



Pluriharmonic mappings in \mathbb{C}^n and complex Banach spaces



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ABSTRACT

In this paper, we obtain a sufficient condition for pluriharmonic mappings on the Euclidean unit ball \mathbb{B}^n to be univalent, sense-preserving, quasiconformal and bi-Lipschitz diffeomorphisms on \mathbb{B}^n and to have linearly connected images. Also, we give a sufficient condition for pluriharmonic mappings on \mathbb{B}^n to have quasiconformal extensions to \mathbb{C}^n . Next, we generalize the harmonic Schwarz lemma to pluriharmonic mappings of the unit ball B_X of a complex Banach space X into the unit ball B^n of \mathbb{C}^n with respect to an arbitrary norm. Further, we obtain a generalization of the harmonic Schwarz–Pick lemma to the case of pluriharmonic mappings of the homogeneous unit ball B_X of a complex Banach space X into the unit ball B^n . We also obtain a version of the holomorphic Schwarz–Pick lemma for the Jacobian determinant on the Euclidean unit ball \mathbb{B}^n to the case of pluriharmonic mappings of the homogeneous unit ball B_X into B^n , in the case that B_X is an open subset of \mathbb{C}^n . Finally, we obtain the Landau and the Bloch theorems for pluriharmonic or holomorphic mappings on finite dimensional homogeneous unit balls.

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1. Introduction and preliminaries

For a real or complex Banach space X with norm $\|\cdot\|$, the open ball $\{z \in X : \|z\| < r\}$ is denoted by $B(0, r)$ and the unit ball $B(0, 1)$ is denoted by B . Also, let $L(X)$ denote the space of continuous linear operators from X into itself with the standard operator norm.

Let X, Y be complex Banach spaces and let X^* and Y^* be the dual space of X and Y , respectively. For $x \in X \setminus \{0\}$, we define

$$T(x) = \{l_x \in X^* : l_x(x) = \|x\|, \|l_x\| = 1\}.$$

Then $T(x) \neq \emptyset$ in view of the Hahn–Banach theorem. Let Ω be a domain in X . Let $H(\Omega, Y)$ denote the set of holomorphic mappings of Ω into Y . A mapping $f \in H(\Omega, Y)$ is said to be biholomorphic if $f(\Omega)$ is

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a domain, the inverse f^{-1} exists and is holomorphic on $f(\Omega)$. Let $\text{Aut}(\Omega)$ denote the set of biholomorphic automorphisms of Ω . A domain Ω is said to be homogeneous if for any $x, y \in \Omega$, there exists some mapping $f \in \text{Aut}(\Omega)$ such that $f(x) = y$.

Definition 1.1. A complex Banach space X is called a *JB*-triple* if there exists a triple product $\{\cdot, \cdot, \cdot\} : X^3 \rightarrow X$ which is conjugate linear in the middle variable, but linear and symmetric in the other variables, and satisfies

- (i) $\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\};$
- (ii) the map $a \square a : x \in X \mapsto \{a, a, x\} \in X$ is hermitian with nonnegative spectrum;
- (iii) $\|\{a, a, a\}\|_X = \|a\|_X^3;$

for $a, b, x, y, z \in X$.

Remark 1.2. It is known that every bounded symmetric domain in a complex Banach space is homogeneous. Conversely, the open unit ball B of a Banach space admits a symmetry $s(z) = -z$ at 0 and if B is homogeneous, then B is a symmetric domain. Banach spaces with a homogeneous open unit ball are precisely the JB*-triples (see [35]). We refer to [10,45] for relevant details of JB*-triples and references.

For every $a \in X$, let $Q_a : X \rightarrow X$ be the conjugate linear operator defined by $Q_a(z) = \{a, z, a\}$. This operator is called the quadratic representation and it satisfies the fundamental formula

$$Q_{Q_a(b)} = Q_a Q_b Q_a$$

for all $a, b \in X$. For every $z, w \in X$, the Bergman operator $B(z, w) \in L(X)$ is defined by

$$B(z, w) = I - 2z \square w + Q_z Q_w, \quad (1.1)$$

where $z \square w(x) = \{z, w, x\}$. Let B be the unit ball of a JB*-triple X . Then, for each $a \in B$, the Möbius transformation g_a defined by

$$g_a(z) = a + B(a, a)^{1/2} (I + z \square a)^{-1} z, \quad (1.2)$$

is a biholomorphic mapping of B onto itself with $g_a(0) = a$, $g_a(-a) = 0$, $g_{-a} = g_a^{-1}$ and $Dg_a(0) = B(a, a)^{1/2}$.

Further, let B be the unit ball of a JB*-triple X and let h_0 be the Bergman metric on X at 0. Also, let $c(B)$ be the following constant (see [26]):

$$c(B) = \frac{1}{2} \sup_{x, y \in B} |h_0(x, y)|. \quad (1.3)$$

Recently, the first author [24, Lemma 2.5] proved the following estimate for the determinant of $B(z, z)$ on the unit ball B of a finite dimensional JB*-triple X :

$$(1 - \|z\|^2)^{2c(B)} \leq \det B(z, z), \quad z \in B, \quad (1.4)$$

and equality holds for $z \neq 0$ such that $z/\|z\|$ is a maximal tripotent in X .

Let \mathbb{C}^n denote the space of n complex variables $z = (z_1, \dots, z_n)$ with respect to an arbitrary norm $\|\cdot\|$. The open ball $\{z \in \mathbb{C}^n : \|z\| < r\}$ is denoted by $B^n(0, r)$ and the unit ball $B^n(0, 1)$ is denoted by B^n . Also, let I_n be the identity in $L(\mathbb{C}^n)$. If Ω is a domain in \mathbb{C}^n , let $H(\Omega) = H(\Omega, \mathbb{C}^n)$. If $f \in H(B^n)$, we say that

f is locally biholomorphic on B^n if $\det Df(z) \neq 0$, $z \in B^n$, where $Df(z)$ is the complex Jacobian matrix of f at z . Also, we say that $f \in H(B^n)$ is normalized if $f(0) = 0$ and $Df(0) = I_n$.

If \mathbb{C}^n is the n -dimensional complex space with respect to the Euclidean inner product $\langle z, w \rangle = \sum_{j=1}^n z_j \bar{w}_j$ and the Euclidean norm $\|z\| = \langle z, z \rangle^{1/2}$, the Euclidean open ball $\{z \in \mathbb{C}^n : \|z\| < r\}$ is denoted by $\mathbb{B}^n(0, r)$ and the Euclidean unit ball $\mathbb{B}^n(0, 1)$ is denoted by \mathbb{B}^n . Clearly, \mathbb{B}^1 is the usual unit disc \mathbb{U} in \mathbb{C} .

A C^2 mapping $f : B^n \rightarrow \mathbb{C}^m$ is said to be pluriharmonic if the restriction of each component f_j to every complex line is harmonic. Then it is easily seen that f is pluriharmonic if and only if

$$\frac{\partial^2}{\partial z_j \partial \bar{z}_k} f(z) = 0, \quad \forall z \in B^n, \quad \forall j, k = 1, 2, \dots, n.$$

Note that every pluriharmonic mapping $f : B^n \rightarrow \mathbb{C}^n$ can be written as $f = h + \bar{g}$, where $g, h \in H(B^n)$, and this representation is unique if $g(0) = 0$. In view of this result, we define the notion of pluriharmonic mappings in infinite dimensional spaces. Let B be the unit ball of a complex Banach space X . A continuous mapping $f : B \rightarrow \mathbb{C}^n$ is said to be pluriharmonic if there exist $h, g \in H(B, \mathbb{C}^n)$ such that $f = h + \bar{g}$.

If $f = h + \bar{g} : B^n \rightarrow \mathbb{C}^n$ is a pluriharmonic mapping such that h is locally biholomorphic on B^n , we denote by J_f the real Jacobian of f and $\omega_f(z) = Dg(z)[Dh(z)]^{-1}$ for $z \in B^n$. Then

$$J_f(z) = \det \begin{pmatrix} Dh(z) & \overline{Dg(z)} \\ Dg(z) & \overline{Dh(z)} \end{pmatrix}, \quad z \in B^n,$$

and it is elementary to deduce that

$$J_f(z) = |\det Dh(z)|^2 \det(I_n - \omega_f(z) \overline{\omega_f(z)}), \quad z \in B^n.$$

Consequently, f (with h locally biholomorphic on B^n) is sense-preserving, i.e., $J_f(z) > 0$ for $z \in B^n$, if and only if $\det(I_n - \omega_f(z) \overline{\omega_f(z)}) > 0$, for all $z \in B^n$. In the case of one complex variable, $\omega_f = g'/h'$ is the dilatation of f . It is known that $f = h + \bar{g}$ is locally univalent and sense-preserving on \mathbb{U} if and only if $|g'(z)| < |h'(z)|$ for $z \in \mathbb{U}$, i.e., h is locally univalent on \mathbb{U} and $|\omega_f(z)| < 1$ for $z \in \mathbb{U}$. In dimension $n \geq 2$, if $f = h + \bar{g} : \mathbb{B}^n \rightarrow \mathbb{C}^n$ is a pluriharmonic mapping such that h is locally biholomorphic on \mathbb{B}^n and $\|\omega_f(z)\| < 1$ for $z \in \mathbb{B}^n$, then f is a sense-preserving locally univalent mapping on \mathbb{B}^n (cf. [15, Theorem 5]).

We give a definition of linearly connected domains in the Euclidean space \mathbb{C}^n (see [11], for $n = 1$, [39]).

Definition 1.3. A domain $\Omega \subseteq \mathbb{C}^n$ is said to be linearly connected if there exists a constant $M > 0$ such that any two points $v_1, v_2 \in \Omega$ can be connected by a smooth curve $\gamma \subset \Omega$ with length $\ell(\gamma) \leq M\|v_1 - v_2\|$.

Remark 1.4. It is clear that $M \geq 1$ in Definition 1.3 and that any convex domain is linearly connected with constant $M = 1$. On the other hand, if $\Omega_j \subseteq \mathbb{C}$ is a linearly connected domain with constant $M_j > 0$ for $j = 1, 2, \dots, n$, then it is easy to see that $\Omega = \prod_{j=1}^n \Omega_j$ is a linearly connected domain in \mathbb{C}^n with constant $M = \sqrt{n} \max_{j=1, \dots, n} M_j$.

In the case of one complex variable, every simply connected bounded linearly connected domain Ω is a Jordan domain (see [39]). Chuaqui and Hernández [12, Theorem 1] proved that if $h \in H(\mathbb{U})$ is a univalent function, then there exists a constant $c > 0$ such that each harmonic mapping $f = h + \bar{g}$ with $|\omega_f| < c$ is univalent on \mathbb{U} if and only if $h(\mathbb{U})$ is a linearly connected domain.

Let Ω, Ω' be domains in \mathbb{R}^m . Let $\|\cdot\|$ be the Euclidean norm on \mathbb{R}^m and $K > 0$ be a constant. A homeomorphism $f : \Omega \rightarrow \Omega'$ is said to be K -quasiconformal if it is differentiable a.e., ACL (absolutely continuous on lines) and

$$\|D(f; x)\|^m \leq K |\det D(f; x)| \quad \text{a.e. in } \Omega,$$

where $D(f; x)$ denotes the (real) Jacobian matrix of f and

$$\|D(f; x)\| = \sup\{\|D(f; x)(a)\| : \|a\| = 1\}.$$

Let Ω, Ω' be domains in \mathbb{C}^n . Let $F : \Omega \rightarrow \Omega'$ be a homeomorphism such that it is differentiable a.e. and ACL. Also, let $F(z) = u(x, y) + iv(x, y)$ for $z = x + iy \in \Omega$. Since

$$\|F_z w + F_{\bar{z}} \bar{w}\| = \left\| D(F; z) \begin{pmatrix} a \\ b \end{pmatrix} \right\|, \quad \text{for } w = a + ib, \text{ a.e. on } \Omega,$$

and

$$\det \begin{pmatrix} F_z & F_{\bar{z}} \\ \bar{F}_z & \bar{F}_{\bar{z}} \end{pmatrix} = \det D(F; z) = J_F(z), \quad \text{a.e. on } \Omega,$$

where

$$D(F; z) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

is the real Jacobian matrix, F is K -quasiconformal if and only if

$$\sup_{\|w\|=1} \|F_z w + F_{\bar{z}} \bar{w}\|^{2n} \leq K \det \begin{pmatrix} F_z & F_{\bar{z}} \\ \bar{F}_z & \bar{F}_{\bar{z}} \end{pmatrix}, \quad \text{a.e. on } \Omega. \quad (1.5)$$

Let Ω be a domain in \mathbb{C}^n and let $K > 0$ be a constant. Also, let $f \in H(\Omega)$. The mapping f is said to be K -quasiregular if

$$\|Df(z)\|^n \leq K |\det Df(z)|, \quad z \in \Omega.$$

Note that a K -quasiregular biholomorphic mapping is K^2 -quasiconformal.

Assume that $f \in H(\mathbb{B}^n)$ is a normalized mapping which satisfies the following condition:

$$\|Df(z) - I_n\| < c, \quad z \in \mathbb{B}^n.$$

If $c = 1$, then f is biholomorphic on \mathbb{B}^n and if $c < 1$, then f is quasiregular on \mathbb{B}^n and extends to a quasiconformal homeomorphism of \mathbb{R}^{2n} onto itself (see Brodskiĭ [3], Graham, Hamada and Kohr [20], Hamada and Kohr [29], cf. [16, 19, 30, 31]; see also [21, Chapter 8] and the references therein). For harmonic mappings on the unit disc \mathbb{U} , similar results were obtained by Avcı and Złotkiewicz [1], Ganczar [18], Hamada, Honda and Shon [28], Jahangiri [34] (see also, [16] and [37], in the case of holomorphic mappings on \mathbb{U}). In this paper, we will generalize the above results to pluriharmonic mappings on \mathbb{B}^n . We obtain a sufficient condition for pluriharmonic mappings on the Euclidean unit ball \mathbb{B}^n to be univalent, sense-preserving, quasiconformal and bi-Lipschitz diffeomorphisms on \mathbb{B}^n and to have linearly connected images. Also, we give a sufficient condition for pluriharmonic mappings on \mathbb{B}^n to have quasiconformal extensions to \mathbb{C}^n . For harmonic or pluriharmonic mappings and their connection with linearly connected domains, see [8, 11, 12].

Next, we will consider about the Schwarz lemma and the Schwarz–Pick lemma. The following sharp inequality which is a harmonic version of the Schwarz lemma is due to Heinz [33] (cf. Duren [14, p. 77]).

Theorem 1.5. *Let f be a complex-valued harmonic mapping in \mathbb{U} such that $f(0) = 0$ and $|f(z)| < 1$, $z \in \mathbb{U}$. Then $|f(z)| \leq \frac{4}{\pi} \arctan |z|$, and this inequality is sharp for each point $z \in \mathbb{U}$.*

The following result is known as the harmonic Schwarz–Pick lemma (see [13]; cf. [6,14]; compare [7]). Note that sharpness of (1.6) is a consequence of [13, Theorem 4].

Theorem 1.6. *Let $f = h + \bar{g} : \mathbb{U} \rightarrow \mathbb{C}$ be a complex-valued harmonic mapping such that $|f(z)| < 1$, $z \in \mathbb{U}$. Then*

$$\Lambda_f(z) \leq \frac{4}{\pi} \frac{1}{1 - |z|^2}, \quad z \in \mathbb{U}, \quad (1.6)$$

where $\Lambda_f(z) = |h'(z)| + |g'(z)|$ is the maximum dilation of f at $z \in \mathbb{U}$. This estimate is sharp.

Chen and Gauthier [6] generalized Theorems 1.5 and 1.6 to pluriharmonic mappings from \mathbb{B}^n into \mathbb{B}^m . They also obtained a Schwarz–Pick lemma for holomorphic mappings [5] of \mathbb{B}^n into \mathbb{B}^m . In this paper, we will generalize the harmonic Schwarz lemma to pluriharmonic mappings from the unit ball of a complex Banach space into the unit ball B^n of \mathbb{C}^n with respect to an arbitrary norm $\|\cdot\|$. We will also generalize the harmonic Schwarz–Pick lemma to pluriharmonic mappings on the homogeneous unit ball of a complex Banach space into the unit ball B^n . Further, we will obtain a version of the holomorphic Schwarz–Pick lemma for the Jacobian determinant on the Euclidean unit ball \mathbb{B}^n to the case of pluriharmonic mappings of the homogeneous unit ball B_X into B^n , in the case that B_X is an open subset of \mathbb{C}^n .

Finally, we will consider about the Landau theorem and the Bloch theorem. The classical Bloch theorem asserts the existence of a positive constant b such that for any holomorphic mapping of the unit disc \mathbb{U} with $f'(0) = 1$, the image $f(\mathbb{U})$, as a covering surface, contains a univalent disc of radius b . Bloch’s constant is defined as the supremum of such constants b . For general holomorphic mappings of higher dimensions, there is no positive Bloch constant (see [32,47]). Chen and Gauthier [5] proved the Landau theorem and the Bloch theorem for quasiregular holomorphic mappings (Wu K -mappings) on \mathbb{B}^n . They [6] also proved the Landau theorem and the Bloch theorem for pluriharmonic K -mappings on \mathbb{B}^n . For other results on the Bloch theorem in several complex variables, see [17,22,23,32,38,42,44,47]. In this paper, we will obtain the Landau and the Bloch theorems for pluriharmonic K -mappings and quasiregular holomorphic mappings (Wu K -mappings) on finite dimensional homogeneous unit balls.

For other results on pluriharmonic mappings on the unit ball in \mathbb{C}^n , see [2,8,15,27].

2. Univalence, quasiconformality and linearly connectedness

In this section, let \mathbb{C}^n be the n -dimensional complex space with respect to the Euclidean norm $\|\cdot\|$. We obtain a sufficient condition for pluriharmonic mappings on \mathbb{B}^n to be univalent, sense-preserving, quasiconformal and bi-Lipschitz diffeomorphisms on \mathbb{B}^n and to have linearly connected images.

Theorem 2.1. *Let $f = h + \bar{g}$ be a pluriharmonic mapping on \mathbb{B}^n such that $h(0) = g(0) = 0$ and $Dh(0) = I_n$. Assume that*

$$\|Dh(z) - I_n\| + \|Dg(z)\| < 1, \quad z \in \mathbb{B}^n. \quad (2.1)$$

Then f is a univalent sense-preserving diffeomorphism on \mathbb{B}^n . Also, h is biholomorphic on \mathbb{B}^n .

Proof. Putting $\tilde{h}(z) = h(z) - z$, we obtain that $\|D\tilde{h}(z)\| + \|Dg(z)\| < 1$ for $z \in \mathbb{B}^n$. For arbitrary $z_1, z_2 \in \mathbb{B}^n$, $z_1 \neq z_2$, we have

$$\begin{aligned}
\|f(z_1) - f(z_2)\| &\geq \|z_1 - z_2\| - \int_{\gamma} (\|D\tilde{h}(z)\| + \|Dg(z)\|) \|dz\| \\
&\geq \|z_1 - z_2\| - c_1 \int_{\gamma} \|dz\| = \|z_1 - z_2\| - c_1 \ell(\gamma) \\
&= (1 - c_1) \|z_1 - z_2\|,
\end{aligned}$$

where $\gamma(t) = z_1 + t(z_2 - z_1)$ for $t \in [0, 1]$, and $c_1 = \sup_{z \in \gamma} (\|D\tilde{h}(z)\| + \|Dg(z)\|) < 1$. Thus, f is univalent on \mathbb{B}^n .

Next, we show that f is sense-preserving. The relation (2.1) implies that h is biholomorphic on \mathbb{B}^n ([43, Theorem 7]; cf. [20, Lemma 2.2]). Next, we show that $\|\omega_f(z)\| = \|Dg(z)[Dh(z)]^{-1}\| < 1$. We may assume that $Dh(z) \neq I_n$. Since $\|Dh(z) - I_n\| < 1$ for $z \in \mathbb{B}^n$, it follows that

$$\begin{aligned}
\|Dg(z)[Dh(z)]^{-1}\| &= \|Dg(z)[I_n - (I_n - Dh(z))]^{-1}\| \\
&\leq \|Dg(z)\| \cdot \|[I_n - (I_n - Dh(z))]^{-1}\| \\
&\leq \frac{\|Dg(z)\|}{1 - \|I_n - Dh(z)\|} < 1, \quad z \in \mathbb{B}^n.
\end{aligned}$$

This completes the proof. \square

Theorem 2.2. *Let $f = h + \bar{g}$ be a pluriharmonic mapping on \mathbb{B}^n such that $h(0) = g(0) = 0$ and $Dh(0) = I_n$. Assume that*

$$\|Dh(z) - I_n\| + \|Dg(z)\| \leq c < 1, \quad z \in \mathbb{B}^n. \quad (2.2)$$

Then f satisfies the following inequality:

$$\|(f(z_1) - f(z_2)) - (z_1 - z_2)\| \leq c \|z_1 - z_2\|, \quad z_1, z_2 \in \mathbb{B}^n. \quad (2.3)$$

Consequently, f is a univalent sense-preserving bi-Lipschitz diffeomorphism on \mathbb{B}^n . Moreover, f is quasiconformal on \mathbb{B}^n and $f(\mathbb{B}^n)$ is a linearly connected domain in \mathbb{C}^n . Also, h is quasiregular biholomorphic on \mathbb{B}^n and extends to a quasiconformal homeomorphism of \mathbb{R}^{2n} onto itself.

Proof. Putting $\tilde{h}(z) = h(z) - z$, we obtain that $\|D\tilde{h}(z)\| + \|Dg(z)\| \leq c$ for $z \in \mathbb{B}^n$. For arbitrary $z_1, z_2 \in \mathbb{B}^n$, $z_1 \neq z_2$, we have

$$\begin{aligned}
\|(f(z_1) - f(z_2)) - (z_1 - z_2)\| &\leq \int_{\gamma} (\|D\tilde{h}(z)\| + \|Dg(z)\|) \|dz\| \\
&\leq c \int_{\gamma} \|dz\| \\
&= c \|z_1 - z_2\|,
\end{aligned}$$

where $\gamma(t) = z_1 + t(z_2 - z_1)$, for $t \in [0, 1]$. Thus, we obtain (2.3).

Next, we will show that f is quasiconformal. Since $f_z = Dh(z)$ and $f_{\bar{z}} = \overline{Dg(z)}$, $\sup_{\|w\|=1} \|f_z w + f_{\bar{z}} \bar{w}\|$ is bounded on \mathbb{B}^n . We also have

$$\|Dg(z)[Dh(z)]^{-1}\| \leq c, \quad z \in \mathbb{B}^n,$$

by using an argument similar to that in the proof of [Theorem 2.1](#). Since the absolute values of the eigenvalues of $Dh(z)$ are greater than or equal to $1 - c$ from [\(2.2\)](#) and those of $I_n - \omega_f(z)\overline{\omega_f(z)}$ are greater than or equal to $1 - c^2$ from the above, there exists a constant $\varepsilon > 0$ such that

$$\left| \begin{array}{cc} f_z & f_{\bar{z}} \\ \bar{f}_z & \bar{f}_{\bar{z}} \end{array} \right| = \left| \begin{array}{cc} Dh(z) & \overline{Dg(z)} \\ Dg(z) & \overline{Dh(z)} \end{array} \right| = |\det Dh(z)|^2 \det(I_n - \omega_f(z)\overline{\omega_f(z)}) \geq \varepsilon$$

for $z \in \mathbb{B}^n$. Thus, the inequality [\(1.5\)](#) is satisfied. This implies that f is quasiconformal on \mathbb{B}^n .

Next, we show that $f(\mathbb{B}^n)$ is linearly connected. Let $w_1, w_2 \in f(\mathbb{B}^n)$ be such that $w_1 \neq w_2$. Then $w_j = f(z_j)$, where $z_j \in \mathbb{B}^n$, $j = 1, 2$. Also, let $\Gamma = f(\gamma)$, where γ is as above. Then Γ is a smooth curve in $f(\mathbb{B}^n)$ between w_1 and w_2 . We prove that

$$\ell(\Gamma) \leq \frac{1+c}{1-c} \|w_1 - w_2\|. \quad (2.4)$$

We obtain that

$$\begin{aligned} \ell(\Gamma) &= \int_{\Gamma} \|dw\| = \int_{\gamma} \|f_z dz + f_{\bar{z}} d\bar{z}\| \\ &\leq \int_{\gamma} (\|I_n\| + \|Dh(z) - I_n\| + \|Dg(z)\|) \|dz\| \\ &\leq (1+c) \int_{\gamma} \|dz\| \\ &= (1+c)\ell(\gamma). \end{aligned}$$

Thus, we obtain that

$$\ell(\Gamma) \leq (1+c)\|z_1 - z_2\|. \quad (2.5)$$

Next, in view of [\(2.3\)](#) and [\(2.5\)](#), we obtain that

$$\ell(\Gamma) \leq (1+c)\|z_1 - z_2\| \leq \frac{1+c}{1-c} \|w_1 - w_2\|.$$

Hence, the relation [\(2.4\)](#) follows, as desired.

The fact that h is quasiregular biholomorphic on \mathbb{B}^n and extends to a quasiconformal homeomorphism of \mathbb{R}^{2n} onto itself is a direct consequence of [\[29, Corollary 4.2\]](#). This completes the proof. \square

Taking into account [Theorem 2.2](#), we obtain the following consequences.

Corollary 2.3. *Let $f = h + \bar{g}$ be a pluriharmonic mapping on \mathbb{B}^n such that $h(0) = g(0) = 0$ and $Dh(0) = I_n$. Assume that f satisfies the relation [\(2.2\)](#). Then f , g and h extend continuously to $\overline{\mathbb{B}^n}$, the relation $f = h + \bar{g}$ holds on $\overline{\mathbb{B}^n}$, and*

$$\|(f(z_1) - f(z_2)) - (z_1 - z_2)\| \leq c\|z_1 - z_2\|, \quad z_1, z_2 \in \overline{\mathbb{B}^n}. \quad (2.6)$$

Proof. The statements follow from [\(2.3\)](#). \square

Corollary 2.4. *Let $f \in H(\mathbb{B}^n)$ be normalized. Assume that*

$$\|Df(z) - I_n\| \leq c < 1, \quad z \in \mathbb{B}^n.$$

Then f satisfies the inequality (2.3). Thus, f is a bi-Lipschitz biholomorphism on \mathbb{B}^n . Moreover, f is quasiconformal on \mathbb{B}^n , extends to a quasiconformal homeomorphism of \mathbb{R}^{2n} onto itself and $f(\mathbb{B}^n)$ is a linearly connected domain in \mathbb{C}^n .

Corollary 2.5. *Let $f = h + \bar{g}$ be a pluriharmonic mapping on \mathbb{B}^n such that $h(z) = z + \sum_{k=2}^{\infty} P_k(z^k)$ and $g(z) = \sum_{k=1}^{\infty} Q_k(z^k)$, for all $z \in \mathbb{B}^n$. If*

$$\|Q_1\| + \sum_{k=2}^{\infty} k(\|P_k\| + \|Q_k\|) < 1,$$

then f is a univalent sense-preserving bi-Lipschitz diffeomorphism on \mathbb{B}^n . Moreover, f is quasiconformal on \mathbb{B}^n and $f(\mathbb{B}^n)$ is a linearly connected domain in \mathbb{C}^n .

Proof. Let

$$c = \|Q_1\| + \sum_{k=2}^{\infty} k(\|P_k\| + \|Q_k\|).$$

Then $c < 1$, and since

$$\begin{aligned} \|Dh(z) - I_n\| + \|Dg(z)\| &\leq \sum_{k=2}^{\infty} k\|P_k\| \cdot \|z\|^{k-1} + \sum_{k=1}^{\infty} k\|Q_k\| \cdot \|z\|^{k-1} \\ &\leq \|Q_1\| + \sum_{k=2}^{\infty} k(\|P_k\| + \|Q_k\|) \\ &= c < 1, \end{aligned}$$

for all $z \in \mathbb{B}^n$, the result follows in view of Theorem 2.2, as desired. \square

3. Quasiconformal extension of pluriharmonic mappings to \mathbb{C}^n

In this section, we obtain a sufficient condition for pluriharmonic mappings on \mathbb{B}^n to have quasiconformal extensions to \mathbb{C}^n .

By direct computations, we obtain the following lemma.

Lemma 3.1. *Let*

$$\varphi(z) = \frac{z}{\|z\|^2}, \quad z \in \mathbb{C}^n \setminus \{0\},$$

where $\|\cdot\|$ denotes the Euclidean norm on \mathbb{C}^n . Then

$$[\varphi_z(z)](w) = \frac{1}{\|z\|^2} \left(w - \left\langle w, \frac{z}{\|z\|} \right\rangle \frac{z}{\|z\|} \right)$$

and

$$[\varphi_{\bar{z}}(z)](w) = \frac{-1}{\|z\|^2} \left\langle w, \frac{\bar{z}}{\|z\|} \right\rangle \frac{z}{\|z\|}.$$

Theorem 3.2. Let $f = h + \bar{g}$ be a pluriharmonic mapping on \mathbb{B}^n such that $h(0) = g(0) = 0$ and $Dh(0) = I_n$.

(i) Assume that f satisfies the relation (2.2).

Let

$$F(z) = \begin{cases} f(z) & \text{for } z \in \mathbb{B}^n \\ z + \tilde{h}\left(\frac{z}{\|z\|^2}\right) + \overline{g\left(\frac{z}{\|z\|^2}\right)} & \text{for } z \in \mathbb{C}^n \setminus \mathbb{B}^n, \end{cases}$$

be an extension of f to \mathbb{C}^n , where $\tilde{h}(z) = h(z) - z$. Then F satisfies

$$(1 - c)\|z_1 - z_2\| \leq \|F(z_1) - F(z_2)\| \leq (1 + c)\|z_1 - z_2\|, \quad \text{for } z_1, z_2 \in \mathbb{C}^n. \quad (3.1)$$

Thus, F is a bi-Lipschitz continuous univalent mapping on \mathbb{C}^n .

(ii) Assume that

$$\|Dh(z) - I_n\| + \|Dg(z)\| \leq c < \frac{1}{2}, \quad z \in \mathbb{B}^n.$$

Then the mapping F defined as above is a quasiconformal homeomorphism of \mathbb{C}^n onto itself.

Proof. (i) Let $G(z) = F(z) - z$. To show (3.1), it suffices to show that

$$\|G(z_1) - G(z_2)\| \leq c\|z_1 - z_2\|, \quad z_1, z_2 \in \mathbb{C}^n. \quad (3.2)$$

Note that (3.2) is satisfied for $z_1, z_2 \in \bar{\mathbb{B}}^n$ by (2.6). For $z_1, z_2 \in \mathbb{C}^n \setminus \mathbb{B}^n$ with $z_1 \neq z_2$, let

$$\gamma(t) = \frac{z_1}{\|z_1\|^2} + t\left(\frac{z_2}{\|z_2\|^2} - \frac{z_1}{\|z_1\|^2}\right), \quad t \in [0, 1]$$

be the line segment joining $\frac{z_1}{\|z_1\|^2}$ and $\frac{z_2}{\|z_2\|^2}$. Then

$$\begin{aligned} \|G(z_1) - G(z_2)\| &\leq \int_{\gamma} (\|I_n - Dh(z)\| + \|Dg(z)\|) \|dz\| \\ &\leq c \int_{\gamma} \|dz\| \\ &= c \left\| \frac{z_2}{\|z_2\|^2} - \frac{z_1}{\|z_1\|^2} \right\| \\ &= c \frac{\|z_1 - z_2\|}{\|z_1\| \|z_2\|} \\ &\leq c \|z_1 - z_2\|. \end{aligned}$$

We consider the case $z_1 \in \mathbb{B}^n$ and $z_2 \in \mathbb{C}^n \setminus \bar{\mathbb{B}}^n$. Let $z_3 \in \partial\mathbb{B}^n \cap [z_1, z_2]$, where $[z_1, z_2]$ is the line segment between z_1 and z_2 . Then we have

$$\begin{aligned} \|G(z_1) - G(z_2)\| &\leq \|G(z_1) - G(z_3)\| + \|G(z_3) - G(z_2)\| \\ &\leq c(\|z_1 - z_3\| + \|z_3 - z_2\|) \\ &= c\|z_1 - z_2\|. \end{aligned}$$

Thus, we obtain (3.1) for all $z_1, z_2 \in \mathbb{C}^n$.

(ii) Let

$$F^r(z) = \frac{1}{r}F(rz), \quad z \in \mathbb{B}^n, \text{ for } r \in (0, 1).$$

We will show that F^r converges uniformly on compact subsets of \mathbb{R}^{2n} to F as $r \rightarrow 1$, is ACL, is differentiable a.e., satisfies the estimation (1.5) a.e. in \mathbb{R}^{2n} for some K which is independent of r , and is a homeomorphism of \mathbb{R}^{2n} onto itself. Then by [46, Theorem 21.7 and Corollary 37.4], F is a quasiconformal homeomorphism of \mathbb{R}^{2n} onto itself, since $F|_{\mathbb{B}^n} = f$ is nonconstant.

Since F^r is Lipschitz continuous on \mathbb{C}^n , it is ACL. It is clear that F^r is differentiable on $\mathbb{C}^n \setminus \partial\mathbb{B}^n$. Since $(F^r)_z(z) = F_z(rz)$ and $(F^r)_{\bar{z}}(z) = F_{\bar{z}}(rz)$, if F satisfies the estimation (1.5) for some $K > 0$, then F^r also satisfies the estimation (1.5) for the same K for all $r \in (0, 1)$. We will show that the estimation (1.5) for F holds on $\mathbb{C}^n \setminus \partial\mathbb{B}^n$ for some $K > 0$. The estimation (1.5) for F in \mathbb{B}^n is proved in Theorem 2.2. Since

$$F_z = I + D\tilde{h}(\varphi(z))[\varphi_z] + \overline{Dg(\varphi(z))[\varphi_{\bar{z}}]},$$

and

$$F_{\bar{z}} = D\tilde{h}(\varphi(z))[\varphi_{\bar{z}}] + \overline{Dg(\varphi(z))[\varphi_z]},$$

$\sup_{\|w\|=1} \|F_z w + F_{\bar{z}} \bar{w}\|$ is bounded on $\mathbb{C}^n \setminus \bar{\mathbb{B}}^n$ by Lemma 3.1. On the other hand, we have

$$\begin{aligned} \|\bar{F}_z\| &= \|D\tilde{h}(\varphi(z))[\varphi_{\bar{z}}] + \overline{Dg(\varphi(z))[\varphi_z]}\| \\ &\leq \|D\tilde{h}(\varphi(z))\| \cdot \|\varphi_{\bar{z}}\| + \|Dg(\varphi(z))\| \cdot \|\varphi_z\| \\ &\leq \|D\tilde{h}(\varphi(z))\| + \|Dg(\varphi(z))\| \leq c \end{aligned}$$

and

$$\begin{aligned} \|[F_z]^{-1}\| &= \|[I + D\tilde{h}(\varphi(z))[\varphi_z] + \overline{Dg(\varphi(z))[\varphi_{\bar{z}}]}\]^{-1}\| \\ &\leq \frac{1}{1 - \|D\tilde{h}(\varphi(z))[\varphi_z] + \overline{Dg(\varphi(z))[\varphi_{\bar{z}}]}\|} \\ &\leq \frac{1}{1 - c} \end{aligned}$$

by Lemma 3.1. Therefore,

$$\|\bar{F}_z[F_z]^{-1}\| \leq \frac{c}{1 - c} < 1.$$

Since

$$\det \begin{pmatrix} F_z & F_{\bar{z}} \\ \bar{F}_z & \bar{F}_{\bar{z}} \end{pmatrix} = |\det F_z|^2 \det(I_n - \omega \bar{\omega}),$$

where $\omega = \bar{F}_z[F_z]^{-1}$, by considering the eigenvalues of F_z and $I_n - \omega \bar{\omega}$ as in the proof of Theorem 2.2, we obtain that there exists a constant $\varepsilon > 0$ such that

$$\det \begin{pmatrix} F_z & F_{\bar{z}} \\ \bar{F}_z & \bar{F}_{\bar{z}} \end{pmatrix} > \varepsilon.$$

Thus, the estimation (1.5) holds on $\mathbb{C}^n \setminus \partial\mathbb{B}^n$ for some $K > 0$. This implies that F^r also satisfies the estimation (1.5) for the same K for all $r \in (0, 1)$. Especially, F^r is sense-preserving on $\mathbb{C}^n \setminus \partial\mathbb{B}^n$. Therefore, F^r is a homeomorphism on $\mathbb{C}^n \setminus \partial\mathbb{B}^n$. It remains to show that F^r is a surjective mapping from \mathbb{C}^n to \mathbb{C}^n . Assume that F^r is not surjective. Then there exists $w \in \mathbb{C}^n \setminus F^r(\mathbb{C}^n)$. Since F^r extends to a homeomorphism from a connected neighborhood of $\bar{\mathbb{B}}^n$ onto a domain U , $\mathbb{C}^n \setminus F^r(\bar{\mathbb{B}}^n)$ is connected. Indeed, for any point $w_1, w_2 \in \mathbb{C}^n \setminus F^r(\bar{\mathbb{B}}^n)$, there exist $t_1, t_2 \in (0, 1)$ such that $t_j w_j \in F^r(\partial\mathbb{B}^n)$ and $(t_j w_j, w_j] \subset \mathbb{C}^n \setminus F^r(\bar{\mathbb{B}}^n)$. Then there exists $\varepsilon > 0$ such that $(t_j + \varepsilon)w_j \in U$. Since $(t_1 + \varepsilon)w_1$ and $(t_2 + \varepsilon)w_2$ can be connected by a curve in $U \setminus F^r(\bar{\mathbb{B}}^n)$, w_1 and w_2 can be connected by a curve in $\mathbb{C}^n \setminus F^r(\bar{\mathbb{B}}^n)$. Thus, there exists a curve Γ in $\mathbb{C}^n \setminus F^r(\bar{\mathbb{B}}^n)$ such that $\Gamma(0) = w_0 \in F^r(\mathbb{C}^n \setminus \bar{\mathbb{B}}^n)$ and $\Gamma(1) = w$. Let $t_0 = \sup\{t \in [0, 1] : \Gamma(t) \in F^r(\mathbb{C}^n \setminus \bar{\mathbb{B}}^n)\}$. Since F^r is a homeomorphism on $\mathbb{C}^n \setminus \bar{\mathbb{B}}^n$, $F^r(\mathbb{C}^n \setminus \bar{\mathbb{B}}^n)$ is an open subset of \mathbb{C}^n . Therefore, $\Gamma(t_0) \in \partial F^r(\mathbb{C}^n \setminus \bar{\mathbb{B}}^n)$. There exists a sequence $(w_j) \subseteq F^r(\mathbb{C}^n \setminus \bar{\mathbb{B}}^n)$ such that $w_j \rightarrow \Gamma(t_0)$ as $j \rightarrow \infty$. Hence, there exists a sequence $(z_j) \subseteq \mathbb{C}^n \setminus \bar{\mathbb{B}}^n$ such that $F^r(z_j) = w_j$. Since (w_j) converges, (z_j) also converges to some $z_\infty \in \mathbb{C}^n$. Since $F^r(z_\infty) = \Gamma(t_0) \notin F^r(\mathbb{C}^n \setminus \bar{\mathbb{B}}^n)$, $z_\infty \in \partial\mathbb{B}^n$. This implies that $\Gamma(t_0) \in F^r(\bar{\mathbb{B}}^n)$. However, this contradicts the assumption that Γ is a curve in $\mathbb{C}^n \setminus F^r(\bar{\mathbb{B}}^n)$. Thus, F^r is surjective. This completes the proof. \square

Taking into account Theorems 2.2 and 3.2(ii), and Corollary 2.4, we obtain the following consequence.

Corollary 3.3. *Let $f = h + \bar{g}$ be a pluriharmonic mapping on \mathbb{B}^n such that $h(0) = g(0) = 0$ and $Dh(0) = I_n$. Assume that there exists $c < 1$ such that the relation (2.2) holds. Then the following statements hold:*

- (i) *The mapping $f_A = h + A\bar{g}$ is univalent, sense-preserving, bi-Lipschitz continuous and quasiconformal on \mathbb{B}^n , and $f_A(\mathbb{B}^n)$ is a linearly connected domain in \mathbb{C}^n , for all $A \in L(\mathbb{C}^n)$, $\|A\| \leq 1$.*
- (ii) *The mapping $F_A = h + Ag$ is biholomorphic on \mathbb{B}^n , for all unitary operators $A \in L(\mathbb{C}^n)$. Moreover, if $Dg(0) = \mathbf{0}_n$, then F_A is a quasiregular biholomorphic mapping on \mathbb{B}^n and extends to a quasiconformal homeomorphism of \mathbb{R}^{2n} onto itself, for all $A \in L(\mathbb{C}^n)$, $\|A\| \leq 1$.*
- (iii) *In addition, if $c < 1/2$, then f_A extends to a quasiconformal homeomorphism of \mathbb{R}^{2n} onto itself, for all $A \in L(\mathbb{C}^n)$, $\|A\| \leq 1$.*

Proof. Fix $A \in L(\mathbb{C}^n)$ with $\|A\| \leq 1$. Since

$$\|Dh(z) - I_n\| + \|\bar{A}Dg(z)\| \leq c < 1, \quad z \in \mathbb{B}^n,$$

by (2.2), it follows that f_A is a univalent, sense-preserving, bi-Lipschitz continuous and quasiconformal pluriharmonic mapping on \mathbb{B}^n and $f_A(\mathbb{B}^n)$ is a linearly connected domain in \mathbb{C}^n , in view of Theorem 2.2.

The fact that F_A is biholomorphic on \mathbb{B}^n , for all unitary operators $A \in L(\mathbb{C}^n)$, follows from [11, Theorem 3.1] and the statement (i). If, in addition, $Dg(0) = \mathbf{0}_n$, then F_A is a normalized mapping. Taking into account the condition (2.2), we deduce that

$$\|DF_A(z) - I_n\| \leq c < 1, \quad z \in \mathbb{B}^n.$$

Hence F_A is a quasiregular biholomorphic mapping on \mathbb{B}^n and extends to a quasiconformal homeomorphism of \mathbb{R}^{2n} onto itself, for all $A \in L(\mathbb{C}^n)$ with $\|A\| \leq 1$, in view of Corollary 2.4.

The statement (iii) follows from Theorem 3.2(ii), applied to the mapping f_A , for all $A \in L(\mathbb{C}^n)$ with $\|A\| \leq 1$. This completes the proof. \square

In view of Corollary 3.3, we obtain the following consequence.

Corollary 3.4. *Let $f = h + \bar{g}$ be a pluriharmonic mapping on \mathbb{B}^n such that $h(z) = z + \sum_{k=2}^{\infty} P_k(z^k)$ and $g(z) = \sum_{k=1}^{\infty} Q_k(z^k)$, for all $z \in \mathbb{B}^n$. If*

$$\|Q_1\| + \sum_{k=2}^{\infty} k(\|P_k\| + \|Q_k\|) < \frac{1}{2},$$

then f has a quasiconformal extension F of \mathbb{R}^{2n} onto itself given in Theorem 3.2. Also, $f_A = h + A\bar{g}$ is univalent, sense-preserving, bi-Lipschitz continuous and quasiconformal on \mathbb{B}^n and extends to a quasiconformal homeomorphism of \mathbb{R}^{2n} onto itself, for all $A \in L(\mathbb{C}^n)$, $\|A\| \leq 1$.

4. The Schwarz lemma and the Schwarz–Pick lemma for pluriharmonic mappings on homogeneous unit balls

In this section, let \mathbb{C}^n be the space of n -complex variables with respect to an arbitrary norm $\|\cdot\|$. Also, let B^n be the unit ball in \mathbb{C}^n with respect to the norm $\|\cdot\|$. First, we generalize the harmonic Schwarz lemma on the unit disc to pluriharmonic mappings of the unit ball of a complex Banach space X into B^n .

The following result was obtained by Chen and Gauthier [6, Theorem 4] in the case of pluriharmonic mappings from the Euclidean unit ball \mathbb{B}^n of \mathbb{C}^n into the Euclidean unit ball \mathbb{B}^m of \mathbb{C}^m .

Theorem 4.1. *Let B_X be the unit ball of a complex Banach space X and let $F : B_X \rightarrow B^n$ be a pluriharmonic mapping such that $F(0) = 0$. Then*

$$\|F(z)\| \leq \frac{4}{\pi} \arctan \|z\|_X, \quad z \in B_X.$$

This inequality is sharp for each $z \in B_X$.

Proof. Let $z \in B_X \setminus \{0\}$ be fixed. We may assume that $F(z) \neq 0$. Let $w = z/\|z\|_X \in \partial B_X$. Let $a \in \partial B^n$ be arbitrarily fixed. Since, for each $l_a \in T(a)$,

$$\varphi(\zeta) = l_a(F(\zeta w)), \quad \zeta \in \mathbb{U}$$

is a harmonic mapping on \mathbb{U} such that

$$\varphi(\mathbb{U}) \subseteq \mathbb{U}, \quad \varphi(0) = 0,$$

we obtain from the harmonic Schwarz lemma on the unit disc that

$$|\varphi(\zeta)| = |l_a(F(\zeta w))| \leq \frac{4}{\pi} \arctan |\zeta|, \quad \zeta \in \mathbb{U}.$$

Especially, let $\zeta = \|z\|_X$. Since $\zeta w = z$, we have

$$|l_a(F(z))| \leq \frac{4}{\pi} \arctan \|z\|_X.$$

Finally, if $a = F(z)/\|F(z)\|$, then we obtain

$$\|F(z)\| \leq \frac{4}{\pi} \arctan \|z\|_X.$$

Next, we prove the sharpness. Let $z_0 \in B_X \setminus \{0\}$ be fixed. Let $w_0 = z_0/\|z_0\|_X$ and $l_{w_0} \in T(w_0)$ be fixed. Let $a \in \partial B^n$ be fixed and

$$F(z) = \frac{2}{\pi} \arg \left\{ \frac{1 + e^{i\theta} l_{w_0}(z)}{1 - e^{i\theta} l_{w_0}(z)} \right\} a, \quad z \in B_X.$$

Then F is a pluriharmonic mapping from B_X to B^n with $F(0) = 0$. Moreover,

$$F(\zeta w_0) = \frac{2}{\pi} \arg \left\{ \frac{1 + e^{i\theta} \zeta}{1 - e^{i\theta} \zeta} \right\} a, \quad \zeta \in \mathbb{U}.$$

By the sharpness of the harmonic Schwarz lemma on the unit disc, there exists a θ_0 such that (note that $z_0 = \|z_0\|_X w_0$)

$$\|F(z_0)\| = \left| \frac{2}{\pi} \arg \left\{ \frac{1 + e^{i\theta_0} \|z_0\|_X}{1 - e^{i\theta_0} \|z_0\|_X} \right\} \right| = \frac{4}{\pi} \arctan \|z_0\|_X.$$

This completes the proof. \square

Next, we will generalize the harmonic Schwarz–Pick lemma to pluriharmonic mappings on the unit ball B_X of a complex Banach space such that B_X is homogeneous. For a pluriharmonic mapping $f : B_X \rightarrow \mathbb{C}^n$ of the form $f = h + \bar{g}$, where $h, g \in H(B_X, \mathbb{C}^n)$, let

$$A_f(z) = \sup_{\|w\|_X=1} \|Dh(z)w + \overline{Dg(z)w}\|, \quad z \in B_X,$$

be the maximum dilation. Then we obtain the following theorem (see [6, Theorem 4] in the case of \mathbb{B}^n).

Theorem 4.2. *Let B_X be the unit ball of a complex Banach space X . Assume that B_X is homogeneous. Let B^n be the unit ball of the space \mathbb{C}^n with respect to an arbitrary norm $\|\cdot\|$. Also, let $f : B_X \rightarrow B^n$ be a pluriharmonic mapping of the form $f = h + \bar{g}$, where $h, g \in H(B_X, \mathbb{C}^n)$. Then*

$$A_f(z) \leq \frac{4}{\pi} \frac{1}{1 - \|z\|_X^2}, \quad z \in B_X. \quad (4.1)$$

This inequality is sharp for each $z \in B_X$.

Proof. First, we will show that

$$\|Dh(0)w + \overline{Dg(0)w}\| \leq \frac{4}{\pi}, \quad w \in X, \quad \|w\|_X = 1. \quad (4.2)$$

Let $w \in X$ with $\|w\|_X = 1$ be fixed. Let $a = Dh(0)w + \overline{Dg(0)w}$. If $a = 0$, then (4.2) holds. So, we may assume that $a \neq 0$. Let

$$\phi(\zeta) = l_a(f(\zeta w)), \quad \zeta \in \mathbb{U},$$

where $l_a \in T(a)$. Then ϕ is a harmonic mapping from \mathbb{U} into \mathbb{U} . By applying the harmonic Schwarz–Pick lemma to ϕ , we obtain

$$\|a\| = |\phi_\zeta(0) + \phi_{\bar{\zeta}}(0)| \leq \frac{4}{\pi}.$$

Thus (4.2) holds.

Next, let $z \in B_X \setminus \{0\}$ be fixed. Since B_X is homogeneous, there exists a Möbius transformation $g_z \in \text{Aut}(B_X)$ such that $g_z(0) = z$ and $g_z^{-1} = g_{-z}$. By [36, Corollary 3.6] (see also [25]),

$$\|Dg_{-z}(z)\|_X = \frac{1}{1 - \|z\|_X^2}.$$

Thus, from (4.2) and this equality, we obtain

$$\Lambda_f(z) \leq \Lambda_{f \circ \varphi}(0) \|Dg_{-z}(z)\|_X \leq \frac{4}{\pi} \frac{1}{1 - \|z\|_X^2},$$

where $\varphi = g_z$.

Finally, we will show that the estimate (4.1) is sharp. Let $z_0 \in B_X \setminus \{0\}$ be fixed. Let $w_0 = z_0/\|z_0\|_X$ and $l_{w_0} \in T(w_0)$ be fixed. Let $a \in \partial B^n \cap \mathbb{R}^n$ be fixed and

$$f(z) = \phi(l_{w_0}(z))a, \quad z \in B_X,$$

where $\phi : \mathbb{U} \rightarrow \mathbb{U}$ is a harmonic mapping which satisfies

$$|\phi_\zeta(\|z_0\|_X)| + |\phi_{\bar{\zeta}}(\|z_0\|_X)| = \frac{4}{\pi} \frac{1}{1 - \|z_0\|_X^2}.$$

(The existence of the mapping ϕ is provided by the sharpness of Theorem 1.6.) Then f is a pluriharmonic mapping from B_X to B^n . Moreover,

$$\begin{aligned} \Lambda_f(z_0) &= \sup_{\|w\|_X=1} |\phi_\zeta(\|z_0\|_X)l_{w_0}(w) + \phi_{\bar{\zeta}}(\|z_0\|_X)\overline{l_{w_0}(w)}| \\ &= |\phi_\zeta(\|z_0\|_X)| + |\phi_{\bar{\zeta}}(\|z_0\|_X)| \\ &= \frac{4}{\pi} \frac{1}{1 - \|z_0\|_X^2}. \end{aligned}$$

If $z_0 = 0$, then for arbitrary $w_0 \in \partial B_X$, by using the above argument, we have

$$\Lambda_f(0) = \frac{4}{\pi}.$$

This completes the proof. \square

In view of Theorem 4.2, we obtain the following Schwarz–Pick estimate for the real Jacobian determinant of pluriharmonic mappings of B_X into B^n , where B_X is an open subset of \mathbb{C}^n such that it is homogeneous.

Corollary 4.3. *Assume that B_X is the unit ball of \mathbb{C}^n such that it is homogeneous. Let B^n be the unit ball of the space \mathbb{C}^n with respect to an arbitrary norm $\|\cdot\|$. Also, let $f : B_X \rightarrow B^n$ be a pluriharmonic mapping of the form $f = h + \bar{g}$, where $h, g \in H(B_X, \mathbb{C}^n)$. Then*

$$J_f(z) \leq \left(\frac{4}{\pi}\right)^{2n} \frac{1}{\det B(z, z)}, \quad z \in B_X, \quad (4.3)$$

where $B(z, z) \in L(\mathbb{C}^n)$ is the Bergman operator given by (1.1). In addition,

$$J_f(z) \leq \left(\frac{4}{\pi}\right)^{2n} \frac{1}{(1 - \|z\|^2)^{2c(B_X)}}, \quad z \in B_X, \quad (4.4)$$

where $c(B_X)$ is the constant given by (1.3).

Proof. For $z = 0$, (4.3) follows from (4.1). For fixed $z \in B_X \setminus \{0\}$, let $g_z \in \text{Aut}(B_X)$ be the Möbius transformation of B_X such that $g_z(0) = z$ and $g_{-z} = g_z^{-1}$. Also, let $\phi = g_z$ and $F = f \circ \phi$. Then $Dg_z(0) = B(z, z)^{1/2}$ and

$$\begin{aligned} J_F(0) &= \det \begin{pmatrix} Dh(z) & \overline{Dg(z)} \\ Dg(z) & \overline{Dh(z)} \end{pmatrix} \det \begin{pmatrix} D\phi(0) & 0 \\ 0 & \overline{D\phi(0)} \end{pmatrix} \\ &= J_f(z) |\det D\phi(0)|^2 = J_f(z) \det B(z, z). \end{aligned}$$

Also, since $F : B_X \rightarrow B^n$ is a pluriharmonic mapping, by the inequality (4.3) for $z = 0$, we have $J_F(0) \leq (4/\pi)^{2n}$. Hence, the relation (4.3) follows, as desired.

Finally, taking into account the relations (4.3) and (1.4), we obtain (4.4), as desired. This completes the proof. \square

If $X = \mathbb{C}^n$ with respect to the Euclidean norm $\|\cdot\|$, then $c(\mathbb{B}^n) = (n+1)/2$ (see [26]). Hence, we obtain the following consequence, in view of the relation (4.4). In the case of holomorphic mappings of \mathbb{B}^n into \mathbb{B}^n , see [41, p. 29, Eq. (6)].

Corollary 4.4. *Let $f = h + \bar{g} : \mathbb{B}^n \rightarrow B^n$ be a pluriharmonic mapping, where $h, g \in H(\mathbb{B}^n, \mathbb{C}^n)$ and B^n is the unit ball of \mathbb{C}^n with respect to an arbitrary norm. Then*

$$J_f(z) \leq \left(\frac{4}{\pi}\right)^{2n} \frac{1}{(1 - \|z\|^2)^{n+1}}, \quad z \in \mathbb{B}^n.$$

If $X = \mathbb{C}^n$ with respect to the maximum norm $\|\cdot\|_\infty$ and $B_X = \mathbb{U}^n$ is the unit polydisc in \mathbb{C}^n , then $c(\mathbb{U}^n) = n$ (see [26]). In view of (4.4), we obtain the following consequence (cf. [9]).

Corollary 4.5. *Let $f = h + \bar{g} : \mathbb{U}^n \rightarrow B^n$ be a pluriharmonic mapping, where $h, g \in H(\mathbb{U}^n, \mathbb{C}^n)$ and B^n is the unit ball of \mathbb{C}^n with respect to an arbitrary norm. Then*

$$J_f(z) \leq \left(\frac{4}{\pi}\right)^{2n} \frac{1}{(1 - \|z\|_\infty^2)^{2n}}, \quad z \in \mathbb{U}^n.$$

Next, we will generalize the Schwarz–Pick lemma to holomorphic mappings on the unit ball B_X of a complex Banach space such that B_X is homogeneous. Let Y be a complex Banach space with respect to a norm $\|\cdot\|_Y$. For $f \in H(B_X, Y)$, let

$$\|Df(z)\| = \sup_{\|w\|_X=1} \|Df(z)w\|_Y, \quad z \in B_X,$$

be the operator norm. Then we obtain the following theorem (see [5, Lemma 3] in the case of \mathbb{B}^n).

Theorem 4.6. *Let B_X be the unit ball of a complex Banach space X . Assume that B_X is homogeneous. Let B_Y be the unit ball of a complex Banach space Y . Let $f : B_X \rightarrow B_Y$ be a holomorphic mapping. Then*

$$\|Df(z)\| \leq \frac{1}{1 - \|z\|_X^2}, \quad z \in B_X. \quad (4.5)$$

This inequality is sharp for each $z \in B_X$.

Proof. First, we will show that

$$\|Df(0)w\|_Y \leq 1, \quad w \in X, \quad \|w\|_X = 1. \quad (4.6)$$

Let $w \in X$ with $\|w\|_X = 1$ be fixed. Let $a = Df(0)w$. If $a = 0$, then (4.6) holds. So, we may assume that $a \neq 0$. Let

$$\phi(\zeta) = l_a(f(\zeta w)), \quad \zeta \in \mathbb{U},$$

where $l_a \in T(a)$. Then ϕ is a holomorphic function from \mathbb{U} into \mathbb{U} . By applying the Schwarz–Pick lemma to ϕ , we obtain $\|a\|_Y = |\phi_\zeta(0)| \leq 1$. Thus (4.6) holds.

Next, let $z \in B_X \setminus \{0\}$ be fixed. Since B_X is homogeneous, there exists a Möbius transformation $g_z \in \text{Aut}(B_X)$ such that $g_z(0) = z$ and $g_z^{-1} = g_{-z}$. By [36, Corollary 3.6] (see also [25]),

$$\|Dg_{-z}(z)\| = \frac{1}{1 - \|z\|_X^2}.$$

Thus, from (4.6) and this equality, we obtain

$$\|Df(z)\| \leq \|D(f \circ \varphi)(0)\| \|D\varphi^{-1}(z)\| \leq \frac{1}{1 - \|z\|_X^2},$$

where $\varphi = g_z$.

Finally, we show that the relation (4.5) is sharp for each $z \in B_X$. To this end, fix $z \in B_X \setminus \{0\}$ and let $w_0 = z_0/\|z_0\|_X$ and $l_{w_0} \in T(w_0)$ be fixed. Let $a \in \partial B_Y$ be fixed and

$$f(z) = \phi(l_{w_0}(z))a, \quad z \in B_X,$$

where

$$\phi(\zeta) = \frac{\zeta - \|z_0\|_X}{1 - \|z_0\|_X \zeta}.$$

Then $\phi : \mathbb{U} \rightarrow \mathbb{U}$ is a holomorphic mapping which satisfies

$$|\phi_\zeta(\|z_0\|_X)| = \frac{1}{1 - \|z_0\|_X^2}.$$

Therefore, f is a holomorphic mapping from B_X to B_Y . Moreover,

$$\begin{aligned} \|Df(z_0)\| &= \sup_{\|w\|_X=1} |\phi_\zeta(\|z_0\|_X) l_{w_0}(w)| \\ &= |\phi_\zeta(\|z_0\|_X)| \\ &= \frac{1}{1 - \|z_0\|_X^2}. \end{aligned}$$

If $z_0 = 0$, then for arbitrary $w_0 \in \partial B_X$, by using the above argument, we have

$$\|Df(0)\| = 1.$$

This completes the proof. \square

Assume that B_X and B_Y are open subsets of \mathbb{C}^n . In view of the relation (1.4), we obtain the following Schwarz–Pick estimate for the Jacobian determinant of holomorphic mappings of the unit ball $B_X \subset \mathbb{C}^n$ into $B_Y \subset \mathbb{C}^n$, where B_X is homogeneous (compare with [40, Lemma 7.5.6] and [41, p. 29, Eq. (6)]). If $B_X = B_Y = \mathbb{B}^n$, see [41, p. 29, Eq. (6)].

Corollary 4.7. Assume that B_X and B_Y are open subsets of \mathbb{C}^n such that B_X is homogeneous. Let $f : B_X \rightarrow B_Y$ be a holomorphic mapping. Then

$$|\det Df(z)| \leq \frac{1}{(1 - \|z\|_X^2)^{c(B_X)}}, \quad z \in B_X, \quad (4.7)$$

where $c(B_X)$ is given by (1.3). If $B_X \subset B_Y$, then this estimate is sharp at each $z \in B_X \setminus \{0\}$ such that $z/\|z\|_X$ is a maximal tripotent in X .

Proof. By an argument similar to that in the proof of Corollary 4.3, we obtain that

$$|\det Df(z)| \leq \frac{1}{[\det B(z, z)]^{1/2}}, \quad z \in B_X,$$

where $B(z, z) \in L(\mathbb{C}^n)$ is the Bergman operator given by (1.1). Next, taking into account (1.4) and the above relation, we obtain (4.7).

To deduce the sharpness of (4.7) in the case that $B_X \subset B_Y$, it suffices to fix $z \in B_X \setminus \{0\}$ such that $e = z/\|z\|_X$ is a maximal tripotent in X . Since B_X is homogeneous, there exists a Möbius transformation $g_z \in \text{Aut}(B_X)$ such that $g_z(0) = z$ and $g_z^{-1} = g_{-z}$. Next, let $f = g_{-z}$. Since $z/\|z\|_X$ is a maximal tripotent in X , we deduce from [24, Corollary 2.6] that

$$|\det Df(z)| = |\det [Dg_z(0)]^{-1}| = \frac{1}{(1 - \|z\|_X^2)^{c(B_X)}},$$

as desired. This completes the proof. \square

5. The Landau and the Bloch theorems for pluriharmonic mappings on finite dimensional homogeneous unit balls

In this section, let B be the homogeneous unit ball of $X = \mathbb{C}^n$, that is B is the unit ball of a finite dimensional JB*-triple X . Let $\|\cdot\|_X$ be the norm on X and $\|\cdot\|_e$ denote the Euclidean norm on \mathbb{C}^n . We also assume that

$$\inf\{\|z\|_e : z \in \partial B\} \geq 1. \quad (5.1)$$

This assumption is not so strong, because the unit polydisc satisfies this condition and for any homogeneous unit ball B in \mathbb{C}^n , there exists a constant $c > 0$ such that cB satisfies the inequality (5.1). For $x, y \in \mathbb{R}^n$, let

$$\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|_X = \|x + iy\|_X, \quad \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|_e = \|x + iy\|_e.$$

Then $\|\cdot\|_X$ and $\|\cdot\|_e$ are norms on \mathbb{R}^{2n} . For $A \in L(\mathbb{R}^{2n})$, let

$$\|A\|_{X,e} = \sup \left\{ \left\| A \begin{pmatrix} x \\ y \end{pmatrix} \right\|_e : \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|_X = 1 \right\}.$$

Lemma 5.1. Let $A \in L(\mathbb{R}^{2n})$. Then the following inequalities hold:

$$\|A\|_{X,e} \geq |\det A|^{1/(2n)}, \quad (5.2)$$

$$\left\| A \begin{pmatrix} a \\ b \end{pmatrix} \right\|_e \geq \frac{|\det A|}{\|A\|_{X,e}^{2n-1}}, \quad a + ib \in \partial B, \quad \text{if } \|A\|_{X,e} > 0. \quad (5.3)$$

If A satisfies $\|A\|_{X,e} \leq K|\det A|^{1/(2n)}$, $K \geq 1$, then

$$\left\| A \begin{pmatrix} a \\ b \end{pmatrix} \right\|_e \geq \frac{|\det A|^{1/(2n)}}{K^{2n-1}}, \quad a + ib \in \partial B. \quad (5.4)$$

Proof. If $A \in L(\mathbb{R}^{2n})$, then $\|A\|_{X,e} \geq \sqrt{\lambda_{2n}}$, where $0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{2n}$ are the eigenvalues of tAA and

$$\sqrt{\lambda_1} \leq \inf \left\{ \left\| A \begin{pmatrix} a \\ b \end{pmatrix} \right\|_e : \left\| \begin{pmatrix} a \\ b \end{pmatrix} \right\|_X = 1 \right\}.$$

Also, $|\det A| = \sqrt{\lambda_1 \cdots \lambda_{2n}} \leq \sqrt{\lambda_1} \lambda_{2n}^{(2n-1)/2} \leq \lambda_{2n}^n$.

Moreover, if A satisfies the relation $\|A\|_{X,e} \leq K|\det A|^{1/(2n)}$, $K \geq 1$, then $|\det A| = \sqrt{\lambda_1 \cdots \lambda_{2n}} \leq \sqrt{\lambda_1} K^{2n-1} |\det A|^{(2n-1)/(2n)}$. From these, we obtain the lemma. \square

For a C^1 mapping f of B into \mathbb{C}^n , the maximum dilation A_f and the minimum dilation λ_f can be written as follows:

$$A_f(z) = \max_{a+ib \in \partial B} \left\| D(f; z) \begin{pmatrix} a \\ b \end{pmatrix} \right\|_e, \\ \lambda_f(z) = \min_{a+ib \in \partial B} \left\| D(f; z) \begin{pmatrix} a \\ b \end{pmatrix} \right\|_e.$$

From (5.3), we have

$$\lambda_f \geq \frac{|J_f|}{A_f^{2n-1}}, \quad \text{if } A_f > 0. \quad (5.5)$$

Now, we can generalize the Landau theorem for pluriharmonic mappings on the Euclidean unit ball \mathbb{B}^n [6, Theorem 5] (cf. [8]) to pluriharmonic mappings on homogeneous unit balls B in \mathbb{C}^n which satisfy the condition (5.1). Using (5.5), Theorem 4.1 and (4.1), we can prove the following theorem.

Theorem 5.2. *Let B be a homogeneous unit ball in \mathbb{C}^n which satisfies the condition (5.1) and let f be a pluriharmonic mapping of B into \mathbb{C}^n such that $f(0) = 0$, $\|f(z)\|_e < M$ for $z \in B$ and $(4M/\pi)^{2n} \geq |J_f(0)| = \alpha > 0$. Let*

$$\rho_0 = \frac{\alpha \pi^{2n+1}}{4m(4M)^{2n}}, \quad R_0 = \frac{\alpha^2 \pi^{4n}}{8m(4M)^{4n-1}},$$

where $m \approx 4.2$ is the minimum of the function $(2-r^2)/r(1-r^2)$ on the interval $(0, 1)$. Then f is univalent on the ball $B(0, \rho_0)$. Moreover, $f(B(0, \rho_0))$ covers the ball $\mathbb{B}^n(0, R_0)$.

Proof. By using arguments similar to those in the proof of [6, Theorem 5], we obtain that f is univalent on the ball $B(0, \rho_0)$ and f maps $\partial B(0, \rho_0)$ outside of the ball $\mathbb{B}^n(0, R_0)$. Since f is continuous and univalent on $B(0, \rho_0)$, f is a homeomorphism on $B(0, \rho_0)$ by the invariance of domain [4]. Thus, $f(B(0, \rho_0))$ covers the ball $\mathbb{B}^n(0, R_0)$. This completes the proof. \square

For a pluriharmonic mapping f on B , $\mathbb{B}^n(a, r)$ is called a schlicht ball contained in $f(B)$ if there exists a domain $G \subset B$ such that f maps G homeomorphically onto $\mathbb{B}^n(a, r)$. Let b_f be the Bloch radius of the mapping f , i.e. b_f is the least upper bound of the radii of all schlicht balls which are contained in $f(B)$.

A pluriharmonic mapping f of B into \mathbb{C}^n is called a K -mapping (cf. [6]) if

$$\|D(f; z)\|_{X,e} \leq K |J_f(z)|^{1/(2n)}, \quad z \in B.$$

For pluriharmonic K -mappings, we can generalize the Bloch theorem on the Euclidean unit ball \mathbb{B}^n [6, Theorem 6] to pluriharmonic K -mappings on homogeneous unit balls in \mathbb{C}^n which satisfy the condition (5.1). By applying Theorem 5.2, we can prove the following theorem. Since the proof is similar to that in the proof of [6, Theorem 6], we omit it.

Theorem 5.3. *Let B be a homogeneous unit ball in \mathbb{C}^n which satisfies the condition (5.1) and let f be a pluriharmonic K -mapping of the unit ball B into \mathbb{C}^n , $n > 1$, such that $J_f(0) = 1$. Then*

$$b_f \geq R_n = \frac{k_n \pi}{8m} \left(\frac{k_n \pi}{4K \log(1/(1 - k_n))} \right)^{4n-1},$$

where $0 < k_n < 1$ is the unique number such that

$$4n \log \frac{1}{1 - k_n} = (4n - 1) \cdot \frac{k_n}{1 - k_n}. \quad (5.6)$$

For bounded pluriharmonic K -mappings, we can generalize the Bloch theorem on the Euclidean unit ball \mathbb{B}^n [6, Theorem 7] to bounded pluriharmonic K -mappings on homogeneous unit balls in \mathbb{C}^n which satisfy the condition (5.1).

Theorem 5.4. *Let B be a homogeneous unit ball in \mathbb{C}^n which satisfies the condition (5.1) and let f be a pluriharmonic K -mapping of the unit ball B into \mathbb{C}^n , $n > 1$, such that $f(0) = 0$, $(4M/\pi)^{2n} \geq |J_f(0)| = \alpha > 0$, and $\|f(z)\|_e < M$ for $z \in B$. Then there exists a domain $\Omega \subset B(0, \rho_0)$ such that $0 \in \Omega$ and f maps Ω onto a ball $\mathbb{B}^n(0, R_0)$ homeomorphically, where*

$$\rho_0 = \frac{\pi^2 \alpha^{1/(2n)}}{16mMK^{2n-1}}, \quad R_0 = \frac{\pi^2 \alpha^{1/n}}{32mMK^{4n-2}},$$

and m is the same number as in Theorem 5.2.

Proof. We use arguments similar to those in the proof of [6, Theorem 5]. First, we show that f is univalent on $B(0, \rho_0)$. For fixed different points $z', z'' \in B(0, \rho_0)$, let $z'' - z' = \|z'' - z'\|_X \theta$ and define the pluriharmonic mapping

$$\phi_\theta(z) = (f_z(z) - f_z(0))\theta + (f_{\bar{z}}(z) - f_{\bar{z}}(0))\bar{\theta}.$$

Let $r_0 \in (0, 1)$ be the number such that $(2 - r^2)/r(1 - r^2)$ attains its minimum m on $(0, 1)$ at $r = r_0$. Then $m \approx 4.2$ and $r_0 \approx 0.66$ (see the proof of [5, Lemma 5]). Therefore, $1/m < r_0$. Also, since $\alpha^{1/(2n)} \leq K^{2n-1}M$ by (5.4), we have $\rho_0 \leq 1/m$. Then, by (4.1), we have

$$\begin{aligned} \|\phi_\theta(z)\|_e &\leq \Lambda_f(0) + \Lambda_f(z) \\ &\leq \frac{4M}{\pi} \left(1 + \frac{1}{1 - \|z\|_X^2} \right) \\ &< \frac{4M(2 - r_0^2)}{\pi(1 - r_0^2)} \quad \text{for } \|z\|_X < r_0. \end{aligned}$$

By Theorem 4.1, we have

$$\|\phi_\theta(z)\|_e \leq \frac{16M(2-r_0^2)}{\pi^2 r_0(1-r_0^2)} \|z\|_X = \frac{16mM}{\pi^2} \|z\|_X \quad \text{for } \|z\|_X < r_0. \quad (5.7)$$

Therefore, as in the proof of [6, Theorem 5], by (5.4) and (5.7), we have

$$\begin{aligned} \|f(z'') - f(z')\|_e &> \frac{|\det Df(0)|^{1/(2n)}}{K^{2n-1}} \|z'' - z'\|_X - \frac{16mM}{\pi^2} \rho_0 \|z'' - z'\|_X \\ &\geq \left(\frac{\alpha^{1/(2n)}}{K^{2n-1}} - \frac{16mM}{\pi^2} \rho_0 \right) \|z'' - z'\|_X = 0. \end{aligned}$$

Thus, f is univalent on $B(0, \rho_0)$.

Next, as in the proof of [6, Theorem 7], we obtain that f maps $\partial B(0, \rho_0)$ outside of the ball $\mathbb{B}^n(0, R_0)$. Since f is a homeomorphism on $B(0, \rho_0)$ by the invariance of domain [4], $f(B(0, \rho_0))$ covers the ball $\mathbb{B}^n(0, R_0)$. This completes the proof. \square

For pluriharmonic K -mappings, we can generalize the Bloch theorem on the Euclidean unit ball \mathbb{B}^n [6, Theorem 8] to pluriharmonic K -mappings on homogeneous unit balls in \mathbb{C}^n which satisfy the condition (5.1). As in the proof of [6, Theorem 8], we obtain the following theorem.

Theorem 5.5. *Let B be a homogeneous unit ball in \mathbb{C}^n which satisfies the condition (5.1) and let f be a pluriharmonic K -mapping of the homogeneous unit ball B into \mathbb{C}^n , $n > 1$, such that $\det J_f(0) = 1$. Then*

$$b_f \geq \frac{1}{134K^{4n-1}}.$$

6. The Landau and the Bloch theorem for holomorphic mappings on finite dimensional homogeneous unit balls

In this section, let B be a homogeneous unit ball of $X = \mathbb{C}^n$ which satisfies the condition (5.1). Let $\|\cdot\|_X$ be the norm on X and $\|\cdot\|_e$ denote the Euclidean norm on \mathbb{C}^n .

For $A \in L(\mathbb{C}^n)$, let

$$\|A\|_{X,e} = \sup\{\|Az\|_e : \|z\|_X = 1\}.$$

Lemma 6.1. *Let $A \in L(\mathbb{C}^n)$. Then the following inequalities hold:*

$$\|A\|_{X,e} \geq |\det A|^{1/n}, \quad (6.1)$$

$$\|Aw\|_e \geq \frac{|\det A|}{\|A\|_{X,e}^{n-1}}, \quad w \in \partial B, \quad \text{if } \|A\|_{X,e} > 0. \quad (6.2)$$

If A satisfies $\|A\|_{X,e} \leq K|\det A|^{1/n}$, $K \geq 1$, then

$$\|Aw\|_e \geq \frac{|\det A|^{1/n}}{K^{n-1}}, \quad w \in \partial B. \quad (6.3)$$

Proof. If $A \in L(\mathbb{C}^n)$, then $\|A\|_{X,e} \geq \sqrt{\lambda_n}$, where $0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ are the eigenvalues of A^*A and

$$\sqrt{\lambda_1} \leq \inf\{\|Aw\|_e : \|w\|_X = 1\}.$$

Also, $|\det A| = \sqrt{\lambda_1 \cdots \lambda_n} \leq \sqrt{\lambda_1} \lambda_n^{(n-1)/2} \leq \lambda_n^{n/2}$.

Moreover, if A satisfies $\|A\|_{X,e} \leq K|\det A|^{1/n}$, $K \geq 1$, then

$$|\det A| = \sqrt{\lambda_1 \cdots \lambda_n} \leq \sqrt{\lambda_1} K^{n-1} |\det A|^{(n-1)/n}.$$

From these, we obtain the lemma. \square

Let f be a holomorphic function on the unit disc \mathbb{U} . If $f(0) = 0$, $f'(0) = \alpha > 0$ and $|f(z)| < 1$ for $z \in \mathbb{U}$, then the classical Landau's theorem asserts that f is univalent in the disc \mathbb{U}_{r_0} of center 0 and radius

$$r_0 = \frac{\alpha}{1 + \sqrt{1 - \alpha^2}} > \frac{\alpha}{2},$$

and $f(\mathbb{U}_{r_0})$ contains a disc \mathbb{U}_{R_0} of center 0 and radius $R_0 = r_0^2$ (see e.g. [5]). For holomorphic mappings of the Euclidean unit ball \mathbb{B}^n into \mathbb{C}^n , Chen and Gauthier [5, Theorem 2] obtained a version of the Landau theorem. Now, we can generalize the Landau theorem by Chen and Gauthier to holomorphic mappings on homogeneous unit balls in \mathbb{C}^n which satisfy the condition (5.1).

Theorem 6.2. *Let B be a homogeneous unit ball in \mathbb{C}^n which satisfies the condition (5.1) and let f be a holomorphic mapping of B into \mathbb{C}^n such that $f(0) = 0$, $|\det Df(0)| = \alpha > 0$ and $\|f(z)\|_e < M$ for $z \in B$. Let*

$$\rho_0 = \frac{\alpha}{mM^n}, \quad R_0 = \frac{\alpha^2}{2mM^{2n-1}} = \frac{\alpha\rho_0}{2M^{n-1}},$$

where $m \approx 4.2$ is the minimum of the function $(2 - r^2)/r(1 - r^2)$ on the interval $(0, 1)$. Then f is univalent on the ball $B(0, \rho_0)$ and $f(B(0, \rho_0))$ covers the ball $\mathbb{B}^n(0, R_0)$.

Proof. We use arguments similar to those in the proof of [6, Theorem 5]. First, we show that f is univalent on $B(0, \rho_0)$. For fixed different points $z', z'' \in B(0, \rho_0)$, let $z'' - z' = \|z'' - z'\|_X \theta$ and define the holomorphic mapping

$$\phi_\theta(z) = (Df(z) - Df(0))\theta.$$

Let $r_0 \in (0, 1)$ be the number such that $(2 - r^2)/r(1 - r^2)$ attains its minimum m on $(0, 1)$ at $r = r_0$. Then $m \approx 4.2$ and $r_0 \approx 0.66$ (see the proof of [5, Lemma 5]). Therefore, $1/m < r_0$. Also, since $\alpha \leq M^n$ by (6.1), we have $\rho_0 \leq 1/m$. Then, by (4.5), we have

$$\begin{aligned} \|\phi_\theta(z)\|_e &\leq \|Df(0)\|_{X,e} + \|Df(z)\|_{X,e} \\ &\leq M \left(1 + \frac{1}{1 - \|z\|_X^2} \right) \\ &< \frac{M(2 - r_0^2)}{1 - r_0^2} \quad \text{for } \|z\|_X < r_0. \end{aligned}$$

By the Schwarz lemma, we have

$$\|\phi_\theta(z)\|_e \leq \frac{M(2 - r_0^2)}{r_0(1 - r_0^2)} \|z\|_X = mM \|z\|_X \quad \text{for } \|z\|_X < r_0. \quad (6.4)$$

Therefore, by (6.2) and (6.4), we have

$$\begin{aligned}
\|f(z'') - f(z')\|_e &= \left\| \int_0^1 Df(z' + t(z'' - z'))(z'' - z') dt \right\|_e \\
&\geq \left\| \int_0^1 Df(0)(z'' - z') dt \right\|_e \\
&\quad - \left\| \int_0^1 (Df(z' + t(z'' - z')) - Df(0))(z'' - z') dt \right\|_e \\
&= \|Df(0)(z'' - z')\|_e \\
&\quad - \left\| \int_0^1 (Df(z' + t(z'' - z')) - Df(0))(z'' - z') dt \right\|_e \\
&> \frac{|\det Df(0)|}{\|Df(0)\|_{X,e}^{n-1}} \|z'' - z'\|_X - mM\rho_0 \|z'' - z'\|_X \\
&\geq \left(\frac{\alpha}{M^{n-1}} - mM\rho_0 \right) \|z'' - z'\|_X = 0.
\end{aligned}$$

Thus, f is univalent on $B(0, \rho_0)$.

Next, for $z \in \partial B(0, \rho_0)$, we obtain

$$\begin{aligned}
\|f(z)\|_e &= \left\| \int_0^1 Df(tz)z dt \right\|_e \\
&\geq \left\| \int_0^1 Df(0)z dt \right\|_e - \left\| \int_0^1 (Df(tz) - Df(0))z dt \right\|_e \\
&\geq \left(\frac{\alpha}{M^{n-1}} - mM\rho_0 \int_0^1 t dt \right) \|z\|_X = R_0.
\end{aligned}$$

Thus, $f(B(0, \rho_0))$ covers the ball $\mathbb{B}^n(0, R_0)$. This completes the proof. \square

Let B be a homogeneous unit ball of $X = \mathbb{C}^n$ which satisfies the condition (5.1). A mapping $f \in H(B)$ is called a Wu K -mapping if (cf. [5], in the case $X = \mathbb{C}^n$ with respect to the Euclidean norm)

$$\|Df(z)\|_{X,e} \leq K |\det Df(z)|^{1/n} \quad z \in B.$$

For bounded Wu K -mappings on B , we can generalize the Landau theorem by Chen and Gauthier [5, Theorem 3] to bounded Wu K -mappings on homogeneous unit balls in \mathbb{C}^n which satisfy the condition (5.1). As in the proof of [5, Theorem 3], we obtain the following theorem.

Theorem 6.3. *Let B be a homogeneous unit ball in \mathbb{C}^n which satisfies the condition (5.1) and let f be a Wu K -mapping of the unit ball B into \mathbb{C}^n such that $f(0) = 0$, $|\det Df(0)| = \alpha > 0$ and $\|f(z)\|_e < M$ for $z \in B$. Then there exists a domain $G \subset B(0, \rho_0)$ such that $0 \in G$ and f maps G onto the ball $\mathbb{B}^n(0, R_0)$ injectively, where*

$$\rho_0 = \frac{\alpha^{1/n}}{2MK^{n-1}}, \quad R_0 = M\rho_0^2 = \frac{\alpha^{2/n}}{4MK^{2n-2}}.$$

For $f \in H(B)$, $\mathbb{B}^n(a, r)$ is called a schlicht ball contained in $f(B)$ if there exists a domain $G \subset B$ such that f maps G biholomorphically onto $\mathbb{B}^n(a, r)$. Let b_f be the Bloch radius of the mapping f , i.e. b_f is the least upper bound of the radii of all schlicht balls which are contained in $f(B)$ [5].

For Wu K -mappings, we can generalize the Bloch theorem by Chen and Gauthier [5, Theorem 4] to Wu K -mappings on homogeneous unit balls in \mathbb{C}^n which satisfy the condition (5.1). As in the proof of [5, Theorem 4], we obtain the following theorem.

Theorem 6.4. *Let B be a homogeneous unit ball in \mathbb{C}^n which satisfies the condition (5.1) and let f be a Wu K -mapping of the unit ball B into \mathbb{C}^n , $n > 1$, such that $\det Df(0) = 1$. Then*

$$b_f \geq \frac{1}{9.83K^{2n-1}} > \frac{1}{10K^{2n-1}}.$$

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