J. Math. Anal. Appl. 424 (2015) 289–303

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Two variable extensions of the Laguerre and disc polynomials

Mourad E.H. Ismail [a,b,1], Jiang Zeng [c,*]

[a] Department of Mathematics, King Saud University, Riyadh, Saudi Arabia
[b] Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA
[c] Université de Lyon, Université Lyon 1, Institut Camille Jordan, UMR 5208 du CNRS, 69622 Villeurbanne Cedex, France

ARTICLE INFO

Article history:
Received 10 July 2014
Available online 11 November 2014
Submitted by K. Driver

Keywords:
$2D$-Laguerre polynomials
(Zernike) disc polynomials
Integrals of products of orthogonal polynomials
Generating functions
Connection relations
Combinatorial interpretations

ABSTRACT

This work contains a detailed study of a one parameter generalization of the $2D$-Hermite polynomials and a two parameter extension of Zernike's disc polynomials. We derive linear and bilinear generating functions, and explicit formulas for our generalizations and study integrals of products of some of these $2D$ orthogonal polynomials. We also establish a combinatorial inequality involving elementary symmetric functions and solve the connection coefficient problem for our polynomials.

1. Introduction

The $2D$-Hermite polynomials

$$H_{m,n}(z_1, z_2) = \sum_{k=0}^{m\wedge n} \binom{m}{k}\binom{n}{k}(-1)^k k! z_1^{m-k} z_2^{n-k} \tag{1.1}$$

were introduced by Ito in [16] and have many applications to physical problems, see [2,4,22,25–27]. Mathematical properties of these polynomials have been developed in [7–9]. A multilinear generating function, of Kibble–Slepian type [17], is proved in [11]. The combinatorics of integrals of products of $2D$-Hermite polynomials has been explored in [13] while the combinatorics of the $2D$-Hermite polynomials, of their generating functions including the Kibble–Slepian type formula is in our forthcoming paper, [14], Ismail

* Corresponding author.
E-mail address: zeng@math.univ-lyon1.fr (J. Zeng).

[1] Research supported by the NPST Program of King Saud University; project number 10-MAT1293-02 and the DSFP at King Saud University.

http://dx.doi.org/10.1016/j.jmaa.2014.11.015

and Zhang [15] gave two q-analogues of the $2D$-Hermite polynomials. They studied these q-polynomials in great detail.

Ismail and Zhang [15] identified a general class of two variable polynomials whose measure is the product of the uniform measure on the circle times a radial measure. This class not only contains the $2D$-Hermite polynomials and their q-analogues but it also contains the generalized Zernike (or disc) polynomials and their q-analogues. This will be formulated in Section 2. The generalized disc polynomials have been known for a long time, see [20,19,18]. More recent papers are [28] and [1]. They form a one parameter generalization of the original Zernike polynomials.

In Section 3 we study a one parameter extension of the $2D$-Hermite polynomials. These polynomials appeared in [15] and are denoted by $Z^{(\beta)}_{m,n}(z_1, z_2)$. We record the definition, orthogonality relation, and the three term recurrence relations in Section 3. In Section 3 we also derive several differential properties of our polynomials. Section 4 contains a two parameter generalization of Zernike polynomials, so they contain one additional parameter. Several authors considered the sign regularity of integrals of products of orthogonal polynomials times certain functions. Some of the literature on this problem is in Askey's classic [3], see also Chapter 9 of [10]. In Section 5 we analyze the positivity of the integrals

$$\int_0^\infty \prod_{j=1}^N L^{(\alpha-n_j)}_{n_j}(-x) e^{-\lambda x} dx.$$

Our analysis leads to a curious rationale symmetric functions with nonnegative integral coefficients. This will be stated as Theorem 5.6.

2. General construction

The general construction given here for $2D$-systems is due to Ismail and Zhang [15]. One starts with a system of orthogonal polynomials $\{\phi_n(r;\alpha)\}$ satisfying the orthogonality relation

$$\int_0^\infty \phi_m(r;\alpha)\phi_n(r;\alpha) r^\alpha d\mu(r) = \zeta_n(\alpha)\delta_{m,n}, \quad \alpha \ge 0. \tag{2.1}$$

It is assumed the μ does not depend on α. Let

$$\phi_n(r;\alpha) = \sum_{j=0}^n c_j(n,\alpha) r^{n-j}, \quad c_j(n,\alpha) \in \mathbb{R}, \tag{2.2}$$

and define polynomials

$$f_{n+\alpha,n}(z_1, z_2) = \sum_{j=0}^n c_j(n,\alpha) z_1^{n+\alpha-j} z_2^{n-j} = z_1^\alpha \phi_n(z_1 z_2;\alpha). \tag{2.3}$$

Also define $f_{n,m}(z_1, z_2) = f_{m,n}(z_2, z_1)$. Thus $\overline{f_{m,n}(z,\bar{z})} = f_{n,m}(z,\bar{z})$.

Theorem 2.1. *For $m \ge n$ the polynomials $\{f_{m,n}(z,\bar{z})\}$ satisfy the orthogonality relation*

$$\int_{\mathbb{R}^2} f_{m,n}(z,\bar{z})\overline{f_{s,t}(z,\bar{z})} \frac{d\theta}{2\pi} d\mu(r^2) = \zeta_n(m-n)\delta_{m,s}\delta_{n,t}. \tag{2.4}$$

We now come to the three term recurrence relations.

Theorem 2.2. *The polynomials $\{f_{m,n}(z,\bar{z})\}$ satisfy the three term recurrence relations*

$$\begin{aligned} \bar{z} f_{m,n}(z,\bar{z}) &= \frac{c_0(n,m-n+1)}{c_0(n+1,m-n)} f_{m+1,n+1}(z,\bar{z}) \\ &\quad - \frac{c_0(n,m-n+1)c_{n+1}(n+1,m-n)}{c_0(n+1,m-n)c_n(n,m-n)} f_{m,n}(z,\bar{z}), \quad m \geq n. \end{aligned} \tag{2.5}$$

It is clear that (2.5) remains valid if $\bar{z}$ is replaced by another complex variable.

Examples. Consider the case of the $2D$-Hermite polynomials when

$$c_j(n,\alpha) = \frac{n!(n+\alpha)!(-1)^j}{j!(n-j)!(n+\alpha-j)!}.$$

The recursion (2.5) leads to the three term recurrence relations

$$\begin{aligned} z_1 H_{m,n}(z_1,z_2) &= n H_{m,n-1}(z_1,z_2) + H_{m+1,n}(z_1,z_2), \\ z_2 H_{m,n}(z_1,z_2) &= m H_{m-1,n}(z_1,z_2) + H_{m,n+1}(z_1,z_2). \end{aligned} \tag{2.6}$$

Similarly the case

$$c_j(n,\alpha) = \frac{(q;q)_n (q;q)_{n+\alpha}!(-1)^j}{(q;q)_j!(q;q)_{n-j}(q;q)_{n+\alpha-j}} q^{\binom{j}{2}}$$

leads to the recurrence relations for a q-analogues of $2D$-Hermite polynomials, see [15].

While writing this paper we realized that there is a second recurrence relation satisfied by general $2D$-systems, that was missed in [15].

Theorem 2.3. *The polynomials $\{f_{m,n}(z,\bar{z})\}$ satisfy the three term recurrence relations*

$$z f_{m,n}(z,\bar{z}) - \frac{c_0(n,m-n)}{c_0(n,m+1-n)} f_{m+1,n}(z,\bar{z}) = u_{m,n} f_{m,n-1}(z,\bar{z}), \tag{2.7}$$

where

$$u_{m,n} = \frac{c_0(n,m+1-n)c_1(n,m-n) - c_0(n,m-n)c_1(n,m+1-n)}{c_0(n-1,m+1-n)c_0(n,m+1-n)}. \tag{2.8}$$

Proof. Consider the vector space of all polynomials of the form $\sum_{j=0}^n a(m,n,j) z_1^{m-j} z_2^{n-j}$ with $m \geq n \geq 0$, and $a(m,n,j) \in \mathbb{R}$. It is clear that $\{f_{m,n}(z_1,z_2) : m \geq n\}$ is a basis for such space. For $m \geq n$ write

$$z f_{m,n}(z,\bar{z}) - \frac{c_0(n,m-n)}{c_0(n,m+1-n)} f_{m+1,n}(z,\bar{z}) = \sum_{m \geq k \geq j \geq 0} b(j,k) f_{k,j}(z,\bar{z}).$$

The left-hand side of (2.7) contains only terms of the form $z^{m+1-s}(\bar{z})^{n-s}$. Hence $m+1-n = k-j$. Therefore $k > j$ and $b(j,k)$ is a nonzero multiple of

$$\int_{\mathbb{R}^2} z f_{m,n}(z,\bar{z}) \overline{f_{k,j}(z,\bar{z})} \frac{d\theta}{2\pi} d\mu(r^2) = \int_{\mathbb{R}^2} f_{m,n}(z,\bar{z}) \overline{\bar{z} f_{k,j}(z,\bar{z})} \frac{d\theta}{2\pi} d\mu(r^2).$$

The above integral vanishes if $k < m$. Therefore the left-hand side of (2.7) is a constant multiple of $f_{m,n-1}(z,\bar{z})$. The constant is then evaluated by equating coefficients of the term $z^m(\bar{z})^{n-1}$. $\square$

Let

$$r\phi_n(r;\alpha) = a_n\phi_{n+1}(r;\alpha) + c_n\phi_n(r;\alpha) + b_n\phi_{n-1}(r;\alpha) \tag{2.9}$$

be the three term recurrence relation satisfied by the polynomials $\{\phi_n(r;\alpha)\}$. Making use of (2.2) and (2.3) we conclude that the above recursion becomes

$$[z\bar{z} - c_n]f_{m,n}(z,\bar{z}) = a_n f_{m+1,n+1}(z,\bar{z}) + b_n f_{m-1,n-1}(z,\bar{z}). \tag{2.10}$$

If we normalize the polynomials $\{\phi_n(r;\alpha)\}$ by $\phi_n(0;\alpha) = 1$ then $-c_n = a_n + b_n$. This normalization makes

$$\begin{gathered} c_n(n,\alpha) = 1, \qquad a_n = \frac{c_0(n,\alpha)}{c_0(n+1,\alpha)}, \qquad a_n + b_n + c_n = 0, \\ b_n = -c_1(n,\alpha) + \frac{c_0(n,\alpha)}{c_0(n+1,\alpha)}\big[c_1(n+1,\alpha) - c_0(n,\alpha)\big]. \end{gathered} \tag{2.11}$$

Remark 2.4. In view of the defining equation (2.3) the question of finding the large m, n asymptotics of the bivariate polynomials $f_{m,n}$ is equivalent to finding the large m, n behavior of $\phi_n(r, m-n)$. It will be of interest to carry out this investigation at least for some special systems, including the Freud type polynomials orthogonal with respect to $x^\alpha \exp(-p(x))$, where p is a polynomial with positive leading term.

In general

$$\left[z_1\frac{\partial}{\partial z_1} - z_2\frac{\partial}{\partial z_2}\right] f_{m,n}(z_1,z_2) = (m-n)f_{m,n}(z_1,z_2), \quad m \ge n, \tag{2.12}$$

holds because the operator on the left-hand side of (2.12) is $-i\frac{\partial}{\partial\theta}$ when $z_1 = z = re^{i\theta}$ and $z_2 = \bar{z}$.

We now discuss generating functions. Assume we know a generating function for the ϕ_n's,

$$\sum_{n=0}^{\infty} \phi_n(r;\alpha)t^n = F(r,t,\alpha). \tag{2.13}$$

We have

$$\begin{aligned} \sum_{m,n=0}^{\infty} f_{m,n}(z_1,z_2)u^m v^n &= \sum_{m\ge n} + \sum_{m\le n} - \sum_{m=n} \\ &= S_1(z_1,z_2,u,v) + S_2(z_1,z_2,u,v) - S_3(z_1,z_2,u,v) \\ &= \sum_{\alpha=0}^{\infty}\big((z_1u)^\alpha + (z_2u)^\alpha\big)F(z_1z_2,uv,\alpha) - F(z_1z_2,uv,0), \end{aligned}$$

because, see (2.3),

$$S_1 = \sum_{\alpha,n=0}^{\infty} z_1^\alpha\phi_n(z_1z_2;\alpha)u^{n+\alpha}v^n = \sum_{\alpha=0}^{\infty}(z_1u)^\alpha F(z_1z_2,uv,\alpha), \tag{2.14}$$

$S_2(z_1,z_2,u,v) = S_1(z_2,z_1,v,u)$ and

$$S_3(z_1, z_2, u, v) = \sum_{n=0}^{\infty} (uv)^n \phi_n(z_1 z_2; 0) = F(z_1 z_2, uv, 0). \tag{2.15}$$

Similarly one can establish the exponential generating function

$$\sum_{m \geq n \geq 0} \frac{u^m v^n}{(m-n)!} f_{m,n}(z_1, z_2) = \sum_{\alpha=0}^{\infty} \frac{(uz_1)^\alpha}{\alpha!} F(z_1 z_2, uv, \alpha). \tag{2.16}$$

Note that Malodano [19] considered polynomials orthogonal with respect to spherically symmetric weights. This is essentially the class of polynomials in this section. He only considered the radial part, and did not record the orthogonality relation, nor did he study the recurrence relations or the structure relation.

3. The polynomials $\{Z^{(\beta)}_{m,n}(z_1, z_2)\}$

Motivated by the class of general 2D-systems in Section 2 we define polynomials $\{Z^{(\beta)}_{m,n}\}$ by

$$Z^{(\beta)}_{m,n}(z_1, z_2) = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \frac{(\beta+1)_m}{(\beta+1)_{m-k}} (-1)^k z_1^{m-k} z_2^{n-k}, \tag{3.1}$$

for $m \geq n$, where $\beta \geq 0$. When $m \leq n$ the polynomials are defined by

$$Z^{(\beta)}_{m,n}(z_1, z_2) = Z^{(\beta)}_{n,m}(z_2, z_1). \tag{3.2}$$

These polynomials arise through the choice $\phi_n(r; \alpha) = (-1)^n L_n^{(\alpha+\beta)}(r)$, where $L_n^{(a)}(x)$ is a Laguerre polynomial, [5,10,21,23], defined by

$$L_n^{(a)}(x) = \frac{(a+1)_n}{n!} \sum_{k=0}^{n} \binom{n}{k} \frac{(-x)^k}{(a+1)_k}. \tag{3.3}$$

When $\beta = 0$, from (1.1) and (3.1) it follows that

$$H_{m,n}(z_1, z_2) = (m \wedge n)! Z^{(0)}_{m,n}(z_1, z_2). \tag{3.4}$$

By (2.3) we have

$$Z^{(\beta)}_{m,n}(z_1, z_2) = (-1)^n z_1^{m-n} L_n^{(\beta+m-n)}(z_1 z_2) \quad (m \geq n). \tag{3.5}$$

Therefore in the notation of Section 2, we have

$$\zeta_n(\alpha) = \frac{\Gamma(\alpha+\beta+n+1)}{n!}, \qquad c_j(n, \alpha) = \frac{(-1)^j (\alpha+\beta+1)_n}{(n-j)! j! (\alpha+\beta+1)_{n-j}}. \tag{3.6}$$

Theorem 3.1. *We have the orthogonality relation*

$$\int_{\mathbb{R}^2} Z^{(\beta)}_{m,n}(z, \bar{z}) \overline{Z^{(\beta)}_{s,t}(z, \bar{z})} r^{2\beta+1} e^{-r^2} dr d\theta = \pi \frac{\Gamma(\beta+m+1)}{n!} \delta_{m,s} \delta_{n,t}, \tag{3.7}$$

for $m \geq n$.

This result follows from Theorem 2.1 and (3.6).

In view of (3.6) the recurrence relations (2.5) become

$$\begin{aligned} z_1 Z_{m,n}^{(\beta)}(z_1,z_2) &= Z_{m+1,n}^{(\beta)}(z_1,z_2) + Z_{m,n-1}^{(\beta)}(z_1,z_2), \quad m \ge n \\ z_2 Z_{m+1,n}^{(\beta)}(z_1,z_2) &= (n+1) Z_{m+1,n+1}^{(\beta)}(z_1,z_2) + (\beta+m+1) Z_{m,n}^{(\beta)}(z_1,z_2), \quad m \ge n. \end{aligned} \tag{3.8}$$

Theorem 3.2. *The polynomials* $\{Z_{m,n}^{(\beta)}(z_1,z_2)\}$ *have the generating functions*

$$\sum_{m\ge n\ge 0} u^m v^n Z_{m,n}^{(\beta)}(z_1,z_2) = (1+uv)^{-\beta-1} \exp\left(\frac{uvz_1z_2}{1+uv}\right) \frac{1+uv}{1+uv-uz_1}, \tag{3.9}$$

$$\sum_{m,n\ge 0} u^m v^n Z_{m,n}^{(\beta)}(z_1,z_2) = (1+uv)^{-\beta-1} \exp\left(\frac{uvz_1z_2}{1+uv}\right) \left[\frac{1+uv}{1+uv-uz_1} + \frac{1+uv}{1+uv-vz_2} - 1\right], \tag{3.10}$$

$$\sum_{m\ge n\ge 0} \frac{u^m v^n}{(m-n)!} Z_{m,n}^{(\beta)}(z_1,z_2) = (1+uv)^{-\beta-1} \exp\left(\frac{uvz_1z_2 + z_1u}{1+uv}\right), \tag{3.11}$$

and

$$\sum_{m,n=0}^{\infty} \frac{u^m v^n}{(|m-n|)!} Z_{m,n}^{(\beta)}(z_1,z_2) = (1+uv)^{-\beta-1} \exp\left(\frac{uvz_1z_2}{1+uv}\right) \left[\exp\left(\frac{z_1u}{1+uv}\right) + \exp\left(\frac{z_2v}{1+uv}\right) - 1\right]. \tag{3.12}$$

Proof. We write the sum at the left-hand side of (3.10) as $S_1 + S_2 - S_3$, where

$$S_1 = \sum_{m\ge n\ge 0}, \qquad S_2 = \sum_{0\le m\le n}, \qquad S_3 = \sum_{m=n\ge 0}.$$

First of all, we note that S_3 is essentially the generating functions of Laguerre polynomials:

$$S_3 = (1+uv)^{-\beta-1} \exp\left(\frac{z_1z_2uv}{1+uv}\right). \tag{3.13}$$

Writing $m = n+k$ with $k \ge 0$, by (3.8) we have

$$\begin{aligned} S_1 &= \sum_{m\ge n\ge 0} (-1)^n z_1^{m-n} L_n^{(\beta+m-n)}(z_1z_2) u^m v^n \\ &= \sum_{k,n\ge 0} (z_1u)^k (uv)^n L_n^{(\beta+k)}(z_1z_2)(-1)^n \\ &= \sum_{k\ge 0} (z_1u)^k (1+uv)^{-\beta-k-1} \exp\left(\frac{z_1z_2uv}{1+uv}\right) \\ &= (1+uv)^{-\beta-1} \exp\left(\frac{z_1z_2uv}{1+uv}\right) \frac{1}{1 - z_1u/(1+uv)}. \end{aligned}$$

This proves (3.9). By symmetry we have

$$S_2 = (1+uv)^{-\beta-1} \exp\left(\frac{z_1z_2uv}{1+uv}\right) \frac{1}{1 - z_2v/(1+uv)}.$$

Combining the above three summations yields the first generating function (3.10). The derivation of (3.11) is almost the same as (3.9). Formula (3.12) follows similarly from (3.13) and (3.11). □

Corollary 3.3. *We have the connection relation*

$$Z_{m,n}^{(\beta)}(z_1,z_2)=\sum_{j=0}^{n}\frac{(\beta-\gamma)_j}{j!}(-1)^j Z_{m-j,n-j}^{(\gamma)}(z_1,z_2)\quad (m\ge n). \tag{3.14}$$

This follows from (3.9) by writing $(1+uv)^{-\beta+1}=(1+uv)^{-\beta-\gamma}(1+uv)^{\gamma-1}$ and applying the binomial theorem to $(1+uv)^{\beta-\gamma}$. Two special cases $\gamma=0$ and $\beta=0$ of Corollary 3.3 are worth noting. For $m\ge n$, they are

$$n!Z_{m,n}^{(\beta)}(z_1,z_2)=\sum_{j=0}^{n}\binom{n}{j}(\beta)_j(-1)^j H_{m-j,n-j}(z_1,z_2), \tag{3.15}$$

$$H_{m,n}(z_1,z_2)=n!\sum_{j=0}^{n}\frac{(-\gamma)_j}{j!}(-1)^j Z_{m-j,n-j}^{(\gamma)}(z_1,z_2). \tag{3.16}$$

Lemma 3.4 *(Hille–Hardy). (See [21].) The Poisson kernel for Laguerre polynomials is essentially*

$$\sum_{n=0}^{\infty}\frac{n!r^n}{(\alpha+1)_n}L_n^{(\alpha)}(x)L_n^{(\alpha)}(y)=(1-r)^{-\alpha-1}\exp\bigl(-r(x+y)/(1-r)\bigr){}_0F_1\bigl(-;\alpha+1;xyr/(1-r)^2\bigr). \tag{3.17}$$

Theorem 3.5. *The polynomials $\{Z_{m,n}^{(\beta)}(z_1,\bar z_1)\}$ have the bilinear generating function*

$$\begin{aligned}&\sum_{m,n\ge 0}\frac{(m\wedge n)!u^m v^n}{(\beta+|m-n|+1)_{m\wedge n}}Z_{m,n}^{(\beta)}(z_1,\bar z_1)Z_{m,n}^{(\beta)}(z_2,\bar z_2)\\ &\quad=(1-uv)^{-\beta-1}\exp\left(-\frac{uv(z_1\bar z_1+z_2\bar z_2)}{1-uv}\right)\\ &\qquad\times\left[\sum_{k\ge 0}\frac{(uz_1\bar z_1)^k+(vz_2\bar z_2)^k}{(1-uv)^k}{}_0F_1\left(\left.\begin{matrix}-\\ \beta+k+1\end{matrix}\,\right|\,\frac{z_1\bar z_1 z_2\bar z_2 uv}{(1-uv)^2}\right)-{}_0F_1\left(\left.\begin{matrix}-\\ \beta+1\end{matrix}\,\right|\,\frac{z_1\bar z_1 z_2\bar z_2 uv}{(1-uv)^2}\right)\right].\end{aligned} \tag{3.18}$$

Proof. We write the sum on the left-hand side as $S_1+S_2-S_3$ where

$$S_1=\sum_{m\ge n\ge 0},\qquad S_2=\sum_{0\le m\le n},\qquad S_3=\sum_{m=n\ge 0}.$$

First by (3.5) and (3.17), we note that S_3 is the Hille–Hardy formula of Laguerre polynomials:

$$S_3=(1-uv)^{-\beta-1}\exp\left(-\frac{uv(z_1\bar z_1+z_2\bar z_2)}{1-uv}\right){}_0F_1\left(\left.\begin{matrix}-\\ \beta+1\end{matrix}\,\right|\,\frac{z_1\bar z_1 z_2\bar z_2 uv}{(1-uv)^2}\right).$$

Writing $m=n+k$ with $k\ge 0$ we have

$$\begin{aligned} S_1 &= \sum_{m,n\geq 0} \frac{n!u^m v^n}{(\beta+m-n+1)_n} Z^{(\beta)}_{m,n}(z_1,\bar z_1) Z^{(\beta)}_{m,n}(z_2,\bar z_2) \\ &= \sum_{k,n\geq 0} \frac{n!(uv)^n (uz_1z_2)^k}{(\beta+k+1)_n} L_n^{(\beta+k)}(z_1\bar z_1) L_n^{(\beta+k)}(z_2\bar z_2) \\ &= (1-uv)^{-\beta-1}\exp\left(-\frac{uv(z_1\bar z_1+z_2\bar z_2)}{1-uv}\right) \\ &\quad \times \sum_{k\geq 0} \frac{(uz_1z_2)^k}{(1-uv)^k}\, {}_0F_1\left(\begin{matrix} - \\ \beta+k+1 \end{matrix} \,\middle|\, \frac{z_1\bar z_1 z_2 \bar z_2 uv}{(1-uv)^2}\right). \end{aligned}$$

By (3.2) and symmetry we have

$$\begin{aligned} S_2 &= (1-uv)^{-\beta-1}\exp\left(-\frac{uv(z_1\bar z_1+z_2\bar z_2)}{1-uv}\right) \\ &\quad \times \sum_{k\geq 0} \frac{(v\bar z_1\bar z_2)^k}{(1-uv)^k}\, {}_0F_1\left(\begin{matrix} - \\ \beta+k+1 \end{matrix} \,\middle|\, \frac{z_1\bar z_1 z_2 \bar z_2 uv}{(1-uv)^2}\right). \end{aligned}$$

Combining the above three summations yields the desired formula. □

Note that the generating function (3.18) remains valid if we replace $\bar z_1$ and $\bar z_2$ by two general complex variables z_3 and z_4.

It must be noted that the bilinear generating function (3.18) is not the Poisson kernel of the polynomials $Z^{(\beta)}_{m,n}(z,\bar z)$ in view of the orthogonality relation (3.7). The Poisson kernel would be a constant multiple of

$$\sum_{m,n=0}^{\infty} \frac{(m\wedge n)!u^m v^n}{(\beta+1)_{m\vee n}} Z^{(\beta)}_{m,n}(z_1,\bar z_1)\overline{Z^{(\beta)}_{m,n}(z_2,\bar z_2)}.$$

It follows from (3.1) that the polynomials $Z^{(\beta)}_{m,n}(z_1,z_2)$ satisfy the differential recurrence relations

$$z_1\frac{\partial}{\partial z_1} Z^{(\beta)}_{m,n}(z_1,z_2) = (m-n)Z^{(\beta)}_{m,n}(z_1,z_2) + z_2 Z^{(\beta)}_{m,n-1}(z_1,z_2), \tag{3.19}$$

$$\frac{\partial}{\partial z_2} Z^{(\beta)}_{m,n}(z_1,z_2) = Z^{(\beta)}_{m,n-1}(z_1,z_2), \tag{3.20}$$

for $m\geq n$. If we eliminate $z_2 Z^{(\beta)}_{m,n-1}(z_1,z_2)$ we recover (2.12) for these polynomials.

Consider the integrals

$$I(\mathbf{m},\mathbf{n},\mathbf{s},\mathbf{t}) := \prod_{j=1}^N (-1)^{n_j+s_j} \int_{\mathbb{R}^2} \prod_{j=1}^N Z^{(\beta)}_{m_j,n_j}(z,\bar z) Z^{(\beta)}_{s_j,t_j}(z,\bar z) \frac{re^{-r^2}drd\theta}{\sqrt{\pi}}. \tag{3.21}$$

We shall assume that $m_j\geq n_j, t_j\geq s_j$ for $1\leq j\leq N$. Our derivation of the generating function for $I(\mathbf{m},\mathbf{n},\mathbf{s},\mathbf{t})$ uses the elementary integral evaluation for complex a, b with $\mathrm{Re}(a)>0$

$$\int_{\mathbb{R}} e^{-ax^2+bx}dx = \sqrt{\frac{\pi}{a}}e^{-b^2/4a}. \tag{3.22}$$

Theorem 3.6. *We have the exponential generating function*

$$\sum_{m_j\ge n_j\ge 0,t_j\ge s_j\ge 0,1\le j\le N} I(\mathbf{m},\mathbf{n},\mathbf{s},\mathbf{t})\prod_{j=1}^{N}\frac{u_j^{m_j}v_j^{n_j}\xi_j^{s_j}\eta_j^{t_j}}{(m_j-n_j)!(t_j-s_j)!}$$
$$=\frac{1}{\sqrt{C}}\prod_{j=1}^{N}(1-u_jv_j)^{-\beta-1}(1-\xi_j\eta_j)^{-\beta-1}\exp\left(-\frac{1}{C}\sum_{\substack{j,k=1\\ j\neq k}}^{N}\frac{u_j}{1-u_jv_j}\frac{\eta_k}{1-\xi_k\eta_k}\right), \tag{3.23}$$

where

$$C=1+\sum_{j=1}^{N}\left\{\frac{u_jv_j}{1-u_jv_j}+\frac{\xi_j\eta_j}{1-\xi_j\eta_j}\right\}. \tag{3.24}$$

Proof. Using the generating function (3.11) and (3.2) we find that

$$\sum_{m_j\ge n_j\ge 0,t_j\ge s_j\ge 0,1\le j\le N} I(\mathbf{m},\mathbf{n},\mathbf{s},\mathbf{t})\prod_{j=1}^{N}u_j^{m_j}v_j^{n_j}\xi_j^{s_j}\eta_j^{t_j}$$
$$=\prod_{j=1}^{N}(1-u_jv_j)^{-\beta-1}(1-\xi_j\eta_j)^{-\beta-1}$$
$$\times\int_{\mathbb{R}^2}\prod_{j=1}^{N}\exp\left[-r^2-\sum_{j=1}^{N}\left\{\frac{r^2u_jv_j}{1-u_jv_j}+\frac{r^2\xi_j\eta_j}{1-\xi_j\eta_j}+\frac{(x+iy)u_j}{1-u_jv_j}+\frac{(x-iy)\eta_j}{1-\xi_j\eta_j}\right\}\right]\frac{rdrd\theta}{\sqrt{\pi}}.$$

We now change variables from polar to Cartesian coordinates and use the integral evaluation (3.22) to obtain (3.23). □

4. The polynomials $\{M_{m,n}^{(\beta,\gamma)}(z_1,z_2)\}$

In this section we consider the $2D$ version of Jacobi polynomials, see [21,10]. Let

$$\phi_n(r,\alpha)=P_n^{(\alpha+\gamma,\beta)}(1-2r), \tag{4.1}$$

which satisfy the orthogonality relation (2.1) with $d\mu(r)=-(1-r)^\beta dr$,

$$\zeta_n(\alpha)=\frac{\Gamma(\alpha+\gamma+n+1)\Gamma(\beta+n+1)}{n!\Gamma(\alpha+\beta+\gamma+n+1)(\alpha+\beta+\gamma+2n+1)}. \tag{4.2}$$

Equivalently, we define the two variable polynomials

$$M_{m,n}^{(\beta,\gamma)}(z,\bar z)=\begin{cases} z^{m-n}P_n^{(\gamma+m-n,\beta)}(1-2z\bar z), & m\ge n,\\ \bar z^{n-m}P_m^{(\gamma+n-m,\beta)}(1-2z\bar z), & n\ge m.\end{cases} \tag{4.3}$$

Therefore by (2.4) for $m\ge n$ we have

$$\int_{|z|\le 1} M_{m,n}^{(\beta,\gamma)}(z,\bar z)\overline{M_{p,q}^{(\beta,\gamma)}(z,\bar z)}r^{2\gamma+1}\left(1-r^2\right)^\beta dr\frac{d\theta}{2\pi}=\frac{\Gamma(\gamma+n+1)\Gamma(\beta+n+1)}{n!\Gamma(\beta+\gamma+n+1)(\beta+\gamma+2n+1)}\delta_{m,p}\delta_{n.q}. \tag{4.4}$$

Theorem 4.1. *The polynomials* $\{M_{m,n}^{(\beta,\gamma)}(z,\bar{z})\}$ *have the generating functions*

$$\sum_{m,n=0}^{\infty} M_{m,n}^{(\beta,\gamma)}(z,\bar{z})u^m v^n = \frac{2^{\beta+\gamma}}{\rho}(1+uv+\rho)^{-\beta}(1-uv+\rho)^{-\gamma} \times \left[\frac{1}{1-2zu/(1+\rho-uv)} + \frac{1}{1-2v\bar{z}/(1+\rho-uv)} - 1\right], \tag{4.5}$$

and

$$\sum_{m\ge n\ge 0} M_{m,n}^{(\beta,\gamma)}(z,\bar{z})\frac{u^m v^n}{(m-n)!} = \frac{2^{\beta+\gamma}}{\rho}(1+uv+\rho)^{-\beta}(1-uv+\rho)^{-\gamma}\exp\left(\frac{2zu}{1-uv+\rho}\right). \tag{4.6}$$

Proof. We shall use the generating function

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)t^n = 2^{\alpha+\beta}\rho^{-1}(1+t+\rho)^{-\beta}(1-t+\rho)^{-\alpha}, \tag{4.7}$$

where $\rho = (1-2xt+t^2)^{1/2}$, [10, (4.3.11)]. In this case $\rho = (1-2uv(1-2r^2)+u^2v^2)^{1/2}$ and by (2.13) and (4.1)

$$F(r^2, uv, \alpha) = 2^{\alpha+\beta+\gamma}\rho^{-1}(1+uv+\rho)^{-\beta}(1-uv+\rho)^{-\alpha-\gamma}. \tag{4.8}$$

This yields

$$S_1 = \frac{2^{\beta+\gamma}}{\rho}(1+uv+\rho)^{-\beta}(1-uv+\rho)^{-\gamma}\frac{1}{1-2zu/(1+\rho-uv)}. \tag{4.9}$$

Using (2.16)–(2.13) we are led to the generating functions (4.5) and (4.6). □

We next verify that (3.10) is a limiting case of (4.5). It is clear that when $m \ge n$ then

$$\lim_{\beta\to\infty} M_{m,n}^{(\beta,\gamma)}(z,\bar{z}/\beta) = z^{m-n}L_n^{(\gamma+m-n)}(z\bar{z}) = (-1)^n Z_{m,n}^{(\gamma)}(z,\bar{z}).$$

If $m \le n$ a similar analysis works with z replaced by z/β. In either case, as $\beta \to \infty$, we have

$$\rho = (1-uv)\left[1 + \frac{z\bar{z}uv}{\beta(1-uv)^2} + \mathcal{O}(\beta^{-2})\right],$$
$$[1+\rho+uv]/2 = 1 + \frac{z\bar{z}uv}{\beta(1-uv)} + \mathcal{O}(\beta^{-2}),$$
$$[1+\rho-uv]/2 = (1-uv)\left[1 + \frac{z\bar{z}uv}{\beta(1-uv)^2} + \mathcal{O}(\beta^{-2})\right].$$

Therefore $[(1+\rho+uv)/2]^\beta \to \exp(z\bar{z}uv/(1-uv))$ and after $v \to -v$ we see that $\sum_{m\ge n\ge 0}$ in (4.5) tends to the corresponding sum in (3.10). The other two sums are similar. One can similarly prove that (4.6) tends to (3.11) if $\bar{z}$ is replaced by $\bar{z}/\beta$ then let $\beta \to \infty$. The proof uses the above large β expansions and will be omitted.

It must be noted that Myrick [20] considered the radial part of the polynomials $\{M_{m,n}(z,\bar{z})\}$. However, he did not record the orthogonality relation, nor did he investigate recursions, or generating functions.

5. Integrals of products of $\{L_n^{(\alpha-n)}(x)\}$

The generating function

$$\sum_{n=0}^{\infty} L_n^{(\alpha-n)}(x)t^n = (1+t)^\alpha e^{-xt}, \tag{5.1}$$

is well known [10, Theorem 4.3.2] and follows easily from the explicit formula (3.3). We take $\alpha < 0$. It is known that the polynomials $\{L_n^{(\alpha-n)}(x)\}$ are not orthogonal with respect to a positive measure. This section is devoted to the sign behavior of the integrals

$$I(\mathbf{n}) := \int_0^\infty \prod_{j=1}^{N} L_{n_j}^{(\alpha-n_j)}(-x)e^{-\lambda x}dx, \tag{5.2}$$

where $\mathbf{n} = (n_1, \dots, n_N)$ and $\lambda > 0$.

Theorem 5.1. *The integrals defined by (5.2) have the generating function*

$$\sum_{n_j=0, 1\le j\le N}^{\infty} I(\mathbf{n}) \prod_{j=1}^{N} u_j^{n_j} = \prod_{j=1}^{N}(1+u_j)^\alpha \left[\lambda - \sum_{j=1}^{N} u_j\right]^{-1}. \tag{5.3}$$

Proof. Apply the generating function (5.1) to see that the left-hand side of (5.3) equals

$$\prod_{j=1}^{N}(1+u_j)^\alpha \int_0^\infty \exp\left(-\lambda x + x\sum_{j=1}^{N} u_j\right) dx = \prod_{j=1}^{N}(1+u_j)^\alpha \left[\lambda - \sum_{j=1}^{N} u_j\right]^{-1}. \qquad \square$$

The simplest case is $\alpha = -1$. Experiments with a symbolic algebra package convinced us that when $0 < \lambda < 1$ some of the coefficients in the expansion of the right-hand side of (5.3) are negative when $\alpha = -1$. If $\lambda > 1$, then

$$\frac{1}{1-(u_1+\cdots+u_N)} = \frac{1}{1-(u_1+\cdots+u_N)/\lambda}\left[1 + \frac{(1-1/\lambda)(u_1+\cdots+u_N)}{1-(u_1+\cdots+u_N)}\right],$$

and we see that a positivity result for $\lambda = 1$ implies the corresponding positivity result for $\lambda > 1$. In the rest of this section we assume $\lambda = 1$.

Lemma 5.2. *Let $n = n_1 + \cdots + n_N$. Then, for $0 \le r \le N$,*

$$\frac{(1-u_1)\dots(1-u_r)}{1-(u_1+\cdots+u_N)} = \sum_{n_1,\dots,n_N\ge 0} \frac{1}{n!}\binom{n}{n_1,\dots,n_N}\left(\sum_{k=0}^{r}(-1)^k(n-k)!e_k(n_1,\dots,n_N)\right) u_1^{n_1}\dots u_N^{n_N}, \tag{5.4}$$

where e_k denotes the k-th elementary symmetric function.

Proof. Note that for $1 \le r \le N$ we have

$$\frac{u_r}{1-(u_1+\cdots+u_N)} = \sum_{n_1,\dots,n_N\ge 0} \frac{n_r}{n}\binom{n}{n_1,\dots,n_N} u_1^{n_1}\dots u_N^{n_N},$$

and more generally, for k distinct integers $i_1, \dots, i_k$ in $\{1, \dots, N\}$,

$$\frac{u_{i_1}u_{i_2}\dots u_{i_k}}{1-(u_1+\dots+u_N)} = \sum_{n_1,\dots,n_N\ge 0} \frac{n_{i_1}n_{i_2}\dots n_{i_k}}{(n-k+1)_k}\binom{n}{n_1,\dots,n_N} u_1^{n_1}\dots u_N^{n_N}.$$

This yields immediately (5.4). □

Lemma 5.3. *Let $\mathbf{X} = (x_1, \dots, x_N)$ be a sequence of N real numbers such that $x_i \ge 1$ for $i = 1, \dots, N$. Then, for $k = 0, 1, \dots,$*

$$(x_1 + \dots + x_N - 2k)e_{2k}(\mathbf{X}) - (2k+1)e_{2k+1}(\mathbf{X}) \ge 0. \tag{5.5}$$

Moreover, the equality holds if and only if $x_i = 1$ for $i = 1, \dots, N$.

Proof. Let $e_j(\mathbf{X}_j^-)$ denote the elementary symmetric function of $\mathbf{X} \setminus \{x_j\}$. Then it is not hard to see that

$$\begin{aligned}
(2k+1)e_{2k+1}(\mathbf{X}) &= \sum_{j=1}^{N} x_j \sum_{\substack{i_1<\dots<i_{2k}\\ j\neq i_1,\dots,i_{2k}}} x_{i_1}\cdots x_{i_{2k}} \\
&= \sum_{j=1}^{N} x_j\big(e_{2k}(\mathbf{X}) - x_j e_{2k-1}(\mathbf{X}_j^-)\big) \\
&= (x_1+\dots+x_N)e_{2k}(\mathbf{X}) - \sum_{j=1}^{N} x_j^2 e_{2k-1}\big(\mathbf{X}_j^-\big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
(x_1+\dots+x_N-2k)e_{2k}(\mathbf{X}) - (2k+1)e_{2k+1}(\mathbf{X}) &= \sum_{j=1}^{N} x_j^2 e_{2k-1}\big(\mathbf{X}_j^-\big) - 2ke_{2k}(\mathbf{X}) \\
&= \sum_{i_1<\dots<i_{2k}} x_{i_1}\cdots x_{i_{2k}}(x_{i_1}+\dots+x_{i_{2k}}-2k),
\end{aligned}$$

which is clearly a sum of nonnegative numbers and equal to zero if and only if $x_i = 1$ for $i = 1, \dots, N$. □

Lemma 5.4. *Let $\mathbf{n} := (n_1, \dots, n_N)$ be a sequence of nonnegative integers and $n = n_1 + \dots + n_N$. Let $\varphi : \mathbb{R}[t] \to \mathbb{R}$ be the Laguerre linear functional such that $\varphi(t^k) = k!$ for $k \ge 0$. Then*

$$\varphi\big(t^{n-N}(t-n_1)\dots(t-n_N)\big) = \sum_{k=0}^{N}(-1)^k(n-k)!e_k(\mathbf{n}) \ge 0. \tag{5.6}$$

Proof. Let $a_k = (n-k)!e_k(\mathbf{n})$ for $k = 0, \dots, N$ and $a_k = 0$ if $k > N$. Then

$$\sum_{k=0}^{N}(-1)^k(n-k)!e_k(\mathbf{n}) = \sum_{0\le 2k\le N}(a_{2k} - a_{2k+1}).$$

Now, by Theorem 5.3 each term

$$a_{2k} - a_{2k+1} = (n-2k-1)!\big((n-2k)e_{2k}(\mathbf{n}) - e_{2k+1}(\mathbf{n})\big)$$

is positive. □

Remark 5.5. When all n_i are equal to 1, Eq. (5.6) reduces to

$$\varphi\big((t-1)^n\big) = \sum_{k=0}^{n}(-1)^k(n-k)!\binom{n}{k},$$

which is the number of derangements of $\{1,\dots,n\}$. It would be interesting to give a combinatorial interpretation of (5.6).

Theorem 5.6. *For $r = 1,\dots,N$ we have*

$$\frac{(1-u_1)\dots(1-u_r)}{1-(u_1+\dots+u_N)} \in \mathbb{N}[u_1,\dots,u_N]. \tag{5.7}$$

Proof. This follows from Lemma 5.2, Lemma 5.3 and Lemma 5.4. □

Corollary 5.7. *For any sequence of N nonnegative integers $n_1,\dots,n_N$, the integral*

$$\int_{-\infty}^{0}\prod_{j=1}^{N} L_{n_j}^{(-1-n_j)}(y)e^y\,dy$$

is nonnegative.

Proof. This integral is the special $\alpha = -1$ and $\lambda = 1$ case of (5.2). Applying (5.3) we obtain the following generating function

$$\frac{1}{\prod_{j=1}^{N}(1-u_j^2)}\frac{\prod_{j=1}^{N}(1-u_j)}{1-(u_1+\dots+u_n)}.$$

The positivity follows then from Theorem 5.6. □

The polynomials $\{L_n^{(\alpha-n)}(-x)\}$ have interesting properties. For $\alpha < 0$ they seem to have only one real zero if n is odd and no real zeros if n is even. This property is shared with the Bessel polynomials [10, Section 4.10]. We also observed that the real zero increases with the degree.

The polynomials $\{L_n^{(\alpha-n)}(-x)\}$ do not seem to satisfy a three term recurrence relation of polynomials orthogonal with respect to a positive or signed measure. Instead we found the recurrence relation for the monic polynomials $p_n^{(\alpha)}(x) = n!L_n^{(-\alpha-n)}(-x)$

$$xp_n^{(\alpha)}(x) = p_{n+1}^{(\alpha)}(x) + \alpha p_n^{(\alpha)}(x) - n\alpha p_{n-1}^{(\alpha+1)}(x). \tag{5.8}$$

Indeed, this is equivalent to the recurrence relation for Laguerre polynomials

$$(n+1)L_{n+1}^{(\alpha)}(x) = (-x+\alpha+n+1)L_n^{(\alpha+1)}(x) - (\alpha+n+1)L_{n-1}^{(\alpha+1)}(x), \tag{5.9}$$

which follows then from two known recurrence relations for Laguerre polynomials [21, p. 203, (8), (10)].

Lemma 5.8. *For nonnegative integer $j \le n$ we have*

$$L_n^{(-j)}(x) = (-1)^j\frac{(n-j)!}{n!}x^jL_{n-j}^{(j)}(x).$$

Proof. By definition (3.3)

$$\begin{aligned} L_n^{(-j)}(x) &= \frac{1}{n!}\lim_{\alpha\to -j}\sum_{k=0}^{n}\frac{(-n)_k}{k!}\frac{(\alpha+1)_n}{(\alpha+1)_k}x^k = \frac{1}{n!}\lim_{\alpha\to -j}\sum_{k=0}^{n-j}\frac{(-n)_{k+j}}{(k+j)!}\frac{(\alpha+1)_n}{(\alpha+1)_{k+j}}x^{k+j} \\ &= \frac{1}{n!}\lim_{\alpha\to -j}\sum_{k=0}^{n-j}\frac{(-n)_{k+j}}{(k+j)!}\frac{(\alpha+j+1)_{n-j}}{(\alpha+j+1)_k}x^{k+j} \\ &= \frac{(-1)^j}{j!}x^j\,{}_1F_1(-n+j;j+1;x) = (-1)^j\frac{(n-j)!}{n!}x^j L_{n-j}^{(j)}(x). \qquad \square \end{aligned}$$

Let $\mathbf{n} = (n_1, \dots, n_N)$. It follows from Lemma 5.8 that

$$\begin{aligned} I(\mathbf{n},\lambda,j) &:= \prod_{k=1}^{N} n_k! \int_0^\infty x^\lambda e^{-x}\prod_{k=1}^{N}(-1)^{n_k}L_{n_k}^{(-j)}(x)dx \\ &= \int_0^\infty x^{\lambda+(N-1)j}x^j e^{-x}\prod_{k=1}^{N}(-1)^{n_k-j}(n_k-j)!L_{n_k-j}^{(j)}(x)dx. \end{aligned}$$

Now, comparing with the following integral, see [12, Eq. (9.3)],

$$A^{(\alpha)}(n_0,\mathbf{n}) = (-1)^{\sum_{j=1}^m n_j}\int_0^\infty x^{n_0}\frac{x^\alpha e^{-x}}{\Gamma(\alpha+1)}\prod_{j=1}^{m} n_j!L_{n_j}^{(\alpha)}(x)dx, \tag{5.10}$$

we derive that

$$I(\mathbf{n},\lambda,j) = A^{(j)}\big(\lambda+(N-1)j,(n_1-j,\dots,n_N-j)\big). \tag{5.11}$$

We can give a combinatorial interpretation of $I(\mathbf{n},\lambda,j)$ when λ is a nonnegative integer. Let $S_0, \dots, S_N$ be $N+1$ disjoint sets (boxes) such that the cardinality of S_0 is $\lambda+(N-1)j$ and the cardinality of S_k is $n_k - j$ for $k = 1, \dots, N$. Let $\mathfrak{S}^*(\mathbf{n},\lambda,j)$ denote the set of permutations of $S_0 \cup \cdots \cup S_N$ such that all the elements in box S_k should not stay in the original box after permutation for $1 \le k \le N$ and the objects in box S_0 are not restricted.

Therefore, by [12, Theorem 9.3], we have the following combinatorial interpretation.

Theorem 5.9. *Let λ and j be nonnegative integers. We have*

$$I(\mathbf{n},\lambda,j) = \sum_{\pi\in\mathfrak{S}^*(\mathbf{n},\lambda,j)}(j+1)^{\mathrm{cyc}(\pi)}, \tag{5.12}$$

where $\mathrm{cyc}(\pi)$ *is the number of cycles of the permutation* π.

Note that there are different interpretations for $I(\mathbf{n},\lambda,j)$ when $\lambda = 0$ and $j = 1$, see Gessel [6] and Taylor [24].

Acknowledgments

The first author would like to thank the Institute Camille Jordan of the Université Claude Bernard Lyon 1 for the hospitality during his visit when part of this work was done. The authors are grateful to the referee for many helpful comments, corrections, and friendly criticism.

References

[1] B. Aharmim, A. El Hamyani, F. El Wassouli, A. Ghanmi, New operational formulas and generating functions for the generalized Zernike polynomials, arXiv:1312.3628.
[2] S.T. Ali, F. Bagarello, G. Honnouvo, Modular structures on trace class operators and applications to Landau levels, J. Phys. A (2010) 105202, 17 pp.
[3] R.A. Askey, Orthogonal Polynomials and Special Functions, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1975.
[4] N. Cotfas, J.P. Gazeau, K. Górska, Complex and real Hermite polynomials and related quantizations, J. Phys. A 43 (2010) 305304, 14 pp.
[5] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, Higher Transcendental Functions, vol. 2, McGraw-Hill, New York, 1953.
[6] I. Gessel, Generalized rook polynomials and orthogonal polynomials, in: q-Series and Partitions, Minneapolis, MN, 1988, in: IMA Vol. Math. Appl., vol. 18, Springer, New York, 1989, pp. 159–176.
[7] A. Ghanmi, A class of generalized complex Hermite polynomials, J. Math. Anal. Appl. 340 (2008) 1395–1406.
[8] A. Ghanmi, Operational formulae for the complex Hermite polynomials $H_{p,q}(z,\bar{z})$, Integral Transforms Spec. Funct. 24 (2013) 884–895.
[9] A. Intissar, A. Intissar, Spectral properties of the Cauchy transform on $L^2(\mathbb{C}; e^{-|z|^2}dz)$, J. Math. Anal. Appl. 31 (2006) 400–418.
[10] M.E.H. Ismail, Classical and Quantum Orthogonal Polynomials in One Variable, paperback edition, Cambridge University Press, Cambridge, 2009.
[11] M.E.H. Ismail, Analytic properties of complex Hermite polynomials, Trans. Amer. Math. Soc. (2014), in press.
[12] M.E.H. Ismail, A. Kasraoui, J. Zeng, Separation of variables and combinatorics of linearization coefficients of orthogonal polynomials, J. Combin. Theory Ser. A 120 (3) (2013) 561–599.
[13] M.E.H. Ismail, P. Simeonov, Complex Hermite polynomials: their combinatorics and integral operators, Proc. Amer. Math. Soc. (2014), in press.
[14] M.E.H. Ismail, J. Zeng, A combinatorial approach to the $2D$-Hermite and $2D$-Laguerre polynomials, Adv. in Appl. Math. (2014), in press.
[15] M.E.H. Ismail, R. Zhang, On some $2D$ orthogonal q-polynomials, in press.
[16] K. Ito, Complex multiple Wiener integral, Jpn. J. Math. 22 (1952) 63–86.
[17] W.F. Kibble, An extension of theorem of Mehler on Hermite polynomials, Math. Proc. Cambridge Philos. Soc. 41 (1945) 12–15.
[18] T.H. Koornwinder, Two variable analogues of the classical orthogonal polynomials, in: R. Askey (Ed.), Theory and Application of Special Functions, Academic Press, New York, 1975, pp. 436–495.
[19] C.D. Maldonado, Note on orthogonal polynomials which are "invariant in form" to rotations of axes, J. Math. Phys. 6 (1965) 1935–1938.
[20] D.R. Myrick, A generalization of the radial polynomials of F. Zernike, SIAM J. Appl. Math. 14 (1966) 476–489.
[21] E.D. Rainville, Special Functions, Macmillan, New York, 1960.
[22] I. Shigekawa, Eigenvalue problems for the Schrödinger operator with the magnetic field on a compact Riemannian manifold, J. Funct. Anal. 75 (1987) 92–127.
[23] G. Szegő, Orthogonal Polynomials, fourth edition, American Mathematical Society, Providence, 1975.
[24] J. Taylor, Counting words with Laguerre polynomials, Electron. J. Combin. 21 (2) (2014), Paper 2.1, 23 pp.
[25] K. Thirulogasanthar, G. Honnouvo, A. Krzyzak, Coherent states and Hermite polynomials on quaternionic Hilbert spaces, J. Phys. A (2010) 385205, 13 pp.
[26] A. Wünsche, Laguerre $2D$-functions and their applications in quantum optics, J. Phys. A 31 (1998) 8267–8287.
[27] A. Wünsche, Transformations of Laguerre $2D$-polynomials and their applications to quasiprobabilities, J. Phys. A 21 (1999) 3179–3199.
[28] A. Wünsche, Generalized Zernike or disc polynomials, J. Comput. Appl. Math. 174 (2005) 135–163.