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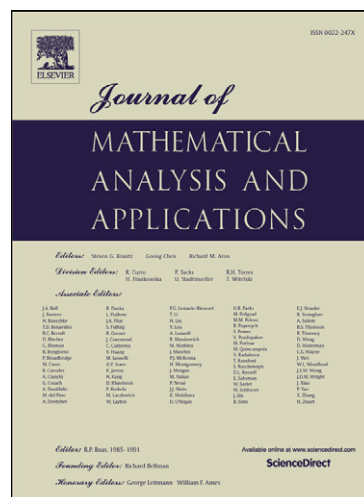
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New Criteria on Persistence in Mean and Extinction for Stochastic Competitive Lotka-Volterra Systems with Regime Switching

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Abstract

This paper is concerned with persistence in mean and extinction for stochastic competitive Lotka-Volterra systems with regime switching. By using some novel stochastic analysis techniques, sufficient criteria for the partial permanence and partial extinction are established. Some novel sufficient conditions on persistence in mean and extinction are also obtained. Nontrivial examples are provided to illustrate our results.

Keywords: Persistence in Mean, Extinction, Partial Permanence and Partial Extinction, Stochastic Competitive Lotka-Volterra Systems, Regime Switching

1. Introduction

One of the most common phenomena considering ecological population is that many species which grow in the same environment compete for the limited resources or in some way inhibit others' growth. It is therefore very important to study the competition models for multi-species. It is well known that one of the most famous models is the following classical Lotka-Volterra competition system

$$\frac{dx_i}{dt} = x_i(b_i - \sum_{j=1}^n a_{ij}x_j), \quad i = 1, \dots, n. \quad (1)$$

where $x_i(t)$ represents the population size of species i at time t , the constant b_i is the growth rate of species i , and a_{ij} represents the effect of interspecific ($i \neq j$) or intraspecific ($i = j$) interaction.

On the other hand, population systems are inevitably affected by environmental noise. One type of environmental noise is color noise, say telegraph noise. The telegraph noise can be illustrated as a switching between two or more regimes of environment, which differs by factors such as nutrition or rain falls. The switching is memoryless and the waiting time for the next switch has an exponential distribution. We can hence model the regime switching by a finite-state Markovian Chain. The other is the well-know white noise described by the Brownian motion. In recent years, the population dynamics under environmental noise have been considered by many authors(see [3], [4], [6]-[11]).

When taking the white and color noise into account, system (1) becomes the stochastic competitive Lotka-Volterra system with regime switching as the following:

$$dx_i = x_i(b_i(r(t)) - \sum_{j=1}^n a_{ij}(r(t))x_j)dt + \sigma_i(r(t))x_i dB_i(t), \quad i = 1, \dots, n, \quad (2)$$

The stochastic competitive Lotka-Volterra model has been extensively studied due to its universal existence and importance(see[1], [2], [5], [6], [15], [16]). In the study of population systems, extinction and permanence, including

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stochastic permanence, persistence in mean, are two important and interesting properties, respectively meaning that the population system will die out or survive in the future, which have received a lot of attention (see [2]-[6]). Li et al. [6] discussed the stochastic general Lotka-Volterra population under regime switching, and sufficient conditions for stochastic permanence and extinction were obtained. Li et al. [4] discussed the stochastic logistic population under regime switching, and sufficient and necessary conditions for stochastic permanence and extinction under some assumption were obtained. More recently, A class of non-autonomous stochastic Lotka-Volterra competitive system was discussed by Li and Mao [5], the sufficient conditions for the stochastic permanence, extinction were obtained. Jiang et al. [2] have studied the stable in time average of the autonomous stochastic Lotka-Volterra competitive system, which implied the persistence in mean.

However, most of the existing criteria are established for stochastic general Lotka-Volterra system. It is well known that these criteria will produce conservatism when dealing with the competitive systems. And there are a few results on the persistence in mean for competitive systems available in the literature. This motivates us to investigate the persistence in mean and extinction for stochastic competitive Lotka-Volterra systems with regime switching.

Moreover, most of the existing criteria are established for total extinction and total permanence. The partial permanence and partial extinction have not been fully investigated, which are very important and useful properties. To the best of our knowledge, results on this problem are rare, which remain an interesting research topic.

We aim to establish new results on persistence in mean and extinction for system (2). By using the Lyapunov methods, and some novel stochastic analysis techniques, sufficient criteria are established which ensure the partial permanence and partial extinction. Sufficient conditions on persistence in mean and extinction for system (2) are also obtained.

2. Notation

Throughout this paper, unless otherwise specified, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous while \mathcal{F}_0 contains all \mathbb{P} -null sets). Let $B(t) = (B_t^1, \dots, B_t^d)$ be a d -dimensional Brownian motion defined on the probability space. Let $r(t)$, $t \geq 0$, be a right-continuous Markovian chain on the probability space taking values in a finite state space $S = \{1, 2, \dots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N \times N}$ given by

$$P\{r(t + \Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & i \neq j \\ 1 + \gamma_{ii}\Delta + o(\Delta) & i = j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} > 0$ is transition rate from i to j if $i \neq j$ while $\gamma_{ii} = -\sum_{i \neq j} \gamma_{ij} < 0$. We assume that the Markovian chain $r(t)$ is independent of the Brownian motion $B(t)$. It is well known that almost every sample path of $r(t)$ is right continuous step function. We also assume that the Markovian chain $r(t)$ is irreducible. Under this condition, the Markov chain has a unique stationary (probability) distribution $\pi = (\pi_1, \pi_2, \dots, \pi_N)$ which can be determined by solving the following linear equation

$$\pi \Gamma = 0, \quad \sum_{k=1}^N \pi_k = 1.$$

And for any vector $f = (f_1, f_2, \dots, f_N)$,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(r(t)) dt = \sum_{k=1}^N \pi_k f_k. \quad (3)$$

For the convenience and simplicity in the following discussion, define

$$\begin{aligned} \hat{b}_i &= \min_{k \in S} b_i(k), \check{b}_i = \max_{k \in S} b_i(k), \hat{\sigma}_i = \min_{k \in S} \sigma_i(k), \check{\sigma}_i = \max_{k \in S} \sigma_i(k), \hat{a}_{ij} = \min_{k \in S} a_{ij}(k), \check{a}_{ij} = \max_{k \in S} a_{ij}(k), \\ M_{ij} &= \frac{\check{a}_{ij}}{\hat{a}_{jj}}, m_{ij} = \frac{\hat{a}_{ij}}{\check{a}_{jj}}, i, j = 1, \dots, n. \end{aligned}$$

In order to obtain our main result, we need the following assumptions.

Assumption 1: For each $k \in S$ and $i, j = 1, \dots, n$ with $i \neq j$, $a_{ii}(k) > 0$, $a_{ij}(k) \geq 0$.

Assumption 2: For each $i = 1, \dots, n$, $\sum_{k \in S} \pi_k(b_i(k) - \frac{\sigma_i^2(k)}{2}) > 0$.

Assumption 3: For each $i = 1, \dots, n$, $\sum_{k \in S} \pi_k((b_i(k) - \frac{\sigma_i^2(k)}{2}) - \sum_{j \neq i} M_{ij}(b_j(k) - \frac{\sigma_j^2(k)}{2})) > 0$.

In the same way as Mao et al.[8] did, we can also show the following result on the existence of global positive solution.

Lemma 2.1: Let Assumption 1 hold. Then for any given initial value $x_0 \in R_+^n, r_0 \in S$, there is a unique solution $x(t, x_0, r_0)$ to system (2) and the solution will remain in R_+^n with probability 1, namely

$$\mathbb{P}\{x(t, x_0, r_0) \in R_+^n, \forall t \geq 0\} = 1,$$

for any $x_0 \in R_+^n, r_0 \in S$.

3. Persistent in Mean

In this section, we will investigate persistence in mean. First we introduce one definition.

Definition 1: The system (2) is said to be persistent in mean, if there exist positive constants α_i and β_i $i = 1, \dots, n$ such that the solution to system (2) has the following property:

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t x_i(s, x_0, r_0) ds \leq \alpha_i, \quad \text{a.s.} \quad \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t x_i(s, x_0, r_0) ds \geq \beta_i, \quad \text{a.s.} \quad i = 1, \dots, n.$$

To this end, we consider two auxiliary stochastic differential equations

$$\begin{cases} dy_i = y_i(b_i(r(t)) - a_{ii}(r(t))y_i)dt + \sigma_i(r(t))y_i dB_i(t), \\ y_i(0) = x_i(0), r(0) = r_0, i = 1, \dots, n. \end{cases} \quad (4)$$

$$\begin{cases} dz_i = z_i(b_i(r(t)) - a_{ii}(r(t))z_i - \sum_{j \neq i} a_{ij}(r(t))y_j)dt + \sigma_i(r(t))z_i dB_i(t), \\ z_i(0) = x_i(0), r(0) = r_0, i = 1, \dots, n. \end{cases} \quad (5)$$

Remark 1: Then it follows from comparison principle(see [14]) that

$$z_i(t, x_0, r_0) \leq x_i(t, x_0, r_0) \leq y_i(t, x_0, r_0), \quad i = 1, \dots, n. \quad (6)$$

And it follows from Itô's formula that

$$\begin{aligned} \frac{1}{z_i(t)} &= \frac{1}{x_i(0)} \exp\left\{\left(\frac{\sigma_i^2}{2} - b_i\right)t - \sigma_i B_i(t) + \sum_{j \neq i} \int_0^t a_{ij} y_j(s) ds\right\} \\ &\quad + a_{ii} \int_0^t \exp\left\{\left(\frac{\sigma_i^2}{2} - b_i\right)(t-s) - \sigma_i(B_i(t) - B_i(s)) + \sum_{j \neq i} \int_s^t a_{ij} y_j(\tau) d\tau\right\} ds. \end{aligned} \quad (7)$$

Lemma 3.1: Let Assumption 1 hold. And assume that $\sum_{k \in S} \pi_k((b_i(k) - \frac{\sigma_i^2(k)}{2})) \geq 0$. Then the solution $y_i(t)$ to system(4) has the following property

$$\lim_{t \rightarrow \infty} \frac{\log y_i(t)}{t} = 0. \quad \text{a.s.} \quad (8)$$

Remark 2: The proof is composed of two parts. The first part is to prove the assertion (8) in the case of $\sum_{k \in S} \pi_k((b_i(k) - \frac{\sigma_i^2(k)}{2})) > 0$, and the proof is essentially a special version of the proof of Lemma 4.1 in Zhu [16]. The second part is to prove the assertion (8) when $\sum_{k \in S} \pi_k((b_i(k) - \frac{\sigma_i^2(k)}{2})) = 0$, in which techniques are similar to the proof of Lemma 4.2 in Section 4.

Lemma 3.2: Let Assumption 1 hold. And assume that

$$\begin{aligned} \sum_{k \in S} \pi_k((b_i(k) - \frac{\sigma_i^2(k)}{2}) - \sum_{j \neq i} M_{ij}(b_j(k) - \frac{\sigma_j^2(k)}{2})) &> 0, \\ \sum_{k \in S} \pi_k((b_i(k) - \frac{\sigma_i^2(k)}{2})) &\geq 0, \forall j \neq i. \end{aligned}$$

Then the solution $z_i(t)$ to system (5) has the following property

$$\lim_{t \rightarrow \infty} \frac{\log z_i(t)}{t} = 0. \quad a.s. \quad (9)$$

Proof: Applying Itô's formula to $\log y_i(t)$ yields,

$$\log y_i(t) = \log y_i(s) + \int_s^t (b_i - \frac{\sigma_i^2}{2}) d\tau - \int_s^t a_{ii}(r(\tau)) y_i(\tau) d\tau + \int_s^t \sigma_i^2 dB_i(\tau),$$

and hence

$$\int_s^t a_{ii}(r(\tau)) y_i(\tau) d\tau = (\log y_i(s) - \log y_i(t)) + \int_s^t (b_i - \frac{\sigma_i^2}{2}) d\tau + \int_s^t \sigma_i^2 dB_i(\tau). \quad (10)$$

It is easy to estimate that

$$m_{ij} \int_s^t a_{jj}(r(\tau)) y_j(\tau) d\tau \leq \int_s^t a_{ij}(r(\tau)) y_j(\tau) d\tau \leq M_{ij} \int_s^t a_{jj}(r(\tau)) y_j(\tau) d\tau.$$

Simple computation shows that

$$\begin{aligned} &\sum_{j \neq i} \int_s^t a_{ij}(r(\tau)) y_j(\tau) d\tau \\ &\leq \sum_{j \neq i} M_{ij} \left((\log y_j(s) - \log y_j(t)) + \int_s^t (b_j - \frac{\sigma_j^2}{2}) d\tau + \check{\sigma}_j(B_j(t) - B_j(s)) \right) \\ &\leq \sum_{j \neq i} M_{ij} \left((\max_{0 \leq s \leq t} \log y_j(s) - \log y_j(t)) + \int_s^t (b_j - \frac{\sigma_j^2}{2}) d\tau + \check{\sigma}_j(B_j(t) - \min_{0 \leq s \leq t} B_j(s)) \right), \end{aligned} \quad (11)$$

Substituting (11) into (7) yields

$$\begin{aligned} \frac{1}{z_i(t)} &\leq \exp \left\{ \sum_{j \neq i} M_{ij} \left(\max_{0 \leq s \leq t} \log y_j(s) - \log y_j(t) \right) + \sum_{j \neq i} M_{ij} \check{\sigma}_j \left(B_j(t) - \min_{0 \leq s \leq t} B_j(s) \right) \right. \\ &\quad \left. + \check{\sigma}_i \left(\max_{0 \leq s \leq t} B(s) - B(t) \right) \right\} \left\{ \exp \left\{ \int_0^t \left[- (b_i - \frac{\sigma_i^2}{2})(r(s)) + \sum_{j \neq i} M_{ij} (b_j - \frac{\sigma_j^2}{2})(r(s)) \right] ds \right\} \frac{1}{x_i(0)} \right. \\ &\quad \left. + \int_s^t \left[- (b_i - \frac{\sigma_i^2}{2})(r(s)) + \sum_{j \neq i} M_{ij} (b_j - \frac{\sigma_j^2}{2})(r(s)) \right] ds \right\} \\ &= \exp \left\{ \sum_{j \neq i} M_{ij} \left(\min_{0 \leq s \leq t} \log y_j(s) - \log y_j(t) \right) + \sum_{j \neq i} M_{ij} \check{\sigma}_j \left(B_j(t) - \max_{0 \leq s \leq t} B_j(s) \right) \right. \\ &\quad \left. + \hat{\sigma}_i \left(\max_{0 \leq s \leq t} B(s) - B(t) \right) \right\} \varphi_i^{-1}(t), \end{aligned} \quad (12)$$

where $\varphi_i(t)$ is the solution to the following system

$$\begin{cases} \dot{\varphi}_i(t) = \varphi_i \left((b_i - \frac{\sigma_i^2}{2})(r(t)) - \sum_{j \neq i} M_{ij}(b_j - \frac{\sigma_j^2}{2})(r(t)) - a_{ii}(r(t))\varphi_i(t) \right), \\ \varphi_i(0) = x_i(0), r(0) = r_0. \end{cases}$$

Similarly, we get

$$\begin{aligned} \frac{1}{z_i(t)} &\geq \exp \left\{ \sum_{j \neq i} m_{ij} \left(\min_{0 \leq s \leq t} \log y_j(s) - \log y_j(t) \right) + \sum_{j \neq i} m_{ij} \hat{\sigma}_j \left(B_j(t) - \max_{0 \leq s \leq t} B_j(s) \right) \right. \\ &\quad \left. + \hat{\sigma}_i \left(\min_{0 \leq s \leq t} B(s) - B(t) \right) \right\} \left\{ \exp \left\{ \int_0^t \left[- (b_i - \frac{\sigma_i^2}{2}) + \sum_{j \neq i} m_{ij} (b_j - \frac{\sigma_j^2}{2}) \right] d\tau \right\} \frac{1}{x_i(0)} \right. \\ &\quad \left. + \int_s^t \left[- (b_i - \frac{\sigma_i^2}{2})(r(s)) + \sum_{j \neq i} m_{ij} (b_j - \frac{\sigma_j^2}{2})(r(s)) \right] ds \right\} \\ &:= \exp \left\{ \sum_{j \neq i} m_{ij} (\min_{0 \leq s \leq t} \log y_j(s) - \log y_j(t)) + \sum_{j \neq i} m_{ij} \hat{\sigma}_j (B_j(t) - \max_{0 \leq s \leq t} B_j(s)) \right. \\ &\quad \left. + \hat{\sigma}_i \left(\min_{0 \leq s \leq t} B(s) - B(t) \right) \right\} \psi_i^{-1}(t), \end{aligned} \quad (13)$$

where $\psi_i(t)$ is the solution to the following system

$$\begin{cases} \dot{\psi}_i(t) = \psi_i \left((b_i - \frac{\sigma_i^2}{2})(r(t)) - \sum_{j \neq i} m_{ij} (b_j - \frac{\sigma_j^2}{2})(r(t)) - a_{ii}(r(t))\psi_i(t) \right), \\ \psi_i(0) = x_i(0), r(0) = r_0. \end{cases}$$

We, therefore, conclude from (12) and (13) that

$$\begin{aligned} \frac{\log z_i(t)}{t} &\geq - \sum_{j \neq i} \frac{\left(M_{ij} (\max_{0 \leq s \leq t} \log y_j(s) - \log y_j(t)) \right)}{t} - \sum_{j \neq i} \frac{M_{ij} \hat{\sigma}_j (B_j(t) - \min_{0 \leq s \leq t} B_j(s))}{t} \\ &\quad - \frac{\hat{\sigma}_i \left(\max_{0 \leq s \leq t} B(s) - B(t) \right)}{t} - \frac{\log \varphi_i(t)}{t}, \end{aligned} \quad (14)$$

$$\begin{aligned} \frac{\log z_i(t)}{t} &\leq - \sum_{j \neq i} \frac{\left(m_{ij} (\min_{0 \leq s \leq t} \log y_j(s) - \log y_j(t)) \right)}{t} - \sum_{j \neq i} \frac{m_{ij} \hat{\sigma}_j (B_j(t) - \max_{0 \leq s \leq t} B_j(s))}{t} \\ &\quad - \frac{\hat{\sigma}_i \left(\min_{0 \leq s \leq t} B(s) - B(t) \right)}{t} - \frac{\log \psi_i(t)}{t}. \end{aligned} \quad (15)$$

Using the property of Brownian motion, we conclude that

$$\lim_{t \rightarrow \infty} \frac{(\max_{0 \leq s \leq t} B_i(s) - B_i(t))}{t} = 0, \lim_{t \rightarrow \infty} \frac{(B_j(t) - \min_{0 \leq s \leq t} B_j(s))}{t} = 0. \quad \text{a.s.} \quad (16)$$

It is easy to see that if $\sum_{k \in \mathcal{S}} ((b_i - \frac{\sigma_i^2}{2}) - \sum_{j \neq i} M_{ij}(b_j - \frac{\sigma_j^2}{2})) > 0$, then we have

$$\lim_{t \rightarrow \infty} \frac{\log \varphi_i(t)}{t} = 0, \lim_{t \rightarrow \infty} \frac{\log \psi_i(t)}{t} = 0. \quad \text{a.s.} \quad (17)$$

Besides, it follows from Lemma 3.1 that for any $j \neq i$

$$\lim_{t \rightarrow \infty} \frac{(\max_{0 \leq s \leq t} \log y_j(s) - \log y_j(t))}{t} = 0, \quad \lim_{t \rightarrow \infty} \frac{(\min_{0 \leq s \leq t} \log y_j(s) - \log y_j(t))}{t} = 0. \quad \text{a.s.} \quad (18)$$

Letting $t \rightarrow \infty$ on both sides of (14), (15) and using condition (16)-(18) yields

$$\limsup_{t \rightarrow \infty} \frac{\log z_i(t)}{t} \leq 0, \quad \liminf_{t \rightarrow \infty} \frac{\log z_i(t)}{t} \geq 0. \quad \text{a.s.}$$

So the assertion (9) must hold. The proof is therefore completed.

Theorem 3.1: *Let Assumption 1-3 hold. Then the solution to system (2) has the following property*

$$\begin{aligned} \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t x_i(s, x_0, r_0) ds &\geq \frac{1}{\hat{a}_{ii}} \sum_{k \in S} \pi_k \left((b_i(k) - \frac{\sigma_i^2(k)}{2}) - \sum_{j \neq i} M_{ij} (b_j(k) - \frac{\sigma_j^2(k)}{2}) \right) \quad i = 1, \dots, n. \\ \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t x_i(s, x_0, r_0) ds &\leq \frac{1}{\hat{a}_{ii}} \sum_{k \in S} \pi_k (b_i(k) - \frac{\sigma_i^2(k)}{2}) \quad i = 1, \dots, n. \end{aligned}$$

That is, the system (2) is persistent in mean.

Proof: Let $x_i(t)$ for $x_i(t, x_0, r_0)$ simplicity. For each $i = 1, \dots, n$, it then follows from (10) that

$$\begin{aligned} &\frac{1}{t} \int_0^t a_{ii}(r(s)) y_i(s) ds \\ &= \frac{1}{t} (\log y_i(t) - \log y_i(0)) + \frac{1}{t} \int_0^t (b_i - \frac{\sigma_i^2}{2})(r(s)) ds + \frac{1}{t} \int_0^t \sigma_i(r(s)) dB_i(s), \quad i = 1, \dots, n. \end{aligned} \quad (19)$$

Noting that, for each $i = 1, \dots, n$, $\frac{1}{t} \int_0^t \sigma_i^2(r(s)) ds < \check{\sigma}_i^2$. By the law of strong large numbers for martingales, we have

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \sigma_i(r(s)) dB_i(s) = 0, \quad \text{a.s.} \quad i = 1, \dots, n.$$

The ergodic property of $r(t)$ implies that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t (b_i - \frac{\sigma_i^2}{2})(r(s)) ds = \sum_{k \in S} \pi_k (b_i(k) - \frac{\sigma_i^2(k)}{2}), \quad i = 1, \dots, n. \quad (20)$$

By Lemma 3.1, letting $t \rightarrow \infty$ on both sides of (19) yields

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t a_{ii}(r(s)) y_i(s) ds = \sum_{k \in S} \pi_k (b_i(k) - \frac{\sigma_i^2(k)}{2}), \quad \text{a.s.} \quad i = 1, \dots, n.$$

Simple computation shows that

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t x_i(s) ds &\leq \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t y_i(s) ds \leq \frac{1}{\hat{a}_{ii}} \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t a_{ii}(r(s)) x_i(s) ds \\ &= \frac{1}{\hat{a}_{ii}} \sum_{k \in S} \pi_k (b_i(k) - \frac{\sigma_i^2(k)}{2}), \quad \text{a.s.} \quad i = 1, \dots, n. \end{aligned}$$

Applying the Itô formula to $\log z_i(t)$ yields

$$\begin{aligned} \log z_i(t) &= \log z_i(0) + \int_0^t (b_i - \frac{\sigma_i^2}{2})(r(s)) ds - \int_0^t a_{ii}(r(s)) z_i(s) ds \\ &\quad - \int_0^t \sum_{j \neq i} a_{ij}(r(s)) y_j(s) ds + \int_0^t \sigma_i(r(s)) dB_i(s), \quad i = 1, \dots, n, \end{aligned}$$

and hence

$$\begin{aligned}
 & \frac{1}{t} \int_0^t a_{ii}(r(s))z_i(s)ds \\
 &= \frac{1}{t}(\log z_i(t) - \log z_i(0)) + \frac{1}{t} \left(\int_0^t (b_i - \frac{\sigma_i^2}{2})(r(s))ds - \int_0^t \sum_{j \neq i} a_{ij}(r(s))y_j(s)ds \right) \\
 & \quad + \frac{1}{t} \int_0^t \sigma_i(r(s))dB_i(s) \\
 & \geq \frac{1}{t}(\log z_i(t) - \log z_i(0)) + \frac{1}{t} \int_0^t (b_i - \frac{\sigma_i^2}{2})(r(s))ds \\
 & \quad - \frac{1}{t} \sum_{j \neq i} M_{ij} \int_0^t a_{jj}(r(s))y_j(s)ds + \frac{1}{t} \int_0^t \sigma_i(r(s))dB_i(s), \quad \text{a.s. } i = 1, \dots, n.
 \end{aligned} \tag{21}$$

By Lemma 3.2 and (20), letting $t \rightarrow \infty$ on both sides of (19) yields

$$\begin{aligned}
 & \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t a_{ii}(r(s))z_i(s)ds \\
 & \geq \sum_{k \in S} \pi_k((b_i(k) - \frac{\sigma_i^2(k)}{2}) - \sum_{j \neq i} M_{ij}(b_j(k) - \frac{\sigma_j^2(k)}{2})), \quad \text{a.s. } i = 1, \dots, n.
 \end{aligned}$$

Simple computation shows that

$$\begin{aligned}
 & \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t x_i(s)ds \geq \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t z_i(s)ds \geq \frac{1}{\check{a}_{ii}} \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t a_{ii}(r(s))z_i(s)ds \\
 & \geq \frac{1}{\check{a}_{ii}} \sum_{k \in S} \pi_k((b_i(k) - \frac{\sigma_i^2(k)}{2}) - \sum_{j \neq i} M_{ij}(b_j(k) - \frac{\sigma_j^2(k)}{2})), \quad \text{a.s. } i = 1, \dots, n.
 \end{aligned}$$

The proof is completed.

4. Extinction

One of the most basic questions one can ask in population dynamics is extinction, which means a species will be doomed. The interesting question is: Can the exponential extinction rate be estimated precisely? In many cases, we need to know the extinction rate of the species in order to have a suitable policy in investment and to have timely measures to protect them from the extinct disaster.

Theorem 4.1: *Let Assumption 1 hold. Assume that there exists a integer $1 \leq m \leq n$, such that*

$$\sum_{k \in S} \pi_k(b_i(k) - \frac{\sigma_i^2(k)}{2}) < 0, \quad i = 1, \dots, m,$$

and

$$\sum_{k \in S} \pi_k(b_i(k) - \frac{\sigma_i^2(k)}{2}) = 0, \quad i = m+1, \dots, n.$$

We then have the following assertions:

i) For $i = 1, \dots, m$, the solution $x_i(t, x_0, r_0)$ to system (2) has the property that

$$\lim_{t \rightarrow \infty} \frac{\log x(t, x_0, r_0)}{t} = \sum_{k \in S} \pi_k(b_i(k) - \frac{\sigma_i^2(k)}{2}). \quad \text{a.s.} \tag{22}$$

That is, for each $i = 1, \dots, m$, the species i will become extinct exponentially with probability one and the exponential extinction rate is $-(\frac{\sigma_i}{2} - b_i)$.

ii) For $i = m + 1, \dots, n$, the solution $x_i(t, x_0, r_0)$ to system (2) has the property that

$$\lim_{t \rightarrow \infty} x(t, x_0, r_0) = 0. \quad \text{a.s.} \quad \lim_{t \rightarrow \infty} \frac{\log x(t, x_0, r_0)}{t} = 0. \quad \text{a.s.} \quad (23)$$

That is, for each $i = m + 1, \dots, n$, the species i still becomes extinct with zero exponential extinction rate.

To prove Theorem 4.1, let us present four lemmas which are essential to the proof.

We first state a Lemma which can be found in [16]. This Lemma plays an important role in this section and here we omit the proof.

Lemma 4.1: Let Assumption 1 hold and $x(t, x_0, r_0)$ be the global solution to system (2) with any positive initial value $x_0 \in \mathbb{R}_+^n, r_0 \in S$. For each $i = 1, \dots, n$, $x_i(t, x_0, r_0)$ is uniformly continuous. a.s.

Lemma 4.2: ([13]) Let $f(t)$ be uniformly continuous on $[0, \infty)$, and $f(t) \in L^1(\mathbb{R}_+; \mathbb{R}_+)$, then $\lim_{t \rightarrow \infty} f(t) = 0$.

Lemma 4.3: Let Assumption 1 hold. If $\sum_{k \in S} \pi_k(b_i(k) - \frac{\sigma_i^2(k)}{2}) < 0$, the solution $x_i(t, x_0, r_0)$ to system (2) has the property that

$$\limsup_{t \rightarrow \infty} \frac{\log x_i(t, x_0, r_0)}{t} \leq - \sum_{k \in S} (\frac{\sigma_i^2(k)}{2} - b_i(k)). \quad \text{a.s.} \quad (24)$$

Proof: Let $x_i(t)$ for $x_i(t, x_0, r_0)$ simplicity. It follows from Itô's formula that

$$\log x_i(t) = \log x_i(0) + \int_0^t (b_i - \frac{\sigma_i^2}{2})(r(s))ds - \int_0^t (\sum_{j=1}^n a_{ij}(r(s))x_j(s))ds + M_i(t),$$

where $M_i(t) = \int_0^t \sigma_i^2(r(s))dB_i(s)$ is the real-valued continuous local martingale vanishing at $t = 0$. Dividing both sides by t yields

$$\frac{\log x_i(t)}{t} = \frac{\log x_i(0)}{t} + \frac{1}{t} \int_0^t (b_i - \frac{\sigma_i^2}{2})(r(s))ds - \frac{1}{t} \int_0^t (\sum_{j \neq i} a_{ij}(r(s))x_j(s))ds + \frac{1}{t} \int_0^t \sigma_i dB_i(s). \quad (25)$$

Using the law of strong large numbers for martingales(see [12]), we can claim that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \sigma_i(r(s))dB_i(s) = 0. \quad \text{a.s.}$$

By letting $t \rightarrow \infty$ yields

$$\limsup_{t \rightarrow \infty} \frac{\log x_i(t)}{t} \leq - \sum_{k \in S} (\frac{\sigma_i^2(k)}{2} - b_i(k)). \quad \text{a.s.} \quad (26)$$

The proof is completed.

Lemma 4.4: Let Assumption 1 and $\sum_{k \in S} \pi_k(b_i(k) - \frac{\sigma_i^2(k)}{2}) = 0$ hold and $x_i(t, x_0, r_0)$ be the solution to system (2) with any positive initial value $x_0 \in \mathbb{R}_+^n, r_0 \in S$. We then have the following assertion:

$$\lim_{t \rightarrow \infty} x_i(t, x_0, r_0) = 0. \quad \text{a.s.} \quad (27)$$

Proof Let $x_i(t)$ for $x_i(t, x_0, r_0)$ simplicity. Decompose the sample space into three mutually exclusive events as follows:

$$E_{i1} = \{\omega : \limsup_{t \rightarrow \infty} |x_i(t)| \geq \liminf_{t \rightarrow \infty} x_i(t) = \gamma_i > 0\};$$

$$E_{i2} = \{\omega : \limsup_{t \rightarrow \infty} x_i(t) > \liminf_{t \rightarrow \infty} x_i(t) = 0\};$$

$$E_{i3} = \{\omega : \lim_{t \rightarrow \infty} x_i(t) = 0\}.$$

We, furthermore, decompose the sample space into the following two mutually exclusive events according to the convergence of $\int_0^\infty x_i(s)ds$:

$$J_{i1} = \{\omega : \int_0^\infty x_i(s)ds < \infty\}, J_{i2} = \{\omega : \int_0^\infty x_i(s)ds = \infty\}.$$

The proof of $\lim_{t \rightarrow \infty} x_i(t) = 0$ is equivalent to show $J_{i1} \subset E_{i3}, J_{i2} \subset E_{i3}$. a.s. The strategy of the proof is as following

- First, using the Lemma 4.1 and 4.2, we show that $J_{i1} \subset E_{i3}$.
- Second, using some novel techniques, we prove that $\mathbb{P}(J_{i2} \cap E_{i1}) = 0$ and $\mathbb{P}(J_{i2} \cap E_{i2}) = 0$, which means $J_{i2} \subset E_{i3}$ a.s.

Now we realize this strategy as following:

Step 1: Let us now show $J_{i1} \subset E_{i3}$. a.s. It follows from Lemma 4.1 that almost every sample path of $x_i(t)$ is locally but uniformly Holder continuous. And therefore almost every sample path of $x_i(t)$ must be uniformly continuous. Combining the definition of J_{i1} and Lemma 4.2, we have

$$\lim_{t \rightarrow \infty} x_i(t) = 0, \text{ a. s.}$$

which means $J_{i1} \subset E_{i3}$ a.s.

Step 2: Now, we turn to prove that $J_{i2} \subset E_{i3}$. a.s. It is sufficient to show $\mathbb{P}(J_{i2} \cap E_{i1}) = 0$ and $\mathbb{P}(J_{i2} \cap E_{i2}) = 0$. We prove by contradiction.

If $\mathbb{P}(J_{i2} \cap E_{i1}) > 0$, for almost sure $\omega \in J_{i2} \cap E_{i1}$, $\varepsilon_0 \in (0, \frac{\gamma_i}{2})$, there exists $T = T(\varepsilon_0, \omega)$ such that

$$x_i(t) > \gamma_i - \varepsilon_0 > \frac{\gamma_i}{2}, \forall t > T.$$

It then follows from (25) that

$$\frac{1}{t} \int_0^t a_{ii}(r(s))x_i(s)ds = \frac{1}{t} \int_0^T a_{ii}(r(s))x_i(s)ds + \frac{1}{t} \int_T^t a_{ii}(r(s))x_i(s)ds \geq \frac{1}{t} \int_0^T a_{ii}(r(s))x_i(s)ds + \hat{a}_{ii} \frac{t-T}{t} (\frac{\gamma_i}{2}).$$

Letting $t \rightarrow \infty$, we obtain that

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t a_{ii}(r(s))x_i(s)ds > \hat{a}_{ii} (\frac{\gamma_i}{2}) > 0.$$

This implies

$$\limsup_{t \rightarrow \infty} \frac{\log x_i(t)}{t} \leq -\hat{a}_{ii} (\frac{\gamma_i}{2}) < 0, \text{ a.s.}$$

which contradicts with the definition of J_{i2} and E_{i1} . So $\mathbb{P}(J_{i2} \cap E_{i1}) = 0$ must hold.

Now we process to show $\mathbb{P}(J_{i2} \cap E_{i2}) > 0$ is false. For this purpose, we need a few more notations as following:

$$A_i^\varepsilon(t) := \{0 \leq s \leq t : x_i(s) \geq \varepsilon\}, d_i^\varepsilon(t) := \frac{m(A_i^\varepsilon(t))}{t}, d_i^\varepsilon := \liminf_{t \rightarrow \infty} d_i^\varepsilon(t), D_i^\varepsilon := \{\omega \in J_{i2} \cap E_{i2} : d_i^\varepsilon > 0\},$$

where $m(A_i^\varepsilon(t))$ indicates the length of $A_i^\varepsilon(t)$. It is easy to see that $D_i^0 = J_{i2} \cap E_{i2}$. For any $\varepsilon_1 < \varepsilon_2$, simple computations show that

$$A_i^{\varepsilon_1}(t) \supset A_i^{\varepsilon_2}(t), m(A_i^{\varepsilon_1}(t)) \geq m(A_i^{\varepsilon_2}(t)), d_i^{\varepsilon_1} = \frac{m(A_i^{\varepsilon_1}(t))}{t} \geq d_i^{\varepsilon_2} = \frac{m(A_i^{\varepsilon_2}(t))}{t},$$

which implies

$$d_i^{\varepsilon_2} \leq d_i^{\varepsilon_1}, \quad D_i^{\varepsilon_2} \subset D_i^{\varepsilon_1}, \forall \varepsilon_1 < \varepsilon_2.$$

It is easy to observe from the continuity of probability that

$$\mathbb{P}(D_i^\varepsilon) \rightarrow \mathbb{P}(D_i^0) = \mathbb{P}(J_{i2} \cap E_{i2}) \quad \varepsilon \rightarrow 0.$$

If $\mathbb{P}(J_{i2} \cap E_{i2}) > 0$, there exists $\epsilon > 0$ such that $\mathbb{P}(D_i^\epsilon) > 0$. For almost sure $\omega \in D_i^\epsilon$, simple computations show that

$$\frac{1}{t} \int_0^t a_{ii}(r(s))x_i(s)ds = \frac{1}{t} \int_{A_i^\epsilon(t)} a_{ii}(r(s))x_i(s)ds + \frac{1}{t} \int_{[0,t] \setminus A_i^\epsilon(t)} a_{ii}(r(s))x_i(s)ds \geq \frac{1}{t} \int_{A_i^\epsilon(t)} a_{ii}(r(s))x_i(s)ds.$$

By letting $t \rightarrow \infty$, we have

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t a_{ii}(r(s))x_i(s)ds \geq \liminf_{t \rightarrow \infty} \frac{1}{t} \int_{A_i^\epsilon(t)} a_{ii}(r(s))x_i(s)ds \geq \liminf_{t \rightarrow \infty} \frac{1}{t} \int_{A_i^\epsilon(t)} \hat{a}_{ii}x_i(s)ds \geq \hat{a}_{ii}d^\epsilon. \quad (28)$$

Substituting (28) into (25), we obtain that

$$\limsup_{t \rightarrow \infty} \frac{\log x_i(t)}{t} \leq -\hat{a}_{ii}d^\epsilon < 0. \quad a.s.$$

This contradicts with the definition of J_2 and E_{i2} . It yields the desired assertion $\mathbb{P}(J_2 \cap E_{i2}) = 0$ immediately. Combining with the fact $J_{i1} \subset E_{i3}$, $\mathbb{P}(J_2 \cap E_{i1}) = 0$ and $\mathbb{P}(J_2 \cap E_{i2}) = 0$, we can claim that

$$\lim_{t \rightarrow \infty} x_i(t) = 0, \quad a.s. \quad (29)$$

which means the species is extinct when $\sigma_i = 2b_i$. The proof is completed.

Proof of Theorem 4.1

Setting $\xi = \min_{1 \leq i \leq m} \sum_{k \in S} \{\frac{\sigma_i^2(k)}{2} - b_i(k)\}$. It follows from Lemma 4.1, for each $i = 1, \dots, m$ and $\epsilon \in (0, \xi)$, there is a positive random variable $T(\epsilon)$ such that, with probability one,

$$x_i(t) \leq e^{-\xi t + \epsilon t}, \quad \forall t > T(\epsilon), \quad a.s. \quad i = 1, \dots, m.$$

It follows that

$$x_i(t) \leq e^{-\alpha \xi t + \alpha \epsilon t}, \quad \forall t > T(\epsilon), \quad a.s. \quad i = 1, \dots, m.$$

which means

$$\int_0^\infty x_i(s)ds < \infty, \quad i = 1, \dots, m.$$

It follows from Lemma 4.4 that for any $m+1 \leq i \leq n$,

$$\lim_{t \rightarrow \infty} x_i(t) = 0, \quad a.s. \quad i = m+1, \dots, n.$$

This implies

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t x_i(s)ds = 0, \quad a.s. \quad i = m+1, \dots, n. \quad (30)$$

By the law of strong large numbers for martingales and (30), letting $t \rightarrow \infty$ on both sides of (25) yields

$$\lim_{t \rightarrow \infty} \frac{\log x_i(t)}{t} = - \sum_{k \in S} \left(\frac{\sigma_i^k(k)}{2} - b_i(k) \right), \quad a.s. \quad i = 1, \dots, m.$$

$$\lim_{t \rightarrow \infty} x_i(t) = 0, \quad a.s. \quad \lim_{t \rightarrow \infty} \frac{\log x_i(t)}{t} = 0. \quad a.s. \quad i = m+1, \dots, n.$$

The proof is completed.

Remark 3: Note, for $n = 1$, the system (2) becomes the following classic stochastic logistic system under regime switching(see [4]).

$$dx = x(b(r(t)) - a(r(t))x)dt + \sigma(r(t))xdB(t). \quad (31)$$

Li et.al. [4] have studied the stochastic permanence and and extinction for the system (31), the sufficient and necessary conditions for stochastic permanence and extinction were obtained under the assumption $\sum_{k \in S} \pi(b(k) - \frac{\sigma^2(k)}{2}) \neq 0$. By using some novel techniques, we show the extinction of the system when $\sum_{k \in S} \pi(b(k) - \frac{\sigma^2(k)}{2}) = 0$.

5. Partial Permanence and Partial Extinction

Theorem 5.1: Let Assumption 1 hold. Assume that there exist two integer $1 \leq m_1 \leq n$, such that

$$\begin{aligned} \sum_{k \in S} \pi_k(b_i(k) - \frac{\sigma_i^2(k)}{2}) &> 0, \quad \sum_{k \in S} \pi_k \left((b_i(k) - \frac{\sigma_i^2(k)}{2}) - M_{ij} \sum_{j \neq i}^{m_1} ((b_j(k) - \frac{\sigma_j^2(k)}{2})) \right) > 0, \quad i = 1, \dots, m_1. \\ \sum_{k \in S} \pi_k(b_i(k) - \frac{\sigma_i^2(k)}{2}) &\leq 0, \quad i = m_1 + 1, \dots, n. \end{aligned}$$

and

$$\forall k \in S, \quad a_{ii}(k) - \sum_{j \neq i}^{m_1} a_{ji}(k) > 0 \quad i = 1, \dots, m_1.$$

We then have the following assertions:

i) For $i = 1, \dots, m_1$, the solution $x_i(t, x_0, r_0)$ to system (2) has the property that

$$\begin{aligned} \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t x_i(s) ds &\geq \frac{1}{\hat{a}_{ii}} \sum_{k \in S} \pi_k \left((b_i(k) - \frac{\sigma_i^2(k)}{2}) - M_{ij} \sum_{j \neq i}^{m_1} (b_j(k) - \frac{\sigma_j^2(k)}{2}) \right), \quad \text{a.s.} \\ \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t x_i(s) ds &\leq \frac{1}{\hat{a}_{ii}} \sum_{k \in S} \pi_k (b_i(k) - \frac{\sigma_i^2(k)}{2}). \quad \text{a.s.} \end{aligned} \quad (32)$$

That is, for each $i = 1, \dots, m_1$, the species i is persistence in mean.

ii) For $i = m_1 + 1, \dots, n$, the solution $x_i(t, x_0, r_0)$ to system (2) has the property that

$$\limsup_{t \rightarrow \infty} \frac{\log x_i(t)}{t} \leq \sum_{k \in S} \pi_k \left[(b_i(k) - \frac{\sigma_i^2(k)}{2}) - \sum_{j=1}^{m_1} m_{ij} \left((b_j(k) - \frac{\sigma_j^2(k)}{2}) - \sum_{l \neq j}^{m_1} M_{jl} \left((b_l(k) - \frac{\sigma_l^2(k)}{2}) \right) \right) \right]. \quad (33)$$

That is, for each $i = m_1 + 1, \dots, n$, the system (2) still becomes extinct with zero exponential extinction rate.

Proof: Let $x_i(t)$ for $x_i(t, x_0, r_0)$ simplicity. As the whole proof is very technical, we will divide it into two steps. The first step is to show the convergence of $(x_i(t) - u_i(t)) \rightarrow 0$, $i = 1, \dots, m_1$, a.s. as $t \rightarrow \infty$, where $u_i(t)$ is the solution to the following auxiliary stochastic differential systems

$$\begin{cases} du_i(t) = u_i(b_i(r(t)) - \sum_{j=1}^{m_1} a_{ij}(r(t))u_j(r(t))dt + \sigma_i(r(t))dB_i(t), \\ u_i(0) = x_i(0), r(0) = r_0, \quad i = 1, \dots, m_1. \end{cases}$$

The second step is to prove the assertion (32) and (33) by Theorem 3.1, Lemma 4.3 and Lemma 4.4.

Step 1 It follows from the Lemma 2.1 that Lemma 4.4 that for any given initial value $x_0 \in R_+^n$, there is a unique positive solution $x(t, x_0, r_0)$ to system (34) and the solution is uniformly continuous on $[0, +\infty)$. Applying Itô's formula to

$$V(t) = \sum_{i=1}^{m_1} |\log x_i(t) - \log u_i(t)| \text{ yields}$$

$$\begin{aligned} D^+ V(t) &= \sum_{i=1}^{m_1} \text{sign} \left(\sum_{i=1}^{m_1} |\log x_i(t) - \log u_i(t)| (d \log x_i(t) - d \log u_i(t)) \right) \\ &\leq \sum_{i=1}^{m_1} \text{sign} \left(\sum_{i=1}^{m_1} |\log x_i(t) - \log u_i(t)| \left(\sum_{j=1}^{m_1} (x_j(t) - u_j(t)) + \sum_{j=m_1+1}^n a_{ij}(r(t)) x_j(t) \right) \right) \\ &= - \sum_{i=1}^{m_1} (a_{ii}(r(t)) - \sum_{j \neq i} a_{ji}(r(t)) |x_i(t) - u_i(t)| + \sum_{j=m_1+1}^n a_{ij}(r(t)) x_j(t)). \end{aligned}$$

This implies

$$V(t) + \sum_{i=1}^{m_1} \int_0^t (a_{ii}(r(s)) - \sum_{j \neq i} a_{ji}(r(s)) |x_i(s) - u_i(s)|) ds \leq V(0) + \sum_{j=m_1+1}^n \int_0^t a_{ij}(r(s)) x_j(s) ds.$$

It follows from Lemma 4.3 that

$$\sum_{j=m_1+1}^n \int_0^\infty x_j(s) ds < \infty, \quad \text{a.s.} \quad i = m_1 + 1, \dots, n.$$

We therefore have

$$\sum_{i=1}^{m_1} \int_0^t (a_{ii}(r(s)) - \sum_{j \neq i} a_{ji}(r(s)) |x_i(s) - u_i(s)|) ds < \infty, \quad \text{a.s.}$$

Therefore from Lemma 4.1 and Lemma 4.2 we obtain that

$$\lim_{t \rightarrow \infty} (x_i(t) - u_i(t)) = 0, \quad \text{a.s.} \quad i = 1, \dots, m_1. \quad (34)$$

Step 2 It follows from Theorem 3.1 that the system (34) is persistent in mean, that is

$$\begin{aligned} \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t u_i(s) ds &\geq \frac{1}{\hat{a}_{ii}} \sum_{k \in S} \pi_k \left((b_i(k) - \frac{\sigma_i^2(k)}{2}) - \sum_{j \neq i} M_{ij}((b_j(k) - \frac{\sigma_j^2(k)}{2})) \right), \\ \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t u_i(s) ds &\leq \frac{1}{\hat{a}_{ii}} \sum_{k \in S} \pi_k (b_i(k) - \frac{\sigma_i^2(k)}{2}), \quad \text{a.s.} \quad i = 1, \dots, m_1. \end{aligned}$$

Simple computation shows that

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t x_i(s) ds &\leq \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t (u_i(s) - x_i(s)) ds + \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t u_i(s) ds \\ &\leq \frac{1}{\hat{a}_{ii}} \sum_{k \in S} \pi_k (b_i(k) - \frac{\sigma_i^2(k)}{2}), \quad \text{a.s.} \quad i = 1, \dots, m_1. \\ \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t x_i(s) ds &\geq \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t (u_i(s) - x_i(s)) ds + \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t u_i(s) ds \\ &\geq \frac{1}{\hat{a}_{ii}} \sum_{k \in S} \pi_k \left((b_i(k) - \frac{\sigma_i^2(k)}{2}) - \sum_{j \neq i} M_{ij}((b_j(k) - \frac{\sigma_j^2(k)}{2})) \right), \quad \text{a.s.} \quad i = 1, \dots, m_1. \end{aligned} \quad (35)$$

So the assertion (32) must hold. It is easy to see from Lemma 4.3 and Lemma 4.4 that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t x_i(s) ds = 0, \quad \text{a.s.} \quad i = m_1 + 1, \dots, n. \quad (36)$$

Letting $t \rightarrow \infty$ on both side of (25) and using (35), (36) yields

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{\log x_i(t)}{t} \\ & \leq \sum_{k \in S} \pi_k \left[\left(b_i(k) - \frac{\sigma_i^2(k)}{2} \right) - \sum_{j=1}^{m_1} m_{ij} \left(\left(b_j(k) - \frac{\sigma_j^2(k)}{2} \right) - \sum_{l \neq j}^{m_1} M_{jl} \left(\left(b_l(k) - \frac{\sigma_l^2(k)}{2} \right) \right) \right) \right], \quad \text{a.s.} \quad i = m_1 + 1, \dots, n. \end{aligned}$$

which is the required assertion (33). The proof is completed.

6. Numerical simulations

In this section, to illustrate the usefulness and flexibility of the theorem developed in previous section, we present two numerical examples.

Example 6.1: Let $r(t)$ be a right-continuous Markovian chain taking values in $S = \{1, 2\}$ with generator

$$\Gamma = \begin{pmatrix} -1 & 1 \\ \frac{1}{4} & -\frac{1}{4} \end{pmatrix}.$$

The stationary distribution $\pi = (\pi_1, \pi_2)$ of $r(t)$ is $(0.2, 0.8)$. Consider a 2-dimensional stochastic Lotka-Volterra system with Markovian switching as follows:

$$\begin{cases} dx_1 = x_1(b_1(r(t)) - 0.8x_1 - 0.3x_2)dt + \sigma_1(r(t))x_1dB_1(t), \\ dx_2 = x_2(b_2(r(t)) - x_2 - 0.2x_1)dt + \sigma_2(r(t))x_2dB_2(t). \end{cases} \quad (37)$$

where $b_1(1) = b_2(1) = 1, b_1(2) = 1.2, b_2(2) = 1.5$. The existence and uniqueness of the solution follows from Lemma 2.1. We consider the solution $x(t, x_0, r_0)$ with initial data $x_1(0) = 0.5, x_2(0) = 0.5$. Let $x(t) = x(t; x(0))$ for simplicity.

i) $\sigma_1(1) = 0.5, \sigma_1(2) = 0.3, \sigma_2(1) = 0.4, \sigma_2(2) = 0.5$: Simple computation shows that

$$\begin{aligned} \sum_{k \in S} \pi_k \left(b_1(k) - \frac{\sigma_1^2(k)}{2} \right) &= 1.0990 > 0, \quad \sum_{k \in S} \pi_k \left(b_2(k) - \frac{\sigma_2^2(k)}{2} \right) = 1.2840 > 0, \\ \sum_{k \in S} \pi_k \left(\left(b_1(k) - \frac{\sigma_1^2(k)}{2} \right) - \frac{a_{12}}{a_{22}} \left(b_2(k) - \frac{\sigma_2^2(k)}{2} \right) \right) &= 0.7138 > 0, \\ \sum_{k \in S} \pi_k \left(\left(b_2(k) - \frac{\sigma_2^2(k)}{2} \right) - \frac{a_{21}}{a_{11}} \left(b_1(k) - \frac{\sigma_1^2(k)}{2} \right) \right) &= 0.8444 > 0. \end{aligned}$$

By Theorem 3.1, the solution to system (37) is persistent in mean. The computer simulations in Figure 1, using the Heun method, support these results clearly.

ii) $\sigma_1(1) = \sigma_1(2) = 1.5, \sigma_2(1) = \sigma_2(2) = 2$: Note that

$$\sum_{k \in S} \pi_k \left(b_1(k) - \frac{\sigma_1^2(k)}{2} \right) = -0.6650 < 0, \quad \sum_{k \in S} \pi_k \left(b_2(k) - \frac{\sigma_2^2(k)}{2} \right) = -0.5250 < 0,$$

by virtue of Theorem 4.1, the system (37) is exponentially extinctive. The computer simulations in Figure 2, using the Heun method, illustrate the extinction of the population.

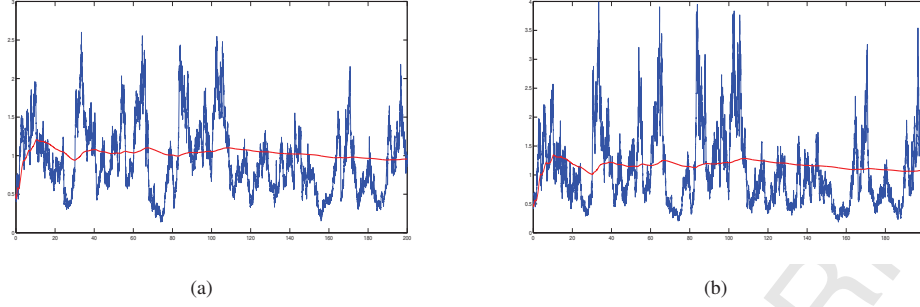


Figure 1: Computer simulation of a single path of $(x_1(t), x_2(t))$ and $(\frac{1}{t} \int_0^t x_1(s)ds, \frac{1}{t} \int_0^t x_2(s)ds)$ for system (37) with initial values $x_1(0) = x_2(0) = 0.5$ using the Heun scheme with time step $\Delta = 2^{-5}$ on $[20, 200]$, and the blue curve and the red curve represent them, respectively. (a) $x_1(t)$ and $\frac{1}{t} \int_0^t x_1(s)ds$. (b) $x_2(t)$ and $\frac{1}{t} \int_0^t x_2(s)ds$

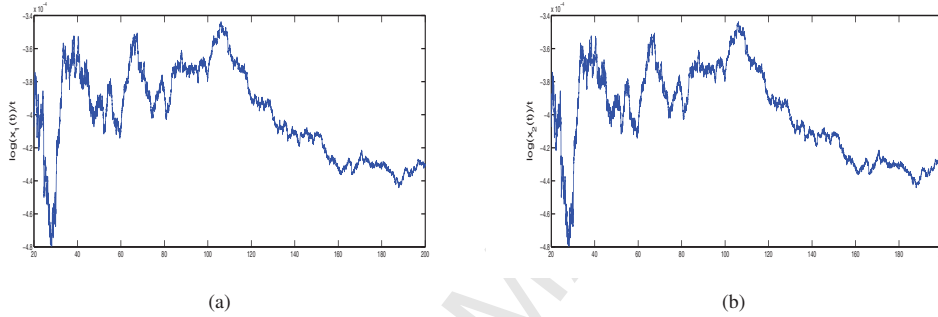


Figure 2: Computer simulation of a single path of $\frac{\log x_1(t)}{t}$ and $\frac{\log x_2(t)}{t}$ with initial values $x_1(0) = x_2(0) = 0.5$ using the Heun scheme with time step $\Delta = 2^{-5}$ on $[20, 200]$.

Example 6.2: Assume that the switching between two seasons is governed by a Markovian chain $r(t)$ on the state space $S = 1, 2$ with generator Γ defined in Example 6.1. Consider a 3-dimensional stochastic Lotka-Volterra system with Markovian switching as follows:

$$\begin{cases} dx_1 = x_1(b_1(r(t)) - 0.8x_1 - 0.3x_2 - 0.4x_3)dt + \sigma_1(r(t))x_1dB_1(t), \\ dx_2 = x_2(b_2(r(t)) - x_2 - 0.2x_1 - 0.4x_3)dt + \sigma_2(r(t))x_2dB_2(t), \\ dx_3 = x_3(b_3(r(t)) - x_3 - 0.5x_1 - 0.5x_2)dt + \sigma_3(r(t))x_3dB_3(t). \end{cases} \quad (38)$$

where $b_1(1) = b_2(1) = 1, b_1(2) = 1.2, b_2(2) = 1.5, b_3(1) = 1, b_3(2) = 0.8; \sigma_1(1) = 0.5, \sigma_2(1) = \sqrt{5}, \sigma_3(1) = 2, \sigma_1(2) = 0.3, \sigma_2(2) = 1.5, \sigma_3(2) = 2$. The existence and uniqueness of the solution follows from Lemma 2.1. We consider the solution $x(t, x_0, r_0)$ with initial data $x_1(0) = 0.5, x_2(0) = 0.5, r(0) = 1$. Let $x(t) = x(t; x(0))$ for simplicity. Simple computation shows that

$$\sum_{k \in S} \pi_k(b_1(k) - \frac{\sigma_1^2(k)}{2}) = 1.0990 > 0, \sum_{k \in S} \pi_k(b_2(k) - \frac{\sigma_2^2(k)}{2}) = 0, \sum_{k \in S} \pi_k(b_3(k) - \frac{\sigma_3^2(k)}{2}) = -1.26 < 0; \\ a(11) - a(21) = 0.6 > 0, a(22) - a(12) = 0.7 > 0.$$

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t x_1(s)ds = 1.3738, \limsup_{t \rightarrow \infty} \frac{\log x_2(t)}{t} \leq -0.2747, \limsup_{t \rightarrow \infty} \frac{\log x_3(t)}{t} \leq -1.9469.$$

By virtue of Theorem 5.1, the species 1 is persistence in mean and species 2 and 3 are exponentially extinctive. The computer simulations in Figure 3 and 4, using the Heun method, support these results clearly.

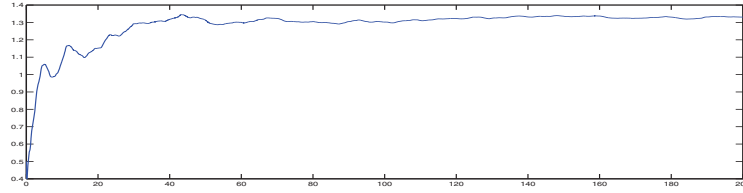


Figure 3: Computer simulation of a single path of $\frac{1}{t} \int_0^t x_1(s)ds$ for system (37) with initial values $x_1(0) = x_2(0) = 0.5$ using the Heun scheme with time step $\Delta = 2^{-5}$ on $[20, 200]$.

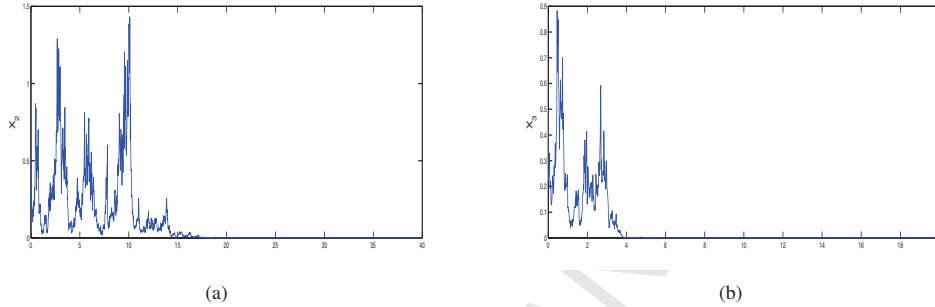


Figure 4: Computer simulation of a single path of $(x_2(t), x_3(t))$ for system (37) with initial values $x_1(0) = x_2(0) = 0.5$ using the Heun scheme with time step $\Delta = 2^{-5}$ on $[20, 200]$.

7. Conclusion

In this paper, we have investigated persistence in mean and extinction for stochastic competitive Lotka-Volterra systems with regime switching. Firstly, by utilizing some novel stochastic analysis techniques and the stochastic comparison principle, the persistence in mean and extinction have been researched. Secondly, by applying Lyapunov methods, sufficient conditions are obtained under which the system is partial permanence and partial extinction. Finally, two numerical examples are provided to illustrate the effectiveness of our results.

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