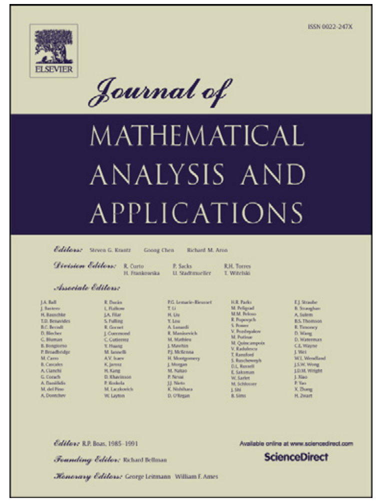
Accepted Manuscript



Invasion of a inferior or superior competitor: A diffusive competition model with a free boundary in heterogeneous environment

Jie Wang, Liang Zhang

PII: S0022-247X(14)00888-9
DOI: 10.1016/j.jmaa.2014.09.055
Reference: YJMAA 18871

To appear in: *Journal of Mathematical Analysis and Applications*

Received date: 1 July 2014

Please cite this article in press as: J. Wang, L. Zhang, Invasion of a inferior or superior competitor: A diffusive competition model with a free boundary in heterogeneous environment, *J. Math. Anal. Appl.* (2015), http://dx.doi.org/10.1016/j.jmaa.2014.09.055

Invasion of a inferior or superior competitor: a diffusive competition model with a free boundary in heterogeneous environment

Jie Wang[a*] and Liang Zhang[a]
[a]School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu 730000, People's Republic of China

September 23, 2014

Abstract

In this paper, we consider the population dynamics of an invasive species and a resident species modeled by a diffusive competition model in a radially symmetric setting with free boundary. We assume that the resident species undergoes diffusion and growth in $\mathbb{R}^n$, and the invasive species exists initially in a finite ball, but invades into the environment with spreading front evolving according to a free boundary. In the case that the invasive species is inferior, we show that if the resident species is already rather established at beginning, then the invader can never invades deep into the underlining habitat, and it dies out before its invading front reaches a certain finite limiting position. While if the invasive species is superior, a spreading-vanishing dichotomy holds, and the sharp criteria for the spreading and vanishing with d_1, μ and u_0 as variable factors is obtained, where d_1, μ and u_0 are dispersal rate, expansion capacity and initial number of the invader, respectively. Specially, we still give some rough estimates of the asymptotic spreading speed when spreading occurs.
Keywords: Diffusive competition model; Invasive population; Selection for dispersal; Free boundary; Spreading-vanishing dichotomy

AMS Subject Classification (1991): 35K20; 35R35; 35J60; 92B05

1 Introduction

There are a variety of models which are used to describe the competition and co-existence dynamics arising in population ecology. Among these, the following Lotka-Volterra competition reaction-diffusion system for two species in a bounded smooth domain $\Omega \subset \mathbb{R}^n$ is

*Corresponding author.

the typical one:

$$\begin{cases} u_t - d_1\Delta u = [a_1(x) - b_1(x)u - c_1(x)v]u, & x \in \Omega, t > 0, \\ v_t - d_2\Delta v = [a_2(x) - b_2(x)u - c_2(x)v]v, & x \in \Omega, t > 0, \\ u(x,t) = v(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) \geq 0,\ v(x,0) = v_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{1.1}$$

where u, v denote the population densities of two competing species, d_1, d_2 are positive and represent the dispersal rates of two species, strictly positive functions $a_i(x)$, $b_i(x)$, $c_i(x)(i = 1,2) \in C^1(\Omega) \cap L^\infty(\overline{\Omega})$ account for the local growth rate of the population or the density of the local resource, self-regulation of species and competition between species, respectively.

In general, the long time dynamics is among the central problems investigated for (1.1) and is quite well understood. The reader is referred to see [4, 5, 14] etc., and references therein for this aspect. Roughly speaking, people find that weak competition allows for coexistence states in (1.1), while stronger competition leads to the extinction of the species with low reproduction rate and large diffusion rate. More precisely, let λ^* be the principal eigenvalue of the operator $-\Delta$ in Ω subject to the homogeneous Dirichlet boundary conditions. Set

$$a_{iL} = \inf_{x\in\overline{\Omega}} a_i(x), \quad a_{iM} = \sup_{x\in\overline{\Omega}} a_i(x)$$

for $i = 1,2$, and b_{iL}, b_{iM}, c_{iL} and $c_{iM}(i = 1,2)$ are defined analogously. Then the following results have been proved for (1.1):

1. if $a_{1L} > d_1\lambda^* + \frac{a_{2M}c_{1M}}{c_{2L}}$ and $a_{2L} > d_2\lambda^* + \frac{a_{1M}b_{2M}}{b_{1L}}$, then there exists a coexistence state for (1.1), that is, a stationary solution (u^*, v^*) with $u^*, v^* > 0$ in Ω;
2. if $\frac{a_{1M}}{a_{2L}} < \min\left\{\frac{c_{1L}}{c_{2M}}, \frac{b_{1L}}{b_{2M}}\right\}$, $a_{2L} \geq a_{1M}$, $d_1 = d_2 = D$ and $a_{2L} > D\lambda^*$, then the species u is eventually driven to extinction, that is, $\lim_{t\to\infty} u(x,t) = 0$ for any $v_0 \not\equiv 0$;
3. if $\frac{a_{1M}}{a_{2L}} > \max\left\{\frac{c_{1L}}{c_{2M}}, \frac{b_{1L}}{b_{2M}}\right\}$, $a_{1L} \geq a_{2M}$, $d_1 = d_2 = D$ and $a_{1L} > D\lambda^*$, then the species v is eventually driven to extinction, that is, $\lim_{t\to\infty} v(x,t) = 0$ for any $u_0 \not\equiv 0$.

Ecologically, in the second case, the competitor u is wiped out by v in the long run and v will win the competition, so we call v the superior competitor, while u the inferior competitor. Analogously, u is the inferior competitor and v is the superior competitor in the third case. The first case is often regarded as the weak competition case, where no competitor wins or loses in the competition.

On the other hand, however, we still note that the model (1.1) is not a realistic model to describe the dynamics of a new competitive invasive species invading into the habitat of a resident species, due to the limited fixed domain and the little information about the precise invading dynamics. Thus, it is necessary to relax these requirement and to consider the precise dynamics by which an invading species spreads spatially into new habitat.

Inspired by the above aim, the current paper is concerned with the impact of spatial feature of environment on dynamics of a new competitor u with a free boundary describing the moving front invading into the habitat of a resident species v. For simplicity, we assume the environment is radially symmetric and investigate the behavior of the positive solution $(u(r,t), v(r,t), h(t))$ with $r := |x| (x \in \mathbb{R}^n)$ to the following variation of the reaction-diffusion problem (1.1)

$$\begin{cases} u_t - d_1 \Delta u = [a_1(r) - b_1(r)u - c_1(r)v]u, & 0 < r < h(t), t > 0, \\ v_t - d_2 \Delta v = [a_2(r) - b_2(r)u - c_2(r)v]v, & 0 < r < \infty, t > 0, \\ u_r(0,t) = v_r(0,t) = 0, u(r,t) = 0, & h(t) \le r < \infty, t > 0, \\ h'(t) = -\mu u_r(h(t), t), & t > 0, \\ u(r,0) = u_0(r), \quad h(0) = h_0, & 0 \le r \le h_0, \\ v(r,0) = v_0(r), & 0 \le r < \infty, \end{cases} \tag{1.2}$$

where $\Delta u = u_{rr} + \frac{n-1}{r} u_r$, $r = h(t)$ denotes the spreading front, that is, the free boundary to be determined, $d_1, d_2 > 0$ are diffusion rates, $\mu > 0$ called the expansion capacity, is the ratio of the expansion speed of the free boundary and the population gradient at the expanding front and accounts for the ability of the invasive species to transmit and dispersal in the new habitat, $h_0 >$ is the initial boundary or survival range, and $a_i(r)$, $b_i(r)$, $c_i(r)(i = 1, 2) \in C^1([0,\infty)) \cap L^\infty([0,\infty))$ are the strictly positive functions as above. We still denote the followings in like manner:

$$a_{iL} = \inf_{r \in [0,\infty)} a_i(r), \quad a_{iM} = \sup_{r \in [0,\infty)} a_i(r)$$

for $i = 1, 2$, and b_{iL}, b_{iM}, c_{iL} and $c_{iM}(i = 1, 2)$ are defined analogously. Throughout this paper, we always assume that these constant are well defined and positive. The initial functions u_0 and v_0 satisfy

$$\begin{cases} u_0 \in C^2([0, h_0]), u_0'(0) = u_0(h_0) = 0, \text{ and } u_0 > 0 \text{ in } [0, h_0); \\ v_0 \in C^2([0,\infty)) \cap L^\infty([0,\infty)), v_0'(0) = 0, \text{ and } v_0 \ge 0 \text{ in } [0,\infty), \text{ and } v_0 \not\equiv 0. \end{cases} \tag{1.3}$$

When $a_i(r)$, $b_i(r)$, $c_i(r)(i = 1, 2) \in C^1([0,\infty)) \cap L^\infty([0,\infty))$ are just positive constant, that is, the environment is assumed to be spatially homogeneous, Du and Lin [9] proved that: in the case that the invasive species is inferior, if the resident species is already rather established at beginning, then the invader can never invades deep into the underlining habitat, and it dies out before its invading front reaches a certain finite limiting position; while if the invasive species is superior, there exists a spreading-vanishing dichotomy for the invasive species, which means that, as time $t \to \infty$, either $h \to \infty$, $u \to \frac{a_1}{b_1}$ and $v \to 0$, namely the invasive species successfully establishes itself in the underlining habitat (called spreading), or $h \to h_\infty \le \infty$, $u \to 0$ and $v \to \frac{a_2}{c_2}$, namely the invasive species fails to establish and vanishes eventually (called vanishing). Moreover, some sharp criteria for the spreading and vanishing presented by μ and h_0, and rough estimates of the asymptotic spreading speed (a concept comes from Aronson and Weinberger [1, 2]) when spreading occurs were also obtained in Du and Lin [9]. We remark that the similar Lotka-Volterra type model with free boundary was first introduced by Lin [15], in which a prey-predator model was studied and only the existence results were given. Some other works on prey-predator model with free boundary can be referred to [20, 22, 24] and references cited therein. Moreover, there were still some studies caring about Lotka-Volterra competition model. See Guo and Wu [13] for weak competition case and Wang and Zhao [23] for some extensions.

Furthermore, if in the problem (1.2), the resident species is absent, namely $v \equiv 0$, we then obtain the following diffusive logistic problem

$$\begin{cases} u_t - d_1\Delta u = [a_1(r) - b_1(r)u]u, & 0 < r < h(t),\ t > 0, \\ u_r(0,t) = u(h(t),t) = 0, & t > 0, \\ h'(t) = -\mu u_r(h(t),t), & t > 0, \\ u(r,0) = u_0(r) > 0,\ h(0) = h_0 > 0, & 0 \le r \le h_0, \end{cases} \tag{1.4}$$

which has been well treated in [8] and [6]. Du and Guo [6] showed that a similar spreading-vanishing dichotomy as above holds. Some sharp criteria for spreading and vanishing by μ and h_0 were presented as well. Specially, Du and Guo [6] further proved that if the spreading occurs, then the spreading front $h(t)$ behaves precisely like $[k_0 + o(1)]t$ for large time, where k_0 is an estimate of the asymptotic spreading speed. In fact, since the work of Du and Lin [8] and Du and Guo [6], there have been many theoretical advances on the free boundary problem in homogeneous or heterogeneous environment. In [11], a two free boundaries problem of single equation with a general reaction term was considered, where the Dirichlet conditions are assumed at free boundaries. Peng and Zhao [18] studied a diffusive logistic model with free boundary and seasonal succession. Later on, Du et al. [7] was concerned with a diffusive logistic equation in time-periodic environment, which was a development of Du and Guo [6] and Peng and Zhao [18].

Since it have been intensively revealed by more and more empirical facts that the real environment is usually heterogeneous, one must consider the more general case like (1.1) and (1.2) to appropriate to the empirical and theoretical requirements. In this paper, we mainly study the dynamics of problem (1.2) in the general heterogeneous environment. For the case that the invasive species is inferior, we will show that it will still dies out before its invading front reaches a certain finite limiting position, provided its competitor is already rather established at beginning. Nextly, for the case that the invasive species is superior, we give some sufficient conditions to ensure that the spreading and vanishing happen, which result in the spreading-vanishing dichotomy. Furthermore, since dispersal is an important, perhaps most, aspect of the life histories of many species, and can influence the persistence of species and mediates interactions between species, we choose the dispersal rete d instead of h_0 used in [9] as the varying parameter to show that there exists a positive threshold D^* such that if $0 < d_u \le D^*$, then the persistence for invasive species always happens whatever the the initial function u_0 or the expansion capacity μ is; while if $d_u > D^*$, there still exists a critical $\mu^* \ge 0$ such that vanishing happens if $0 < \mu \le \mu^*$, and spreading happens if $\mu > \mu^*$. Hence, sharp criteria for the invasive species spreading and vanishing are obtained for problem (1.2). A similar result related to the initial function u_0 is also presented.

The rest parts of this paper are organized as follows. Section 2 is devoted to some fundamental results, including the global existence and uniqueness of the solution of (1.2). Moreover, some rough a priori estimates are given, as well as the comparison principle in the moving domain. In Section 3, we study an eigenvalue problem under some general assumptions, whose principal eigenvalue will play important roles in our later analysis. Section 4 treats the case that the invasive species is inferior. Section 5 is all devoted to the case that the invasive species is superior, where a spreading-vanishing dichotomy is established, and a sharp criterion to distinguish the dichotomy and some rough estimates of the asymptotic spreading speed when spreading occurs are also given.

2 Preliminaries results

In this section, we prove the existence and uniqueness of the solution to (1.2) for all $t > 0$. The proof of the local existence and uniqueness can be done similarly as in [6, 9] by some minor modifications the arguments there. So we omit the details.

Theorem 2.1 *For any given (u_0, v_0) satisfying (1.3) and any $\alpha \in (0, 1)$, there is a constant $T > 0$ such that problem (1.2) admits a unique solution which satisfies*

$$(u, v, h) \in C^{1+\alpha, \frac{1+\alpha}{2}}(D_T) \times C^{1+\alpha, \frac{1+\alpha}{2}}(D_T^\infty) \times C^{1+\frac{\alpha}{2}}([0, T]);$$

moreover,

$$\|u\|_{C^{1+\alpha, \frac{1+\alpha}{2}}(D_T)} + \|v\|_{C^{1+\alpha, \frac{1+\alpha}{2}}(D_T^\infty)} + \|h\|_{C^{1+\frac{\alpha}{2}}([0,T])} < C,$$

where

$$D_T = \{(r,t) \in \mathbb{R}^2 : r \in [0, h(t)], t \in [0,T]\},$$

$$D_T^\infty = \{(r,t) \in \mathbb{R}^2 : r \in [0, \infty), t \in [0,T]\},$$

and the constants T and C only depend on h_0, α, $\|u_0\|_{C^2([0,h_0])}$ and $\|v_0\|_{C^2([0,\infty))}$.

To prove the global existence of the solution of (1.2) obtained in 2.1, we need the following estimates.

Lemma 2.2 *Let (u,v,h) be a solution to problem (1.2) defined for $t \in (0,T)$ for some $T \in (0,\infty]$. Then there exist constants M_1, M_2 and M_3 independent of T such that*

$$\begin{aligned}
&0 < u(r,t) \le M_1, \ \ for \ 0 < r < h(t), 0 < t \le T,\\
&0 < v(r,t) \le M_2, \ \ for \ 0 < r < \infty, 0 < t \le T,\\
&0 < h'(t) \le M_3, \quad for \ 0 < t \le T.
\end{aligned}$$

Proof. It follows from the strong maximum principle that $u > 0$ in $[0, h(t)) \times (0,T]$ and $v > 0$ in $[0,\infty) \times (0,T]$. Since $u(r,t)$ satisfies

$$\begin{cases}
u_t - d_1 \Delta u = [a_1(r) - b_1(r)u - c_1(r)v]u, & 0 < r < h(t), 0 < t < T,\\
u_r(0,t) = u(h(t),t) = 0, & 0 < t < T,\\
u(r,0) = u_0(r), & 0 \le r < h_0,
\end{cases}$$

we obtain that $u(r,t) \le \max\{\|u_0\|_{L^\infty([0,h_0])}, \frac{a_{1M}}{b_{1L}}\} \triangleq M_1$. Similarly, since $v(r,t)$ satisfies

$$\begin{cases}
v_t - d_2 \Delta v = [a_2(r) - b_2(r)u - c_2(r)v]v, & 0 < r < \infty, 0 < t < T,\\
v_r(0,t) = 0, & 0 < t < T,\\
v(r,0) = v_0(r), & 0 \le r < \infty,
\end{cases}$$

we obtain that $v(r,t) \le \max\{\|v_0\|_{L^\infty([0,\infty))}, \frac{a_{2M}}{c_{2M}}\} \triangleq M_2$. Moreover, the strong maximum principle also yields the inequalities $u_r(t, h(t)) < 0$, which implies that $h'(t) > 0$ in $(0,T]$.

It remains to verify that $h'(t)$ is bounded from above. To this end, we define

$$\Omega_M \triangleq \{(r,t) : h(t) - \frac{1}{M} < r < h(t), \ 0 < t < T\}$$

and construct the following auxiliary function

$$\overline{u} = M_1[2M(h(t) - r) - M^2(h(t) - r)^2]$$

in Ω_M. In what follows, we will choose some $M(> \frac{1}{h_0})$ such that $u \le \overline{u}$ in Ω_M.

It follows from some direct calculations that

$$\begin{cases} \overline{u}_t - d_1\Delta\overline{u} \geq 2d_1M_1M^2 \geq [a_1(r) - b_1(r)u - c_1(r)v]u, \\ \overline{u}(h(t) - \frac{1}{M}, t) = M_1 \geq u(h(t) - \frac{1}{M}, t), \\ \overline{u}(h(t), t) = u(h(t), t), \end{cases}$$

if $M \geq \sqrt{\frac{a_{1M}}{2d_1}}$. On the other hand, we have

$$u_0(r) = u_0(r) - u_0(h_0) = -\int_r^{h_0} u_0'(s)\,\mathrm{d}s \leq (h_0 - r)\|u_0'\|_{C([0,h_0])}$$

and

$$\overline{u}(r, 0) = M_1[2M(h_0 - r) - M^2(h_0 - r)^2] \geq M_1M(h_0 - r)$$

in $[h_0 - \frac{1}{M}, h_0]$. Therefore, if $M \geq \frac{\|u_0'\|_{C([0,h_0])}}{M_1}$, we then have $u_0(r) \leq \overline{u}(r, 0)$ in $[h_0 - \frac{1}{M}, h_0]$.

Let $M = \max\left\{\sqrt{\frac{a_{1M}}{2d_1}}, \frac{\|u_0'\|_{C([0,h_0])}}{M_1}, \frac{1}{h_0}\right\}$. We now can apply the maximum principle to $\overline{u} - u$ over Ω_M to obtain that $u \leq \overline{u}$ for $(r, t) \in \Omega_M$. It follows that

$$u_r(h(t), t) \geq \overline{u}_r(h(t), t) = -2M_1M$$

and

$$h'(t) = -\mu u_r(h(t), t) \leq 2\mu M_1M \triangleq M_3.$$

The proof is complete. □

The following theorem guarantees the global existence. Since its proof is standard, we omit it here and one can refer to [9] or [24] for a similar proof.

Theorem 2.3 *The solution of problem* (1.2) *uniquely exists, and it can be extended to* $[0, \infty)$.

In what follows, we present the comparison principle for (1.2) which can be proved similarly as [9, Lemma 2.6].

Theorem 2.4 *(The Comparison Principle). Suppose that* $T \in (0, \infty)$, $\underline{h}$, $\overline{h} \in C^1([0, T])$, $\underline{u} \in C(\overline{D_T^*}) \cap C^{2,1}(D_T^*)$ *with* $D_T^* := \{(r, t) \in \mathbb{R}^2 : r \in (0, \underline{h}(t)), t \in (0, T]\}$, $\overline{u} \in C(\overline{D_T^{**}}) \cap C^{2,1}(D_T^{**})$ *with* $D_T^{**} := \{(r, t) \in \mathbb{R}^2 : r \in (0, \overline{h}(t)), t \in (0, T]\}$, $\underline{v}, \overline{v} \in (L^\infty \cap C)([0, \infty) \times$

$[0,T]) \cap C^{1,2}([0,\infty)\times(0,T])$ *and*

$$\begin{cases} \underline{u}_t - d_1\Delta\underline{u} \le [a_1(r) - b_1(r)\underline{u} - c_1(r)\overline{v}]\underline{u}, & 0 < r < \underline{h}(t),\ 0 < t \le T, \\ \overline{u}_t - d_1\Delta\overline{u} \ge [a_1(r) - b_1(r)\overline{u} - c_1(r)\underline{v}]\overline{u}, & 0 < r < \overline{h}(t),\ 0 < t \le T, \\ \underline{v}_t - d_2\Delta\underline{v} \le [a_2(r) - b_2(r)\overline{u} - c_2(r)\underline{v}]\underline{v}, & 0 < r < \infty,\ 0 < t \le T, \\ \overline{v}_t - d_2\Delta\overline{v} \ge [a_2(r) - b_2(r)\underline{u} - c_2(r)\overline{v}]\overline{v}, & 0 < r < \infty,\ 0 < t \le T, \\ \underline{u}_r(0,t) = \overline{v}_r(0,t) = 0, \underline{u}(r,t) = 0, & \underline{h}(t) \le r < \infty,\ 0 < t \le T, \\ \overline{u}_r(0,t) = \underline{v}_r(0,t) = 0, \overline{u}(r,t) = 0, & \overline{h}(t) \le r < \infty,\ 0 < t \le T, \\ \underline{h}'(t) \le -\mu\underline{u}_r(\underline{h}(t),t),\ \overline{h}'(t) \ge -\mu\overline{u}_r(\overline{h}(t),t), & 0 < t \le T, \\ \underline{h}(0) \le h(0) \le \overline{h}(0), & \\ \underline{u}(r,0) \le u_0(r) \le \overline{u}(r,0), & 0 \le r \le h_0, \\ \underline{v}(r,0) \le v_0(r) \le \overline{v}(r,0), & 0 \le r < \infty. \end{cases}$$

Let (u,v,h) *be the unique solution of the free boundary problem* (1.2)*, then*

$$h(t) \ge \underline{h}(t) \text{ in } (0,T], u(r,t) \ge \underline{u}(r,t), v(r,t) \le \overline{v}(r,t) \text{ for } (r,t) \in [0,\infty)\times(0,T],$$
$$h(t) \le \overline{h}(t) \text{ in } (0,T], u(r,t) \le \overline{u}(r,t), v(r,t) \ge \underline{v}(t) \text{ for } (r,t) \in [0,\infty)\times(0,T].$$

In the sequel, we will call the triples $(\underline{u},\overline{v},\underline{h})$ and $(\overline{u},\underline{v},\overline{h})$ an lower solution and supper solution of (1.2), respectively.

We next fix u_0, v_0, d_1, d_2, $a_i(r)$, $b_i(r)$, $c_i(r)$ $(i = 1,2)$ and examine the dependence of the solution on μ. It follows from the uniqueness of the solution to free boundary problem (1.2) and a standard compactness argument that the unique solution (u,v,h) depends continuously on μ, and we write (u^μ, v^μ, h^μ) to emphasize this dependence. The following corollary results directly from Theorem 2.4.

Corollary 2.5 *For fixed* u_0*,* v_0*,* d_1*,* d_2*,* $a_i(r)$*,* $b_i(r)$ *and* $c_i(r)$ $(i = 1,2)$*. If* $\mu_1 \le \mu_2$*, then* $u^{\mu_1}(r,t) \le u^{\mu_2}(r,t)$*,* $v^{\mu_1}(r,t) \ge v^{\mu_2}(r,t)$ *for* $r \in [0, h^{\mu_1}(t))$*,* $t \in (0,\infty)$ *and* $h^{\mu_1}(t) \le h^{\mu_2}(t)$ *in* $(0,\infty)$*.*

Finally, for fixed v_0, d_1, d_2, $a_i(r)$, $b_i(r)$, $c_i(r)$ $(i = 1,2)$ and μ, we set $u_0 = \delta\theta(r)$ and examine the dependence of the solution on δ, and we write $(u^\delta, v^\delta, h^\delta)$ to emphasize this dependence. Similar to Corollary 2.5, the following corollary results directly from Theorem 2.4.

Corollary 2.6 *For fixed* v_0*,* d_1*,* d_2*,* $a_i(r)$*,* $b_i(r)$*,* $c_i(r)$ $(i = 1,2)$ *and* μ*. If* $\delta_1 \le \delta_2$*, then* $u^{\delta_1}(r,t) \le u^{\delta_2}(r,t)$*,* $v^{\delta_1}(r,t) \ge v^{\delta_2}(r,t)$ *for* $r \in [0, h^{\delta_1}(t))$*,* $t \in (0,\infty)$ *and* $h^{\delta_1}(t) \le h^{\delta_2}(t)$ *in* $(0,\infty)$*.*

3 Some eigenvalue problems

In this section, we mainly study an eigenvalue problem and analyze the property of its principal eigenvalue. These results play an important role in later sections.

Consider the following eigenvalue problem with $r := |x|(x \in \mathbb{R}^n)$:

$$\begin{cases} D\Delta\phi + \alpha(r)\phi + \lambda\phi = 0, & x \in B_R, \\ \phi = 0, & x \in \partial B_R, \end{cases} \tag{3.1}$$

where D, R are positive constants, and $\alpha(r) \in C^1([0,\infty)) \cap L^\infty([0,\infty))$ is positive somewhere in $(0, R)$. Let $\lambda_1(D, R, \alpha)$ denote the principal eigenvalue of the problem (3.1). It is well known that $\lambda(D, R, \alpha)$ uniquely exists and the corresponding eigenfunction, denoted by ϕ_1, can be chosen positive in B_R and normalized by $\|\phi_1\|_{L^2} = 1$. Moreover, it is obviously that ϕ_1 is radially symmetric and $(\phi_1)_r(0) = 0$. Since the operator $D\Delta + \alpha(|x|)I$ is self-adjoint, $\lambda_1(D, R, \alpha)$ can be characterized by the following variational form:

$$\lambda_1(D, R, \alpha) = \inf\left\{ \int_{B_R} [D|\nabla\phi|^2 - \alpha(r)\phi^2]\,\mathrm{d}x : \phi \in H_0^1(B_R) \ and \int_{B_R} \phi^2\,\mathrm{d}x = 1 \right\}. \tag{3.2}$$

For fixed R and varying D, we write $\lambda_1(D, R, \alpha) = \lambda_1(D, \alpha)$ for brevity. Similarly, we still write $\lambda_1(D, R, \alpha) = \lambda_1(R, \alpha)$ for fixed D and varying R. We first present the property of $\lambda_1(D, \alpha)$.

Theorem 3.1 *For fixed R, the following conclusions about $\lambda_1(D, \alpha)$ hold:*

(i) *$\lambda_1(D, \alpha)$ is a strictly monotone increasing function of D;*

(ii) *$\lambda_1(D, \alpha) \to \widehat{\lambda} \triangleq -\max_{r\in[0,R]} \alpha(r) < 0$ as $D \to 0$;*

(iii) *$\lambda_1(D, \alpha) \to +\infty$ as $D \to +\infty$.*

Theorem 3.1 can be proved similarly as [25, Theorem 3.1] by making little minor modifications, so we omit it here.

The above theorem implies the following result.

Corollary 3.2 *For any fixed R, there exists a $D^*(R, \alpha) > 0$ such that $\lambda_1(D, \alpha) < 0$ if $0 < D < D^*(R, \alpha)$, $\lambda_1(D, \alpha) = 0$ if $D = D^*(R, \alpha)$, $\lambda_1(D, \alpha) > 0$ if $D > D^*(R, \alpha)$.*

Next, we fix D and let R change. The results below is the counterpart of Theorem 3.1, and we recommend [16] or [25] for a detailed proof.

Theorem 3.3 *For fixed D, the following conclusions about $\lambda_1(R, \alpha)$ hold:*

(i) $\lambda_1(R,\alpha)$ *is a strictly monotone decreasing function of* R.

(ii) $\lambda_1(R,\alpha) \to +\infty$ *as* $R \to 0$.

The following corollary is a direct consequence of Theorem 3.3, [25, Theorem 3.2(c)] and [21, Remark 3.1]

Corollary 3.4 *For any fixed* D, *if* $\alpha(r) > 0$ *for* $r \geq 0$, *then there exists a* $R^*(D,\alpha) > 0$ *such that* $\lambda_1(R,\alpha) > 0$ *if* $0 < R < R^*(D,\alpha)$, $\lambda_1(R,\alpha) = 0$ *if* $R = R^*(D,\alpha)$, $\lambda_1(R,\alpha) < 0$ *if* $R > R^*(D,\alpha)$.

By Lemma 2.2, it is easy to see that the habitat of the specie u in the free boundary problem (1.2) increases with time. Under this case, we replace B_R by $B_{h(t)}$, and introduce

$$\lambda_1(D,h(t),\alpha) = \inf\left\{\int_{B_{h(t)}} [D|\nabla\phi|^2 - \alpha(r)\phi^2]\,\mathrm{d}x : \phi \in H_0^1(B_{h(t)}) \ \textit{and} \int_{B_{h(t)}} \phi^2\,\mathrm{d}x = 1\right\}.$$

Let $\lambda_1(h(t),\alpha)$ denote $\lambda_1(D,h(t),\alpha)$ for fixed D. From Lemma 2.2, Theorem 3.3 and Corollary 3.4, we have the following result.

Corollary 3.5 *For any fixed* D, $\lambda_1(h(t),\alpha)$ *is a strictly monotone decreasing function of time* t. *Moreover, if* $\alpha(r) > 0$ *for* $r \geq 0$ *and* $h_\infty \triangleq \lim_{t\to\infty} h(t) = +\infty$, *then* $\lambda_1(h(t),\alpha) < 0$ *for sufficient large* t.

4 Invasion of a inferior competitor

In this section, we examine the case that the invader u is an inferior competitor, which means that

$$\frac{a_{1M}}{a_{2L}} < \min\left\{\frac{c_{1L}}{c_{2M}}, \frac{b_{1L}}{b_{2M}}\right\}. \tag{4.1}$$

Firstly, consider the following logistic equation on the entire space:

$$-d_2\Delta v = [a_2(r) - c_2(r)v]v, \ \ x \in \mathbb{R}^n. \tag{4.2}$$

It follows from [10, Theorem 2.3] that problem (4.2) has a unique positive (radial) solution, denoted by $V(r)$, and the symbol $V(r)$ will be always used in the sequel. We have the following consequence.

Theorem 4.1 *If* (4.1) *holds and* $v_0 \geq \delta$ *for some* $\delta > 0$, *then* $h_\infty < \infty$, $\lim_{t\to\infty} u(r,t) = 0$ *uniformly for* $r \in [0,\infty)$, *and* $\lim_{t\to\infty} v(r,t) = V(r)$ *locally uniformly for* $r \in [0,\infty)$.

Proof. *Step 1*: Proof of $\lim_{t\to\infty} u(r,t) = 0$ uniformly for $r \in [0,\infty)$.

It follows from Lemma 2.2 and Theorem 2.3 that there exist positive M_1 and M_2 such that

$$0 \le u(r,t) \le M_1,\ \ in\ [0,h(t)] \times [0,\infty),$$
$$0 \le v(r,t) \le M_2,\ \ in\ [0,\infty) \times [0,\infty).$$

We now consider the following auxiliary problem

$$\begin{cases} z_t = (a_{1M} - b_{1L}z - c_{1L}w)z, & t > 0, \\ w_t = (a_{2L} - b_{2M}z - c_{2M}w)w, & t > 0, \\ z(0) = M_1,\ w(0) = \delta. \end{cases}$$

The comparison principle implies that $u(r,t) \le z(t)$ and $v(r,t) \ge w(t)$ in $[0,\infty) \times [0,\infty)$. Since (4.1) holds, it is well-known (see, for example, [17]) that $(z,w) \to (0, \frac{a_{2L}}{c_{2M}})$ as $t \to \infty$. As a result, we find that $\lim_{t\to\infty} u(r,t) = 0$ uniformly for $r \in [0,\infty)$.

Step 2: Proof of $\lim_{t\to\infty} v(r,t) = V(r)$ locally uniformly for $r \in [0,\infty)$.

We use a squeezing argument introduced in [12]. Consider the following Dirichlet problem

$$\begin{cases} -d_2\Delta\underline{v} = [a_2(r)(1-\epsilon) - c_2(r)\underline{v}]\underline{v}, & x \in B_R, \\ \underline{v} = 0, & x \in \partial B_R, \end{cases}$$

and the following boundary blow-up problem

$$\begin{cases} -d_2\Delta\overline{v} = [a_2(r) - c_2(r)\overline{v}]\overline{v}, & x \in B_R, \\ \overline{v} = +\infty, & x \in \partial B_R. \end{cases}$$

It is well known that these problems have positive radial solutions, denoted by $\underline{V}_R^\epsilon$ and $\overline{V}_R$ respectively, when $\epsilon > 0$ is small and $R > 0$ is large. By the comparison principle given by [12, Lemma 2.1], we see that as $\epsilon \to 0^+$ and $R \to +\infty$, $\underline{V}_R^\epsilon$ increases to the unique positive solution V of (4.2) and $\overline{V}_R$ decreases to V, respectively. We thus can choose a decreasing sequence $\{\epsilon_k\}$ and an increasing sequence $\{R_k\}$ such that $\epsilon_k \to 0^+$, $\{R_k\} \to +\infty$ as $k \to \infty$, and $\lambda_1(d_2, R_k, m(1-\epsilon_k)) < 0$ holds for all $k \in \mathbb{N}$. It follows from Step 1 that for each ϵ_k, there exists $T_k > 0$ such that

$$u(r,t) < \epsilon_k \frac{a_{2L}}{b_{2M}} \le \epsilon_k \frac{a_2(r)}{b_2(r)}$$

for all $t \ge T_k$ and $r \in [0,\infty)$. For such T_k, we obtain from the choice of ϵ_k, R_k and [4, Proposition 3.3] that the following problem

$$\begin{cases} \underline{v}_t - d_2\Delta\underline{v} = [a_2(r)(1-\epsilon_k) - c_2(r)\underline{v}]\underline{v}, & 0 < r < R_k, t > T_k, \\ \underline{v}_r(0,t) = \underline{v}(R_k,t) = 0, & t > T_k, \\ \underline{v}(r,T_k) = v(r,T_k), & 0 \le r \le R_k \end{cases}$$

admits a unique positive solution $\underline{v}_k(r,t)$ satisfying

$$\underline{v}_k(r,t) \to \underline{V}_{R_k}^{\epsilon_k}(r) \ \text{ uniformly for } r \in [0, R_k] \text{ as } t \to \infty.$$

It follows from the comparison principle that for each $k \in \mathbb{N}$,

$$v(r,t) \geq \underline{v}_k(r,t), \quad \text{for } r \in [0, R_k] \text{ and } t \geq T_k.$$

Hence, we have

$$\liminf_{t\to\infty} v(r,t) \geq \underline{V}_{R_k}^{\epsilon_k}(r) \ \text{ uniformly for } r \in [0, R_k].$$

By setting $k \to \infty$ in the above inequality, we obtain that

$$\liminf_{t\to\infty} v(r,t) \geq V(r) \ \text{ locally uniformly for } r \in [0, \infty).$$

Analogously, by arguments similar to those in the proof of [12, Theorem 4.1], one can prove

$$\limsup_{t\to\infty} v(r,t) \leq \overline{V}_{R_k}(r) \ \text{ uniformly for } r \in [0, R_k],$$

which implies (by sending $k \to \infty$) that

$$\limsup_{t\to\infty} v(r,t) \leq V(r) \ \text{ locally uniformly for } r \in [0, \infty).$$

The desired result would then follow directly from the above inequalities.

Step 3: Proof of $h_\infty < \infty$.

We first note that since $h(t)$ is monotonic increasing, the limit h_∞ always exists.

Now, Step 1 shows that $\lim_{t\to\infty} u(r,t) = 0$ uniformly for $r \in [0,\infty)$. On the other hand, it follows from Step 2 that $\lim_{t\to\infty} v(r,t) = V(r)$ locally uniformly for $r \in [0,\infty)$. Since $V(r)$ satisfies (4.2), the comparison principle indicates that $\frac{a_{2L}}{c_{2M}} \leq V(r) \leq \frac{a_{2M}}{c_{2L}}$ for all $r \in [0,\infty)$. We thus see that

$$\frac{a_{2L}}{c_{2M}} \leq \lim_{t\to\infty} v(r,t) \leq \frac{a_{2M}}{c_{2L}}$$

for all $r \in [0,\infty)$. Hence,

$$\lim_{t\to\infty} [a_1(r) - b_1(r)u - c_1(r)v] \leq a_1(r) - c_1(r)\frac{a_{2L}}{c_{2M}} < 0$$

for all $r \in [0,\infty)$. There thus exists $T \gg 1$ so large that $[a_1(r) - b_1(r)u - c_1(r)v] \leq 0$ for all $r \in [0,\infty)$, $t \geq T$.

Some direct calculations yields that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{h(t)} r^{n-1} u(r,t)\,\mathrm{d}r$$

$$
\begin{aligned}
&= \int_0^{h(t)} r^{n-1} u_t(r,t)\,\mathrm{d}r + h^{n-1}(t)h'(t)u(h(t),t) \\
&= \int_0^{h(t)} r^{n-1}\{d_1 \triangle u + [a_1(r) - b_1(r)u - c_1(r)v]\}\,\mathrm{d}r \\
&= -\frac{d_1}{\mu} h^{n-1}(t)h'(t) + \int_0^{h(t)} r^{n-1}[a_1(r) - b_1(r)u - c_1(r)v]\,\mathrm{d}r.
\end{aligned}
$$

Integrating from T to t yields

$$
\begin{aligned}
&\int_0^{h(t)} r^{n-1} u(r,t)\,\mathrm{d}r \\
= &\int_0^{h(T)} r^{n-1} u(r,T)\,\mathrm{d}r - \frac{d_1}{n\mu} h^n(t) + \frac{d_1}{n\mu} h^n(T) \\
&+ \int_T^t \int_0^{h(s)} r^{n-1}[a_1(r) - b_1(r)u - c_1(r)v]\,\mathrm{d}r\,\mathrm{d}s.
\end{aligned}
$$

Since $\int_0^{h(t)} r^{n-1}u(r,t)\,\mathrm{d}r \geq 0$ and $[a_1(r) - b_1(r)u - c_1(r)v] \leq 0$ for all $r \in [0,\infty)$, $t \geq T$, we have

$$
h^n(t) \leq \frac{n\mu}{d_1} \int_0^{h(T)} r^{n-1} u(r,T)\,\mathrm{d}r + h^n(T) < \infty
$$

for all $t \geq T$. It follows that

$$
h_\infty < \infty.
$$

The proof is complete. □

Theorem 4.1 implies that if the resident species is already rather established at beginning, then an inferior competitor can never invades deep into the underlining habitat, and it dies out before its invading front reaches a certain finite limiting position.

5 Invasion of a superior competitor

In this section, we examine the case that the invader u is an inferior competitor, which means that

$$
\frac{a_{1L}}{a_{2M}} > \min\left\{\frac{c_{1M}}{c_{2L}}, \frac{b_{1M}}{b_{2L}}\right\}. \tag{5.1}
$$

The important investigation of ours is the spreading-vanishing dichotomy. In fact, Lemma 2.2 implies that $r = h(t)$ is monotonic increasing, thus, there exists $h_\infty \in (0,\infty]$ such that $\lim_{t\to\infty} h(t) = h_\infty$. Let (u,v,h) be the solution to problem (1.2), we have the following definitions:

(A1) *Spreading of the invasive species* u if

$$h_\infty = \infty \text{ and } \liminf_{t\to\infty} \|u(\cdot,t)\|_{C([0,h(t)])} > 0;$$

(A2) *Vanishing of the invasive species* u if

$$h_\infty < \infty \text{ and } \lim_{t\to\infty} \|u(\cdot,t)\|_{C([0,h(t)])} = 0.$$

We remark that in the case spreading, the invasive species can survive and spread to the whole space; in the case vanishing, however, the invasive species will be limited in a finite region and finally goes to extinction.

5.1 Spreading-vanishing dichotomy

In this subsection, we prove the spreading-vanishing dichotomy. The following consequence indicates that if the invasive species cannot spread into the whole space, then it will vanish eventually.

Lemma 5.1 *If* $h_\infty < \infty$*, then* $\lim_{t\to\infty} \|u(\cdot,t)\|_{C([0,h(t)])} = 0$*, and* $\lim_{t\to\infty} v(r,t) = V(r)$ *locally uniformly for* $r \in [0,\infty)$.

Proof. We first show that if $h_\infty < \infty$, then $\lim_{t\to\infty} \|u(\cdot,t)\|_{C([0,h(t)])} = 0$ locally uniformly for $r \in [0,\infty)$.

Using the same arguments as in Theorem 2.1, one can obtain that for any $\alpha \in (0,1)$, there exists a constant $\widetilde{C}$ depending on α, (u_0,v_0), h_0 and h_∞ such that

$$\|u\|_{C^{1+\alpha,\frac{1+\alpha}{2}}([0,h(t)]\times[0,\infty))} + \|v\|_{C^{1+\alpha,\frac{1+\alpha}{2}}([0,h(t)]\times[0,\infty))} + \|h\|_{C^{1+\frac{\alpha}{2}}([0,\infty))} < \widetilde{C}. \tag{5.2}$$

In what follows, we use the contradiction argument. Assume that

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{C([0,h(t)])} = \delta > 0$$

Then there exists a sequence $(r_k,t_k) \in [0,h(t)) \times (0,\infty)$ with $t_k \to \infty$ as $k \to \infty$ such that $u(r_k,t_k) \ge \frac{\delta}{2}$ for all $k \in \mathbb{N}$. Since $0 \le r_k < h(t_k) < h_\infty < \infty$, passing to a subsequence if necessary, it follows that $r_k \to r_0 \in [0,h_\infty)$, as $k \to \infty$.

Define

$$u_k(r,t) = u(r,t+t_k),\ v_k(r,t) = v(r,t+t_k),$$

for $r \in [0,h(t+t_k)]$ and $t \in (-t_k,\infty)$. It follows from (5.2) and the standard parabolic regularity that $\{(u_k,v_k)\}$ has a subsequence $\{(u_{k_i},v_{k_i})\}$ such that $(u_{k_i},v_{k_i}) \to (\widetilde{u},\widetilde{v})$ as $k_i \to \infty$, where $(\widetilde{u},\widetilde{v})$ satisfies

$$\begin{cases} u_t - d_1\Delta u = [a_1(r) - b_1(r)u - c_1(r)v]u, & (r,t) \in (0,h_\infty) \times (-\infty,\infty), \\ v_t - d_2\Delta v = [a_2(r) - b_2(r)u - c_2(r)v]v, & (r,t) \in (0,h_\infty) \times (-\infty,\infty). \end{cases}$$

Since $\widetilde{u}(r_0, 0) = \lim_{k_i\to\infty} u_{k_i}(r_{k_i}, 0) = \lim_{k_i\to\infty} u(r_{k_i}, t_{k_i}) \geq \frac{\delta}{2}$, it follows from the maximum principle that $\widetilde{u} > 0$ in $[0, h_\infty) \times (-\infty, \infty)$. Thus we can apply the Hopf boundary lemma at the point $(h_\infty, 0)$, where $\widetilde{u}(h_\infty, 0) = 0$, to conclude that

$$\widetilde{u}_r(h_\infty, 0) \leq -\sigma < 0$$

for some $\sigma > 0$. From the above fact and (5.2), we find that

$$u_r(h(t_{k_i}), t_{k_i}) = \partial_r u_{k_i}(h(t_{k_i}), 0) \leq -\frac{\sigma}{2} < 0$$

for all large k_i, which together with the Stefan condition implies that $h'(t_{k_i}) \geq \frac{\mu\sigma}{2} > 0$

On the other hand, combining (4.2) with $h_\infty < \infty$, we find that $h'(t) \to 0$ as $t \to \infty$. This contradiction shows that we must have

$$\lim_{t\to\infty} \|u(\cdot, t)\|_{C([0,h(t)])} = 0.$$

The proof of $\lim_{t\to\infty} v(r, t) = V(r)$ locally uniformly for $r \in [0, \infty)$ is just same as in Theorem 4.1, so we omit it. the proof is complete. □

Set $R^*(d_1, a_1 - c_1V)$ the positive constant determined by Corollary 3.4, which satisfies $\lambda_1(d_1, R^*(d_1, a_1 - c_1V), a_1 - c_1V) = 0$. The following result gives a sufficient condition of spreading and an estimate of h_∞ when $h_\infty < \infty$.

Lemma 5.2 *If $h_\infty < \infty$, then $h_\infty \leq R^*(d_1, a_1 - c_1V)$. Hence, $h_0 \geq R^*(d_1, a_1 - c_1V)$ implies $h_\infty = \infty$.*

Proof. Arguing indirectly, we suppose that $h_\infty > R^*(d_1, a_1 - c_1V)$ to get a contradiction. It follows from Lemma 5.1 and the continuity of $R^*(d_1, \alpha)$ in α that for any given $0 < \epsilon \ll 1$, there is $T_\epsilon > 0$ such that

$$h(T_\epsilon) > \max\{h_0, R^*(d_1, a_1 - c_1V - \epsilon)\},$$

and

$$v(r, t) \leq V(r) + \frac{\epsilon}{c_1(r)}$$

for any $t \geq T_\epsilon$ and $r \in [0, h_\infty]$. Setting $R = h(T_\epsilon)$, we then have $R > R^*(d_1, a_1 - c_1V - \epsilon)$. Consider the following auxiliary problem

$$\begin{cases} \underline{u}_t - d_1\Delta\underline{u} = [a_1(r) - c_1(r)V - \epsilon - \underline{u}]\underline{u}, & 0 < r < R, t > T_\epsilon, \\ \underline{u}_r(0, t) = \underline{u}(R, t) = 0, & t > T_\epsilon, \\ \underline{u}(r, T_\epsilon) = u(r, T_\epsilon), & 0 \leq r \leq R, \end{cases}$$

It follows from the choice of R and [4, Proposition 3.3] that the above problem admits a unique positive solution $\underline{u}(r,t)$ satisfying

$$\lim_{t\to\infty} \underline{u}(r,t) = \underline{U}^{\epsilon}(r) \quad \text{uniformly for } r \in [0,R],$$

where $\underline{U}^{\epsilon}(r)$ is the unique positive solution of

$$\begin{cases} -d_1\Delta \underline{u} = [a_1(r) - c_1(r)V - \epsilon - \underline{u}]\underline{u}, \ 0 < r < R, \\ \underline{u}_r(0) = \underline{u}(R) = 0. \end{cases}$$

By the comparison principle ([6, Lemma 2.6]), we have

$$u(r,t) \geq \underline{u}(r,t), \quad \text{for all } r \in [0,R], \ t \geq T_\epsilon,$$

which implies that

$$\liminf_{t\to\infty} \|u(r,t)\|_{C([0,h(t)])} \geq \underline{U}^{\epsilon}(0) > 0.$$

The above fact makes a contradiction with Lemma 5.1, and the proof is finished. □

Let $U(r)$ denote the unique positive (radial) solution to the following equation determined by [10, Theorem 2.3]

$$-d_1\Delta u = [a_1(r) - b_1(r)u]u, \ x \in \mathbb{R}^n. \tag{5.3}$$

We have the following consequence.

Lemma 5.3 *If $h_\infty = \infty$, then spreading occurs, and $\lim_{t\to\infty} u(r,t) = U(r)$, $\lim_{t\to\infty} v(r,t) = 0$ locally uniformly for $r \in [0,\infty)$.*

Proof. We first prove that $\lim_{t\to\infty} v(r,t) = 0$ locally uniformly for $r \in [0,\infty)$.

To begin with, the comparison principle implies that $v(r,t) \leq v^*(t)$ for $r \in [0,\infty)$, $t > 0$, where $v^*(t)$ satisfies

$$\begin{cases} v^*_t = (a_{2M} - c_{2L}v^*)v^*, \ t > 0, \\ v^*(0) = \|v_0\|_{L^\infty([0,\infty))}. \end{cases}$$

Since $v^*(t) \to \frac{a_{2M}}{c_{2L}}$ as $t \to \infty$, we thus have

$$\limsup_{t\to\infty} v(r,t) \leq \frac{a_{2M}}{c_{2L}} \quad \text{uniformly for } r \in [0,\infty).$$

Therefore for $\epsilon_1 = \frac{1}{2}(\frac{a_{1L}}{c_{1M}} - \frac{a_{2M}}{c_{2L}})$, there exists $T_1 > 0$ such that

$$v(r,t) \leq \frac{a_{2M}}{c_{2L}} + \epsilon_1$$

for any $t \geq T_\epsilon$ and $r \in [0, \infty)$. It follows that u satisfies

$$\begin{cases} u_t - d_1 \Delta u \geq (c_{1L}\epsilon_1 - b_{1M} u)u, \ 0 < r < h(t), t > T_1, \\ u_r(0,t) = u(h(t),t) = 0, & t > T_1, \\ h'(t) = -\mu u_r(h(t),t), & t > T_1, \\ u(r, T_1) > 0, \ 0 < r < h(t). \end{cases}$$

We claim that for any $R > R^*(d_1, c_{1L}\epsilon_1)$, there exists $T_R \geq T_1$ such that $u(r,t) \geq \frac{c_{1L}\epsilon_1}{b_{1M}}$ for $0 \leq r \leq R$, $t \geq T_R$. In fact, since $h_\infty = \infty$, there always exists $T_2 \geq T_1$ such that $h(T_2) \geq R$. Consider the following auxiliary problem

$$\begin{cases} \underline{u}_t - d_1 \Delta \underline{u} = (c_{1L}\epsilon_1 - b_{1M}\underline{u})\underline{u}, & 0 < r < \underline{h}(t), t > T_2, \\ \underline{u}_r(0,t) = \underline{u}(\underline{h}(t),t) = 0, & t > T_2, \\ \underline{h}'(t) = -\mu \underline{u}_r(\underline{h}(t),t), & t > T_2, \\ \underline{u}(r, T_2) = u(r, T_2), \ \underline{h}(T_2) = h(T_2), \ 0 < r < h(T_2). \end{cases}$$

Since $h(T_2) \geq R > R^*(d_1, c_{1L}\epsilon_1)$, it follows from [6, Theorem 2.1] that the above problem has a unique solution $(\underline{u}, \underline{h})$ existing for all $t \geq T$. Moreover, [6, Theorem 2.5] implies that $\underline{h}(t) \to \infty$ and $\underline{u}(r,t) \to \frac{c_{1L}\epsilon_1}{b_{1M}}$ as $t \to \infty$ uniformly in $[0, R]$. Taking $(\overline{u}, \overline{h})$ as supper solution, we obtain from the comparison principle [6, Lemma 2.6] that $u(r,t) \geq \underline{u}(r,t)$ for $r \in [0, \underline{h}(t)]$, $t \in [T_2, \infty)$, and $h(t) \geq \underline{h}(t)$ for $t \in [T_2, \infty)$. These facts indicate our claim hold for some $T_R \geq T_1$.

We thus find that (u, v) satisfies the following

$$\begin{cases} u_t - d_1 \Delta u = [a_1(r) - b_1(r)u - c_1(r)v]u, \ 0 \leq r \leq R, \ t \geq T_R, \\ v_t - d_2 \Delta v = [a_2(r) - b_2(r)u - c_2(r)v]v, \ 0 \leq r \leq R, \ t \geq T_R, \\ u_r(0,t) = v_r(0,t) = 0, & t \geq T_R, \\ u(r,t) \geq \frac{c_{1L}\epsilon_1}{b_{1M}}, \ v(r,t) \leq \frac{a_{2M}}{c_{2L}} + \epsilon_1, & 0 \leq r \leq R, \ t \geq T_R. \end{cases}$$

Since $h_\infty = \infty$, the comparison principle yields $u \geq \underline{u}$ and $v \leq \overline{v}$ for $0 \leq r \leq R$, $t \geq T_R$, where $(\underline{u}, \overline{v})$ is the solution of the following system

$$\begin{cases} \underline{u}_t - d_1 \Delta \underline{u} = [a_1(r) - b_1(r)\underline{u} - c_1(r)\overline{v}]\underline{u}, \ 0 \leq r \leq R, \ t \geq T_R, \\ \overline{v}_t - d_2 \Delta \overline{v} = [a_2(r) - b_2(r)\underline{u} - c_2(r)\overline{v}]\overline{v}, \ 0 \leq r \leq R, \ t \geq T_R, \\ \underline{u}_r(0,t) = \overline{v}_r(0,t) = 0, & t \geq T_R, \\ \underline{u}(R,t) = \frac{c_{1L}\epsilon_1}{b_{1M}}, \ \overline{v}(R,t) = \frac{a_{2M}}{c_{2L}} + \epsilon_1, & t > T_R \\ \underline{u}(r,T_R) = \frac{c_{1L}\epsilon_1}{b_{1M}}, \ \overline{v}(r,T_R) = \frac{a_{2M}}{c_{2L}} + \epsilon_1, & 0 \leq r \leq R. \end{cases}$$

The above system generates a monotone dynamical system with respect to the competition order, which means that

$$(u_1, v_1) \leq_C (u_2, v_2) \text{ if and only if } u_1 \leq u_2 \text{ and } v_1 \geq v_2.$$

Specially, the initial value $(\frac{c_{1L}\epsilon_1}{b_{1M}}, \frac{a_{2M}}{c_{2L}} + \epsilon_1)$ is a lower solution with respect to the competition order. The theory of monotone dynamical systems (see, for example, [19, Corollary 7.3.6]) yields that $\lim_{t\to\infty} \underline{u}(r,t) = \underline{u}_R(r)$ and $\lim_{t\to\infty} \overline{v}(r,t) = \overline{v}_R(r)$ uniformly in $[0, R]$, where $(\underline{u}_R(r), \overline{v}_R(r))$ is the minimal solution above $(\frac{c_{1L}\epsilon_1}{b_{1M}}, \frac{a_{2M}}{c_{2L}} + \epsilon_1)$ of the following system under the competition order

$$\begin{cases} -d_1\Delta \underline{u}_R = [a_1(r) - b_1(r)\underline{u}_R - c_1(r)\overline{v}_R]\underline{u}_R, \ 0 \le r \le R, \\ -d_2\Delta \overline{v}_R = [a_2(r) - b_2(r)\underline{u}_R - c_2(r)\overline{v}_R]\overline{v}_R, \ 0 \le r \le R, \\ \partial_r \underline{u}_R(0) = \partial_r \overline{v}_R(0) = 0, \\ \underline{u}_R(R) = \frac{c_{1L}\epsilon_1}{b_{1M}}, \ \overline{v}_R(R) = \frac{a_{2M}}{c_{2L}} + \epsilon_1. \end{cases}$$

Let $R \to \infty$. It follows from the classical elliptic regularity theory and a diagonal procedure that $(\underline{u}_R(r), \overline{v}_R(r))$ converges locally uniformly to $(\underline{u}_\infty(r), \overline{v}_\infty(r))$, which satisfies

$$\begin{cases} -d_1\Delta \underline{u}_\infty = [a_1(r) - b_1(r)\underline{u}_\infty - c_1(r)\overline{v}_\infty]\underline{u}_\infty, \ 0 \le r < \infty, \\ -d_2\Delta \overline{v}_\infty = [a_2(r) - b_2(r)\underline{u}_\infty - c_2(r)\overline{v}_\infty]\overline{v}_\infty, \ 0 \le r < \infty, \\ \partial_r \underline{u}_\infty(0) = \partial_r \overline{v}_\infty(0) = 0, \\ \underline{u}_\infty(r) \ge \frac{c_{1L}\epsilon_1}{b_{1M}}, \ \overline{v}_\infty(r) \le \frac{a_{2M}}{c_{2L}} + \epsilon_1, \qquad 0 \le r < \infty. \end{cases}$$

In sequel, we show that $\overline{v}_\infty(r) = 0$. Consider the following ODE system

$$\begin{cases} z_t = (a_{1L} - b_{1M}z - c_{1M}w)z, \ t > 0, \\ w_t = (a_{2M} - b_{2L}z - c_{2L}w)w, \ t > 0, \\ z(0) = \frac{c_{1L}\epsilon_1}{b_{1M}}, \ w(0) = \frac{a_{2M}}{c_{2L}} + \epsilon_1. \end{cases}$$

Since (5.1) holds, we have $(z(t), w(t)) \to (\frac{a_{1L}}{b_{1M}}, 0)$ as $t \to \infty$. Hence, the solution $(Z(r,t), W(r,t))$ of the following system

$$\begin{cases} Z_t - d_1\Delta Z = (a_{1L} - b_{1M}Z - c_{1M}W)Z, & 0 \le r < \infty, \ t > 0, \\ W_t - d_2\Delta W = (a_{2M} - b_{2L}Z - c_{2L}W)W, & 0 \le r < \infty, \ t > 0, \\ Z_r(0,t) = W_r(0,t) = 0, & t > 0, \\ Z(r,0) = \frac{c_{1L}\epsilon_1}{b_{1M}}, \ W(r,0) = \frac{a_{2M}}{c_{2L}} + \epsilon_1, & 0 \le r < \infty. \end{cases}$$

satisfies $(Z(r,t), W(r,t)) \to (\frac{a_{1L}}{b_{1M}}, 0)$ as $t \to \infty$ uniformly for $r \in [0, \infty)$. On the other hand, the comparison principle implies that $\underline{u}_\infty(r) \ge Z(r,t)$ and $\overline{v}_\infty(r) \le W(r,t)$ for $r \in [0, \infty)$, $t > 0$. We immediately obtain that $\overline{v}_\infty(r) = 0$

Now, since $\liminf_{t\to\infty} u(r,t) \ge \underline{u}_R(r)$ and $\limsup_{t\to\infty} v(r,t) \le \overline{v}_R(r)$ in $[0, R]$, it follows that $\limsup_{t\to\infty} v(r,t) \le 0$ in $[0, R]$, which directly yields that $\lim_{t\to\infty} v(r,t) = 0$ locally uniformly for $r \in [0, \infty)$.

Since $\lim_{t\to\infty} v(r,t) = 0$ locally uniformly for $r \in [0, \infty)$, by the similar arguments used in Theorem 4.1, we can show that $\lim_{t\to\infty} u(r,t) = U(r)$ locally uniformly for $r \in [0, \infty)$, and we omit it for brevity. The proof is complete. □

Combining Lemmas 5.1, 5.2 and 5.3, we immediately have the following spreading-vanishing dichotomy theorem.

Theorem 5.4 *Let (u, v, h) be the solution to problem (1.2). Then the following alternative holds: either*

(i) *Spreading: $h_\infty = \infty$ and $\lim_{t\to\infty} u(r,t) = U(r)$ locally uniformly for $r \in [0,\infty)$;*

or

(ii) *Vanishing: $h_\infty \le R^*(d_1, a_1 - c_1V)$ and $\lim_{t\to\infty} \|u(\cdot,t)\|_{C([0,h(t)])} = 0$.*

5.2 Sharp criteria for the spreading and vanishing

This subsection is devoted to establish sharp criterion by choosing d_1, μ and initial function u_0 as varying factors to distinguish the spreading-vanishing dichotomy for the invasive species claimed by Theorem 5.4.

Let $D^*(h_0, a_1 - c_1V)$ be the positive constant determined by Corollary 3.4 which satisfies $\lambda_1(D^*(h_0, a_1 - c_1V), h_0, a_1 - c_1V) = 0$. The following result implies that the spreading will occur for small d_1, which means that for the invasive species, there exists a unconditional selection for slow dispersal rate.

Theorem 5.5 *If $0 < d_1 \le D^*(h_0, a_1 - c_1V)$, then spreading occurs.*

Proof. We will first prove the case $0 < d_1 < D^*(h_0, a_1 - c_1V)$. To begin with, define

$$K = \frac{c_{2M}}{a_{2L}}\|v_0\|_{L^\infty([0,\infty))},\ \omega = \frac{a_{2L}c_{2L}}{c_{2M}},\ \overline{v}(r,t) = (1 + Ke^{-\omega t})V(r). \tag{5.4}$$

Since $\frac{a_{2L}}{c_{2M}} \le V(r) \le \frac{a_{2M}}{c_{2L}}$ for any $r \ge 0$. It follows from direct calculations that

$$\begin{aligned}
\overline{v}_t - d_2\Delta\overline{v} - [m(r) - u - \overline{v}]\overline{v} &\ge Ke^{-\omega t}V(r)[-\omega + (1 + Ke^{-\omega t})c_2(r)V(r)] \\
&\ge Ke^{-\omega t}V(r)[\frac{a_{2L}c_{2L}}{c_{2M}} - \omega] \\
&\ge 0
\end{aligned}$$

for any $u \ge 0$, $r \ge 0$ and $t > 0$, and $\overline{v}(r,0) = (1+K)V(r) > \|v_0\|_{L^\infty([0,\infty))} \ge v_0(r)$ for any $r \ge 0$. Since $\lim_{t\to\infty} \overline{v}(r,t) = V(r)$ uniformly in $[0,\infty)$, it follows that for any given $0 < \epsilon \ll 1$, there exists $T_\epsilon > 0$ such that

$$\overline{v}(r,t) \le (1+\epsilon)V(r),$$

for any $r \in [0,\infty)$, $t \ge T_\epsilon$. The comparison principle still yields that $v(r,t) \le (1+\epsilon)V(r)$ for any $r \in [0,\infty)$, $t \ge T_\epsilon$.

On the other hand, use λ_1 and ϕ_1 to denote the principal eigenvalue and the corresponding eigenfunction of problem (3.1) with $D = d_1$, $R = h(T_\epsilon)$ and $\alpha(r) = a_1(r) - c_1(r)V(r)$, respectively. Since $\lambda_1(D^*(h_0, a_1 - c_1V), h_0, a_1 - c_1V) = 0$ and $h(t)$ is strictly increasing with respect to t, it follows from Corollary 3.5 that $\lambda_1 < 0$. We now set

$$\underline{u}(r,t) = \begin{cases} \delta\phi_1(r), & r \in [0, h(T_\epsilon)], \ t \geq T_\epsilon; \\ 0, & r \in (h(T_\epsilon), \infty), \ t \geq T_\epsilon. \end{cases}$$

By some direct calculations, we have

$$\begin{cases} \underline{u}_t - d_1\Delta\underline{u} - [a_1(r) - b_1(r)\underline{u} - c_1(r)(1+\epsilon)V]\underline{u} = (\lambda_1 + \delta b_1(r)\phi_1 + \epsilon c_1(r)V)\delta\phi_1, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 < r < h(T_\epsilon), t > T_\epsilon, \\ \underline{u}_r(0,t) = \underline{u}(r,t) = 0, \qquad\qquad\qquad\qquad\qquad r \geq h(T_\epsilon), \ t > T_\epsilon, \\ \frac{d}{dt}h(T_\epsilon) = 0 \leq -\mu\underline{u}_r(h(T_\epsilon), t), \qquad\qquad\qquad t > T_\epsilon, \\ \underline{u}(r, T_\epsilon) = \delta\phi_1(r), \qquad\qquad\qquad\qquad\qquad\qquad 0 \leq r \leq h(T_\epsilon). \end{cases}$$

Choosing ϵ and δ so small that

$$\lambda_1 + \delta b_1(r)\phi_1 + \epsilon c_1(r)V \leq 0$$

and

$$\delta\phi_1(r) \leq u(r, T_\epsilon)$$

for all $r \in [0, h(T_\epsilon)]$, we thus can use Theorem 2.4 with $(\underline{u}, (1+\epsilon)V, h(T_\epsilon))$ as lower solution to (1.2) to get $u(r,t) \geq \underline{u}(r,t)$ for $r \in [0, h(T_\epsilon)]$, $t \in [T_\epsilon, \infty)$. It follows that

$$\liminf_{t\to\infty} \|u(r,t)\|_{C([0,h(t)])} \geq \delta\phi_1(0) > 0.$$

In view of Lemma 5.1, we conclude that $h_\infty = \infty$, that is, spreading occurs.

When it comes to the case $d_1 = D^*(h_0, a_1 - c_1V)$, we see that

$$\lambda_1 = \lambda_1(d_1, h_0, a_1 - c_1V) = 0.$$

Using the monotonicity of $h(t)$ again, we can choose $t^* > 0$ such that $h(t^*) > h_0$. It follows from Corollary 3.5,

$$\lambda_1(d_1, h(t^*), a_1 - c_1V) < \lambda_1(d_1, h_0, a_1 - c_1V) = 0.$$

Therefore, after replacing h_0 with $h(t^*)$, the same course as above could be done to obtain the desired result again. The proof is finished. $\square$

Let $D^*(h_0, a_1)$ be the positive constant determined by Corollary 3.4 which satisfies $\lambda_1(D^*(h_0, m), h_0, a_1) = 0$. It follows from [4, Corollary 2.2] that

$$D^*(h_0, a_1) > D^*(h_0, a_1 - c_1V).$$

The following consequence indicates that the vanishing will conditionally occur for large d_1.

Lemma 5.6 *If $d_1 > D^*(h_0, a_1)$, then then for any given u_0 satisfying* (1.3)*, there exists $\underline{\mu} > 0$ depending on u_0 and h_0 such that vanishing occurs if $0 < \mu \leq \underline{\mu}$.*

Proof. It is clearly that (u, h) satisfies the following problem

$$\begin{cases} u_t - d_1\Delta u \leq [a_1(r) - b_1(r)u]u, & 0 < r < h(t),\ t > 0, \\ u_r(0,t) = u(h(t),t) = 0, & t > 0, \\ h'(t) = -\mu u_r(h(t),t), & t > 0, \\ u(r,0) = u_0(r), & 0 \leq r \leq h_0, \end{cases}$$

which together with the comparison principle [6, Lemma 2.6] implies that (u, h) is a lower solution to the problem

$$\begin{cases} \overline{u}_t - d_1\Delta\overline{u} = [a_1(r) - b_1(r)\overline{u}]\overline{u}, & 0 < r < \overline{h}(t),\ t > 0, \\ \overline{u}_r(0,t) = \overline{u}(\overline{h}(t),t) = 0, & t > 0, \\ \overline{h}'(t) = -\mu\overline{u}_r(t,\overline{h}(t)), & t > 0, \\ \overline{u}(r,0) = u_0(r), \overline{h}(0) = h_0, & 0 \leq r \leq \overline{h}_0, \end{cases} \tag{5.5}$$

Since $d_1 > D^*(h_0, a_1)$, by the similar arguments applied in [25, Theorem 4.3] or [16, Lemma 5.3] with some minor modifications, we can prove that there exists $\underline{\mu} > 0$ depending on u_0 such that $\overline{h}_\infty < \infty$ if $0 < \mu \leq \underline{\mu}$. The above fact immediately yields that that vanishing occurs if $0 < \mu \leq \underline{\mu}$, and the proof is complete. □

In contrast with Lemma 5.6, the following result indicates that the spreading will still conditionally occur for large d_1.

Lemma 5.7 *If $d_1 > D^*(h_0, a_1 - c_1V)$, then for any given (u_0, v_0) satisfying* (1.3)*, there exists $\overline{\mu} > 0$ depending on u_0, v_0 and h_0 such that spreading occurs if $\mu \geq \overline{\mu}$.*

Proof. One first note that since $\frac{a_{2L}}{c_{2M}} \leq V(r) \leq \frac{a_{2M}}{c_{2L}}$ for any $r \geq 0$ and (5.1) holds, it follows from Theorem 3.3 (ii) that

$$\lim_{T\to\infty} \lambda_1(d_1, \sqrt{T}, a_1 - c_1V) = -\sup_{r\in[0,\infty)} [a_1(r) - c_1(r)V(r)] < 0.$$

Therefore there exists $T^* > 0$ such that $\lambda_1(d_1, \sqrt{T^*}, a_1 - c_1V) < 0$.

Next we are going to construct a suitable lower solution to (1.2) and then apply Theorem 2.4. Define $\overline{v}(r,t)$ as in (5.4). We thus find that

$$\overline{v}_t - d_2\Delta\overline{v} - [a_2(r) - b_2(r)u - c_2(r)\overline{v}]\overline{v} \geq 0$$

for any $u \geq 0$, $r \geq 0$ and $t > 0$, and $\overline{v}(r,0) \geq v_0(r)$ for any $r \geq 0$.

On the other hand, inspired by Lei et al. [16], we set λ be the eigenvalue of

$$\begin{cases} -d_1\Delta\psi - \frac{1}{2}\psi' = \lambda\psi, \ 0 < r < 1, \\ \psi'(0) = \psi(1) = 0, \end{cases}$$

with the corresponding eigenfunction $\psi > 0$, $\psi' \leq 0$ in $[0,1)$, and $\|\psi\|_{L^\infty([0,1))} = 1$. Define

$$\begin{aligned} \underline{h}(t) &= \sqrt{t+\sigma}, \ t \geq 0 \\ \underline{u}(r,t) &= \frac{\rho}{(t+\sigma)^k}\psi(\frac{r}{\sqrt{t+\sigma}}), \ 0 \leq r \leq \sqrt{t+\sigma}, \ t \geq 0, \end{aligned}$$

where σ, k and ρ are positive constants to be chosen later. By Lemma 2.2, we have $0 < u(r,t) \leq M_1$ and $0 < v(r,t) \leq M_2$ for any $0 \leq r \leq h(t)$ and $t \geq 0$. This fact together with $a_{1L} \leq a_1(r) \leq a_{1M}$, $b_{1L} \leq b_1(r) \leq b_{1M}$, and $c_{1L} \leq c_1(r) \leq c_{1M}$ implies that there exists a positive constant L such that $f(r,u,v) \geq -Lu$ for any $0 \leq r \leq h(t)$ and $0 < u \leq M_1$, where $f(r,u) \triangleq (a_1(r) - b_1(r)u - c_1(r)v)u$. We now first choose $0 < \sigma \leq \min\{1, h_0^2\}$, $k > \lambda + L(T^*+1)$, and $\rho > 0$ small such that $\underline{u}(r,0) = \frac{\rho}{\sigma^k}\psi(\frac{r}{\sqrt{\sigma}}) < u_0(r)$ for $0 \leq r \leq \sqrt{\sigma}$. For these k and ρ, we then further select $\mu > 0$ large, say $\mu \geq \overline{\mu} \triangleq -\frac{(T^*+1)^k}{\rho\psi'(1)}$. By the choices of σ, k, ρ and μ, we obtain from direct calculations that

$$\begin{aligned} & \underline{u}_t - d_1\Delta\underline{u} - [a_1(r) - b_1(r)\underline{u} - b_1(r)\overline{v}]\underline{u} \\ = \ & -\frac{\rho}{(t+\sigma)^{k+1}}\Big[k\psi + \frac{r}{2\sqrt{t+\sigma}}\psi' + d_1\Delta\psi - L(t+\sigma)\psi\Big] \\ \leq \ & -\frac{\rho}{(t+\sigma)^{k+1}}\Big[d_1\Delta\psi + \frac{1}{2}\psi' + k\psi - L(t+\sigma)\psi\Big] \\ \leq \ & -\frac{\rho}{(t+\sigma)^{k+1}}\Big[d_1\Delta\psi + \frac{1}{2}\psi' + k\psi - \lambda\psi\Big] = 0, \end{aligned}$$

for $0 < r < \underline{h}(t)$ and $0 < t \leq T^*$, and

$$\underline{h}'(t) + \mu\underline{u}_r(\underline{h}(t),t) = \frac{1}{2\sqrt{t+\sigma}} + \frac{\mu\rho}{(t+\sigma)^{k+\frac{1}{2}}}\psi'(1) \leq 0$$

for $0 < t \leq T^*$.

Since $\underline{h}(0) = \sqrt{\sigma} \leq h_0$, we thus see that $(\underline{u}, \overline{v}, \underline{h})$ is a lower solution to (1.2). By applying Theorem 2.4, we conclude that $h(t) \geq \underline{h}(t)$ for $0 < t \leq T^*$. Specially, it follows that

$$h(T^*) \geq \underline{h}(T^*) = \sqrt{T^*+\sigma} \geq \sqrt{T^*}.$$

Since $\lambda_1(d_1, \sqrt{T^*}, a_1 - c_1V) < 0$, we deduce from Theorem 3.3 that

$$h(T^*) \geq \sqrt{T^*} > R^*(d_1, a_1 - c_1V),$$

which implies that $h_\infty > R^*(d_1, a_1 - c_1 V)$. It follows from Lemma 5.2 that $h_\infty = \infty$, that is, spreading occurs. The proof is complete. □

Lemmas 5.6 and 5.7 imply that for the invasive species, there exists a conditional selection for fast dispersal rate.

Now we can derive the sharp criterion spreading-vanishing for the invasive species.

Theorem 5.8 *For any $d_1 > 0$ and given (u_0, v_0) satisfying (1.3), there exists $\mu^* \geq 0$ depending on u_0, v_0, h_0 and d_1 such that spreading occurs if $\mu > \mu^*$, and vanishing occurs if $0 < \mu \leq \mu^*$. Moreover, $\mu^* = 0$ if $0 < d_1 \leq D^*(h_0, a_1 - c_1 V)$, $\mu^* \geq 0$ if $d_1 > D^*(h_0, a_1 - c_1 V)$, and $\mu^* > 0$ if $d_1 > D^*(h_0, a_1)$.*

We remark that in the above consequences, parameter μ can be replaced by the initial function $u_0(r)$. In fact, we have the following results.

Lemma 5.9 *If $d_1 > D^*(h_0, a_1)$, then for any given (u_0, v_0) satisfying (1.3), vanishing occurs if $\|u_0\|_{L^\infty([0,\infty))}$ sufficiently small.*

Lemma 5.10 *If $d_1 > D^*(h_0, a_1 - c_1 V)$, then for any given (u_0, v_0) satisfying (1.3), spreading occurs if $\|u_0\|_{L^\infty([0,\infty))}$ sufficiently large.*

We note that Lemmas 5.9 and 5.10 are the counterparts of [25, Theorem 4.2] and [16, Lemma 5.1] respectively, and they can be proved by the similar arguments used in the proofs of Lemmas 5.6 and 5.7, so we omit them here.

The following theorem is the direct result of Lemmas 5.9 and 5.10, and can be proved by the similar arguments used in the proofs of Theorem 5.8, so we omit it again.

Theorem 5.11 *For any $d_1 > 0$ and given v_0 satisfying (1.3), if $u_0(r) = \delta\theta(r)$ for some $\delta > 0$ and $\theta(r)$ such that u_0 satisfying (1.3), then there exists δ^* depending on θ, v_0 and d_1 such that spreading occurs if $\delta > \delta^*$, and vanishing occurs if $0 < \delta \leq \delta^*$. Moreover, $\delta^* = 0$ if $0 < d_1 \leq D^*(h_0, a_1 - c_1 V)$, $\delta^* \geq 0$ if $d_1 > D^*(h_0, a_1 - c_1 V)$, and $\delta^* > 0$ if $d_1 > D^*(h_0, a_1)$.*

5.3 Estimates of spreading speed

In this subsection, we will give some rough estimates on the spreading speed of the free boundary $r = h(t)$ for the spreading case. To this aim, let us first denote the followings

$$a_{i\infty} \triangleq \liminf_{r\to\infty} a_i(r), \quad a_i^\infty \triangleq \limsup_{r\to\infty} a_i(r)$$

for $i = 1, 2$, and $b_{i\infty}$, b_i^∞, $c_{i\infty}$ and $c_i^\infty (i = 1, 2)$ are defined analogously. Obviously, we have $a(b,c)_{iL} \le a(b,c)_{i\infty} \le a(b,c)_i^\infty \le a(b,c)_{iM}$ for $i = 1, 2$.

The following proposition is taken from [6], whose complete proof is given in [3].

Proposition 5.12 *Let D, a and b be given positive constants. Then for any $K \in [0, 2\sqrt{aD})$, the following problem*

$$\begin{cases} -DU'' + KU' = aU - bU^2, \ 0 < r < \infty, \\ U(0) = 0, \ U(\infty) = \frac{a}{b}, \end{cases} \tag{5.6}$$

admits a unique positive solution $U = U_K = U_{a,b,D,K}$ satisfying $U_K'(r) > 0$ for $r \ge 0$, $U_{K_1}'(0) > U_{K_2}'(0)$, $U_{K_1}(r) > U_{K_2}(r)$ for $r \ge 0$ and $K_1 > K_2$, and for each $\mu > 0$, there exists a unique $K_0 = K_0(\mu, a, b, D) \in [0, 2\sqrt{aD})$ such that $\mu U_{K_0}'(0) = K_0$. Moreover,

$$\lim_{\frac{a\mu}{bD} \to \infty} \frac{K_0}{\sqrt{aD}} = 2, \quad \lim_{\frac{a\mu}{bD} \to 0} \frac{K_0}{\sqrt{aD}} \frac{bD}{a\mu} = \frac{\sqrt{3}}{3}.$$

We now have the following estimates for the spreading speed of the free boundary $r = h(t)$.

Theorem 5.13 *If $h_\infty = \infty$, that is, the spreading occurs, then*

$$K_0(\mu, a_{1L} - \frac{a_{2M} c_{1M}}{c_{2L}}, b_{1M}, d_1) \le \liminf_{t\to\infty} \frac{h(t)}{t} \le \limsup_{t\to\infty} \frac{h(t)}{t} \le K_0(\mu, a_{1M}, b_{1L}, d_1).$$

Furthermore,

$$K_0(\mu, a_{1\infty} - \frac{a_2^\infty c_1^\infty}{c_{2\infty}}, b_1^\infty, d_1) \le \liminf_{t\to\infty} \frac{h(t)}{t} \le \limsup_{t\to\infty} \frac{h(t)}{t} \le K_0(\mu, a_1^\infty, b_{1\infty}, d_1).$$

Proof. Since the method is similar, we only prove the second estimates.

We first show that $\limsup_{t\to\infty} \frac{h(t)}{t} \le K_0(\mu, a_1^\infty, b_{1\infty}, d_1)$. It follows from the comparison principle [6, Lemma 2.6] that (u, h) is a lower to the following auxiliary problem

$$\begin{cases} \overline{u}_t - d_1 \Delta \overline{u} = [a_1(r) - b_1(r)\overline{u}]\overline{u}, & 0 < r < \overline{h}(t), t > 0, \\ \overline{u}_r(0,t) = \overline{u}(\overline{h}(t), t) = 0, & t > 0, \\ \overline{h}'(t) = -\mu \overline{u}_r(\overline{h}(t), t), & t > 0, \\ \overline{u}(r, 0) = u(r, 0), \qquad \overline{h}(0) = h_0, & 0 < r < h_0. \end{cases}$$

We thus see that $\overline{h}(t) \ge h(t) \to \infty$ as $t \to \infty$. By [6, Theorem 3.6], we have

$$\limsup_{t\to\infty} \frac{\overline{h}(t)}{t} \le K_0(\mu, a_1^\infty, b_{1\infty}, d_1),$$

which directly yields our upper estimate.

On the other hand, similar in the proof of Theorem 5.5, we find that for any given $0 < \epsilon < \epsilon_2 = \frac{a_{1L}}{c_{1M}}/\frac{a_{2M}}{c_{2L}} - 1$, there exists $T_\epsilon > 0$ such that $v(r,t) \leq (1+\epsilon)V(r)$ for any $r \in [0,\infty)$, $t \geq T_\epsilon$. Since $h\infty = \infty$, there still exists $T'_\epsilon > 0$ such that $h(T'_\epsilon) > R^*(d_1, a_1 - (1+\epsilon)c_1 V)$. By choosing $\overline{T}_\epsilon \triangleq \max\{T_\epsilon, T'_\epsilon\}$, we see that (u,h) satisfies

$$\begin{cases} u_t - d_1\Delta u \geq [a_1(r) - c_1(r)(1+\epsilon)V - b_1(r)u]u, & 0 < r < h(t), t > \overline{T}_\epsilon, \\ u_r(0,t) = u(h(t),t) = 0, & t > \overline{T}_\epsilon, \\ h'(t) = -\mu u_r(h(t),t), & t > \overline{T}_\epsilon, \\ u(r,\overline{T}_\epsilon) > 0, & 0 < r < h(\overline{T}_\epsilon). \end{cases}$$

The comparison principle [6, Lemma 2.6] implies that (u,h) is an upper solution to the following problem

$$\begin{cases} \underline{u}_t - d_1\Delta \underline{u} = [a_1(r) - c_1(r)(1+\epsilon)V - b_1(r)\underline{u}]\underline{u}, & 0 < r < \underline{h}(t), t > \overline{T}_\epsilon, \\ \underline{u}_r(0,t) = \underline{u}(\underline{h}(t),t) = 0, & t > \overline{T}_\epsilon, \\ \underline{h}'(t) = -\mu \underline{u}_r(\underline{h}(t),t), & t > \overline{T}_\epsilon, \\ \underline{u}(r,\overline{T}_\epsilon) = u(r,\overline{T}_\epsilon), \ \underline{h}(\overline{T}_\epsilon) = h(\overline{T}_\epsilon), & 0 < r < h(\overline{T}_\epsilon). \end{cases}$$

Since $h(\overline{T}_\epsilon) > R^*(d_1, a_1 - (1+\epsilon)c_1 V)$, [6, Lemma 2.2] yields that $\underline{h} \to \infty$ as $t \to \infty$. Moreover, it is well known (see, e.g. [10]) that $\limsup_{r\to\infty} V(r) \leq \frac{a_2^\infty}{c_{2\infty}}$. By [6, Theorem 3.6], we have $K_0(\mu, a_{1\infty} - (1+\epsilon)\frac{a_2^\infty c_1^\infty}{c_{2\infty}}, b_1^\infty, d_1) \leq \liminf_{t\to\infty} \frac{h(t)}{t}$, which implies that

$$K_0(\mu, a_{1\infty} - (1+\epsilon)\frac{a_2^\infty c_1^\infty}{c_{2\infty}}, b_1^\infty, d_1) \leq \liminf_{t\to\infty} \frac{h(t)}{t}$$

for any $0 < \epsilon \leq \epsilon_1$. Due to the arbitrariness of ϵ and the continuity of K_0 with respect to its arguments, we thus obtain the desired lower estimate. $\square$

We still note that since there do not exist appropriate comparison functions, for example, some traveling waves, due to the heterogeneous environment assumption, thus only the weaker estimates of spreading speed as above can be obtained.

References

[1] D. G. Aronson, H. F. Weinberger, Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation, in Partial Differential Equations and Related Topics, J. A. Goldstein, ed., Lecture Notes in Mathematics Ser., 446. Springer-Verlag, 1975, 5-49.

[2] D. G. Aronson, H. F. Weinberger, Multidimensional nonlinear diffusion arising in population dynamics, Adv. Math., 30 (1978), 33-76.

[3] G. Bunting, Y. Du, K. Krakowski, Spreading speed revisited: Analysis of a free boundary model, Netw. Heterog. Media, 7(2012), 583-603.

[4] R.S. Cantrell, C. Cosner, Spatial Ecology via Reaction-Diffusion Equations, John Wiley & Sons Ltd., 2003.

[5] C. Cosner and A. C. Lazer, Stable coexistence states in the Volterra-Lotka competition model with diffusion, SIAM J. Appl. Math., 44(1984), 1112-1132.

[6] Y.H. Du, Z. Guo, Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary II, J. Differential Equations, 250(2011), 4336-4366.

[7] Y.H. Du, Z.M. Guo, R. Peng, A diffusive logistic model with a free boundary in time-periodic environment, J. Funct. Anal., 265(2013), 2089-2142.

[8] Y. Du, Z.G. Lin, Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary, SIAM J. Math. Anal., 42(2010), 377-405.

[9] Y.H Du, Z.G Lin, The diffusive competition model with a free boundary: Invasion of a superior or inferior competitor, to appear in Discrete Cont. Dyn. Syst.-B (Special issue for Chris Cosner)(2013), arXiv1303.0454.

[10] Y.H Du, L.S. Liu, Remarks on the uniqueness problem for the logistic equation on the entire space, Bull. Aust. Math. Soc., 73(2006), 129-137.

[11] Y.H. Du, B.D. Lou, Spreading and vanishing in nonlinear diffusion problems with free boundaries, arXiv:1301.5373, 2013.

[12] Y.H. Du, L. Ma, Logistic type equations on $\mathbb{R}^N$ by a squeezing method involving boundary blow-up solutions, J. Lond. Math. Soc., 64(2)(2001), 107-124.

[13] J. S. Guo, C. H. Wu, On a free boundary problem for a two-species weak com- petition system, J. Dynam. Differential Equations, 24(2012), 873-895.

[14] G. Hetzer and W. Shen, Uniform persistence, coexistence, and extinction in almost periodic/nonautonomous competition diffusion systems, SIAM J. Math. Anal., 34(2002), 204-227.

[15] Z.G. Lin, A free boundary problem for a predator-prey model, Nonlinearity, 20(2007), 1883-1892.

[16] C.X. Lei, Z.G. Lin and Q.Y. Zhang,The spreading front of invasive species in favorable habitat or unfavorable habitat, J. Differential Equations, 257(2014), 145-166..

[17] Y. Morita, K. Tachibana, An entire solution to the Lotka-Volterra competition-diffusion equations, SIAM J. Math. Anal., 40(2009), 2217-2240.

[18] R. Peng, X.Q. Zhao, The diffusive logistic model with a free boundary and seasonal succession, Discrete Contin. Dyn. Syst., 33(2013), 2007-2031.

[19] H. L. Smith, Monotone Dynamical Systems, American Math. Soc., Providence, 1995.

[20] M.X. Wang, On some free boundary problems of the Lotka-Volterra type prey-predator model, J. Differential Equations, 256(2014), 3365-3394.

[21] M.X. Wang, The diffusive logistic equation with a free boundary and sign-changing coefficient, arXiv:1404.4565v3, 2014.

[22] M.X. Wang, J.F. Zhao, A free boundary problem for a predator-prey model with double free boundaries, arXiv:1312.7751, 2013.

[23] M.X. Wang, J.F. Zhao, Free boundary problem for a Lotka-Volterra competition system, J. Dynam. Differential Equations, DOI 10.1007/s10884-014-9363-4, 2014.

[24] J.F. Zhao, M. X. Wang, A free boundary problem of a predator-prey model with higher dimension and heterogeneous environment, Nonlinear Anal.: Real World Appl., 16(2014), 250-263.

[25] P. Zhou, D.M Xiao, The diffusive logistic model with a free boundary in heterogeneous environment, J. Differential Equations, 256(2014), 1927-1954.