Tx - p\| \leq \|x - p\| \quad \text{for all } x \in \mathcal{H}, p \in \text{Fix}(T);$$

equivalently, for all  $x \in \mathcal{H}$  and  $p \in \text{Fix}(T)$ ,

$$\langle x - Tx, p - x \rangle \leq -\frac{1}{2}\|x - Tx\|^2;$$

- *strongly quasi-nonexpansive* if  $T$  is quasi-nonexpansive and

$$x_n - Tx_n \rightarrow 0$$

whenever  $\{x_n\}$  is a bounded sequence in  $\mathcal{H}$  and  $\|x_n - p\| - \|Tx_n - p\| \rightarrow 0$  for some  $p \in \text{Fix}(T)$ ;

- *monotone* if

$$\langle Tx - Ty, x - y \rangle \geq 0 \quad \text{for all } x, y \in \mathcal{H}.$$

**Proposition 2.2.** If  $T : \mathcal{H} \rightarrow \mathcal{H}$  is a nonexpansive operator, then the following inequality holds for all  $x, y \in \mathcal{H}$

$$\langle x - y, (I - T)x - (I - T)y \rangle \geq \frac{1}{2}\|(I - T)x - (I - T)y\|^2.$$

**Proof.** Since  $T$  is nonexpansive, we have

$$\begin{aligned} \|x - y\|^2 &\geq \|Tx - Ty\|^2 \\ &= \|(I - T)x - (I - T)y - (x - y)\|^2 \\ &= \|(I - T)x - (I - T)y\|^2 - 2\langle x - y, (I - T)x - (I - T)y \rangle + \|x - y\|^2. \end{aligned}$$

Therefore we get

$$\langle x - y, (I - T)x - (I - T)y \rangle \geq \frac{1}{2}\|(I - T)x - (I - T)y\|^2. \quad \square$$

**Corollary 2.3.** Let  $S : \mathcal{H} \rightarrow \mathcal{H}$  be a quasi-nonexpansive operator and

$$T := (1 - \alpha)I + \alpha S,$$

for some  $\alpha \in ]0, 1]$ . Then, for all  $x \in \mathcal{H}$  and  $p \in \text{Fix}(T)$ , we have the following inequality

$$\langle x - Tx, p - x \rangle \leq -\frac{1}{2\alpha} \|x - Tx\|^2.$$

**Proof.** Obviously,  $\text{Fix}(T) = \text{Fix}(S)$ . It follows from Proposition 2.2 that

$$\langle x - Tx, p - x \rangle = \alpha \langle x - Sx, p - x \rangle \leq -\frac{\alpha}{2} \|x - Sx\|^2 = -\frac{1}{2\alpha} \|x - Tx\|^2.$$

The proof is finished.  $\square$

### 3. Main results

Let us recall first the result proved by the second author.

**Theorem 3.1.** (See [13].) Let  $C$  be a closed and convex subset of a Hilbert space  $\mathcal{H}$  and let  $T : C \rightarrow C$  be a strongly quasi-nonexpansive operator such that  $I - T$  is demiclosed at zero. Suppose that  $x_0 \in C$  and  $\{x_n\}$  is a sequence generated iteratively by  $x_1 \in C$  and

$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)Tx_n,$$

where  $\{\alpha_n\}$  is a sequence in  $]0, 1[$  such that  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ . Then  $\{x_n\}$  converges strongly to a fixed point  $P_{\text{Fix}(T)}x_0$  of  $T$ .

Recall that an operator  $T$  is demiclosed at zero [15] if

$$Tx = 0 \quad \text{whenever } x_n \rightharpoonup x \text{ and } Tx_n \rightarrow 0.$$

#### 3.1. The split common fixed point problem

Throughout this paper, let  $\Gamma := \{x \in \text{Fix}(U) : Ax \in \text{Fix}(T)\}$ . It is clear that  $\Gamma$  is closed and convex.

**Theorem 3.2.** Let  $U : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  be a strongly quasi-nonexpansive operator and  $T : \mathcal{H}_2 \rightarrow \mathcal{H}_2$  be a quasi-nonexpansive operator such that both  $I - U$  and  $I - T$  are demiclosed at zero. Let  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  be a bounded linear operator with  $L = \|A^*A\|$ . Suppose that  $\Gamma \neq \emptyset$ . Let  $\{x_n\} \subset \mathcal{H}_1$  be a sequence generated by

$$\begin{cases} x_0 \in \mathcal{H}_1, \\ x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)U(x_n + \gamma A^*(T - I)Ax_n), \end{cases}$$

where the parameter  $\gamma$  and the sequence  $\{\alpha_n\}$  satisfy the following conditions:

- (a)  $\gamma \in ]0, \frac{1}{L}[$ ,
- (b)  $\{\alpha_n\} \subset ]0, 1[$ ,  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=0}^{\infty} \alpha_n = \infty$ .

Then  $x_n \rightarrow P_{\Gamma}x_0$ .

The following lemma is extracted from Lemma 6.2 of [6] which is needed for proving our main result.

**Lemma 3.3.** Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be Hilbert spaces. Let  $T : \mathcal{H}_2 \rightarrow \mathcal{H}_2$  be a nonexpansive operator and  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  be a bounded linear operator with  $L = \|A^*A\|$ . For a positive real number  $\gamma$ , define the operator  $W : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  by

$$W := I + \gamma A^*(T - I)A.$$

Then the following hold:

- For all  $x, y \in \mathcal{H}_1$ ,

$$\|Wx - Wy\|^2 \leq \|x - y\|^2 + \gamma(\gamma L - 1) \|(T - I)Ax - (T - I)Ay\|^2.$$

In addition, if  $T := (1 - \alpha)I + \alpha S$  where  $S : \mathcal{H}_2 \rightarrow \mathcal{H}_2$  is a nonexpansive operator, then

$$\|Wx - Wy\|^2 \leq \|x - y\|^2 + \gamma \left( \gamma L - \frac{1}{\alpha} \right) \|(T - I)Ax - (T - I)Ay\|^2.$$

- If  $Ax \in \text{Fix}(T)$ , then  $x \in \text{Fix}(W)$  and the converse holds provided that  $\gamma \in ]0, \frac{1}{L}[$ .

**Proof.** • Let  $x, y \in \mathcal{H}_1$ . Then we have

$$\begin{aligned} \|Wx - Wy\|^2 &= \|(x + \gamma A^*(T - I)Ax) - (y + \gamma A^*(T - I)Ay)\|^2 \\ &= \|(x - y) + \gamma A^*((T - I)Ax - (T - I)Ay)\|^2 \\ &= \|x - y\|^2 + 2\gamma \langle x - y, A^*((T - I)Ax - (T - I)Ay) \rangle + \gamma^2 \|A^*((T - I)Ax - (T - I)Ay)\|^2 \\ &= \|x - y\|^2 + 2\gamma \langle Ax - Ay, (T - I)Ax - (T - I)Ay \rangle \\ &\quad + \gamma^2 \langle A^*((T - I)Ax - (T - I)Ay), A^*((T - I)Ax - (T - I)Ay) \rangle \\ &= \|x - y\|^2 + 2\gamma \langle Ax - Ay, (T - I)Ax - (T - I)Ay \rangle \\ &\quad + \gamma^2 \langle AA^*((T - I)Ax - (T - I)Ay), (T - I)Ax - (T - I)Ay \rangle \\ &\leq \|x - y\|^2 + 2\gamma \langle Ax - Ay, (T - I)Ax - (T - I)Ay \rangle \\ &\quad + \gamma^2 \|AA^* \|(T - I)Ax - (T - I)Ay\|^2. \end{aligned}$$

Therefore we have

$$\begin{aligned} \|Wx - Wy\|^2 &\leq \|x - y\|^2 + 2\gamma \langle Ax - Ay, (T - I)Ax - (T - I)Ay \rangle \\ &\quad + \gamma^2 L \|(T - I)Ax - (T - I)Ay\|^2. \end{aligned} \tag{2}$$

It follows from Proposition 2.2 that

$$\|Wx - Wy\|^2 \leq \|x - y\|^2 + \gamma(\gamma L - 1) \|(T - I)Ax - (T - I)Ay\|^2.$$

Furthermore, if  $T := (1 - \alpha)I + \alpha S$  where  $S$  is a nonexpansive operator, then

$$\begin{aligned} \langle Ax - Ay, (T - I)Ax - (T - I)Ay \rangle &= \alpha \langle Ax - Ay, (S - I)Ax - (S - I)Ay \rangle \\ &\leq \frac{-\alpha}{2} \|(I - S)Ay - (I - S)Ax\|^2 \\ &= -\frac{1}{2\alpha} \|(I - T)Ay - (I - T)Ax\|^2. \end{aligned}$$

Hence from (2) and Proposition 2.2, we obtain

$$\|Wx - Wy\|^2 \leq \|x - y\|^2 + \gamma \left( \gamma L - \frac{1}{\alpha} \right) \|(T - I)Ax - (T - I)Ay\|^2.$$

• It is obvious that  $Ax \in \text{Fix}(T)$  implies  $x \in \text{Fix}(W)$ . To see the converse, let  $\gamma \in ]0, \frac{1}{L}[$ . Let  $x \in \text{Fix}(W)$  and  $z \in \mathcal{H}_1$  be such that  $Az \in \text{Fix}(T)$ . It follows that  $z \in \text{Fix}(W)$  and hence we get

$$\|x - z\|^2 = \|Wx - Wz\|^2 \leq \|x - z\|^2 + \gamma(\gamma L - 1) \|(T - I)Ax\|^2.$$

Since  $\gamma L < 1$ , we have  $(T - I)Ax = 0$ , that is,  $Ax \in \text{Fix}(T)$ .  $\square$

**Corollary 3.4.** Let  $T : \mathcal{H}_2 \rightarrow \mathcal{H}_2$  be a quasi-nonexpansive operator and  $A, W$  be operators defined as in Lemma 3.3. Then

$$\|Wx - z\|^2 \leq \|x - z\|^2 + \gamma(\gamma L - 1) \|(T - I)Ax\|^2,$$

for all  $x \in \mathcal{H}_1$  and  $z \in \mathcal{H}_1$  such that  $Az \in \text{Fix}(T)$ .

**Lemma 3.5.** Let  $U : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  be a strongly quasi-nonexpansive operator and  $T : \mathcal{H}_2 \rightarrow \mathcal{H}_2$  be a quasi-nonexpansive operator. Let  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  be a bounded linear operator with  $L = \|A^*A\|$ . Define the operator  $W : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  as in Lemma 3.3 where  $\gamma L < 1$ . Suppose that  $\text{Fix}(U) \cap \text{Fix}(W) \neq \emptyset$  and  $\{x_n\}$  is a bounded sequence in  $\mathcal{H}_1$ . Then the following are equivalent:

- (a)  $UWx_n - Wx_n \rightarrow 0$  and  $Wx_n - x_n \rightarrow 0$ ;
- (b)  $UWx_n - x_n \rightarrow 0$ ;
- (c)  $\|x_n - p\| - \|UWx_n - p\| \rightarrow 0$  for some  $p \in \text{Fix}(U) \cap \text{Fix}(W)$ .

**Proof.** It is obvious that (a)  $\Rightarrow$  (b)  $\Rightarrow$  (c). We now show that (c)  $\Rightarrow$  (a). Suppose that  $\|x_n - p\| - \|UWx_n - p\| \rightarrow 0$  for some  $p \in \text{Fix}(U) \cap \text{Fix}(W)$ . By using Corollary 3.4 and the quasi-nonexpansiveness of  $U$ , we get

$$\|UWx_n - p\| \leq \|Wx_n - p\| \leq \|x_n - p\|.$$

Therefore we have  $\|Wx_n - p\| - \|UWx_n - p\| \rightarrow 0$ . Since  $U$  is strongly quasi-nonexpansive, we have  $UWx_n - Wx_n \rightarrow 0$ . Notice that  $\|x_n - p\|^2 - \|UWx_n - p\|^2 \rightarrow 0$ . Using Corollary 3.4 again gives

$$\gamma(1 - \gamma L) \|(T - I)Ax_n\|^2 \leq \|x_n - p\|^2 - \|UWx_n - p\|^2 \rightarrow 0.$$

Since  $\gamma L < 1$ , we get  $Wx_n - x_n = \gamma A^*(T - I)Ax_n \rightarrow 0$ . Then (a) is satisfied and the proof is finished.  $\square$

**Proof of Theorem 3.2.** To conclude the result, by using Theorem 3.1, it suffices to show that:

- (♠) the operator  $UW$  is strongly quasi-nonexpansive, where  $W := I + \gamma A^*(T - I)A$ ;
- (♡)  $I - UW$  is demiclosed at zero.

We first note that  $\Gamma = \text{Fix}(U) \cap \text{Fix}(W) = \text{Fix}(UW)$ . Indeed, it follows from Lemma 3.3 that

$$\begin{aligned} \Gamma &= \{x \in \mathcal{H}_1 : x \in \text{Fix}(U) \text{ and } Ax \in \text{Fix}(T)\} \\ &= \{x \in \mathcal{H}_1 : x \in \text{Fix}(U) \text{ and } x \in \text{Fix}(W)\} \\ &= \text{Fix}(U) \cap \text{Fix}(W). \end{aligned}$$

Then  $\text{Fix}(U) \cap \text{Fix}(W) \neq \emptyset$ . We next show that  $\text{Fix}(U) \cap \text{Fix}(W) = \text{Fix}(UW)$ . To see this, it suffices to show  $\text{Fix}(UW) \subset \text{Fix}(U) \cap \text{Fix}(W)$ . Then let  $p \in \text{Fix}(U) \cap \text{Fix}(W)$  and  $x \in \text{Fix}(UW)$ . By using Lemma 3.5 with  $x_n \equiv x$ , we get that  $Wx = x$  and  $UWx = Wx$ , that is,  $x \in \text{Fix}(U) \cap \text{Fix}(W)$ . So our assertion is obtained. Combining this fact with Lemma 3.5, we have  $(\spadesuit)$ . To prove  $(\heartsuit)$ , let  $\{x_n\}$  be a sequence such that  $x_n - UWx_n \rightarrow 0$  and  $x_n \rightharpoonup x$  for some  $x \in \mathcal{H}_1$ . It follows from Lemma 3.5 that  $x_n - Wx_n \rightarrow 0$  and  $y_n - Uy_n \rightarrow 0$  where  $y_n \equiv Wx_n$ . Notice that  $y_n \rightharpoonup x$ . Since  $I - U$  and  $I - T$  are demiclosed at zero, we have  $x \in \text{Fix}(W) \cap \text{Fix}(U) = \text{Fix}(UW)$ .  $\square$

#### 4. Another split problems deduced from SCFP

##### 4.1. The split variational inequality problem

Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be two real Hilbert spaces. Given operators  $f : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  and  $g : \mathcal{H}_2 \rightarrow \mathcal{H}_2$ , a bounded linear operator  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  and nonempty closed convex subsets  $C \subset \mathcal{H}_1$  and  $Q \subset \mathcal{H}_2$ , the *split variational inequality problem* (SVIP) is the problem of finding a point  $\hat{x} \in \text{VIP}(C, f)$  such that  $A\hat{x} \in \text{VIP}(Q, g)$ , that is,

$$\begin{cases} \hat{x} \in C & \text{such that } \langle f(\hat{x}), x - \hat{x} \rangle \geq 0 \text{ for all } x \in C, \\ \hat{y} := A\hat{x} \in Q & \text{such that } \langle g(\hat{y}), y - \hat{y} \rangle \geq 0 \text{ for all } y \in Q. \end{cases}$$

This is equivalent to the problem of finding  $\hat{x} \in \text{Fix}(P_C(I - \lambda f))$  such that  $A\hat{x} \in \text{Fix}(P_Q(I - \lambda g))$  where  $\lambda > 0$ . We denote the set of solutions by  $\text{SVIP}(A, C, Q, f, g)$ . Therefore SVIP can be viewed as SCFP. Under appropriate conditions of the operators  $f$  and  $g$ , we can apply our result for SVIP.

In the work of Censor et al. [6], the operators  $f$  and  $g$  are assumed to be  $\alpha$ -inverse strongly monotone where  $\alpha > 0$ , that is,

$$\langle x - y, f(x) - f(y) \rangle \geq \alpha \|f(x) - f(y)\|^2 \quad \text{and} \quad \langle u - v, g(u) - g(v) \rangle \geq \alpha \|g(u) - g(v)\|^2,$$

for all  $x, y \in \mathcal{H}_1$  and  $u, v \in \mathcal{H}_2$ . It is known that if  $f$  is  $\alpha$ -inverse strongly monotone and  $\lambda \in ]0, 2\alpha[$  then  $P_C(I - \lambda f)$  is strongly quasi-nonexpansive and  $I - P_C(I - \lambda f)$  is demiclosed at zero. Hence their result becomes a special case of ours. However, since every  $\alpha$ -inverse strongly monotone operator is monotone and Lipschitz continuous, the latter class of operators is then more general. It is worth noting that there exists a monotone Lipschitz continuous operator  $f$  such that  $P_C(I - \lambda f)$  fails to be quasi-nonexpansive [7]. Thanks to the extragradient method introduced by Korpelevič [8], we obtain a slight modification of such operators and prove a strong convergence theorem for SVIP in the case when  $f$  and  $g$  are monotone and Lipschitz continuous. More precisely, the following corollary is established.

**Corollary 4.1.** *Let  $C$  and  $Q$  be nonempty closed convex subsets of Hilbert spaces  $\mathcal{H}_1$  and  $\mathcal{H}_2$ , respectively. Let  $f : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  and  $g : \mathcal{H}_2 \rightarrow \mathcal{H}_2$  be monotone and  $\kappa$ -Lipschitz continuous operators on  $C$  and  $Q$ , respectively and  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  a bounded linear operator with  $\|A^*A\| = L$ . Suppose that  $\text{SVIP}(A, C, Q, f, g) \neq \emptyset$ . Define an iterative sequence  $\{x_n\} \subset \mathcal{H}_1$  by*

$$\begin{cases} x_0 \in \mathcal{H}_1, \\ x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)U(x_n + \gamma A^*(T - I)Ax_n), \end{cases}$$

where  $\gamma \in ]0, \frac{1}{L}[$ ,

$$\begin{aligned} U &:= P_C(I - \lambda f P_C(I - \lambda f)), \\ T &:= P_Q(I - \lambda g P_Q(I - \lambda g)), \end{aligned} \tag{3}$$

and  $\lambda \in ]0, \frac{1}{\kappa}[$ . Then the sequence  $\{x_n\}$  converges strongly to  $\hat{x} \in \text{SVIP}(A, C, Q, f, g)$ .

Before giving a proof, we present the following two lemmas.

**Lemma 4.2.** *Let  $f : \mathcal{H} \rightarrow \mathcal{H}$  be monotone and  $\kappa$ -Lipschitz continuous on  $C$ . Let  $S := P_C(I - \lambda f)$  where  $\lambda > 0$ . If  $\{x_n\}$  is a sequence in  $C$  satisfying  $x_n \rightharpoonup \hat{x}$  and  $x_n - Sx_n \rightarrow 0$ , then  $\hat{x} \in \text{VIP}(C, f)$ .*

**Proof.** Since  $f$  is monotone and continuous, we have (see e.g., [14])

$$\hat{x} \in \text{VIP}(C, f) \iff \langle f(x), x - \hat{x} \rangle \geq 0 \quad \text{for all } x \in C.$$

Let  $x \in C$ . Note that

$$\langle x_n - \lambda f(x_n) - Sx_n, Sx_n - x \rangle \geq 0 \quad \text{for all } n \in \mathbb{N}.$$

Next, we consider

$$\begin{aligned} \langle \lambda f(x), x_n - x \rangle &\leq \langle \lambda f(x_n), x_n - x \rangle \\ &= \langle \lambda f(x_n), x_n - Sx_n \rangle + \langle \lambda f(x_n), Sx_n - x \rangle \\ &= \langle \lambda f(x_n), x_n - Sx_n \rangle - \langle x_n - \lambda f(x_n) - Sx_n, Sx_n - x \rangle + \langle x_n - Sx_n, Sx_n - x \rangle \\ &\leq \langle \lambda f(x_n), x_n - Sx_n \rangle + \langle x_n - Sx_n, Sx_n - x \rangle \\ &\leq \lambda \|f(x_n)\| \|x_n - Sx_n\| + \|x_n - Sx_n\| \|Sx_n - x\|. \end{aligned}$$

Hence

$$\langle f(x), x_n - x \rangle \leq \|f(x_n)\| \|x_n - Sx_n\| + \frac{1}{\lambda} \|x_n - Sx_n\| \|Sx_n - x\|.$$

Since  $\{f(x_n)\}$  is bounded,  $x_n - Sx_n \rightarrow 0$  and  $x_n \rightharpoonup \hat{x}$ , we have

$$\langle f(x), \hat{x} - x \rangle = \lim_{n \rightarrow \infty} \langle f(x), x_n - x \rangle \leq 0.$$

The proof is finished.  $\square$

The following lemma is extracted from [12].

**Lemma 4.3.** *Let  $f : \mathcal{H} \rightarrow \mathcal{H}$  be a monotone and  $\kappa$ -Lipschitz operator on  $C$  and  $\lambda$  be a positive number. Let  $V := P_C(I - \lambda f)$  and  $S := P_C(I - \lambda fV)$ . Then, for all  $q \in \text{VIP}(C, f)$ , we have*

$$\|Sx - q\|^2 \leq \|x - q\|^2 - (1 - \lambda^2 \kappa^2) \|x - Vx\|^2.$$

In particular, if  $\kappa\lambda < 1$ , then  $S$  is a strongly quasi-nonexpansive operator and  $\text{Fix}(S) = \text{Fix}(V) = \text{VIP}(C, f)$ .

**Proof.** Let  $q \in \text{VIP}(C, f)$ . Note that

$$\begin{aligned} \|Sx - q\|^2 &\leq \|(x - \lambda f(Vx)) - q\|^2 - \|(x - \lambda f(Vx)) - Sx\|^2 \\ &= \|x - q\|^2 + 2\lambda \langle q - Sx, f(Vx) \rangle - \|x - Sx\|^2 \\ &= \|x - q\|^2 + 2\lambda \langle q - Vx, f(Vx) - f(q) \rangle \\ &\quad + 2\lambda \langle q - Vx, f(q) \rangle + 2\lambda \langle Vx - Sx, f(Vx) \rangle - \|x - Sx\|^2 \\ &\leq \|x - q\|^2 + 2\lambda \langle Vx - Sx, f(Vx) \rangle - \|x - Sx\|^2 \end{aligned}$$



$$\begin{aligned} &= \|x - q\|^2 + 2\lambda \langle Vx - Sx, f(Vx) \rangle - \|x - Vx\|^2 - 2\langle x - Vx, Vx - Sx \rangle - \|Vx - Sx\|^2 \\ &= \|x - q\|^2 - \|x - Vx\|^2 - \|Vx - Sx\|^2 + 2\langle x - \lambda f(Vx) - Vx, Sx - Vx \rangle. \end{aligned}$$

Now we estimate the last term of the preceding expression

$$\begin{aligned} \langle x - \lambda f(Vx) - Vx, Sx - Vx \rangle &= \langle x - \lambda f(x) - Vx, Sx - Vx \rangle + \langle \lambda f(x) - \lambda f(Vx), Sx - Vx \rangle \\ &\leq \langle \lambda f(x) - \lambda f(Vx), Sx - Vx \rangle \\ &\leq \lambda \kappa \|x - Vx\| \|Sx - Vx\|. \end{aligned}$$

So we have

$$\begin{aligned} \|Sx - q\|^2 &\leq \|x - q\|^2 - \|x - Vx\|^2 - \|Vx - Sx\|^2 + 2\lambda \kappa \|x - Vx\| \|Sx - Vx\| \\ &\leq \|x - q\|^2 - \|x - Vx\|^2 - \|Vx - Sx\|^2 + \lambda^2 \kappa^2 \|x - Vx\|^2 + \|Sx - Vx\|^2 \\ &= \|x - q\|^2 - (1 - \lambda^2 \kappa^2) \|x - Vx\|^2. \end{aligned}$$

Assume further that  $\kappa\lambda < 1$  and let  $\{x_n\}$  be a sequence in  $\mathcal{H}$  such that  $\|Sx_n - q\| - \|x_n - q\| \rightarrow 0$  for some  $q \in \text{Fix}(S)$ . It follows from the above inequality that  $x_n - Vx_n \rightarrow 0$  which can be easily deduced to  $x_n - Sx_n \rightarrow 0$ . Therefore  $S$  is strongly quasi-nonexpansive and it is not difficult to see that  $\text{Fix}(S) = \text{Fix}(V) = \text{VIP}(C, f)$ .  $\square$

**Proof of Corollary 4.1.** It follows from Lemma 4.3 that both operators  $U$  and  $T$  defined in (3) are strongly quasi-nonexpansive. We next show that  $I - U$  is demiclosed at zero. Let  $\{x_n\}$  be a sequence in  $\mathcal{H}_1$  such that  $x_n - Ux_n \rightarrow 0$  and  $x_n \rightharpoonup x$ . Notice that  $\|x_n - q\|^2 - \|Ux_n - q\|^2 \rightarrow 0$  for some  $q \in \text{VIP}(C, f)$ . Using Lemma 4.3, we get

$$(1 - \lambda^2 \kappa^2) \|x_n - P_C(I - \lambda f)x_n\|^2 \leq \|x_n - q\|^2 - \|Ux_n - q\|^2 \rightarrow 0.$$

Thus  $x_n - P_C(I - \lambda f)x_n \rightarrow 0$ . Therefore, by Lemma 4.2, we get  $x \in \text{VIP}(C, f) = \text{Fix}(U)$ . Similarly,  $I - T$  is also demiclosed at zero. Then the result follows from Theorem 3.2.  $\square$

## 4.2. The split common null point problem

Given two set-valued operators  $B_1 \subset \mathcal{H}_1 \times \mathcal{H}_1$  and  $B_2 \subset \mathcal{H}_2 \times \mathcal{H}_2$  and a bounded linear operator  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ , the *split common null point problem* (SCNP) is the problem of finding

$$\hat{x} \in \mathcal{H}_1 \quad \text{such that} \quad 0 \in B_1(\hat{x}) \text{ and } 0 \in B_2(A\hat{x}). \quad (4)$$

Recently, Byrne et al. [3] proposed a strong convergence theorem for finding such a solution  $\hat{x}$  when  $B_1$  and  $B_2$  are maximal monotone. Recall that  $B \subset \mathcal{H} \times \mathcal{H}$  is:

- *monotone* if  $\langle x - y, u - v \rangle \geq 0$  for all  $(x, u) \in B_1$  and  $(y, v) \in B_2$ ;
- *maximal monotone* if it is monotone and its graph is not properly contained in the graph of any other monotone operator.

For a maximal monotone operator  $B \subset \mathcal{H} \times \mathcal{H}$  and  $\lambda > 0$ , we can define a single-valued operator

$$J_\lambda^B =: (I + \lambda B)^{-1} : \mathcal{H} \rightarrow \mathcal{H}.$$

It is known that  $J_\lambda^B$  is *firmly nonexpansive*, that is, for all  $x, y \in \mathcal{H}$ ,

$$\langle x - y, J_\lambda^B x - J_\lambda^B y \rangle \geq \|J_\lambda^B x - J_\lambda^B y\|^2,$$

and

$$0 \in B(\hat{x}) \iff \hat{x} \in \text{Fix}(J_\lambda^B).$$

Therefore, the problem (4) is equivalent to the problem of finding

$$\hat{x} \in \mathcal{H}_1 \quad \text{such that} \quad \hat{x} \in \text{Fix}(J_\lambda^{B_1}) \text{ and } A\hat{x} \in \text{Fix}(J_\lambda^{B_2}),$$

where  $\lambda$  is a positive real number, that is, the SCNP reduces to the SCFP.

The result of Byrne et al. [3] is a consequence of our [Theorem 3.2](#).

**Corollary 4.4.** (See [3].) Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be two Hilbert spaces. Given two set-valued maximal monotone operators  $B_1 : \mathcal{H}_1 \rightarrow 2^{\mathcal{H}_1}$  and  $B_2 : \mathcal{H}_2 \rightarrow 2^{\mathcal{H}_2}$  and a bounded linear operator  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  with  $L = \|A^*A\|$ , we define an iterative sequence  $\{x_n\}$  by

$$\begin{cases} x_0 \in \mathcal{H}_1, \\ x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n), \end{cases} \quad (5)$$

where the parameters  $\lambda, \gamma$  and the sequence  $\{\alpha_n\}$  satisfy the following conditions:

- (a)  $\lambda > 0, \gamma \in ]0, \frac{2}{L}[$ ,
- (b)  $\{\alpha_n\} \subset ]0, 1[, \lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=0}^\infty \alpha_n = \infty$ .

Suppose that the solution set of (4), says  $\Gamma$ , is nonempty. Then  $x_n \rightarrow \hat{x} \in \Gamma$ .

**Remark 4.5.**

- (1) Notice that [Corollary 4.4](#) can be viewed as a corollary of our [Theorem 3.2](#) for the following reasons.
  - (a) For a maximal monotone  $B$  and  $\lambda > 0$ , it is known that  $J_\lambda^B$  is firmly nonexpansive and hence nonexpansive. Moreover,  $I - J_\lambda^B$  is demiclosed at zero [1] and

$$J_\lambda^B = \frac{1}{2}I + \frac{1}{2}S,$$

for some nonexpansive operator  $S : \mathcal{H} \rightarrow \mathcal{H}$ .

- (b) For  $B_2$  and  $A$  defined as in [Corollary 4.4](#), it follows from [Lemma 3.3](#) with  $\alpha = \frac{1}{2}$  that

$$\|Wx - y\|^2 \leq \|x - y\|^2 + \gamma(\gamma L - 2)\|(J_\lambda^{B_2} - I)Ax\|^2,$$

for all  $x \in \mathcal{H}_1$  and  $y \in \mathcal{H}_2$  such that  $Ay \in \text{Fix}(J_\lambda^{B_2})$  where  $W := I + \gamma A^*(J_\lambda^{B_2} - I)A$ . So, in this case, the parameter  $\gamma$  can be relaxed, that is,  $\gamma \in ]0, \frac{2}{L}[$  instead of  $]0, \frac{1}{L}[$ .

- (2) Our [Theorem 3.2](#) allows the parameter  $\lambda$  for  $J_\lambda^{B_1}$  and  $J_\lambda^{B_2}$  in [Corollary 4.4](#) to be chosen differently.
- (3) The strong limit  $\hat{x}$  of the sequence  $\{x_n\}$  generated by (5) is indeed the nearest point projection of  $x_0$  onto the solution set  $\Gamma$ .

### 4.3. Moudafi's split feasibility problem

Let  $\mathcal{H}_1$ ,  $\mathcal{H}_2$  and  $\mathcal{H}_3$  be Hilbert spaces and  $C \subset \mathcal{H}_1$ ,  $Q \subset \mathcal{H}_2$  be nonempty closed convex sets. Let  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_3$ ,  $B : \mathcal{H}_2 \rightarrow \mathcal{H}_3$  be bounded linear operators. Moudafi's split feasibility problem [10,11] is the problem of finding

$$x \in C \text{ and } y \in Q \text{ such that } Ax = By. \quad (6)$$

We will transform this problem into the original SFP. Let us denote

$$\begin{aligned} \mathbf{H}_1 &:= \mathcal{H}_1 \times \mathcal{H}_2, \\ \mathbf{H}_2 &:= \mathcal{H}_3 \times \mathcal{H}_3, \\ \mathbf{C} &:= C \times Q \subset \mathbf{H}_1, \\ \mathbf{Q} &:= \{(z, w) \in \mathbf{H}_2 : z = w\}. \end{aligned}$$

Define a linear operator  $\mathbf{A} : \mathbf{H}_1 \rightarrow \mathbf{H}_2$  by

$$\mathbf{A}(x, y) = (Ax, By) \quad \text{for all } (x, y) \in \mathbf{H}_1.$$

If the set  $\Gamma := \{(x, y) \in \mathbf{C} : \mathbf{A}(x, y) \in \mathbf{Q}\}$  is nonempty, then  $(x, y) \in \mathbf{H}_1$  solves (6) if and only if

$$(x, y) = P_{\mathbf{C}}(I + \gamma \mathbf{A}^*(P_{\mathbf{Q}} - I)\mathbf{A})(x, y).$$

Note that:

- $P_{\mathbf{C}}(x, y) = (P_C x, P_Q y)$  for all  $(x, y) \in \mathbf{H}_1$ ;
- $P_{\mathbf{Q}}(z, w) = (\frac{z+w}{2}, \frac{z+w}{2})$  for all  $(z, w) \in \mathbf{H}_2$ ;
- $\mathbf{A}^*(z, w) = (A^* z, B^* w)$  for all  $(z, w) \in \mathbf{H}_2$ .

As a consequence of our Theorem 3.2, the following iterative sequence  $\{(x_n, y_n)\}$  defined by

$$\begin{cases} x_0 \in \mathcal{H}_1, \\ y_0 \in \mathcal{H}_2, \\ x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) P_C \left( x_n + \frac{\gamma}{2} A^*(By_n - Ax_n) \right), \\ y_{n+1} = \alpha_n y_0 + (1 - \alpha_n) P_Q \left( y_n + \frac{\gamma}{2} B^*(Ax_n - By_n) \right), \end{cases}$$

converges strongly to  $(\hat{x}, \hat{y})$  which simultaneously solves Moudafi's split feasibility problem (6) and is nearest to the initial guess  $(x_0, y_0)$ .

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