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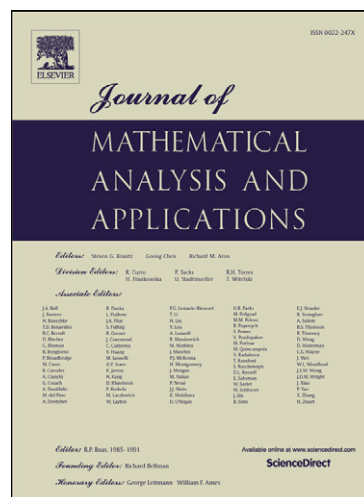
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Vitali-type theorems for filter convergence related to vector lattice-valued modulars and applications to stochastic processes[☆]

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Abstract

A Vitali-type theorem for vector lattice-valued modulars with respect to filter convergence is proved. Some applications are given to modular convergence theorems for moment operators in the vector lattice setting, and also for the Brownian motion and related stochastic processes.

Keywords: Vector lattice, filter convergence, modular, Vitali convergence theorem, moment operator, Brownian motion, Itô integral

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Introduction

The Vitali theorem has been widely studied in the literature. For a historical survey, see for instance [30] and the bibliography therein. This theorem is a well-known milestone in integration theory, directly related with L^p convergence and weak compactness. However it has also many applications in several branches of Mathematics, for example in reconstruction of signals, integral and discrete operators (see [10, 11, 13, 7, 8, 9, 42, 31, 12]). The Vitali theorem has been investigated even in the context of vector lattices with respect to different types of convergence (see [19] and its bibliography) and in abstract convergence groups, not necessarily endowed with an order structure (see [26]).

Also the modular spaces (see for instance [13, 36]) are a rich field of research. They are a natural generalization of the Lebesgue spaces L^p and contain as particular cases the Orlicz and the Musielak-Orlicz spaces (see e.g. [37, 39]).

In modular spaces the Vitali theorem is often used in the problem of approximating a real-valued function f by means of Urysohn-type or sampling operators $(T_n f)_n$, (see also [10, 11, 14, 15, 16, 17, 18, 41, 1, 2, 3, 4, 5, 21, 22, 6, 43, 32, 33, 44]). These operators are particularly

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useful in order to approximate a continuous or analog signal by means of discrete samples, and therefore they are widely applied for example in reconstructing images and videos.

In the classical literature, a signal f is considered as a real-valued function, defined on a (bounded or not) time interval. In these situations, usually it happens that the original signal or image is unknown, and it must be reconstructed from some sampled values of the type $f(\frac{k}{w})$, where $k \in \mathbb{Z}$, and w is a fixed rate. So it would be advisable to consider the studied signal, for example, as a map defined on a (finite or not) time interval and taking values in a suitable space of random variables (for more details related with these topics see also [13] and the literature therein). In this context, it is natural to consider the tool of *vector lattices*. Moreover, also domains with infinite measure, like for instance an infinite time interval, can be treated.

In this paper we consider vector lattices with convergences generated by filters (see also [24, 29, 23]), which generalize classical ones. For example, when the filter involved is the class of all subsets of \mathbb{N} having asymptotic density one, we obtain the so-called statistical convergence. We introduce the theory of modular spaces in the vector lattice context: integration theory can be considered as a particular case of the theory of modulars, even in our setting, so we give a general version of the Vitali and Lebesgue dominated convergence theorems for modular convergence, with applications to integrals for vector lattice-valued functions, defined on sets of possibly infinite measure. In Section 4 we deal with moment kernels in the vector lattice setting, in order to construct a suitable sequence of convolution type operators; moreover in Subsection 3.4, among the applications, we give convergence results for the Itô integral with respect to Brownian motion. Finally, the last Section is devoted to the proof of the main theoretical result (Theorem 3.13) of Section 4.

1. Preliminaries

We begin with recalling some basic properties of vector lattices and filter convergence.

1.1. Vector lattices

A vector lattice (also called Riesz space) \mathbf{X} is said to be *Dedekind complete* iff every nonempty subset B of \mathbf{X} , bounded from above, has a lattice supremum in \mathbf{X} , denoted by $\bigvee B$.

From now on \mathbf{X} denotes a Dedekind complete vector lattice, \mathbf{X}^+ is the set of all strictly positive elements of \mathbf{X} , and we set $\mathbf{X}_0^+ = \mathbf{X}^+ \cup \{0\}$. For all $x \in \mathbf{X}$, let $|x| := x \vee (-x)$. We add to \mathbf{X} an extra element $+\infty$, extending order and operations in a natural way, and we set $\overline{\mathbf{X}} = \mathbf{X} \cup \{+\infty\}$, $\overline{\mathbf{X}}_0^+ = \mathbf{X}_0^+ \cup \{+\infty\}$ and assume by convention $0 \cdot (+\infty) = 0$.

A sequence $(p_n)_n$ in \mathbf{X} is called *(o)-sequence* iff it is decreasing and $\bigwedge_n p_n = 0$.

A *strong unit* of \mathbf{X} is an element e , such that for every $x \in \mathbf{X}$ there is a positive real number c with $|x| \leq c e$.

Let $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}$ be three Dedekind complete vector lattices. We say that $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X})$ is a *product triple* iff a map $\cdot : \mathbf{X}_1 \times \mathbf{X}_2 \rightarrow \mathbf{X}$ (*product*) is given, satisfying natural conditions of compatibility, see [20, Assumption 2.1] and [25]. From now on we suppose that:

(H_0) $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X})$ is a product triple, where \mathbf{X}, \mathbf{X}_1 are endowed with strong units e, e_1 respectively.

1.2. Filter convergence

Let Z be any fixed countable set. A class \mathcal{F} of subsets of Z is called a *filter* of Z iff $\emptyset \notin \mathcal{F}$, $A \cap B \in \mathcal{F}$ whenever $A, B \in \mathcal{F}$ and for each $A \in \mathcal{F}$ and $B \supset A$ we get $B \in \mathcal{F}$. We denote by $\mathcal{F}_{\text{cofin}}$ the filter of all cofinite subsets of Z . A filter of Z is said to be *free* iff it contains $\mathcal{F}_{\text{cofin}}$. An example of free filter (different from $\mathcal{F}_{\text{cofin}}$) is the family \mathcal{F}_d of all subsets of \mathbb{N} , having asymptotic density one. In this paper we only deal with free filters.

We now give the notion of filter convergence in the vector lattice setting (see also [24]).

Definition 1.1. Let \mathcal{F} be any filter of Z . A sequence $(x_z)_{z \in Z}$ in \mathbf{X} ($o_{\mathcal{F}}$)-converges to $x \in \mathbf{X}$ ($x_z \xrightarrow{o_{\mathcal{F}}} x$) iff there exists an (o) -sequence $(\sigma_p)_p$ in \mathbf{X} such that for all $p \in \mathbb{N}$ the set $\{z \in Z : |x_z - x| \leq \sigma_p\}$ is an element of \mathcal{F} .

A sequence $(x_z)_{z \in Z}$ in \mathbf{X} ($r_{\mathcal{F}}$)-converges to $x \in \mathbf{X}$ ($x_z \xrightarrow{r_{\mathcal{F}}} x$) iff there exist a positive element $u \in \mathbf{X}$ and an (o) -sequence $(\varepsilon_p)_p \in \mathbb{R}^+$ such that for all $p \in \mathbb{N}$ the set $\{z \in Z : |x_z - x| \leq \varepsilon_p u\}$ is an element of \mathcal{F} .

Clearly a sequence $(x_z)_z$ in \mathbf{X} (o)-converges to $x \in \mathbf{X}$ (in the classical sense) iff it ($o_{\mathcal{F}_{\text{cofin}}}$)-converges to x , and similarly with (r) -convergence.

2. Convergence theorems

Let G be any infinite set, $\Sigma \subset \mathcal{P}(G)$ be an algebra and $\mu : \Sigma \rightarrow \overline{(\mathbf{X}_2)_0^+}$ be a finitely additive measure. We shall denote by

Σ_b the family of all sets $B \in \Sigma$ with $\mu(B) \in \mathbf{X}_2$. For every $A \in \Sigma$ and $r \in \mathbf{X}_1$, we shall denote by $r \cdot 1_A : G \rightarrow \mathbf{X}_1$ the function whose values $r \cdot 1_A(t)$ are r when $t \in A$ and 0 otherwise.

2.1. Modularity

We now introduce the modularity in the setting of vector lattices (for the classical case and a related literature, see e.g. [13, 36]). Let T be a linear sublattice of \mathbf{X}_1^G , such that

- $e_1 \cdot 1_A \in T$ for every $A \in \Sigma_b$;
- if $f \in T$ and $A \in \Sigma$, then $f \cdot 1_A \in T$.

A functional $\rho : T \rightarrow \overline{\mathbf{X}_0^+}$ is said to be a *modular* on T iff it satisfies the following properties:

- (ρ_0) $\rho(0) = 0$;
- (ρ_1) $\rho(-f) = \rho(f)$ for every $f \in T$;
- (ρ_2) $\rho(\alpha_1 f + \alpha_2 h) \leq \rho(f) + \rho(h)$ for every $f, h \in T$ and for any non negative real numbers α_1, α_2 with $\alpha_1 + \alpha_2 = 1$.

We say that:

- (ρ_m) A modular ρ is *monotone* iff $\rho(f) \leq \rho(h)$ for every $f, h \in T$ with $|f| \leq |h|$. In this case, if $f \in T$, then $|f| \in T$ and hence $\rho(f) = \rho(|f|)$.
- (ρ_f) A modular ρ is *finite* iff for every $A \in \Sigma_b$ and every (o) -sequence $(\varepsilon_p)_p$ in \mathbb{R}^+ , the sequence $(\rho(e_1 \varepsilon_p 1_A))_p$ is (r) -convergent to 0. (For the case $\mathbf{X} = \mathbf{X}_1 = \mathbf{X}_2 = \mathbb{R}$, see also [13]).

We now prove the following

Proposition 2.1. *Let $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X})$ be a product triple satisfying (H_0) , and assume that ρ is a finite modular, $(B_m)_m$ is a sequence in Σ_b and $(\varepsilon_p)_p$ is any (o) -sequence in \mathbb{R} . Then there exists a strictly increasing mapping $m \mapsto p(m)$ from \mathbb{N} to \mathbb{N} , with*

$$\rho(e_1 \varepsilon_{p(m)} 1_{B_m}) \leq \frac{e}{m} \quad \text{for every } m \in \mathbb{N}.$$

Proof: Fix a sequence $(B_m)_m$ in Σ_b , and any (o) -sequence $(\varepsilon_p)_p$ in \mathbb{R} . By finiteness of ρ , we get

$$(r) \quad \lim_{p \rightarrow +\infty} \rho(e_1 \varepsilon_p 1_{B_m}) = 0 \quad \text{for every } m \in \mathbb{N}.$$

This means that a sequence $(u_m)_m$ exists in \mathbf{X}^+ with the following property: for every $m, p \in \mathbb{N}$ there is $\lambda(m, p) \in \mathbb{R}^+$ such that $(\lambda(m, p))_p$ is an (o) -sequence for all m , and

$$0 \leq \rho(e_1 \varepsilon_p 1_{B_m}) \leq \lambda(m, p) u_m \quad \text{for all } m, p. \quad (1)$$

For every $m \in \mathbb{N}$ there is a positive real number h_m such that $u_m \leq h_m e$. Therefore, from (1) we also have

$$0 \leq \rho(e_1 \varepsilon_p 1_{B_m}) \leq h_m \lambda(m, p) e \quad \text{whenever } m, p \in \mathbb{N}. \quad (2)$$

Now, for each integer m , a corresponding integer $p(m)$ can be found, such that the sequence $(p(m))_m$ is strictly increasing and

$$h_m \lambda(m, p(m)) \leq \frac{1}{m} \quad \text{for all } m. \quad (3)$$

The assertion follows from (2) and (3). \square

We now give the concept of (equi)-absolute continuity in the context of modulars and filter convergence: this is one of the crucial tools in the Vitali theorem (for a classical modular version, see [13, Theorem 2.1]).

(a_ρ) We say that $f \in T$ is $(o_{\mathcal{F}})$ -absolutely continuous with respect to the modular ρ (or in short *absolutely continuous*) iff there is a positive real constant α , satisfying the following properties:

$(a_\rho(1))$ for each (o) -sequence $(\sigma_p)_p$ in \mathbf{X}_2^+ there exists an (o) -sequence $(w_p)_p$ in \mathbf{X}^+ such that for all $p \in \mathbb{N}$ and whenever $\mu(B) \leq \sigma_p$ we get $\rho(\alpha f 1_B) \leq w_p$;

$(a_\rho(2))$ there is an (o) -sequence $(z_m)_m$ in \mathbf{X}^+ such that to each $m \in \mathbb{N}$ there corresponds a set $B_m \in \Sigma_b$ with $\rho(\alpha f 1_{G \setminus B_m}) \leq z_m$.

(ac_ρ) Given a modular ρ and any free filter \mathcal{F} of Z , we say that a sequence $(f_z)_{z \in Z}$ in \mathbf{X}_1^G is ρ - \mathcal{F} -equi-absolutely continuous, or in short *equi-absolutely continuous*, iff there is $\alpha \in \mathbb{R}^+$, satisfying the following two conditions:

$(ac_\rho(1))$ for every (o) -sequence $(\sigma_p)_p$ in \mathbf{X}_2^+ there are an (o) -sequence $(w_p)_p$ in \mathbf{X}^+ and a sequence $(\Xi^p)_p$ in \mathcal{F} , with $\rho(\alpha f_z 1_B) \leq w_p$ whenever $z \in \Xi^p$ and $\mu(B) \leq \sigma_p$, $p \in \mathbb{N}$;

$(ac_\rho(2))$ there are an (o) -sequence $(r_m)_m$ in \mathbf{X}^+ and a sequence $(B_m)_m \in \Sigma_b$ such that

$$\Lambda^m := \{z \in Z : \rho(\alpha f_z 1_{G \setminus B_m}) \leq r_m\} \in \mathcal{F} \quad \text{for all } m \in \mathbb{N}. \quad (4)$$

In [10] some sufficient conditions are given for (equi)-absolute continuity with respect to modulars.

2.2. A Vitali-type theorem

As we already pointed out in the Introduction, the Vitali theorem has a fundamental importance in Approximation Theory, also when $\mu(G) = +\infty$: several applications can be found in [11, 13, 28, 35, 37, 41] and the literature therein. Together with equi-absolute continuity, also some notions of convergence are needed, so we recall the concepts of filter uniform convergence and convergence in measure (see also [19]). Let \mathcal{F} be any fixed free filter of Z .

Definitions 2.2.

(2.2.1) A sequence of functions $(f_z)_{z \in Z}$ in \mathbf{X}_1^G $(r_{\mathcal{F}})$ -converges uniformly (shortly converges uniformly) to f , iff there exists an (o) -sequence $(\varepsilon_p)_p$ in \mathbb{R}^+ with

$$\{z \in Z : \bigvee_{t \in G} |f_z(t) - f(t)| < \varepsilon_p e_1\} \in \mathcal{F} \quad \text{for any } p \in \mathbb{N}.$$

In this case we write:

$$(r_{\mathcal{F}}) \lim_z \left(\bigvee_{t \in G} |f_z(t) - f(t)| \right) = 0.$$

(2.2.2) Given a sequence $(f_z)_z$ in \mathbf{X}_1^G and $f \in \mathbf{X}_1^G$, we say that $(f_z)_z$ $(r_{\mathcal{F}})$ -converges in measure (shortly μ -converges) to f , iff there are two (o) -sequences, $(\varepsilon_p)_p$ in \mathbb{R}^+ , $(\sigma_p)_p$ in \mathbf{X}_2^+ , and a double sequence $(A_z^p)_{(z,p) \in Z \times \mathbb{N}}$ in Σ such that

- $A_z^p \supset \{t \in G : |f_z(t) - f(t)| \not\leq \varepsilon_p e_1\}$ for every $z \in Z$ and $p \in \mathbb{N}$, and
- $\{z \in Z : \mu(A_z^p) \leq \sigma_p\} \in \mathcal{F}$ for all $p \in \mathbb{N}$.

It is easy to see that (2.2.1) implies (2.2.2). If $(f_z)_z, (h_z)_z$ μ -converge to f, h respectively, then $(f_z \pm h_z)_z$ μ -converges to $f \pm h$ and $(|f_z|)_z$ μ -converges to $|f|$, (see also [19]). Moreover, when $\mathbf{X}_1 = \mathbf{X}_2 = \mathbb{R}$ and the convergence involved is the usual one, these notions coincide with the classical ones.

We now prove a Vitali-type theorem, which links the theories of modulars and vector lattices, in the context of $(o_{\mathcal{F}})$ -convergence. We observe that a first version of Vitali's theorem in modular spaces was given in [36], in case of real-valued functions, measures and modulars.

Theorem 2.3. (Vitali) Let ρ be a monotone and finite modular, and \mathcal{F} be a fixed free filter of Z . If $(f_z)_z$ is a sequence in T , μ -convergent to 0 and equi-absolutely continuous, then there is a positive number α with

$$(o_{\mathcal{F}}) \lim_z \rho(\alpha f_z) = 0.$$

Proof: Let $\alpha \in \mathbb{R}^+$ be related with equi-absolute continuity, and $(r_m)_m$ in \mathbf{X}^+ be an (o) -sequence associated with $(ac_{\rho}(2))$; let moreover $(\sigma_p)_p$ in \mathbf{X}_2^+ , $(\varepsilon_p)_p$ in \mathbb{R}^+ be related with μ -convergence of the sequence $(f_z)_z$ to 0, and $(w_p)_p$ in \mathbf{X}^+ be associated with $(\sigma_p)_p$ and $(ac_{\rho}(1))$.

Choose any sequence $(B_m)_m$ in Σ_b , and any corresponding sequence $(\Lambda^m)_m$ in \mathcal{F} , satisfying (4). Without loss of generality, we assume that the f_z 's are positive. For every $z \in Z$, $p \in \mathbb{N}$, let $A_z^p \in \Sigma$ be such that $A_z^p \supset \{t \in G : f_z(t) \not\leq \varepsilon_p e_1\}$, according to μ -convergence. For every $z \in Z$, $m, p \in \mathbb{N}$ and $t \in B_m$ we have

$$\frac{\alpha}{3} f_z(t) = \frac{1}{3} \left(\alpha f_z(t) 1_{G \setminus B_m}(t) + \alpha f_z(t) 1_{B_m \setminus \Lambda_z^p}(t) + \alpha f_z(t) 1_{B_m \cap \Lambda_z^p}(t) \right). \quad (5)$$

By the properties of the modular ρ we obtain (with obvious meaning of the notations):

$$\begin{aligned} \rho\left(\frac{\alpha}{3}f_z\right) &\leq \rho(\alpha f_z 1_{G \setminus B_m}) + \rho(\alpha f_z 1_{B_m \setminus A_z^p}) + \rho(\alpha f_z 1_{B_m \cap A_z^p}) = \\ &= I_1 + I_2 + I_3 \quad \text{for all } z \in Z, m, p \in \mathbb{N}. \end{aligned} \quad (6)$$

By (4) we get

$$I_1 = \rho(\alpha f_z 1_{G \setminus B_m}) \leq r_m \quad \text{for all } z \in \Lambda^m. \quad (7)$$

Moreover, since $\mu(A_z^p) \leq \sigma_p$ by convergence in measure, then, thanks to property $(ac_\rho(1))$ of equi-absolute continuity, for each $p \in \mathbb{N}$ there exists an element $\Xi^p \in \mathcal{F}$, with

$$I_3 = \rho(\alpha f_z 1_{B_m \cap A_z^p}) \leq \rho(\alpha f_z 1_{A_z^p}) \leq w_p \quad \text{whenever } m \in \mathbb{N} \text{ and } z \in \Xi^p. \quad (8)$$

In order to estimate I_2 , observe that by Proposition 2.1 there is a strictly increasing mapping $m \mapsto p(m)$ from \mathbb{N} to \mathbb{N} , with

$$\rho(\alpha \varepsilon_{p(m)} e_1 1_{B_m}) \leq \frac{e}{m} \quad \text{for all } m;$$

now, for all $z \in Z$, $m \in \mathbb{N}$ and $t \in B_m$ we see that

$$0 \leq \alpha f_z(t) 1_{B_m \setminus A_z^{p(m)}}(t) \leq \alpha \varepsilon_{p(m)} e_1 1_{B_m}(t)$$

and, consequently,

$$I_2 = \rho(\alpha f_z 1_{B_m \setminus A_z^{p(m)}}) \leq \frac{e}{m}. \quad (9)$$

Thus from (6), (7), (8) and (9) it follows that for every m there are $\Lambda^m, \Xi^{p(m)} \in \mathcal{F}$ such that

$$\rho\left(\frac{\alpha}{3}f_z\right) \leq \rho(\alpha f_z 1_{G \setminus B_m}) + \rho(\alpha f_z 1_{B_m \setminus A_z^{p(m)}}) + \rho(\alpha f_z 1_{B_m \cap A_z^{p(m)}}) \leq r_m + \frac{e}{m} + w_{p(m)}$$

whenever $z \in \Lambda^m \cap \Xi^{p(m)}$, and so, since $m \mapsto r_m + \frac{e}{m} + w_{p(m)}$ is an (o) -sequence, we conclude

$$(o_{\mathcal{F}}) \lim_z \rho\left(\frac{\alpha}{3}f_z\right) = 0,$$

that is the assertion. \square

An easy consequence of Theorem 2.3 is the following

Theorem 2.4. (Lebesgue dominated convergence theorem) *Let $\mathbf{X}, \rho, \mathcal{F}$ be as in Theorem 2.3 and assume that $(f_z)_z$ is a sequence in T , μ -convergent to 0. Suppose that there are an absolutely continuous function g in T and an element $F_0 \in \mathcal{F}$, such that $|f_z(t)| \leq g(t)$ for all $z \in F_0$ and $t \in G$. Then there is a positive real number α with $(o_{\mathcal{F}}) \lim_z \rho(\alpha f_z) = 0$.*

3. Examples and applications

We give here some applications of the results obtained above in different fields of Mathematics, as integration theory, moment operators, stochastic processes and Brownian motion.

We begin with giving an example, which deals with modulars generating Orlicz spaces. Observe that this concept is extended in a natural way even in the context of Riesz spaces.

3.1. Riesz space-valued modulars

Let us consider $\mathcal{F} = \mathcal{F}_{\text{cofin}}$ and $(o) = (o_{\mathcal{F}_{\text{cofin}}})$.

(★) Let $\varphi : (R_1)_0^+ \rightarrow (R_1)_0^+$ be an increasing function, such that

$$\varphi(x \vee y) \leq \varphi(x) + \varphi(y) \quad \text{for all } x, y \in (R_1)_0^+. \quad (10)$$

Clearly, (10) is fulfilled when $R_1 = \mathbb{R}$ and φ is monotone, and even when $R_1 = L_1([0, 1])$ and $\varphi(f)(x) = \int_0^x f(t)dt$. Indeed for all $0 \leq f, g \in L_1([0, 1])$ and $x \in [0, 1]$ we get:

$$\varphi(f \vee g)(x) = \int_0^x [f(t) \vee g(t)] dt \leq \int_0^x [f(t) + g(t)] dt = \varphi(f)(x) + \varphi(g)(x).$$

We denote by \widetilde{L} the space of all integrable functions (see also Definition 3.4), and by L_φ the set

$$L_\varphi := \{f : G \rightarrow R_1 : \varphi(|f|) \in \widetilde{L}\}.$$

We now construct a modular generated by a suitable Riesz space-valued function. More precisely, we prove the following

Proposition 3.1. *Let φ be as in (★). For every $f \in L_\varphi$ the operator*

$$\rho^\varphi(f) = \int_G \varphi(|f(t)|) d\mu(t)$$

is a monotone modular. Moreover, if φ is also convex, and we consider positive functions in L_φ , then ρ^φ is convex too.

Proof: We first claim that $\varphi(ax+by) \leq \varphi(x) + \varphi(y)$ for every $x, y \in (R_1)_0^+$ and $a, b \in \mathbb{R}_0^+, a+b=1$. Indeed, first of all we have

$$ax + by \leq a(x \vee y) + b(x \vee y) = x \vee y. \quad (11)$$

From (10) and (11), since φ is increasing and takes values in $(R_1)_0^+$, we get

$$\varphi(ax + by) \leq \varphi(x \vee y) \leq \varphi(x) + \varphi(y).$$

Thus for all functions $f, h \in R_1^G$ we have

$$\varphi(a|f(t)| + b|h(t)|) \leq \varphi(|f(t)|) + \varphi(|h(t)|) \quad (12)$$

for each fixed $t \in G$. Thanks to additivity of the integral given in [19] we get the first assertion. Convexity for positive functions follows immediately by convexity of φ and linearity and monotonicity of the integral. \square

3.2. Integration theory

Let G, Σ and μ be as above. We now extend the integration theory investigated in [19] to the case in which μ can assume the value $+\infty$, in the setting of $(o_{\mathcal{F}})$ -convergence.

Definitions 3.2. A function $f \in \mathbf{X}_1^G$ is said to be *simple* iff $f(G)$ is a finite set and $f^{-1}(x) \in \Sigma$ for every $x \in \mathbf{X}_1$. The space of all simple functions is denoted by \mathcal{S} .

Let L^* be the set of all simple functions $f \in \mathcal{S}$ vanishing outside a set of μ -finite measure. If $f \in L^*$, we denote its elementary integral with $\int_G f(t) d\mu(t)$, as usual.

Remark 3.3. It is clear that the functional $\iota : L^* \rightarrow \mathbf{X}$ defined as

$$\iota(f) := \int_G |f(t)| d\mu(t), \quad f \in L^*, \quad (13)$$

is a monotone modular, and it is also linear and additive on positive functions and constants.

Also finiteness of ι is easy to see: indeed, let $A \in \Sigma_b$ and $(\varepsilon_p)_p$ be any (o) -sequence in \mathbb{R}^+ . According to our definitions, $\iota(\varepsilon_p e_1 1_A) = \varepsilon_p \mu(A) e_1$ for all $p \in \mathbb{N}$: since $\mu(A)e_1$ is a positive element in \mathbf{X} , it is obvious that the condition of finiteness is satisfied.

Definition 3.4. A positive function $f \in \mathbf{X}_1^G$ is *integrable* iff there exist an equi-absolutely continuous sequence of functions $(f_n)_n$ in L^* , μ -convergent to f , and a map $l : \Sigma \rightarrow \mathbf{X}$, with

$$(o_{\mathcal{F}}) \lim_n \bigvee_{A \in \Sigma} \left| \int_A f_n(t) d\mu(t) - l(A) \right| = 0 \quad (14)$$

(the sequence $(f_n)_n$ is said to be *defining*). Note that, in this case, we get

$$l(A) = (o_{\mathcal{F}}) \lim_n \int_A f_n(t) d\mu(t) \quad \text{uniformly with respect to } A \in \Sigma.$$

If $f \in \mathbf{X}_1^G$, then we say that f is *integrable* iff f^+ and f^- are integrable, where $f^+(t) = f(t) \vee 0$, $f^-(t) = (-f(t)) \vee 0$, $t \in G$. In this case, we set

$$\int_A f(t) d\mu(t) := \int_A f^+(t) d\mu(t) - \int_A f^-(t) d\mu(t), \quad A \in \Sigma. \quad (15)$$

Similarly as in [19, Proposition 3.11], we now prove that the integral in Definition 3.4 is well-defined.

Theorem 3.5. *In the setting described above, let l be as in (14). Then the quantity $l(A)$ is independent of the choice of the defining sequence $(f_n)_n$.*

Proof: Let $(f_n^j)_n$, $j = 1, 2$, be two defining sequences for f , and put

$$\begin{aligned} l_j(A) &= (o_{\mathcal{F}}) \lim_n \int_A f_n^j(t) d\mu(t), \quad A \in \Sigma, \quad j = 1, 2; \\ h_n(t) &= |f_n^1(t) - f_n^2(t)|, \quad t \in G, \quad n \in \mathbb{N}. \end{aligned}$$

Since $(f_n^j)_n$, $j = 1, 2$, are equi-absolutely continuous, then $(h_n)_n$ is too. As $(f_n^1)_n$ and $(f_n^2)_n$ μ -converge to f , $(h_n)_n$ μ -converges to 0. Thus, by Theorem 2.3 applied to the modular $\iota : L^* \rightarrow \mathbf{X}$ defined in Remark 3.3, we obtain:

$$(\mathcal{O}_{\mathcal{F}}) \lim_n \int_G h_n(t) d\mu(t) = 0.$$

Therefore for all $A \in \Sigma$ and $n \in \mathbb{N}$ we get:

$$|l_1(A) - l_2(A)| \leq \left| \int_A f_n^1(t) d\mu(t) - l_1(A) \right| + \left| l_2(A) - \int_A f_n^2(t) d\mu(t) \right| + \int_G h_n(t) d\mu(t).$$

By letting n tend to infinity, we obtain $|l_1(A) - l_2(A)| = 0$, namely $l_1(A) = l_2(A)$ for all $A \in \Sigma$. \square

Remark 3.6. Let L be the space of all integrable functions in the sense of Definition 3.4. It is clear that in this way the elementary integral, defined in L^* , can be uniquely extended to a linear monotone functional $\tau : L \rightarrow \mathbf{X}$ defined on the whole of L . In particular, it turns out that, fixed a measurable subset $B \subset G$ and a mapping $f \in \mathbf{X}_1^G$, then f is integrable in the whole of G iff both mappings $f \cdot 1_B$ and $f \cdot 1_{G \setminus B}$ are (and of course the integral is additive in this case).

Remark 3.7. If $f \in \mathbf{X}^G$ is defined by $f(t) = h(t) u$, where $u \in \mathbf{X}^+$ is fixed and $h : G \rightarrow \mathbb{R}$ is Lebesgue integrable, then f belongs to $L(\lambda)$ and $\tau(f) = (\int_G h d\mu)u$.

Remark 3.8. Integrability of f (according with Definition 3.3) implies also absolute continuity of its integral function. Indeed, since the integral $l(A) := \int_A f d\mu$ is defined by

$$l(A) = (\mathcal{O}_{\mathcal{F}}) \lim_n \int_A f_n(t) d\mu(t) \quad \text{uniformly with respect to } A \in \Sigma,$$

where the functions $(f_n)_n$ are an equi-absolutely continuous sequence in L^* , μ -convergent to f , then it is not difficult to deduce that f , when (positive and) integrable, is absolutely continuous w.r. to the natural extension of ι to the space of all integrable functions, defined as

$$\iota(f) := \int_G |f(t)| d\mu(t), \quad f \in L. \quad (16)$$

From this, one might easily prove that, in case (A_n) is a sequence of measurable subsets of G , whose measure (\mathcal{O}) -converges to 0, then the integrals $\int f 1_{A_n} d\mu$ $(\mathcal{O}_{\mathcal{F}})$ -converge to 0, thanks to the Vitali Theorem (2.3), when applied to the modular above.

3.3. The moment operator

Let $\mathbf{X}_1 = \mathbf{X}$ be a vector lattice with a strong unit e , $\mathbf{X}_2 = \mathbb{R}$, $G = \mathbb{R}^+$, \mathcal{B} be the σ -algebra of all measurable subsets of \mathbb{R}^+ , $\lambda : \mathcal{B} \rightarrow \mathbb{R}^+$ be the Lebesgue measure and $\mathcal{F} = \mathcal{F}_{\text{cofin}}$.

From now on we consider functions $f : \mathbb{R}^+ \rightarrow \mathbf{X}$, and the so-called *moment kernels*, $M_n : \mathbb{R}^+ \rightarrow \mathbb{R}$, defined by

$$M_n(w) = w^n \cdot n \cdot 1_{[0,1]}(w), \quad n \in \mathbb{N}, w \in \mathbb{R}^+.$$

Let $\widehat{L}(\lambda)$ be the set defined by

$$\widehat{L}(\lambda) := \{f \in \mathbf{X}^{\mathbb{R}^+} : f \in L(\lambda) \text{ and } M_n\left(\frac{\cdot}{s}\right)f(\cdot) \in L(\mu) \forall s \in \mathbb{R}^+, n \in \mathbb{N}\} \quad (17)$$

where $\mu(B) := \int_B \frac{dt}{t}$ for every $B \in \mathcal{B}$. We consider the modular $\rho(\cdot) = \int_{\mathbb{R}^+} |\cdot| ds$.

In order to define and study the moment operators we need the following notions and results.

Definition 3.9. We say that $f : \mathbb{R}^+ \rightarrow \mathbf{X}$ is *uniformly continuous* iff there are a positive element $u \in \mathbf{X}$ and two (o) -sequences $(\sigma_p)_p$ and $(\delta_p)_p$ in \mathbb{R}^+ such that $|f(x_1) - f(x_2)| \leq \sigma_p u$ for all $x_1, x_2 \in \mathbb{R}^+$ and $p \in \mathbb{N}$ satisfying $|x_1 - x_2| < \delta_p$.

We remark here that any uniformly continuous mapping f is *locally bounded*, i.e. the set $\{|f(x)| : x \in (a, b)\}$ is bounded in \mathbf{X} , for any bounded interval $(a, b) \subset \mathbb{R}^+$.

Definition 3.10. We say that $f : \mathbb{R}^+ \rightarrow \mathbf{X}$ is *Lipschitz*, iff there is $K \in \mathbf{X}^+$ with $|f(x_1) - f(x_2)| \leq K|x_1 - x_2|$ for all $x_1, x_2 \in \mathbb{R}^+$. If f is Lipschitz, then f is uniformly continuous.

Definition 3.11. Let $I \subset \mathbb{R}$ be a connected set, and $f : I \rightarrow \mathbf{X}$. We say that f is *uniformly differentiable on I* iff there exist a bounded function $g : I \rightarrow \mathbf{X}$ and two (o) -sequences, $(\sigma_p)_p$ and $(\delta_p)_p$ in \mathbb{R}^+ , with

$$\bigvee \left\{ \left| \frac{f(v) - f(u)}{v - u} - g(x) \right| : u \leq x \leq v, 0 < v - u \leq \delta_p \right\} \leq \sigma_p e$$

for every $p \in \mathbb{N}$. In this case we say that g is the *uniform derivative* of f in I .

Note that in general the concept of uniform derivative is a strict strengthening of the classical one. However if $f \in C^1(I, \mathbb{R})$ in the classical sense, then f is uniformly differentiable and $g = f'$.

We see that uniform differentiability and uniform continuity in the vector lattice setting satisfy analogous properties as the corresponding classical ones.

Proposition 3.12. Let $f : [a, b] \rightarrow \mathbf{X}$, $g : [a, b] \rightarrow \mathbb{R}$. Then we have:

- (3.12.1) if f, g are uniformly continuous, then $f \cdot g$ is uniformly continuous too;
- (3.12.2) if f is uniformly differentiable, then f is uniformly continuous;
- (3.12.3) if f, g are uniformly differentiable, with uniform derivatives f', g' respectively, then $f \cdot g$ is uniformly differentiable too, and its uniform derivative is $f'g + g'f$;
- (3.12.4) if f is uniformly continuous, then f is bounded and belongs to $L(\lambda)$. Moreover its integral function F is uniformly differentiable in $[a, b]$, with uniform derivative f .

Proof: The proofs of statements (3.12.j) for $j = 1, 2, 3$ are straightforward. We prove now (3.12.4).

Boundedness has already been pointed out. So we prove just integrability. Choose $(\sigma_p)_p$ and $(\delta_p)_p$ in \mathbb{R}^+ according to uniform continuity of f . For each $p \in \mathbb{N}$, let $D_p := \{a = x_0 < x_1 < \dots <$

$x_{q_p} = b\}$ be a division of $[a, b]$, with mesh $\delta(D_p) \leq \delta_p$, and take D_{p+1} in such a way that D_{p+1} is a refinement of D_p . Now, set

$$f_p(t) := \sum_{i=0}^{q_p-1} \bigwedge_{t \in [x_i, x_{i+1}[} f(t) \cdot 1_{[x_i, x_{i+1}[}(t).$$

So, there exists a suitable positive real number K such that $|f_p(t)| \leq Ke$ for all $p \in \mathbb{N}$ and $t \in [a, b]$. For every $p \in \mathbb{N}$ and $t \in [a, b]$ we have $|f(t) - f_p(t)| \leq \sigma_p e$, and so the sequence $(f_p)_p$ converges uniformly, and then also in μ -measure to f . Since $[a, b]$ has finite measure, and

$$\int_B |f_p(t)| dt \leq \lambda(B) K e,$$

for every Borel set $B \subset [a, b]$ and every integer p , the sequence $(f_p)_p$ is equi-absolutely continuous. Moreover, since

$$\int_a^b |f_{p+r}(t) - f_p(t)| dt \leq (b-a) \sigma_p e$$

for all $p, r \in \mathbb{N}$, we can obtain a mapping $l : \Sigma \rightarrow \mathbf{X}$ as in Definition 3.4. Indeed, if for all $A \in \Sigma$ we set

$$l_1(A) := \liminf_p \int_A f_p(t) dt, \quad l_2(A) := \limsup_p \int_A f_p(t) dt,$$

then we easily get

$$l_2(A) - \int_A f_p(t) dt \leq \sup_r \int_A (f_{p+r}(t) - f_p(t)) dt \leq (b-a) \sigma_p e$$

and, for all $A \in \Sigma$ and $p \in \mathbb{N}$

$$\int_A f_p(t) dt - l_1(A) \leq \int_A f_p(t) dt - \inf_r \int_A f_{p+r}(t) dt \leq \sup_r \int_A (f_p(t) - f_{p+r}(t)) dt \leq (b-a) \sigma_p e.$$

This clearly implies that $l_1(A) = l_2(A) := l(A)$ for all $A \in \Sigma$, and moreover

$$\bigvee_{A \in \Sigma} |l(A) - \int_A f_p(t) dt| \leq (b-a) \sigma_p e$$

for all p , which proves formula (14) and hence integrability of f .

Finally, the proof of differentiability of $F(x) = \int_a^x f(t) dt$, $x \in [a, b]$ is straightforward. \square

We are now in position to introduce the moment operators.

For every $f \in \widehat{L}(\lambda)$ (see (17)) let us consider the operators $T_n f : \mathbb{R}^+ \rightarrow \mathbf{X}$ defined by

$$T_n f(s) = \int_0^{+\infty} M_n\left(\frac{t}{s}\right) f(t) \frac{dt}{t}, \quad n \in \mathbb{N}, s \in \mathbb{R}^+. \quad (18)$$

In what follows, we shall prove that the sequence $(T_n f - f)_n$ satisfies the hypotheses of Theorem 2.3, and therefore it is modularly $(o_{\mathcal{F}})$ -convergent to 0. The proof of this result and of the next Theorem 3.13 will be given in Subsection 4.

Theorem 3.13. Let f be a function with compact support $C \subset [a, b]$ with $a > 0$ and belonging to $\widehat{L}(\lambda)$. Then we have the following:

(3.13.1) if f is bounded, and $a > 1$, then the functions $T_n f$ are Lipschitz and integrable for every $n \geq 2$; moreover, if f is uniformly continuous, we have

$$\int_0^{+\infty} T_n f(s) ds = \frac{n}{n-1} \int_a^b f(s) ds; \quad (19)$$

(3.13.2) if f is uniformly continuous, then $(T_n f)_n$ converges uniformly to f ;

(3.13.3) if f is uniformly continuous and $a > 1$, then

$$(3.13.3.1) \quad (o) \lim_n \int_0^{+\infty} |T_n f(s) - f(s)| ds = 0;$$

(3.13.3.2) for every $n \in \mathbb{N}$, $T_n f$ is uniformly differentiable and its uniform derivative satisfies the following relation:

$$\frac{s}{n} (T_n f)'(s) + T_n f(s) = s (T_1 f)'(s) + T_1 f(s) = f(s)$$

Remark 3.14. Replacing n with a real positive constant v , we can see that, also in our general setting, the linear differential equation

$$s\varphi'(s) + v\varphi(s) = v f(s) \quad (20)$$

(with f uniformly continuous) can be solved by setting

$$\varphi(s) = \frac{1}{s^v} \left(c + v \int_0^s f(t) t^{v-1} dt \right),$$

where c is an arbitrary constant value.

Remark 3.15. A first application can be formulated in terms of weak convergence of the derivative mappings $(T_n f)'$. Let $f : [0, +\infty[\rightarrow \mathbb{R}$ be any uniformly continuous map, vanishing outside the interval $[a, T]$, with $a > 1$, and fix any C^1 mapping $w : [0, +\infty[\rightarrow \mathbb{R}$, with compact support. For each uniformly continuous mapping $g : [0, +\infty[\rightarrow \mathbb{R}$, define

$$\rho(g) := \int_0^{+\infty} |g(s)| |w'(s)| ds.$$

It is easy to see that ρ is a monotone and real-valued modular, and that $\lim_n \rho(T_n f - f) = 0$ thanks to (3.13.3.1). From this we deduce that

$$\lim_n \int_0^{+\infty} T_n f(s) w'(s) ds = \int_0^{+\infty} f(s) w'(s) ds,$$

and also, integrating by parts:

$$\lim_n \int_0^{+\infty} w(s) (T_n f)'(s) ds = \int_0^{+\infty} w(s) df(s)$$

(where the last integral is intended in the Riemann-Stieltjes sense). Thus we can conclude that, though the mappings $(T_n f)'$ do not converge in general, we always have

$$\lim_n \int_0^{+\infty} w(s) d(T_n f(s)) = \int_0^{+\infty} w(s) df(s),$$

for all C^1 mappings w , with compact support.

3.4. Brownian motion and stochastic processes

In order to obtain a more concrete application in Stochastic Integration, we shall assume that $\mathbf{B} := (B_t)_{0 \leq t < T}$ (with $T < +\infty$) is the standard Brownian Motion defined on a probability space (Ω, Σ, P) . Of course, we can consider \mathbf{B} as a mapping from $[0, T]$ into $L_2(\Omega)$: for this reason, and for uniformity of notations, from now on we shall replace the (more common) symbol B_t with $B(t)$ to denote the value of \mathbf{B} at t . Since L_2 has not a strong unit in general, in order to apply the previous theory we shall suitably restrict it: indeed, thanks to the well-known Maximum Principle (see e.g. [27]), we see that there is a suitable positive element $Z \in L_2$ such that $|B(t)| \leq Z$ for all $t \in [0, T]$. Moreover, we remark that, (see e.g. [38] and [34]), there exists a positive random variable W in L_2 such that

$$|B(t+h) - B(t)| \leq |h|^{1/4} W$$

holds, as soon as $t, t+h \in [0, T]$. Thus, taking \mathbf{X} as the (complete) subspace of L_2 generated by all elements dominated by some real multiple of $W + Z$, we see that \mathbf{X} has a strong unit (i.e. $W + Z$), that \mathbf{B} is \mathbf{X} -valued and is also a uniformly continuous mapping from $[0, T]$ to \mathbf{X} , in the sense of our definition.

In order to establish the next results, we introduce a definition.

Definition 3.16. Let $Y : [0, T] \rightarrow \mathbf{X}$ be any stochastic process, predictable with respect to \mathbf{B} , and with continuous trajectories. If we assume that $\sup_{t \in [0, T]} E(Y(t)^2) = K < +\infty$, then it is well-known that Y is integrable in the Itô's sense, with respect to \mathbf{B} (see e.g. [40]). Any process Y of this kind will be called a *regular* process. The Itô integral of Y will be denoted as usual with $(I) \int_0^T Y(t) dB(t)$.

Proposition 3.17. If the process Y is regular, and $\sup_{t \in [0, T]} E(Y(t)^2) = K$, then

$$\left\| (I) \int_0^T Y(t) dB(t) \right\|_2^2 \leq KT.$$

Proof: Let D be any decomposition of $[0, T]$ obtained by means of the points $t_0 = 0 < t_1 < \dots < t_n = T$, and consider the Itô sum:

$$S(Y, D) = \sum_{j=0}^{n-1} Y(t_j)(B(t_{j+1}) - B(t_j)).$$

We easily see that $E(S(Y, D)) = 0$, since the increment $B(t_{i+1}) - B(t_i)$ is independent of $Y(t_i)$ for all i . Similarly, we deduce that $\text{cov}[Y(t_i)(B(t_{i+1}) - B(t_i)), Y(t_j)(B(t_{j+1}) - B(t_j))] = 0$ as soon as $i \neq j$. So,

$$\begin{aligned} E(S(Y, D)^2) &= V(S(Y, D)) = \sum_{i=0}^{n-1} V(Y(t_i)(B(t_{i+1}) - B(t_i))) = \\ &= \sum_{i=0}^{n-1} E([Y(t_i)(B(t_{i+1}) - B(t_i))]^2) = \sum_{i=0}^{n-1} E(Y(t_i)^2)E[(B(t_{i+1}) - B(t_i))^2] \leq \\ &\leq K \sum_{i=0}^{n-1} (t_{i+1} - t_i) = KT. \end{aligned}$$

Since the sums $S(Y, D)$ are norm-convergent to the Itô integral $(I) \int_0^T Y(t)dB(t)$, the assertion is obvious. \square

The following corollary is an immediate consequence.

Corollary 3.18. *Let $(Y_n)_n$ be any sequence of regular stochastic processes. Assume that the processes Y_n converge to a regular process Y uniformly in L_2 , i.e. for every real $\varepsilon > 0$ an integer N exists, such that*

$$\sup_{t \in [0, T]} \|Y_n(t) - Y(t)\|_2 \leq \varepsilon$$

for all $n \geq N$. Then the Itô integrals $(I) \int_0^T Y_n(t)dB(t)$ converge in L_2 to the Itô integral $(I) \int_0^T Y(t)dB(t)$.

We shall apply these results to the process $f : [0, +\infty[\rightarrow \mathbf{X}$ defined as follows:

$$f(t) := \begin{cases} 0 & t \notin [a, T] \\ (t - T)(B(t) - B(a)), & t \in [a, T] \end{cases}$$

where a and T are fixed positive numbers, $1 < a < T$. (So f is a process similar to the well-known *Brownian Bridge*).

Theorem 3.19. *Let $f(t) = (t - T)(B(t) - B(a))$ be the process defined above. Then f is clearly \mathbf{X} -valued and uniformly continuous, and we have*

$$\lim_n \int_0^T T_n f(s)dB(s) = (I) \int_0^T f(s)dB(s).$$

(We remark that the integral on the left-hand side is in the Riemann-Stieltjes sense, since the mappings $T_n f$ are more regular than f).

Proof: It is clear that f is predictable; also $T_n f$ is for all n , since $T_n f(s) = n \int_0^s f(st)dt$, so the values of $T_n f(s)$ depend only on the values of $f(\tau)$ for $\tau \leq s$; then it is sufficient to apply 3.17 to the sequence $Y_n := T_n f - f$, which converges uniformly to 0. \square

We note that, in the previous results, (o) -convergence is not mentioned, since we don't know whether the integrals $\int_0^T T_n f(s) dB(s)$ are dominated by an element in L_2 . However, this problem can be solved by replacing the sequence $n \mapsto \int_0^T T_n f(s) dB(s)$ with the following one:

$$Y_n := \begin{cases} \left(\int_0^T T_n f(s) dB(s) \right) \wedge M, & \text{if } \int_0^T T_n f(s) dB(s) > 0 \\ \left(\int_0^T T_n f(s) dB(s) \right) \vee (-M), & \text{if } \int_0^T T_n f(s) dB(s) \leq 0, \end{cases}$$

where $M \in \mathbf{X}$ is any positive mapping larger than $\left| (I) \int_0^T f(s) dB(s) \right|$.

Indeed, $(Y_n)_n$ is obviously dominated, and we always have

$$\left| Y_n - (I) \int_0^T f(s) dB(s) \right| \leq \left| \int_0^T T_n f(s) dB(s) - (I) \int_0^T f(s) dB(s) \right|,$$

which proves L_2 -convergence of $(Y_n)_n$ to the Itô integral of f with respect to \mathbf{B} . \square

The following pictures show a comparison among some trajectories of the mapping f and the corresponding ones of the mappings $T_n f$.

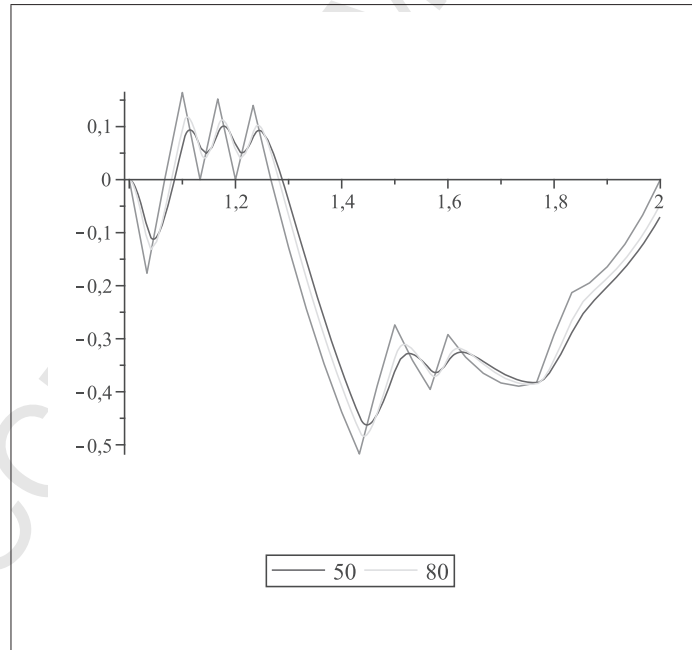


Figure 1: f (gray), $T_{50}f$ (black), $T_{80}f$ (light gray)

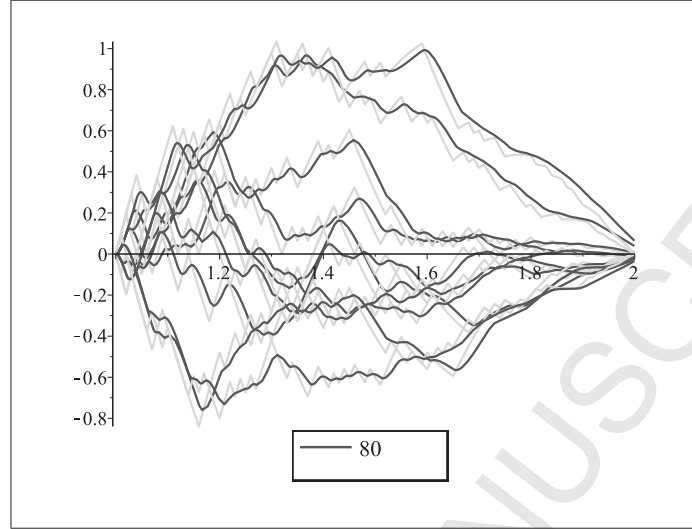


Figure 2: f (ten trajectories in gray) and $T_{80}f$ (black)

4. Proof of Theorem 3.13

Let U_f denote a fixed real positive number such that $\bigvee_{t \in [a,b]} |f(t)| \leq U_f e$ and set $U = n b^n U_f$.

Proof of (3.13.1) We first show that $T_n f$ is Lipschitz and then we will prove integrability of $T_n f$ on \mathbb{R}^+ . For each $s \in \mathbb{R}^+$ we can write:

$$T_n f(s) = n \int_0^{+\infty} \left(\frac{t}{s}\right)^n 1_{(0,1)}\left(\frac{t}{s}\right) \frac{f(t)}{t} dt = \frac{n}{s^n} \int_0^s t^{n-1} f(t) dt. \quad (21)$$

We now estimate the quantity $|T_n f(s_1) - T_n f(s_2)|$. Let $s_1, s_2 \in \mathbb{R}^+$, with $s_1 < s_2$. If $s_2 \leq a$, then $T_n f(s_1) - T_n f(s_2) = 0$.

If $s_2 > a$, then without loss of generality we can suppose $s_1 \geq a$. We get:

$$\begin{aligned} T_n f(s_1) - T_n f(s_2) &= \frac{n}{s_1^n} \int_0^{s_1} t^{n-1} f(t) dt - \frac{n}{s_2^n} \int_0^{s_1} t^{n-1} f(t) dt - \frac{n}{s_2^n} \int_{s_1}^{s_2} t^{n-1} f(t) dt = \\ &= n \left[\left(\frac{1}{s_1^n} - \frac{1}{s_2^n} \right) \int_0^{s_1} t^{n-1} f(t) dt - \frac{1}{s_2^n} \int_{s_1}^{s_2} t^{n-1} f(t) dt \right]. \end{aligned}$$

Then

$$|T_n f(s_1) - T_n f(s_2)| \leq U \left(\frac{1}{s_1^n} - \frac{1}{s_2^n} \right) e + U(s_2 - s_1)e \leq U(s_2 - s_1) \left(1 + \frac{n}{\tau^{n+1}} \right) e$$

for a suitable τ with $1 < a \leq s_1 < \tau < s_2$, thanks to the Lagrange Theorem applied to the real function $s \mapsto s^{-n}$. So, we see that $|T_n f(s_1) - T_n f(s_2)| \leq U(1+n)(s_2 - s_1)e$, thus proving that $T_n f$ is Lipschitz.

We now prove integrability. Let $n \geq 2$ be fixed. First of all observe that for every $s > b$ we have

$T_n f(s) = \frac{n}{s^n} K_n$, where $K_n \in \mathbf{X}$ is the integral of $t \mapsto t^{n-1} f(t)$ in $[0, b]$. Since $n \geq 2$, the real-valued mapping $s \mapsto \frac{n}{s^n}$ is clearly integrable in $[b, +\infty[$, and therefore it is easy to see that $T_n f$ is, in the same halflife, thanks to Remark 3.7. Moreover, since $T_n f$ is uniformly continuous, it is also integrable in $[0, b]$, in view of (3.12.4). Therefore we get integrability on the whole of $[0, +\infty[$ (see Remark 3.6). Let us now turn to prove (19), in the case that f is uniformly continuous. In order to compute the integral of $T_n f$, $n \geq 2$, we first point out that

$$\int_b^{+\infty} T_n f(s) ds = K_n \int_b^{+\infty} \frac{n}{s^n} ds = \frac{n}{(n-1)b^{n-1}} K_n, \quad (22)$$

where K_n is the constant above. Now we evaluate the integral in $[a, b]$. To this aim, we can observe that, by uniform continuity of f and of the mapping $t \mapsto nt^{n-1}f(t)$ in $[a, b]$, there exist an (o) -sequence $(\sigma_p)_p$ in \mathbb{R}^+ and a strictly increasing sequence $(H_p)_p$ in \mathbb{N} , such that

$$|f(t) - f(u)| + |nt^{n-1}f(t) - nu^{n-1}f(u)| < \sigma_p e$$

for all $p \in \mathbb{N}$ and $t, u \in [a, b]$ satisfying $|t - u| \leq (b-a)/H_p$.

Now, fix $p \in \mathbb{N}$ and let $a := s_0 < s_1 < s_2 < \dots < s_{H_p} := b$ be the division of $[a, b]$ obtained with precisely H_p subintervals of the same length, i.e. $(b-a)/H_p$. By additivity of the integral we have

$$\int_a^b T_n f(s) ds = \sum_{j=0}^{H_p-1} \int_{s_j}^{s_{j+1}} T_n f(s) ds = \sum_{j=0}^{H_p-1} \int_{s_j}^{s_{j+1}} \frac{1}{s^n} \left[\int_a^s nt^{n-1}f(t) dt \right] ds.$$

For each fixed $s > a$ let us denote by $j(s)$ the maximum index such that $s_{j(s)} < s$, and set also $j(s_0) = j(a) = 0$. We have then:

$$\begin{aligned} & \left| \int_a^s nt^{n-1}f(t) dt - \sum_{j=0}^{j(s)} ns_j^{n-1}f(s_j) \frac{b-a}{H_p} \right| \leq \int_{s_{j(s)}}^s nt^{n-1}|f(t)| dt + ns_{j(s)}^{n-1}|f(s_{j(s)})| \frac{b-a}{H_p} + \\ & + \sum_{j=0}^{j(s)-1} \int_{s_j}^{s_{j+1}} |nt^{n-1}f(t) - ns_j^{n-1}f(s_j)| dt \leq 2nb^{n-1}U_f \frac{b-a}{H_p} + \sum_{j=0}^{j(s)-1} \sigma_p (s_{j+1} - s_j) e = \\ & = \left(\frac{2nb^n U_f}{H_p} + s_{j(s)} \sigma_p \right) e \leq \left(\frac{2nb^n U_f}{H_p} + b \sigma_p \right) e. \end{aligned}$$

Thus we see that

$$\begin{aligned} & \left| \int_a^b T_n f(s) ds - \sum_{j=0}^{H_p-1} \int_{s_j}^{s_{j+1}} \frac{1}{s^n} \sum_{j=0}^{j(s)} ns_j^{n-1}f(s_j) \frac{b-a}{H_p} ds \right| = \\ & = \left| \sum_{j=0}^{H_p-1} \int_{s_j}^{s_{j+1}} \frac{1}{s^n} \left[\int_a^s nt^{n-1}f(t) dt - \sum_{j=0}^{j(s)} ns_j^{n-1}f(s_j) \frac{b-a}{H_p} \right] ds \right| \leq \\ & \leq \sum_{j=0}^{H_p-1} \int_{s_j}^{s_{j+1}} \frac{1}{s^n} \left(\frac{2nb^n U_f}{H_p} + b \sigma_p \right) e ds = \left(\frac{2nb^n U_f}{H_p} + b \sigma_p \right) e \int_a^b \frac{1}{s^n} ds = \\ & = \frac{b^{n-1} - a^{n-1}}{n-1} \left(\frac{2nb^n U_f}{H_p} + b \sigma_p \right) e. \end{aligned}$$

Letting $\lambda_p := \frac{b^{n-1} - a^{n-1}}{n-1} \left(\frac{2nb^n U_f}{H_p} + b\sigma_p \right)$, we can see that $(\lambda_p)_p$ is an (o) -sequence in \mathbb{R} , and summarize the previous result in the following way:

$$\left| \int_a^b T_n f(s) ds - \sum_{j=0}^{H_p-1} \int_{s_j}^{s_{j+1}} \frac{1}{s^n} \sum_{j=0}^{j(s)} n s_j^{n-1} f(s_j) \frac{b-a}{H_p} ds \right| \leq \lambda_p e. \quad (23)$$

Now, by changing order in the summations, we get

$$\begin{aligned} & \sum_{j=0}^{H_p-1} \int_{s_j}^{s_{j+1}} \frac{1}{s^n} \sum_{j=0}^{j(s)} n s_j^{n-1} f(s_j) \frac{b-a}{H_p} ds = \frac{n}{n-1} \frac{b-a}{H_p} \left(\int_{s_0}^{s_1} (n-1) s^{-n} s_0^{n-1} f(s_0) ds + \right. \\ & + \left. \int_{s_1}^{s_2} [(n-1) s^{-n} s_0^{n-1} f(s_0) + (n-1) s^{-n} s_1^{n-1} f(s_1)] ds + \dots \right) = \\ & = \frac{n}{n-1} \frac{b-a}{H_p} \left(\int_{s_0}^b (n-1) s^{-n} s_0^{n-1} f(s_0) ds + \int_{s_1}^b (n-1) s^{-n} s_1^{n-1} f(s_1) + \dots + \right. \\ & + \left. \int_{s_{H_p-1}}^b (n-1) s^{-n} s_{H_p-1}^{n-1} f(s_{H_p-1}) ds \right) = \frac{n}{n-1} \frac{b-a}{H_p} \left(\left[f(s_0) - \frac{s_0^{n-1} f(s_0)}{b^{n-1}} \right] + \right. \\ & + \left[f(s_1) - \frac{s_1^{n-1} f(s_1)}{b^{n-1}} \right] + \dots + \left[f(s_{H_p-1}) - \frac{s_{H_p-1}^{n-1} f(s_{H_p-1})}{b^{n-1}} \right] \Bigg) = \\ & = \frac{n}{n-1} \sum_{j=0}^{H_p-1} f(s_j) (s_{j+1} - s_j) - \frac{n}{n-1} \sum_{j=0}^{H_p-1} \frac{s_j^{n-1} f(s_j)}{b^{n-1}} (s_{j+1} - s_j). \end{aligned}$$

Now, since

$$\frac{n}{n-1} \left(\sum_{j=0}^{H_p-1} f(s_j) (s_{j+1} - s_j) - \int_a^b f(s) ds \right) \leq \frac{n}{n-1} \sum_{j=0}^{H_p-1} \sigma_p (s_{j+1} - s_j) e = \frac{n}{n-1} (b-a) \sigma_p e,$$

and

$$\begin{aligned} & \left| \frac{n}{n-1} \sum_{j=0}^{H_p-1} \frac{s_j^{n-1} f(s_j)}{b^{n-1}} (s_{j+1} - s_j) - \frac{n}{(n-1)b^{n-1}} \int_a^b s^{n-1} f(s) ds \right| \leq \\ & \leq \frac{n}{(n-1)b^{n-1}} \sum_{j=0}^{H_p-1} \sigma_p (s_{j+1} - s_j) e \leq \frac{n}{(n-1)b^{n-2}} \sigma_p e, \end{aligned}$$

we obtain

$$\left| \sum_{j=0}^{H_p-1} \int_{s_j}^{s_{j+1}} \frac{1}{s^n} \sum_{j=0}^{j(s)} n s_j^{n-1} f(s_j) \frac{b-a}{H_p} ds - \frac{n}{n-1} \left(\int_a^b f(s) ds - \frac{1}{b^{n-1}} \int_a^b s^{n-1} f(s) ds \right) \right| \leq h \sigma_p e, \quad (24)$$

where $h = 2(b - a) + 2$. In conclusion, from (22), (23) and (24) we deduce that

$$\begin{aligned} & \left| \int_a^{+\infty} T_n f(s) ds - \frac{n}{n-1} \int_a^b f(s) ds \right| = \\ & = \left| \int_a^b T_n f(s) ds - \frac{n}{n-1} \int_a^b f(s) ds + \frac{n}{(n-1)b^{n-1}} K_n \right| \leq \lambda_p e + h \sigma_p e \end{aligned}$$

holds, for all $p \in \mathbb{N}$. Letting p tend to $+\infty$, we obtain finally

$$\left| \int_a^{+\infty} T_n f(s) ds - \frac{n}{n-1} \int_a^b f(s) ds \right| = 0. \quad \square$$

Proof of (3.13.2) Let f be as in the hypothesis. For every $n \in \mathbb{N}$ and $s \in \mathbb{R}^+$, we have

$$|T_n f(s) - f(s)| = \left| \frac{n}{s^n} \int_0^s t^{n-1} f(t) dt - \frac{n}{s^n} \int_0^s t^{n-1} f(s) dt \right| \leq \frac{n}{s^n} \int_0^s t^{n-1} |f(t) - f(s)| dt.$$

Since f is uniformly continuous, there exist two (o) -sequences $(\sigma_p)_p$ and $(\gamma_p)_p$ in $]0, 1[$ such that $|f(u) - f(v)| \leq \sigma_p e$ as soon as $|u - v| \leq \gamma_p$ holds, for all positive integers p . Now, for every $p \in \mathbb{N}$ we get

$$\begin{aligned} |T_n f(s) - f(s)| & \leq \frac{n}{s^n} \int_0^{s-\gamma_p} t^{n-1} |f(t) - f(s)| dt + \frac{n}{s^n} \int_{s-\gamma_p}^s t^{n-1} |f(t) - f(s)| dt \leq \\ & \leq \frac{2U_f}{s^n} (s - \gamma_p)^n e + \frac{n}{s^n} \sigma_p e \int_0^s t^{n-1} dt = 2U_f \left(1 - \frac{\gamma_p}{s}\right)^n e + \sigma_p e. \end{aligned}$$

Now, for all $s \geq 2b$ we see that

$$|T_n f(s) - f(s)| = |T_n f(s)| = \left| \frac{n}{s^n} \int_0^b t^{n-1} f(t) dt \right| \leq U_f \left(\frac{b}{s}\right)^n e \leq \frac{U_f}{2^n} e,$$

while for $s \leq 2b$ we get

$$|T_n f(s) - f(s)| \leq 2U_f \left(1 - \frac{\gamma_p}{2b}\right)^n e + \sigma_p e;$$

in any case, we find:

$$\bigvee_{s>0} |T_n f(s) - f(s)| \leq 2U_f \left(1 - \frac{\gamma_p}{2b}\right)^n e + \sigma_p e + \frac{U_f}{2^n} e,$$

hence

$$\limsup_n \bigvee_{s>0} |T_n f(s) - f(s)| \leq \sigma_p e$$

holds, for all p : this obviously implies that $\limsup_n \bigvee_{s>0} |T_n f(s) - f(s)| = 0$. \square

Proof of (3.13.3)

(3.13.3.1) By (3.13.1) $T_n f \in L(\lambda)$ for every $n \in \mathbb{N}$, by (3.13.2) $(T_n f - f)_n$ μ -converges to zero and so it is enough to prove equi-absolute continuity of $(T_n f - f)_n$ with respect to the modular $\rho(\cdot) = \int_{\mathbb{R}^+} |\cdot| ds$, in order to apply Theorem 2.3. From uniform convergence of $T_n f$ to f we see that, setting $\zeta_n := \sup_{p \geq n} \bigvee_{s \in \mathbb{R}^+} |T_p f(s) - f(s)|$, $(\zeta_n)_n$ is an (o) -sequence. Then

$$\rho([T_n f - f] 1_B) = \int_B |T_n f(s) - f(s)| ds \leq \zeta_n \lambda(B)$$

for every n and whenever $B \in \Sigma$. This clearly implies $ac_\rho(1)$. We now turn to $(ac_\rho(2))$. For each $m \in \mathbb{N}$, let $B_m = [0, 2b]$. As we already pointed out, when $s > 2b$ it holds: $|T_n f(s)| \leq U_f (b/s)^n e$. So, for $m, n \in \mathbb{N}$ we have

$$\rho([T_n f - f] 1_{\mathbb{R}^+ \setminus B_m}) \leq U_f b^n e \int_{2b}^{+\infty} \frac{1}{s^n} ds \leq U_f b^n \frac{1}{(2b)^{n-1}} e = \frac{U_f b}{2^{n-1}} e.$$

This clearly means that $(r) \lim_n \rho([T_n f - f] 1_{\mathbb{R}^+ \setminus B_m}) = 0$ uniformly with respect to m , and hence $(ac_\rho(2))$. So, all hypotheses of the Vitali theorem are satisfied and the assertion follows from Theorem 2.3. \square

(3.13.3.2) We shall use the following expression for $T_n f$:

$$T_n f(s) = \frac{n}{s^n} \int_0^s t^{n-1} f(t) dt. \quad (25)$$

We first observe that the integrand $(\cdot)^{n-1} f(\cdot) : [0, s] \rightarrow \mathbf{X}$ satisfies (3.12.1) and hence its integral function is uniformly differentiable by (3.12.4). So $T_n f$ is the product of two functions which satisfy (3.12.3), i.e. $n s^{-n}$ and $s \mapsto \int_0^s t^{n-1} f(t) dt$. By Proposition 3.12 we obtain:

$$(T_n f)'(s) = -\frac{n^2}{s^{n+1}} \int_0^s t^{n-1} f(t) dt + \frac{n}{s} f(s) = \frac{n}{s} (f(s) - T_n(f)(s)). \quad (26)$$

Formula (26) shows that f is uniquely determined by each of the functions $T_n f$'s, in particular

$$f(s) = s(T_1 f)'(s) + T_1 f(s)$$

holds for all s , and therefore all functions $T_n f$'s can be easily obtained from one another. \square

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