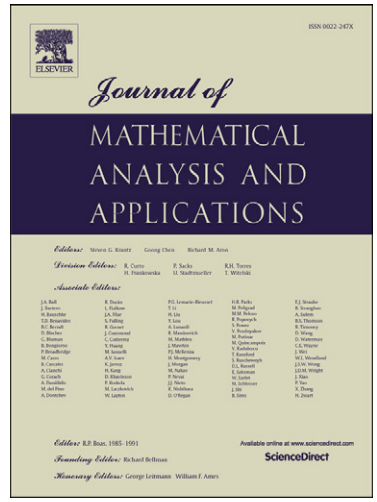
Accepted Manuscript



Some properties of classes of real self–reciprocal polynomials

Vanessa Botta, Cleonice F. Bracciali, Junior A. Pereira

PII: S0022-247X(15)00765-9
DOI: http://dx.doi.org/10.1016/j.jmaa.2015.08.038
Reference: YJMAA 19735

To appear in: *Journal of Mathematical Analysis and Applications*

Received date: 14 May 2015

Some properties of classes of real self–reciprocal polynomials

Vanessa Botta[a,*], Cleonice F. Bracciali[b], Junior A. Pereira[c]

[a] *UNESP – Univ Estadual Paulista, FCT, Departamento de Matemática e Computação, 19060-900, Presidente Prudente, SP, Brazil*
[b] *UNESP – Univ Estadual Paulista, IBILCE, Departamento de Matemática Aplicada, 15054-000, São José do Rio Preto, SP, Brazil*
[c] *UNESP – Univ Estadual Paulista, FCT, Programa de Pós-Graduação em Matemática Aplicada e Computacional, 19060-900, Presidente Prudente, SP, Brazil*

Abstract

The purpose of this paper is twofold. Firstly we investigate the distribution, simplicity and monotonicity of the zeros around the unit circle and real line of the real self–reciprocal polynomials $R_n^{(\lambda)}(z) = 1 + \lambda(z + z^2 + \cdots + z^{n-1}) + z^n$, $n \geq 2$ and $\lambda \in \mathbb{R}$. Secondly, as an application of the first results we give necessary and sufficient conditions to guarantee that all zeros of the self–reciprocal polynomials $S_n^{(\lambda)}(z) = \sum_{k=0}^{n} s_{n,k}^{(\lambda)} z^k$, $n \geq 2$, with $s_{n,0}^{(\lambda)} = s_{n,n}^{(\lambda)} = 1$, $s_{n,n-k}^{(\lambda)} = s_{n,k}^{(\lambda)} = 1 + k\lambda$, $k = 1, 2, ..., \lfloor n/2 \rfloor$ when n is odd, and $s_{n,n-k}^{(\lambda)} = s_{n,k}^{(\lambda)} = 1 + k\lambda$, $k = 1, 2, ..., n/2 - 1$, $s_{n,n/2}^{(\lambda)} = (n/2)\lambda$ when n is even, lie on the unit circle. Solving then an open problem given by Kim and Park in 2008.

Keywords: Self–reciprocal polynomials, unit circle, zeros, monotonicity, interlacing.

1. Introduction

Let the polynomial $P(z) = \sum_{i=0}^{n} a_i z^i$, $a_i \in \mathbb{C}$. Define the polynomial

$$P^*(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)} = \bar{a}_0 z^n + \bar{a}_1 z^{n-1} + \cdots + \bar{a}_n = \bar{a}_0 \prod_{j=1}^{n} (z - z_j^*),$$

whose zeros z_k^* are the inverses of the zeros z_k of $P(z)$, that is, $z_k^* = 1/\bar{z}_k$.

If $P^*(z) = uP(z)$ with $|u| = 1$, then $P(z)$ is said to be a self–inversive polynomial, see [19]. If $P(z) = z^n P(1/z)$, then $P(z)$ is said to be self–reciprocal. If $a_i \in \mathbb{R}$,

*Corresponding author
Email addresses: `botta@fct.unesp.br` (Vanessa Botta), `cleonice@ibilce.unesp.br` (Cleonice F. Bracciali), `junior.gusto@hotmail.com` (Junior A. Pereira)

then $P(z)$ is called real self–reciprocal polynomials, see [14]. Notice that real self–reciprocal polynomials are also self–inversive polynomials. It is clear that if $P(z)$ is a self–reciprocal polynomial, then $a_i = a_{n-i}$, for $i = 0, 1, ..., n$.

The properties of self–reciprocal polynomials are interesting topics to study and have many applications in some areas of mathematics, see for example [9, 10, 11, 13].

It is not difficult to verify that if a polynomial has all its zeros on the unit circle, then it is a self–inversive polynomial. The reciprocal is not always true, since self–inversive polynomials can have zeros that are symmetric with respect to the unit circle. The most famous result about the conditions for a self–inversive polynomial to have all its zeros on the unit circle is due to Conh, see [19, p. 18]: A necessary and sufficient condition for all the zeros of $P(z)$ to lie on the unit circle is that $P(z)$ is self–inversive and that all zeros of $P'(z)$ lie in or on this circle. In [4], Chen has given more flexible conditions than the Cohn's result. Choo and Kim in [7] gave an extension of Chen's result, to guarantee that the zeros on the unit circle are simple. Many authors have investigated special classes of self–inversive polynomials, see for example [14, 15, 16, 17].

In [14], Kim and Park investigate the distribution of zeros around the unit circle of real self–reciprocal polynomials of even degree with five terms whose absolute values of middle coefficients equal the sum of all other coefficients. As a consequence of this study, they present a result related to the location of the zeros of the real self–reciprocal polynomial $S_n^{(\lambda)}(z) = \sum_{k=0}^{n} s_k^{(\lambda)} z^k$, with $s_k^{(\lambda)} = 1 + k\lambda$, for $k = 1, 2, ..., \lfloor n/2 \rfloor$, for n odd and some values of $\lambda \in \mathbb{R}$ (see [14, Th. 7]). The authors remarked that for the three cases "$2 < \lambda < 2 + \frac{2}{\lfloor n/2 \rfloor}$ for $\lfloor n/2 \rfloor$ odd", "$\lambda = 2 + \frac{2}{\lfloor n/2 \rfloor}$ for $\lfloor n/2 \rfloor$ odd" and "$\lambda = -\frac{2}{\lfloor n/2 \rfloor}$" the location of the zeros of $S_n^{(\lambda)}(z)$ remain as an open problem. Here, we give a complete proof about the location of the zeros of $S_n^{(\lambda)}(z)$ in the case n odd and $\lambda \in \mathbb{R}$, answering the open problems of [14, Th. 7] and we present a new result when n is even. Precisely, we will deal with the polynomials

$$S_n^{(\lambda)}(z) = \sum_{k=0}^{n} s_{n,k}^{(\lambda)} z^k, \quad n \geq 2, \tag{1}$$

with $s_{n,0}^{(\lambda)} = s_{n,n}^{(\lambda)} = 1$ and

$$\begin{aligned} s_{n,k}^{(\lambda)} &= s_{n,n-k}^{(\lambda)} = 1 + k\lambda, \ k = 1, 2, ..., \left\lfloor \tfrac{n}{2} \right\rfloor, \quad \text{if } n \text{ is odd}, \\ s_{n,k}^{(\lambda)} &= s_{n,n-k}^{(\lambda)} = 1 + k\lambda, \ k = 1, 2, ..., \tfrac{n}{2} - 1, \ s_{n,n/2}^{(\lambda)} = \tfrac{n}{2}\lambda, \quad \text{if } n \text{ is even}. \end{aligned} \tag{2}$$

The proofs of these results are obtained using properties of the polynomials

$$R_n^{(\lambda)}(z) = 1 + \lambda(z + z^2 + \cdots + z^{n-1}) + z^n, \quad n \geq 2, \tag{3}$$

with $\lambda \in \mathbb{R}$, studied in [1].

We denote the unit circle by $\mathcal{C} = \{z : z = e^{i\theta}, 0 \leq \theta \leq 2\pi\}$. For $z = e^{i\theta}$ with $0 \leq \theta \leq 2\pi$, we consider the transformation

$$x = x(z) = \frac{z^{1/2} + z^{-1/2}}{2} = \cos(\theta/2). \tag{4}$$

In the context of orthogonal polynomials, see [6, 12], the transformation (4) was first used by Delsarte and Genin in [8], and later, was further explored by Zhedanov in [23]. We also consider and present some properties of the zeros of the polynomials $W_n^{(\lambda)}(x)$ defined by

$$W_n^{(\lambda)}(x) = W_n^{(\lambda)}(x(z)) = z^{-n/2} R_n^{(\lambda)}(z), \quad \text{for} \quad n \geq 1. \tag{5}$$

For the general case, the relation (5) has been used in [3] for an application of real orthogonal polynomials in the frequency analysis problem, also it has been used in [9] and [10] for classes of hypergeometric polynomials with zeros on the unit circle.

In [22] we find the following results. Let the sequence of polynomials $\{Q_m\}$ be generated by the three term recurrence relation

$$Q_{m+1}(z) = (z + \beta_{m+1})Q_m(z) - \alpha_{m+1} z Q_{m-1}(z), \quad m \geq 1, \tag{6}$$

with $Q_0(z) = 1$ and $Q_1(z) = z + \beta_1$, where the complex numbers α_m and β_m are such that $\alpha_m \neq 0$, $m \geq 2$ and $\beta_m \neq 0$, $m \geq 1$.

Theorem 1 ([22])**.** *Let $\beta_k = \beta > 0$ for $k = 1, 2, ..., n$ and $\alpha_k > 0$ for $k = 2, ..., n$. Then the zeros of any $Q_k(z)$, $1 \leq k \leq n$, are distinct (except for a possible double zero at $z = \beta$) and lie on $\mathscr{C}(\beta) \cup (0, \infty)$, where $\mathscr{C}(\beta) \equiv \{z : z = \beta e^{i\theta}, 0 < \theta < 2\pi\}$. In particular, if $\left\{\frac{\alpha_{k+1}}{4\beta}\right\}_{k=1}^{n-1}$ is a positive chain sequence then all the zeros are distinct and lie on the open circle $\mathscr{C}(\beta)$.*

We use this result to verify the location of zeros of the polynomials $R_n^{(\lambda)}(z)$, see Section 3. Notice that $\mathcal{C} = \mathscr{C}(1) \cup \{1\}$.

If there exists a sequence of real numbers $\{g_k\}$ such that $0 \leq g_0 < 1$, $0 < g_k < 1$ for $k \geq 1$ and $a_k = (1 - g_{k-1})g_k$ for $k \geq 1$, then $\{a_k\}_{k=1}^{\infty}$ is said to be a positive chain sequence and $\{g_k\}_{k=0}^{\infty}$ is called parameter sequence of the sequence $\{a_k\}_{k=1}^{\infty}$.

This manuscript is organized as follow. In Section 2 we write explicitly our main results about the behaviour of the zeros of the real self–reciprocal polynomials $R_n^{(\lambda)}(z)$ and $S_n^{(\lambda)}(z)$, with respect to the parameter λ. In Section 3 we give some old and new properties of the polynomials $R_n^{(\lambda)}(z)$ for different choices of parameter λ, that are necessary to show the main results. Using the transformation (4) we prove, in Section 4, results about the location and monotonicity behaviour of the zeros of the polynomials $R_n^{(\lambda)}(z)$ and $W_n^{(\lambda)}(x)$. Finally in Section 5 we show completely the location of the zeros of $S_n^{(\lambda)}(z)$ in the case n odd, answering the open problems of [14, Th. 7] and we show a new result when n is even. As consequence of this last result, in Section 5, we also provide some relations between polynomials $R_n^{(\lambda)}(z)$ and $S_n^{(\lambda)}(z)$, for different choices of parameter λ.

2. Main results

In Sections 3 and 4, we develop all the necessary preliminaries to show our main result about the distribution, simplicity and monotonicity of the zeros of polynomial $R_n^{(\lambda)}(z)$, defined by (3), on the unit circle, that it is:

Theorem 2. *For* $-\frac{2}{n-1} \leq \lambda \leq 2$ *(n even) or* $-\frac{2}{n-1} \leq \lambda \leq 2+\frac{2}{n-1}$ *(n odd), the zeros of* $R_n^{(\lambda)}(z)$*, defined by (3), are represented by* $z_{n,r}^{(\lambda)} = e^{i\theta_{n,r}^{(\lambda)}}$*, with* $\theta_{n,r}^{(\lambda)} = 2\arccos(\xi_{n,r}^{(\lambda)})$*, where* $\xi_{n,r}^{(\lambda)}$*,* $r = 1, 2, ..., \lfloor n/2 \rfloor$ *are the non–negative zeros of* $W_n^{(\lambda)}(x)$*, defined by (5). For n odd,* $r = 1, 2, ..., \lfloor n/2 \rfloor + 1$ *and* $\theta_{n,\lfloor n/2\rfloor+1}^{(\lambda)} = \pi$*. Furthermore,*

$$0 \leq \theta_{n,1}^{(\lambda)} < \theta_{n,2}^{(\lambda)} < \cdots < \theta_{n,\lfloor n/2\rfloor}^{(\lambda)} \leq \pi,$$

$\theta_{n,\lfloor n/2\rfloor+1}^{(\lambda)} = \pi$ *(n odd) and* $\theta_{n,r}^{(\lambda)}$*,* $r = 1, 2, ..., \lfloor n/2 \rfloor$*, are increasing functions of* λ*.*

Also the main result about the monotonicity behaviour of the two positive zeros of $R_n^{(\lambda)}(z)$ and behaviour of the two negative zeros of $R_n^{(\lambda)}(z)$, when these zeros exist, that it is:

Theorem 3.

1. *If* $\lambda < -\frac{2}{n-1}$*,* $n > 1$*,* $R_n^{(\lambda)}(z)$ *has two positive zeros* $z_k^{(\lambda)}$ *and* $1/z_k^{(\lambda)}$*, where* $z_k^{(\lambda)} \in (1, \infty)$ *and* $1/z_k^{(\lambda)} \in (0, 1)$*. Moreover,* $z_k^{(\lambda)}$ *is a decreasing function of* λ *and, consequently,* $1/z_k^{(\lambda)}$ *is an increasing function of* λ*.*
2. *If* $\lambda > 2$ *(when* $n > 1$ *is even) or* $\lambda > 2+\frac{2}{n-1}$ *(when* $n > 1$ *is odd),* $R_n^{(\lambda)}(z)$ *has two negative zeros* $z_k^{(\lambda)}$ *and* $1/z_k^{(\lambda)}$*, where* $z_k^{(\lambda)} \in (-\infty, -1)$ *and* $1/z_k^{(\lambda)} \in (-1, 0)$*. Moreover,* $z_k^{(\lambda)}$ *is a decreasing function of* λ *and, consequently,* $1/z_k^{(\lambda)}$ *is an increasing function of* λ*.*

Another new result of this manuscript is the location of the zeros of the polynomial $S_n^{(\lambda)}(z)$ defined by (1) and (2), proved in Section 5, answering the open problem of [14, Th. 7] when n is odd and also presenting a new result when n is even.

Theorem 4. *The zeros of the polynomial* $S_n^{(\lambda)}(z)$*,* $n \geq 2$*, defined by (1) and (2) lie on the unit circle if and only if*

$$-\frac{2}{\lfloor n/2\rfloor} \leq \lambda \leq 2, \quad \text{when } \lfloor n/2\rfloor \text{ is odd;}$$
$$-\frac{2}{\lfloor n/2\rfloor} \leq \lambda \leq 2 + \frac{2}{\lfloor n/2\rfloor}, \quad \text{when } \lfloor n/2\rfloor \text{ is even.}$$

Furthermore,

- *if* $\lambda \in \left(-\infty, -\frac{2}{\lfloor n/2\rfloor}\right)$*,* $S_n^{(\lambda)}(z)$ *has two positive zeros* $z_k^{(\lambda)} \in (1, +\infty)$ *and* $1/z_k^{(\lambda)} \in (0, 1)$ *and the other zeros are located on the unit circle;*
- *if* $\lambda \in (2, +\infty)$ $\left(\lambda \in \left(2 + \frac{2}{\lfloor n/2\rfloor}, +\infty\right)\right)$ *and* $\lfloor n/2\rfloor$ *is odd (*$\lfloor n/2\rfloor$ *is even),* $S_n^{(\lambda)}(z)$ *has two negative zeros* $z_k^{(\lambda)} \in (-\infty, -1)$ *and* $1/z_k^{(\lambda)} \in (-1, 0)$ *and the other zeros are located on the unit circle.*

3. Some properties of the polynomial $R_n^{(\lambda)}(z)$

Firstly we give some results about the zeros of the polynomials $R_n^{(\lambda)}(z)$ and, for specific values of λ, how they can be factorized.

Lemma 5. *Let $R_n^{(\lambda)}(z) = 1 + \lambda(z + z^2 + \cdots + z^{n-1}) + z^n$, $n \geq 2$, be a polynomial of degree n and*

$$
\begin{aligned}
A_l^{(\lambda)}(z) &= \sum_{k=0}^{l} a_k^{(\lambda)} z^k, \text{ with } l \text{ natural and } a_k^{(\lambda)} = \begin{cases} 1, & k \text{ even} \\ \lambda - 1, & k \text{ odd} \end{cases}, \\
B_{n-2}(z) &= \sum_{k=0}^{n-2} b_k z^k, \text{ with } b_k = \frac{(k+1)(n-(k+1))}{(n-1)}, \\
C_{n-3}(z) &= \sum_{k=0}^{n-3} c_k z^k, \text{ with } c_k = \begin{cases} \frac{(k+2)(n-1-k)}{2(n-1)}, & k \text{ even} \\ -\frac{(k+1)(n-1-(k+1))}{2(n-1)}, & k \text{ odd} \end{cases}.
\end{aligned}
$$

1. *If $\lambda = -\frac{2}{n-1}$, $n > 1$, then $z = 1$ is a zero of $R_n^{(\lambda)}(z)$ of multiplicity 2 and $R_n^{(\lambda)}(z) = (z-1)^2 B_{n-2}(z)$.*
2. *If n is odd, then $z = -1$ is a zero of $R_n^{(\lambda)}(z)$ and $R_n^{(\lambda)}(z) = (z+1)A_{n-1}^{(\lambda)}(z)$.*
3. *If n is odd and $\lambda = 2 + \frac{2}{n-1}$, $n > 1$, then $z = -1$ is a zero of $R_n^{(\lambda)}(z)$ of multiplicity 3 and $R_n^{(\lambda)}(z) = (z+1)^3 C_{n-3}(z)$.*
4. *If n is even and $\lambda = 2$, $n > 1$, then $z = -1$ is a zero of $R_n^{(\lambda)}(z)$ of multiplicity 2 and $R_n^{(\lambda)}(z) = (z+1)^2 A_{n-2}^{(1)}(z)$.*

The proof of items 1 (when n is odd), 2 and 3 of Lemma 5 can be found in [18]. By simple manipulations we can prove the other results.

In [1], one can found the following result about necessary and sufficient conditions to guarantee that all the zeros of polynomials $R_n^{(\lambda)}(z)$ lie on the unit circle.

Theorem 6 ([1]). *The zeros of the polynomial $R_n^{(\lambda)}(z) = 1+\lambda(z+z^2+\cdots+z^{n-1})+z^n$, $\lambda \in \mathbb{R}$, of degree $n \geq 2$, lie on the unit circle if and only if*

$-\dfrac{2}{n-1} \leq \lambda \leq 2$, *when n is even;*

$-\dfrac{2}{n-1} \leq \lambda \leq 2 + \dfrac{2}{n-1}$, *when n is odd.*

Observe that it is considered $n > 1$ in the conditions of Theorem 6. If $n = 1$, $z = -1$ is a single root of $R_1^{(\lambda)}(z) = 0$, for $\lambda \in \mathbb{R}$.

Also, it can be proved that

1. If $\lambda \in (-\infty, -\frac{2}{n-1})$, $R_n^{(\lambda)}$ has two positive zeros $z_k^{(\lambda)}$ and $1/z_k^{(\lambda)}$ and the other zeros are located on the unit circle.

2. If $\lambda \in (2, \infty)$ (even case) and $\lambda \in (2+\frac{2}{n-1}, \infty)$ (odd case), $R_n^{(\lambda)}$ has two negative zeros $z_k^{(\lambda)}$ and $1/z_k^{(\lambda)}$ and the other zeros are located on the unit circle.

Indeed:

1. For $\lambda < -\dfrac{2}{n-1}$,

$$\lim_{z\to 0} R_n^{(\lambda)}(z) > 0, \quad \lim_{z\to 1} R_n^{(\lambda)}(z) = 2 + (n-1)\lambda < 0 \quad \text{and} \quad \lim_{z\to +\infty} R_n^{(\lambda)}(z) > 0.$$

   That is, when $\lambda < -\dfrac{2}{n-1}$, there is a sign change of $R_n^{(\lambda)}(z)$ in $(0,1)$ and $(1,\infty)$. Thus, there is an odd number of zeros of $R_n^{(\lambda)}(z)$ in $(0,1)$ and $(1,\infty)$. From [16, Th. 2.1 item (a)(ii)] follows that there is exactly one zero of $R_n^{(\lambda)}$ in $(0,1)$, one zero of $R_n^{(\lambda)}$ in $(1,\infty)$ and $n-2$ zeros of $R_n^{(\lambda)}(z)$ are located on the unit circle.
2. For n even, if $\lambda > 2$,

$$\lim_{z\to -\infty} R_n^{(\lambda)}(z) > 0, \quad \lim_{z\to -1} R_n^{(\lambda)}(z) = 2 - \lambda < 0 \quad \text{and} \quad \lim_{z\to 0} R_n^{(\lambda)}(z) > 0.$$

   Hence, there is a sign change of $R_n^{(\lambda)}(z)$ in $(-\infty,-1)$ and $(-1,0)$. Using the same arguments as above and [16, Th. 2.1 item (b)(ii)] follows the result.
   For n odd, if $\lambda > 2 + \dfrac{2}{n-1}$, since $R_n^{(\lambda)}(z) = (z+1)A_{n-1}^{(\lambda)}(z)$ we have

$$\lim_{z\to -\infty} A_{n-1}^{(\lambda)}(z) > 0, \quad \lim_{z\to -1} A_{n-1}^{(\lambda)}(z) = n - \frac{n-1}{2}\lambda < 0 \quad \text{and} \quad \lim_{z\to 0} A_{n-1}^{(\lambda)}(z) > 0.$$

   That is, there is a sign change of $A_{n-1}^{(\lambda)}(z)$ in $(-\infty,-1)$ and $(-1,0)$, and, from [16, Th. 2.1 item (b)(ii)], $R_n^{(\lambda)}(z)$ has one real zero in $(-\infty,-1)$, another in $(-1,0)$ and $n-2$ on the unit circle.

It is also easy to verify, by mathematical induction, that the sequence of polynomials $\{R_n^{(\lambda)}(z)\}_{n=0}^{\infty}$ is generated by the three term recurrence relation

$$R_{n+1}^{(\lambda)}(z) = (z+1)R_n^{(\lambda)}(z) - \alpha_{n+1} z R_{n-1}^{(\lambda)}(z), \quad n \geq 1, \tag{7}$$

where $R_0(z) = 1$, $R_1(z) = z+1$, $\alpha_2 = 2-\lambda$, $\lambda \in \mathbb{R}$, and $\alpha_n = 1$ for $n \geq 3$.

Using results from [22] and the recurrence relation (7) we can show that for any $n \geq 1$, the two consecutive polynomials $R_n^{(\lambda)}(z)$ and $R_{n+1}^{(\lambda)}(z)$ do not have common zeros.

Since the sequence of polynomials $\{R_n^{(\lambda)}(z)\}$ satisfies the three term recurrence relation of type (6) with $\beta_n = 1$ for $n \geq 1$, $\alpha_2 = 2-\lambda$, and $\alpha_n = 1$ for $n \geq 3$, we can apply Theorem 1. If $\lambda < 2$ and, consequently, $\alpha_2 = 2-\lambda > 0$, then all zeros of $R_n^{(\lambda)}(z)$ are distinct (except for a possible double zero at $z = 1$ that occur when

$\lambda = -\frac{2}{n-1}$) and lie on $\mathscr{C}(1) \cup (0, \infty)$, where $\mathscr{C}(1) \equiv \{z : z = e^{i\theta}, 0 < \theta < 2\pi\}$. In particular, from Theorem 1, if $0 \le \lambda < 2$, $\left\{\frac{2-\lambda}{4}, \frac{1}{4}, \frac{1}{4}, ...\right\}$ is a positive chain sequence, whose parameter sequence is $\left\{\frac{\lambda}{2}, \frac{1}{2}, \frac{1}{2}, ...\right\}$. Then, for $0 \le \lambda < 2$ all the zeros of $R_n^{(\lambda)}(z)$ are distinct and lie on the unit circle. These facts were partially given in the proof of Theorem 6, in [1], using different techniques.

Now we consider $z = e^{i\theta}$ with $0 \le \theta \le 2\pi$, and the transformation (4). For $\theta \in [0, 2\pi]$ then $x \in [-1, 1]$, and $z = z(x) = 2x^2 - 1 + 2x\sqrt{x^2 - 1}$.

The polynomials $W_n^{(\lambda)}(x)$, defined in (5), are given by

$$W_n^{(\lambda)}(x) = U_n(x) - (1 - \lambda)U_{n-2}(x), \quad n \ge 1, \tag{8}$$

where the polynomials $U_n(x), n \ge 0$, with $U_{-1}(x) = 0$, are the Chebyshev polynomials of second kind. Indeed,

$$\begin{aligned}
R_n^{(\lambda)}(z) &= 1 + \lambda(z + z^2 + \cdots + z^{n-1}) + z^n \\
&= 1 + z + \cdots + z^n + (\lambda - 1)z(1 + z + \cdots + z^{n-2}) \\
&= \frac{z^{n+1} - 1}{z - 1} + (\lambda - 1)z\frac{z^{n-1} - 1}{z - 1} \\
&= z^{n/2}\frac{z^{(n+1)/2} - z^{-(n+1)/2}}{z^{1/2} - z^{-1/2}} + (\lambda - 1)z^{n/2}\frac{z^{(n-1)/2} - z^{-(n-1)/2}}{z^{1/2} - z^{-1/2}}.
\end{aligned}$$

Using that $(z^{1/2} + z^{-1/2})/2 = \cos(\theta/2) = x$, we get

$$\begin{aligned}
W_n^{(\lambda)}(x) &= z^{-n/2}R_n^{(\lambda)}(z) = \frac{z^{(n+1)/2} - z^{-(n+1)/2}}{z^{1/2} - z^{-1/2}} + (\lambda - 1)\frac{z^{(n-1)/2} - z^{-(n-1)/2}}{z^{1/2} - z^{-1/2}} \\
&= \frac{\sin((n+1)\theta/2)}{\sin(\theta/2)} + (\lambda - 1)\frac{\sin((n-1)\theta/2)}{\sin(\theta/2)} \\
&= U_n(x) - (1 - \lambda)U_{n-2}(x).
\end{aligned}$$

Since $U_n(x) - U_{n-2}(x) = 2T_n(x)$, for $n \ge 2$, see [20], where the polynomial $T_n(x)$ is the Chebyshev polynomial of first kind, we also can write

$$W_n^{(\lambda)}(x) = \lambda U_n(x) + 2(1 - \lambda)T_n(x). \tag{9}$$

4. Zeros of the polynomials $R_n^{(\lambda)}(z)$ and $W_n^{(\lambda)}(x)$

From relation (8) the polynomials $\{W_n^{(\lambda)}(x)\}$ are quasi-orthogonal polynomials associated with the polynomials $\{U_n(x)\}$, see [5, 6]. Furthermore, $W_n^{(\lambda)}(x)$ has at least $n - 2$ real distinct zeros in $(-1, 1)$ for $\lambda \ne 1$, see [21].

The trivial case $\lambda = 1$ in relation (8) means that $W_n^{(1)}(x) = U_n(x)$ and all zeros of $W_n^{(1)}(x)$ are real, distinct and they are located in $(-1, 1)$. The following results provide the location of the zeros of the polynomial $W_n^{(\lambda)}(x)$ for different choices of the parameter λ.

Lemma 7. *For $\lambda \in \mathbb{R}$ we have the following results about the zeros of $W_n^{(\lambda)}(x)$.*

1. *At least $n-2$ zeros of $W_n^{(\lambda)}(x)$ are real, distinct and are located in $(-1,1)$.*
2. *If $-\frac{2}{n-1} < \lambda < 2$ $\left(-\frac{2}{n-1} < \lambda < 2 + \frac{2}{n-1}\right)$ then all the zeros of $W_n^{(\lambda)}(x)$ are distinct and located in $(-1,1)$, when n is even (odd).*
3. *If $\lambda = 2$ $\left(\lambda = 2 + \frac{2}{n-1}\right)$, then $x = 0$ is zero of multiplicity 2 (3) of $W_n^{(\lambda)}(x)$, when n is even (odd).*
4. *If $\lambda > 2$ $\left(\lambda > 2 + \frac{2}{n-1}\right)$, then $W_n^{(\lambda)}(x)$ has two purely imaginary zeros, when n is even (odd).*
5. *If $\lambda = -\frac{2}{n-1}$, then $x = 1$ and $x = -1$ are zeros of $W_n^{(\lambda)}(x)$.*
6. *If $\lambda < -\frac{2}{n-1}$, $W_n^{(\lambda)}(x)$ has two real zeros ($\xi_{n,k}^{(\lambda)}$ and $-\xi_{n,k}^{(\lambda)}$) outside of the interval $[-1,1]$.*

Proof. 1. Follows directly, since $W_n^{(\lambda)}(x)$ is quasi–orthogonal polynomial of order 2 with respect to $w(x) = \sqrt{1-x^2}$ on $(-1,1)$ when $\lambda - 1 \neq 0$, and when $\lambda = 1$, $W_n^{(1)}(x) = U_n(x)$.

2. For these values of λ, $R_n^{(\lambda)}(z)$ has all its zeros on the unit circle. Consequently, from relation (5) all the zeros of $W_n^{(\lambda)}(x(z))$ are in $(-1,1)$. Observe that $z = 1$ is not a zero of $R_n^{(\lambda)}(z)$ and $z(1) = z(-1) = 1$. Hence, $x = 1$ and $x = -1$ are not zeros of $W_n^{(\lambda)}(x(z))$. For this reason, we consider the open interval $(-1,1)$. Furthermore, in the even case, from Theorem 1 follows that all the zeros of $R_n^{(\lambda)}(z)$ are distinct and, consequently, all the zeros of $W_n^{(\lambda)}(x)$ are distinct. For n odd, the result that the zeros of $W_n^{(\lambda)}(x)$ are distinct follow by the facts that $x = 0$ is a simple zero of $W_n^{(\lambda)}(x)$, $n-2$ zeros of $W_n^{(\lambda)}(x)$ are distinct in the interval $(-1,1)$ and $W_n^{(\lambda)}(x) = (-1)^n W_n^{(\lambda)}(-x)$ (i.e., the real zeros of $W_n^{(\lambda)}(x)$ are symmetric with respect to the origin).
3. In this case, $z = -1$ is zero of multiplicity 2 (n even) and multiplicity 3 (n odd) of $R_n^{(\lambda)}(z)$. Observe that $x(-1) = 0$. Hence, $x = 0$ is zero of multiplicity 2 (3) of $W_n^{(\lambda)}(x)$ when n is even (odd).
4. If $\lambda > 2$ $\left(\lambda > 2 + \frac{2}{n-1}\right)$, from Theorem 6 we know that $R_n^{(\lambda)}(z)$ has two negative zeros $z_k^{(\lambda)}$ and $1/z_k^{(\lambda)}$ and the remaining $n-2$ zeros are on the unit circle. We can write $z_k^{(\lambda)} = r_k^{(\lambda)} e^{i\pi}$ for $r_k^{(\lambda)} > 1$, and

$$\begin{aligned} x(z_k^{(\lambda)}) = x(r_k^{(\lambda)} e^{i\pi}) &= \frac{(r_k^{(\lambda)} e^{i\pi})^{1/2} + (r_k^{(\lambda)} e^{i\pi})^{-1/2}}{2} \\ &= \left(\frac{(r_k^{(\lambda)})^{1/2} - (r_k^{(\lambda)})^{-1/2}}{2} \right) i = \beta_{n,k}^{(\lambda)} i, \end{aligned}$$

also

$$x\left(\frac{1}{z_k^{(\lambda)}}\right) = x\left(\frac{e^{i\pi}}{r_k^{(\lambda)}}\right) = \frac{(e^{i\pi}/r_k^{(\lambda)})^{1/2} + (e^{i\pi}/r_k^{(\lambda)})^{-1/2}}{2} = -\beta_{n,k}^{(\lambda)} i,$$

then $W_n^{(\lambda)}(x)$ has two zeros that are purely imaginary complex numbers and the remaining $n-2$ zeros are in $(-1,1)$.

5. If $\lambda = -\frac{2}{n-1}$, $z=1$ is zero of multiplicity 2 of $R_n^{(\lambda)}(z)$. Observe that $z(1) = z(-1) = 1$. Hence, $x = 1$ and $x = -1$ are zeros of $W_n^{(\lambda)}(x)$.
6. If $\lambda < -\frac{2}{n-1}$, from Theorem 6 we know that $R_n^{(\lambda)}(z)$ has two positive zeros $z_k^{(\lambda)}$ and $1/z_k^{(\lambda)}$ and $n-2$ zeros on the unit circle. Observe that if $z_k^{(\lambda)} \in (1,\infty)$, then $x\left(z_k^{(\lambda)}\right) = x\left(1/z_k^{(\lambda)}\right) \in (1,\infty)$. As the zeros of $W_n^{(\lambda)}(x)$ are symmetric with respect to the origin, we conclude that $-x\left(z_k^{(\lambda)}\right)$ is zero of $W_n^{(\lambda)}(x)$. Hence, $W_n^{(\lambda)}(x)$ has two zeros outside $[-1,1]$.

□

Remark 8. *From Lemma 7 there follows that all the zeros of $W_n^{(\lambda)}(x)$ are distinct, except in the cases $\lambda = 2$ for n even, and $\lambda = 2 + \frac{2}{n-1}$ for n odd.*

Let $x_{n,1}, x_{n,2}, ..., x_{n,n}$ and $x_{n-2,1}, x_{n-2,2}, ..., x_{n-2,n-2}$ be zeros of $U_n(x)$ and $U_{n-2}(x)$, respectively. We know that the zeros of $U_n(x)$ and $U_{n-2}(x)$ are real, simple and they are located in $(-1,1)$. Furthermore, since $U_n(x) = (-1)^n U_n(-x)$ and their zeros are symmetric with respect to the origin, it suffices to consider only their positive zeros, i.e., $x_{n,1} > x_{n,2} > \cdots > x_{n,\lfloor n/2 \rfloor} > 0$ and $x_{n-2,1} > x_{n-2,2} > \cdots > x_{n-2,\lfloor (n-2)/2 \rfloor} > 0$, here denoted in decreasing order. It is very well known that these zeros satisfy the interlacing property

$$x_{n,1} > x_{n-2,1} > \cdots > x_{n-2,\lfloor (n-2)/2 \rfloor} > x_{n,\lfloor n/2 \rfloor} > 0.$$

Observe, from relation (8), that also $W_n^{(\lambda)}(x) = (-1)^n W_n^{(\lambda)}(-x)$. Let $\xi_{n,1}^{(\lambda)}, \xi_{n,2}^{(\lambda)}, ..., \xi_{n,n}^{(\lambda)}$ be zeros of $W_n^{(\lambda)}(x)$. As $W_n^{(\lambda)}(x)$ is an even (odd) polynomial for n even (odd), their real zeros are symmetric with respect to the origin and also it suffices to consider only the positive zeros, $\xi_{n,1}^{(\lambda)}, \xi_{n,2}^{(\lambda)}, ..., \xi_{n,\lfloor n/2 \rfloor -1}^{(\lambda)}$. If n is odd, $\xi_{n,\lfloor n/2 \rfloor +1}^{(\lambda)} = 0$ is zero of $W_n^{(\lambda)}(x)$. We also denote the positive zeros in decreasing order, i.e.,

$$\xi_{n,1}^{(\lambda)} > \xi_{n,2}^{(\lambda)} > \cdots > \xi_{n,\lfloor n/2 \rfloor -1}^{(\lambda)} > 0.$$

The following results deal with the location of the zeros of polynomials $W_n^{(\lambda)}(x)$ with respect to the zeros of polynomials $U_n(x)$ and $U_{n-2}(x)$.

Lemma 9. *For $\lambda \in \mathbb{R}$, we have the following results about the zeros of $W_n^{(\lambda)}(x)$:*

1. *If $\lambda < 1$, then*

$$x_{n,1} < \xi_{n,1}^{(\lambda)} \quad \textit{and} \quad x_{n,r} < \xi_{n,r}^{(\lambda)} < x_{n-2,r-1},\ r = 2,3,...,\lfloor n/2 \rfloor.$$

Furthermore,

- *if $-\dfrac{2}{n-1} < \lambda < 1$, then $\xi_{n,1}^{(\lambda)} < 1$.*

• *If* $\lambda = -\dfrac{2}{n-1}$, *then* $\xi_{n,1}^{(\lambda)} = 1$.
• *If* $\lambda < -\dfrac{2}{n-1}$, *then* $\xi_{n,1}^{(\lambda)} > 1$.

2. *If* $\lambda > 1$, *then*

$$x_{n-2,r} < \xi_{n,r}^{(\lambda)} < x_{n,r}, \quad r = 1, 2, ..., \lfloor n/2 \rfloor - 1.$$

Furthermore,
• *if* $1 < \lambda < 2$ $\left(1 < \lambda < 2 + \frac{2}{n-1}\right)$ *for* n *even (odd), then* $\xi_{n,\lfloor n/2 \rfloor}^{(\lambda)} < x_{n,\lfloor n/2 \rfloor} < 1$.
• *If* $\lambda = 2$ $\left(\lambda = 2 + \frac{2}{n-1}\right)$ *for* n *even (odd), then* $0 = \xi_{n,\lfloor n/2 \rfloor}^{(\lambda)} < x_{n,\lfloor n/2 \rfloor}$, *that is,* 0 *is zero of multiplicity* 2 (3) *of* $W_n^{(\lambda)}(x)$.
• *If* $\lambda > 2$ $\left(\lambda > 2 + \frac{2}{n-1}\right)$ *for* n *even (odd), then the other zero is* $\xi_{n,\lfloor n/2 \rfloor}^{(\lambda)} = \beta_{n,\lfloor n/2 \rfloor}^{(\lambda)} i$.

Proof. 1. The proof of this interlacing property of zeros of combination of polynomials, for general case, can be found in [2, Lemma 2]. For the sake of the completeness of this work we give the proof for the zeros of $U_n(x)$, $U_{n-2}(x)$ and $W_n^{(\lambda)}(x)$.

For $\lambda < 1$ and $r = 2, 3, ..., \lfloor n/2 \rfloor$,

$$sign(W_n^{(\lambda)}(x_{n,r+1})) = -sign(U_{n-2}(x_{n,r+1})) = (-1)^{r-1}$$

and

$$sign(W_n^{(\lambda)}(x_{n-2,r})) = sign(U_n(x_{n-2,r})) = (-1)^r.$$

Hence there exist zeros $\xi_{n,r}^{(\lambda)}$, $r = 2, 3, ..., \lfloor n/2 \rfloor$, of $W_n^{(\lambda)}(x)$ such that $x_{n,r} < \xi_{n,r}^{(\lambda)} < x_{n-2,r-1}$. Furthermore,

$$sign(W_n^{(\lambda)}(x_{n,1})) = -sign(U_{n-2}(x_{n,1})) = -1 \;\; and \;\; \lim_{x\to\infty} W_n^{(\lambda)}(x) = \infty.$$

Hence, $\xi_{n,1}^{(\lambda)}$ is a real zero of $W_n^{(\lambda)}(x)$ and $x_{n,1} < \xi_{n,1}^{(\lambda)}$. Furthermore,
• if $-\dfrac{2}{n-1} < \lambda < 1$, from Lemma 7 item 2 it follows that $\xi_{n,1}^{(\lambda)} < 1$;
• if $\lambda = -\dfrac{2}{n-1}$, from Lemma 7 item 5 it follows that $\xi_{n,1}^{(\lambda)} = 1$;
• if $\lambda < -\dfrac{2}{n-1}$, from Lemma 7 item 6 it follows that $\xi_{n,1}^{(\lambda)} > 1$.

2. Similarly we have for $r = 1, 2..., \lfloor n/2 \rfloor - 1$, that

$$sign(W_n^{(\lambda)}(x_{n,r})) = sign(U_{n-2}(x_{n,r})) = (-1)^{r-1}$$

and

$$sign(W_n^{(\lambda)}(x_{n-2,r})) = sign(U_n(x_{n-2,r})) = (-1)^r.$$

Hence there exist zeros $\xi_{n,r}^{(\lambda)}$, $r = 1, 2, ..., \lfloor n/2 \rfloor - 1$, of $W_n^{(\lambda)}(x)$ such that $x_{n-2,r} < \xi_{n,r}^{(\lambda)} < x_{n,r}$. Furthermore,
• If $1 < \lambda < 2$ and n even, for $r = \lfloor n/2 \rfloor$ we get $sign(W_n^{(\lambda)}(x_{n,\lfloor n/2 \rfloor})) = (-1)^{n/2-1}$ and $sign(W_n^{(\lambda)}(0)) = (-1)^{n/2-1} sign(-2+\lambda)$.
Hence, if $n/2$ is even,

$$sign(W_n^{(\lambda)}(x_{n,\lfloor n/2 \rfloor})) = -1 \text{ and } sign(W_n^{(\lambda)}(0)) = 1,$$

and, for $n/2$ odd,

$$sign(W_n^{(\lambda)}(x_{n,\lfloor n/2 \rfloor})) = 1 \text{ and } sign(W_n^{(\lambda)}(0)) = -1.$$

Then, $\xi_{n,\lfloor n/2 \rfloor}^{(\lambda)}$ is zero of $W_n^{(\lambda)}(x)$ and $\xi_{n,\lfloor n/2 \rfloor}^{(\lambda)} < x_{n,\lfloor n/2 \rfloor}$.
Similarly, if $1 < \lambda < 2 + \frac{2}{n-1}$ and n odd, for $r = \lfloor n/2 \rfloor$, we get

$$sign(W_n^{(\lambda)}(x_{\lfloor n/2 \rfloor}^{(\lambda)})) = (-1)^{\lfloor n/2 \rfloor - 1} \text{ and } W_n^{(\lambda)}(0) = 0.$$

Then, we need to analyse the behaviour of $[W_n^{(\lambda)}(0)]'$. Firstly, we have

$$[W_n^{(\lambda)}(0)]' = (-1)^{\lfloor n/2 \rfloor}(\lambda(1-n) + 2n) = (-1)^{\lfloor n/2 \rfloor}(n-1)\left(2 + \frac{2}{n-1} - \lambda\right).$$

Since $(n-1)\left(2 + \frac{2}{n-1} - \lambda\right) > 0$, it follows that $sign([W_n^{(\lambda)}(0)]') = (-1)^{\lfloor n/2 \rfloor}$. Hence, $W_n^{(\lambda)}(x)$ is an increasing (decreasing) function at the point $x = 0$ when $\lfloor n/2 \rfloor$ is even (odd). Consequently, $\xi_{n,\lfloor n/2 \rfloor}^{(\lambda)}$ is real zero of $W_n^{(\lambda)}(x)$ and $\xi_{n,\lfloor n/2 \rfloor}^{(\lambda)} < x_{n,\lfloor n/2 \rfloor}$.
• If $\lambda = 2$ $\left(\lambda = 2 + \frac{2}{n-1}\right)$ and n even (odd), for $r = \lfloor n/2 \rfloor$, we have that $\xi_{n,\lfloor n/2 \rfloor}^{(\lambda)} = 0$ is zero of multiplicity 2 (3) of $W_n^{(\lambda)}(x)$ and $0 = \xi_{n,\lfloor n/2 \rfloor}^{(\lambda)} < x_{n,\lfloor n/2 \rfloor}$, see Lemma 7 item 3.
• If $\lambda > 2$ $\left(\lambda > 2 + \frac{2}{n-1}\right)$ and n even (odd), from Lemma 7 item 4 it follows that $\xi_{n,\lfloor n/2 \rfloor}^{(\lambda)} = \beta_{n,\lfloor n/2 \rfloor}^{(\lambda)} i$ is zero of $W_n^{(\lambda)}(x)$.

□

Lemma 10. *Every positive zero $\xi_{n,r}^{(\lambda)}$ of $W_n^{(\lambda)}(x)$, for $r = 1, 2, ..., \lfloor n/2 \rfloor$, is an increasing function of $1 - \lambda$ (consequently, decreasing function of λ).*

The proof of this result, for general case, can be found in [2].

From items 2 and 5 of Lemma 7 we know that if $-\frac{2}{n-1} \leq \lambda < 2$ for n even or $-\frac{2}{n-1} \leq \lambda < 2 + \frac{2}{n-1}$ for n odd, the zeros of $W_n^{(\lambda)}(x)$ are real, distinct and lie in the interval $[-1, 1]$. In this case we are denoting the positive zeros of $W_n^{(\lambda)}(z)$ by $\xi_{n,r}^{(\lambda)}$, $r = 1, 2, ..., \lfloor n/2 \rfloor$ and if n is odd $\xi_{n,\lfloor n/2 \rfloor+1}^{(\lambda)} = 0$. Observe that, from Lemma 7 item

3, if $\lambda = 2$ $\left(\lambda = 2 + \frac{2}{n-1}\right)$, $x = 0$ is zero of multiplicity 2 (3) of $W_n^{(\lambda)}(x)$, when n is even (odd). Hence, in these cases, we are considering $\xi_{n,\lfloor n/2 \rfloor}^{(\lambda)} = 0$.

Now we are able to prove Theorem 2 with results about the distribution, simplicity and monotonicity of the zeros of polynomial $R_n^{(\lambda)}(z)$ on the unit circle. Also we show Theorem 3 that deals with the monotonicity behaviour of the two positive zeros of $R_n^{(\lambda)}(z)$ with respect to the parameter λ, when $\lambda < -\frac{2}{n-1}$ and $n > 1$, and with the behaviour of the two negative zeros of $R_n^{(\lambda)}(z)$, when $\lambda > 2$ for n even or when $\lambda > 2 + \frac{2}{n-1}$ for n odd, and $n > 1$.

Proof of Theorem 2. For $-\frac{2}{n-1} \leq \lambda \leq 2$ (n even) or $-\frac{2}{n-1} \leq \lambda \leq 2 + \frac{2}{n-1}$ (n odd), we know from Theorem 6 that all the zeros of $R_n^{(\lambda)}(z)$ lie on the unit circle. From (5) and using the mapping (4) the zeros $z_{n,r}^{(\lambda)}$ of the polynomial $R_n^{(\lambda)}(z)$ are represented by $z_{n,r}^{(\lambda)} = e^{i\theta_{n,r}^{(\lambda)}}$, with $\theta_{n,r}^{(\lambda)} = 2\arccos(\xi_{n,r}^{(\lambda)})$, where $\xi_{n,r}^{(\lambda)}$, $r = 1, 2, ..., \lfloor n/2 \rfloor$ are the non–negative zeros of $W_n^{(\lambda)}(x)$. For n odd, since $W_n^{(\lambda)}(x)$ has a zero at $\xi_{n,\lfloor n/2 \rfloor+1}^{(\lambda)} = 0$, then $z_{n,\lfloor n/2 \rfloor+1}^{(\lambda)} = -1$ is zero of $R_n^{(\lambda)}(z)$ and $\theta_{n,\lfloor n/2 \rfloor+1}^{(\lambda)} = \pi$.

Since $\xi_{n,1}^{(\lambda)} > \xi_{n,2}^{(\lambda)} > \cdots > \xi_{n,\lfloor n/2 \rfloor}^{(\lambda)} \geq 0$ and $\theta_{n,r}^{(\lambda)} = 2\arccos(\xi_{n,r}^{(\lambda)})$ is a decreasing function in $[-1, 1]$, it follows that $0 \leq \theta_{n,1}^{(\lambda)} < \theta_{n,2}^{(\lambda)} < \cdots < \theta_{n,\lfloor n/2 \rfloor}^{(\lambda)} \leq \pi$.

For $\lambda_j < \lambda_l$, from Lemma 10 it follows that $\xi_{n,r}^{(\lambda_j)} > \xi_{n,r}^{(\lambda_l)}$. Hence, since $\theta_{n,r}^{(\lambda)} = 2\arccos(\xi_{n,r}^{(\lambda)})$ is a decreasing function in $[-1, 1]$, we have $\theta_{n,r}^{(\lambda_j)} < \theta_{n,r}^{(\lambda_l)}$. Then, for $\lambda_j < \lambda_l$, $\theta_{n,r}^{(\lambda_j)} < \theta_{n,r}^{(\lambda_l)}$.

□

Proof of Theorem 3.

1. We consider $\epsilon \geq 0$, such that $\lambda + \epsilon < -\frac{2}{n-1}$ (to guarantee the existence of two positive zeros) and

$$R_{n,\epsilon}^{(\lambda)}(z) = 1 + (\lambda + \epsilon)(z + z^2 + \cdots + z^{n-1}) + z^n,$$

with its real zeros are represented by $z_k^{(\lambda)}(\epsilon)$ and $1/z_k^{(\lambda)}(\epsilon)$.
It is clear that $z_k^{(\lambda)} = z_k^{(\lambda)}(0)$ and $R_{n,\epsilon}^{(\lambda)}(z) = R_n^{(\lambda)}(z) + \epsilon(z + z^2 + \cdots + z^{n-1})$.
Thus $R_{n,\epsilon}^{(\lambda)}(z_k^{(\lambda)}) = \epsilon(z_k^{(\lambda)} + (z_k^{(\lambda)})^2 + \cdots + (z_k^{(\lambda)})^{n-1})$ and then, for $\epsilon > 0$,

$$sign(R_{n,\epsilon}^{(\lambda)}(z_k^{(\lambda)})) = 1.$$

Hence, $z_k^{(\lambda)}(0) > z_k^{(\lambda)}(\epsilon)$, showing that $z_k^{(\lambda)}$ is a decreasing function of λ. Consequently, $1/z_k^{(\lambda)}$ is an increasing function of λ.
2. The result follows using the same idea of the proof of the previous item and the fact that $sign(R_{n,\epsilon}^{(\lambda)}(z_k^{(\lambda)})) = (-1)^{n-1}$.

□

Now we can give other information about the monotonicity of the complex zeros of polynomials $W_n^{(\lambda)}(x)$, when they exist, with respect to parameter λ.

Theorem 11. *For $\lambda > 2$ $\left(\lambda > 2 + \frac{2}{n-1}\right)$ and n even (odd), let $\pm\beta_{n,k}^{(\lambda)} i$ be the purely imaginary zeros of $W_n^{(\lambda)}(x)$. Then $\beta_{n,k}^{(\lambda)}$ is an increasing function of λ.*

Proof. From item 4 of Lemma 7, the purely imaginary zeros $\pm\beta_k^{(\lambda)} i$ of $W_n^{(\lambda)}(x)$ are represented by

$$x(z_k^{(\lambda)}) = \left(\frac{(r_k^{(\lambda)})^{1/2} - (r_k^{(\lambda)})^{-1/2}}{2}\right) i \quad \text{and} \quad x\left(\frac{1}{z_k^{(\lambda)}}\right) = \left(\frac{(r_k^{(\lambda)})^{-1/2} - (r_k^{(\lambda)})^{1/2}}{2}\right) i,$$

where $z_k^{(\lambda)} \in (-\infty, -1)$ is a negative zero of $R_n^{(\lambda)}(z)$, i.e., $z_k^{(\lambda)} = -r_k^{(\lambda)}$ with $r_k^{(\lambda)} > 1$, and also $\beta_{n,k}^{(\lambda)} = ((r_k^{(\lambda)})^{1/2} - (r_k^{(\lambda)})^{-1/2})/2 > 0$.

Observe that $\beta_{n,k}^{(\lambda)}$ is a decreasing function of $z_k^{(\lambda)}$ in the interval $(-\infty, 0)$. Hence, from item 2 of Theorem 3, if $\lambda_j < \lambda_l$ then $z_k^{(\lambda_j)} > z_k^{(\lambda_l)}$ and, consequently, $x(z_k^{(\lambda_j)}) < x(z_k^{(\lambda_l)})$. Hence, $\beta_{n,k}^{(\lambda)}$ is an increasing function of λ. □

Remark 12. *From Lemma 5 and Theorems 2, 3 and 11, it follows that the zeros of $R_n^{(\lambda)}(z)$ are distinct, except in the cases: $\lambda = -\frac{2}{n-1}$ ($z = 1$ is a zero of $R_n^{(\lambda)}(z)$ of multiplicity 2); n even and $\lambda = 2$ ($z = -1$ is a zero of $R_n^{(\lambda)}(z)$ of multiplicity 2); n odd and $\lambda = 2 + \frac{2}{n-1}$ ($z = -1$ is a zero of $R_n^{(\lambda)}(z)$ of multiplicity 3).*

Furthermore, we observe that for any $\tilde{x} \in \mathbb{R}$ there exists a parameter $\tilde{\lambda}$ such that $W_n^{(\tilde{\lambda})}(\tilde{x}) = 0$. Indeed,

- If $\tilde{x}$ is such that $U_{n-2}(\tilde{x}) = 0$, i.e., $\tilde{x} = x_{n-2,r}$ for $r = 1, 2, ..., n-2$, since

$$\frac{W_n^{(\tilde{\lambda})}(\tilde{x})}{\tilde{\lambda}} = U_n(\tilde{x}) + 2\left(\frac{1}{\tilde{\lambda}} - 1\right) T_n(\tilde{x}),$$

then, when $\tilde{\lambda} \longrightarrow +\infty$ or $\tilde{\lambda} \longrightarrow -\infty$, the $n-2$ real zeros of the polynomial $W_n^{(\tilde{\lambda})}(x)$ tend, respectively, to the $n-2$ zeros of the polynomial $U_n(x) - 2T_n(x) = U_{n-2}(x)$.

- If $\tilde{x}$ is such that $U_{n-2}(\tilde{x}) \neq 0$, then we may choose

$$\tilde{\lambda} = 1 - \frac{U_n(\tilde{x})}{U_{n-2}(\tilde{x})}$$

and from (8) we have that $W_n^{(\tilde{\lambda})}(\tilde{x}) = U_n(\tilde{x}) - (1 - \tilde{\lambda})U_{n-2}(\tilde{x}) = 0$.

From the three term recurrence relation for the Chebyshev polynomial of second kind, $U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$, for $n \geq 1$, with $U_0(x) = 1$ and $U_1(x) = 2x$, one can easily show that $U_n(1) = (-1)^n U_n(-1) = n + 1$. Furthermore that, for $n \geq 2$,

$$U_n(x) \; > \; \frac{n+1}{n} U_{n-1}(x) \; > \; \frac{n+1}{n-1} U_{n-2}(x), \quad \text{for } x > 1.$$

Notice that if $\tilde{x} = 1$ (or $\tilde{x} = -1$), then

$$\tilde{\lambda} = 1 - \frac{U_n(1)}{U_{n-2}(1)} = 1 - \frac{(-1)^n U_n(-1)}{(-1)^{n-2} U_{n-2}(-1)} = 1 - \frac{n+1}{n-1} = -\frac{2}{n-1}.$$

From Remark 12, when $\tilde{\lambda} = -\frac{2}{n-1}$, the value $\tilde{z} = 1$ is a zero of $R_n^{(\tilde{\lambda})}(z)$ of multiplicity 2, that correspond to $\tilde{x} = 1$ and $\tilde{x} = -1$.

If $\tilde{x} > 1$ (or $\tilde{x} < -1$), then

$$\tilde{\lambda} = 1 - \frac{U_n(\tilde{x})}{U_{n-2}(\tilde{x})} = 1 - \frac{(-1)^n U_n(-\tilde{x})}{(-1)^{n-2} U_{n-2}(-\tilde{x})} < 1 - \frac{n+1}{n-1} = -\frac{2}{n-1}.$$

Also, $\tilde{z} \in (1, \infty)$ and $1/\tilde{z} \in (0, 1)$ are zeros of $R_n^{(\lambda)}(z)$.

Consequently, using the transformation (4) and the relation (8) we can observe the following.

Remark 13. *For any $\tilde{x} \in \mathbb{R}$ there exists a parameter $\tilde{\lambda}$ such that $W_n^{(\tilde{\lambda})}(\tilde{x}) = 0$. For any $\tilde{\theta} \in [0, 2\pi]$ there exists a parameter $\tilde{\lambda}$ such that $R_n^{(\tilde{\lambda})}(e^{i\tilde{\theta}}) = 0$. It means that the unit circle is covered by zeros of $R_n^{(\lambda)}(z)$ for different choices of the parameter $\lambda \in (-\infty, \infty)$.*

4.1. Special cases

For $-\frac{2}{n-1} \leq \lambda \leq 2$ (n even) or $-\frac{2}{n-1} \leq \lambda \leq 2 + \frac{2}{n-1}$ (n odd), we know that the zeros of $R_n^{(\lambda)}(z)$ located in the first and second quadrants are represented by $z_{n,r}^{(\lambda)} = e^{i\theta_{n,r}^{(\lambda)}}$, $r = 1, 2, ..., \lfloor n/2 \rfloor$, $0 \leq \theta_{n,r}^{(\lambda)} \leq \pi$, with $\theta_{n,r}^{(\lambda)} = 2\arccos(\xi_{n,r}^{(\lambda)})$, where $\xi_{n,r}^{(\lambda)}$ are the non–negative zeros of $W_n^{(\lambda)}(x)$. If n is odd, $r = 1, 2, ..., \lfloor n/2 \rfloor + 1$ and $\theta_{n,\lfloor n/2 \rfloor+1}^{(\lambda)} = \pi$.

If $\lambda = 0$, from equation (9) we have $z^{-n/2} R_n^{(0)}(z) = W_n^{(0)}(x) = 2T_n(x)$. Then, the zeros of $W_n^{(0)}(x)$ are represented by $\xi_{n,r}^{(0)} = \cos\left(\frac{(2r-1)\pi}{2n}\right)$, $r = 1, 2..., \lfloor n/2 \rfloor$. Hence,

$$\theta_{n,r}^{(0)} = 2\arccos(\xi_{n,r}^{(0)}) = \frac{(2r-1)\pi}{n}$$

and, for n odd, $\theta_{n,\lfloor n/2 \rfloor+1}^{(0)} = \pi$.

If $\lambda = 1$, from equation (8) we have $W_n^{(1)}(x) = U_n(x)$. Then, the zeros of $W_n^{(1)}(x)$ are represented by $\xi_{n,r}^{(1)} = \cos\left(\frac{r\pi}{n+1}\right)$, $r = 1, 2, ..., \lfloor n/2 \rfloor$. Hence,

$$\theta_{n,r}^{(1)} = 2\arccos(\xi_{n,r}^{(1)}) = \frac{2r\pi}{n+1}$$

and, for n odd, $\theta_{n,\lfloor n/2 \rfloor+1}^{(1)} = \pi$.

If $\lambda = 2$, from equation (8) we have $W_n^{(2)}(x) = U_n(x) - T_n(x)$. Then, the zeros of $W_n^{(2)}(x)$ are represented by $\xi_{n,r}^{(2)} = \cos\left(\frac{r\pi}{n}\right)$, $r = 1, 2, ..., \lfloor n/2 \rfloor$ and $\xi_{n,\lfloor n/2\rfloor+1}^{(2)} = \cos\left(\frac{\pi}{2}\right)$ (for n odd). Hence,

$$\theta_{n,r}^{(2)} = 2\arccos(\xi_{n,r}^{(2)}) = \frac{2r\pi}{n}, r = 1, 2, ..., \lfloor n/2 \rfloor \quad \text{and} \quad \theta_{n,\lfloor n/2\rfloor+1}^{(2)} = \pi \text{ (for } n \text{ odd).}$$

Observe that, if $\lambda = 2$ and n is even, from Lemma 5 follows that $z = -1$ is zero of multiplicity 2 of $R_n^{(\lambda)}(z)$ and, consequently, $\theta_{n,\lfloor n/2\rfloor}^{(2)} = \theta_{n,\lfloor n/2\rfloor+1}^{(2)} = \pi$.

5. An application: zeros of the polynomials $S_n^{(\lambda)}(z)$

We give here the proof of Theorem 4, about the location of the zeros of the polynomial $S_n^{(\lambda)}(z)$ defined by (1) and (2).

Proof of Theorem 4. Observe that, if n is even,

$$S_n^{(\lambda)}(z) = \left(\frac{z^{n/2}-1}{z-1}\right) R_{n/2+1}^{(\lambda)}(z) = R_{n/2-1}^{(1)}(z) R_{n/2+1}^{(\lambda)}(z). \tag{10}$$

Hence, we need to analyse the zeros of $R_{n/2-1}^{(1)}(z)$ and $R_{n/2+1}^{(\lambda)}(z)$.

From Theorem 6 and the results presented in Section 4.1, we have that the zeros of $R_{n/2-1}^{(1)}(z)$ are located on the unit circle and are given by $z_{n/2-1,r}^{(1)} = e^{i\theta_{n/2-1,r}^{(1)}}$, where $\theta_{n/2-1,r}^{(1)} = \frac{4r\pi}{n}$, $r = 1, 2, ..., \frac{n/2-1}{2} = \frac{n-2}{4}$, if $n/2-1$ is even, and $r = 1, 2, ..., \frac{n/2-1}{2}+1 = \frac{n+2}{4}$, if $n/2-1$ is odd. Notice that $0 \leq \theta_{n/2-1,r}^{(1)} \leq \pi$ (we are considering just the zeros on the first and second quadrants; the other zeros are the complex conjugate ones). Notice, also, that the zeros of $R_{n/2-1}^{(1)}(z)$ are fixed.

Also, from Theorem 6 we know that the zeros of $R_{n/2+1}^{(\lambda)}(z)$ are located on the unit circle if

$$-\frac{2}{n/2} \leq \lambda \leq 2, \quad \text{when } \frac{n}{2} \text{ is odd}$$

or

$$-\frac{2}{n/2} \leq \lambda \leq 2 + \frac{2}{n/2}, \quad \text{when } \frac{n}{2} \text{ is even.}$$

Furthermore, if $\lambda \in \left(-\infty, -\frac{2}{n/2}\right)$, $R_{n/2+1}^{(\lambda)}(z)$ has two positive zeros $z_k^{(\lambda)} \in (1, +\infty)$ and $1/z_k^{(\lambda)} \in (0, 1)$ and the other zeros are located on the unit circle. In the same way, if $\lambda \in (2, +\infty)$ $\left(\text{when } \frac{n}{2} \text{ odd}\right)$ and $\lambda \in \left(2 + \frac{2}{n/2}, +\infty\right)$ $\left(\text{when } \frac{n}{2} \text{ even}\right)$, $R_{n/2+1}^{(\lambda)}(z)$ has two negative zeros $z_k^{(\lambda)} \in (-\infty, -1)$ and $1/z_k^{(\lambda)} \in (-1, 0)$ and the other zeros are located on the unit circle.

In the case that n is odd,

$$S_n^{(\lambda)}(z) = \left(\frac{z^{\lfloor n/2\rfloor+1}-1}{z-1}\right) R_{\lfloor n/2\rfloor+1}^{(\lambda)}(z) = R_{\lfloor n/2\rfloor}^{(1)}(z) R_{\lfloor n/2\rfloor+1}^{(\lambda)}(z). \tag{11}$$

Hence, we need to analyse the zeros of $R^{(1)}_{\lfloor n/2\rfloor}(z)$ and $R^{(\lambda)}_{\lfloor n/2\rfloor+1}(z)$.

From Theorem 6 and the results presented in Section 4.1, we have that the zeros of $R^{(1)}_{\lfloor n/2\rfloor}(z)$ are located on the unit circle and they are given by $z^{(1)}_{\lfloor n/2\rfloor,r} = e^{i\theta^{(1)}_{\lfloor n/2\rfloor,r}}$, where $\theta^{(1)}_{\lfloor n/2\rfloor,r} = \frac{2r\pi}{\lfloor n/2\rfloor+1}$, for $r = 1,2,...,\frac{\lfloor n/2\rfloor}{2}$, if $\left\lfloor \frac{n}{2}\right\rfloor$ is even, and for $r = 1,2,...,\frac{\lfloor n/2\rfloor}{2}+1$, if $\left\lfloor \frac{n}{2}\right\rfloor$ is odd. Here $0 \le \theta^{(1)}_{\lfloor n/2\rfloor,r} \le \pi$ (again we are considering just the zeros on the first and second quadrants).

Also, from Theorem 6 we have that the zeros of $R^{(\lambda)}_{\lfloor n/2\rfloor+1}(z)$ are located on the unit circle if

$$-\frac{2}{\lfloor n/2\rfloor} \le \lambda \le 2, \quad \text{when } \left\lfloor \frac{n}{2}\right\rfloor \text{ is odd}$$

or

$$-\frac{2}{\lfloor n/2\rfloor} \le \lambda \le 2+\frac{2}{\lfloor n/2\rfloor}, \quad \text{when } \left\lfloor \frac{n}{2}\right\rfloor \text{ is even.}$$

Furthermore, if $\lambda \in \left(-\infty, -\frac{2}{\lfloor n/2\rfloor}\right)$, $R^{(\lambda)}_{\lfloor n/2\rfloor+1}(z)$ has two positive zeros $z^{(\lambda)}_k \in (1,+\infty)$ and $1/z^{(\lambda)}_k \in (0,1)$ and the other zeros are located on the unit circle. In the same way, if $\lambda \in (2,+\infty)$ (when $\lfloor n/2\rfloor$ odd) and $\lambda \in \left(2+\frac{2}{\lfloor n/2\rfloor},+\infty\right)$ (when $\lfloor n/2\rfloor$ even), $R^{(\lambda)}_{\lfloor n/2\rfloor+1}(z)$ has two negative zeros $z^{(\lambda)}_k \in (-\infty,-1)$ and $1/z^{(\lambda)}_k \in (-1,0)$ and the other zeros are located on the unit circle. $\square$

As consequence of Lemma 5 and Theorem 4 (relations (10) and (11)) we have the following results:

Corollary 14. *Considering the polynomial $S^{(\lambda)}_n(z), n \ge 3$ with n odd, then*

1. *if $\left\lfloor \frac{n}{2}\right\rfloor$ is odd,*
$$S^{(\lambda)}_n(z) = (z+1)A^{(1)}_{\lfloor n/2\rfloor-1}(z)R^{(\lambda)}_{\lfloor n/2\rfloor+1}(z);$$
2. *if $\left\lfloor \frac{n}{2}\right\rfloor$ is odd and $\lambda = 2$,*
$$S^{(2)}_n(z) = (z+1)A^{(1)}_{\lfloor n/2\rfloor-1}(z)(z+1)^2A^{(1)}_{\lfloor n/2\rfloor-1}(z) = (z+1)^3\left(A^{(1)}_{\lfloor n/2\rfloor-1}(z)\right)^2;$$
3. *if $\left\lfloor \frac{n}{2}\right\rfloor$ is even,*
$$S^{(\lambda)}_n(z) = (z+1)R^{(1)}_{\lfloor n/2\rfloor}(z)A^{(\lambda)}_{\lfloor n/2\rfloor}(z);$$
4. *if $\left\lfloor \frac{n}{2}\right\rfloor$ is even and $\lambda = 2$,*
$$S^{(2)}_n(z) = (z+1)R^{(1)}_{\lfloor n/2\rfloor}(z)A^{(2)}_{\lfloor n/2\rfloor}(z) = (z+1)\left(R^{(1)}_{\lfloor n/2\rfloor}(z)\right)^2;$$
5. *if $\left\lfloor \frac{n}{2}\right\rfloor$ is even and $\lambda = 2+\frac{2}{\lfloor n/2\rfloor}$,*
$$S^{(2+2/\lfloor n/2\rfloor)}_n(z) = (z+1)^3R^{(1)}_{\lfloor n/2\rfloor}(z)C_{\lfloor n/2\rfloor-2}(z).$$

Corollary 15. *Considering the polynomial $S_n^{(\lambda)}(z), n \geq 2$ with n even, then*

1. *if $\frac{n}{2}$ is even,*
$$S_n^{(\lambda)}(z) = (z+1)^2 A_{n/2-2}^{(1)}(z) A_{n/2}^{(\lambda)}(z);$$
2. *if $\frac{n}{2}$ is even and $\lambda = 2 + \frac{2}{n/2}$,*
$$S_n^{(2+2/(n/2))}(z) = (z+1)^4 A_{n/2-2}^{(1)}(z) C_{n/2-2}(z);$$
3. *if $\frac{n}{2}$ is odd and $\lambda = 2$,*
$$S_n^{(2)}(z) = (z+1)^2 R_{n/2-1}^{(1)}(z) A_{n/2-1}^{(1)}(z).$$

Corollary 16. *Considering the polynomial $S_n^{(\lambda)}(z), n \geq 2$ with $\lambda = -\frac{2}{\lfloor n/2 \rfloor}$, then*

1. *if n is even,*
$$S_n^{(-2/\lfloor n/2 \rfloor)}(z) = (z-1)^2 R_{n/2-1}^{(1)}(z) B_{n/2-1}(z);$$
2. *if n is odd,*
$$S_n^{(-2/\lfloor n/2 \rfloor)}(z) = (z-1)^2 R_{\lfloor n/2 \rfloor}^{(1)}(z) B_{\lfloor n/2 \rfloor -1}(z).$$

Remark 17. *The zeros of $S_n^{(\lambda)}(z), n \geq 2$, are distinct, except in the following cases*

1. *n even and $\lfloor n/2 \rfloor$ even (Corollary 15 item 1);*
2. *$\lambda = -\frac{2}{\lfloor n/2 \rfloor}$ (Corollary 16);*
3. *n odd, $\lfloor n/2 \rfloor$ even and $\lambda = 2$ (Corollary 14 item 4);*
4. *$\lfloor n/2 \rfloor$ odd and $\lambda = 2$ (Corollary 14 item 2 and Corollary 15 item 3);*
5. *$\lfloor n/2 \rfloor$ even and $\lambda = 2 + \frac{2}{\lfloor n/2 \rfloor}$ (Corollary 14 item 5 and Corollary 15 item 2).*

Acknowledgments

We would like to thank the referee for providing us with constructive comments and suggestions.

The first author is supported by funds (#2013/08012-8) from São Paulo Research Foundation (FAPESP) of Brazil. The second author is supported by funds from FAPESP (#2014/22571-2) and CNPq (#305148/2012-5) of Brazil. The third author is supported by grant from CAPES of Brazil.

References

[1] V. Botta, L.F. Marques, M. Meneguette, Palindromic and perturbed polynomials: zeros location, Acta Math. Hungar. 143 (2014) 81–87.

[2] C.F. Bracciali, D.K. Dimitrov, A. Sri Ranga, Chain sequences and symmetric generalized orthogonal polynomials, J. Comput. Appl. Math. 143 (2002) 95–106.

[3] C.F. Bracciali, X. Li, A. Sri Ranga, Real orthogonal polynomials in frequency analysis, Math. Comp. 74 (2005) 341–362.

[4] W.Y. Chen, On the polynomials with all their zeros on the unit circle, J. Math. Anal. Appl. 190 (1995) 714–724.

[5] T.S. Chihara, On quasi–orthogonal polynomials, Proc. Amer. Math. Soc. 8 (1957) 765–767.

[6] T.S. Chihara, An Introduction to Orthogonal Polynomials, Gordon and Breach, New York, 1978.

[7] Y. Choo, Y–J. Kim, On the zeros of self–inversive polynomials, Int. J. Math. Anal. 7 (2013) 187–193.

[8] P. Delsarte, Y.V. Genin, The split Levinson algorithm, IEEE Trans. Acoust. Speech Signal Process, 34 (1986) 470–478.

[9] D.K. Dimitrov, M.E.H. Ismail, A. Sri Ranga, A class of hypergeometric polynomials with zeros on the unit circle: extremal and orthogonal properties and quadrature formulas, Appl. Numer. Math. 65 (2013) 41–52.

[10] D.K. Dimitrov, A. Sri Ranga, Zeros of a family of hypergeometric para–orthogonal polynomials on the unit circle, Math. Nachr. 286 (2013) 1778–1791.

[11] E. Ghate, E. Hironaka, The arithmetic and geometry of Salem numbers, Bull. Amer. Math. Soc. 38 (2001) 293–314.

[12] M.E.H. Ismail, Classical and Quantum Orthogonal Polynomials in One Variable, Cambridge Univ. Press, Cambridge, 2005.

[13] H.Y. Kim, J.Y. Park, J.H. Cheon, J.H. Park, J.H. Kim, S.G. Hahn, Fast elliptic curve point counting using Gaussian normal basis, Algorithmic Number Theory, Lecture Notes in Computer Science, Vol. 2369 (2002) 292–307.

[14] S–H. Kim, C.W. Park, On the zeros of certain self–reciprocal polynomials, J. Math. Anal. Appl. 339 (2008) 240–247.

[15] D.Y. Kwon, Reciprocal polynomials with all zeros on the unit circle, Acta Math. Hungar. 131 (2011) 285–294.

[16] D.Y. Kwon, Reciprocal polynomials with all but two zeros on the unit circle, Acta Math. Hungar. 134 (2012) 472–480.

[17] M.N. Lalín, C.J. Smyth, Unimodularity of zeros of self–inversive polynomials, Acta Math. Hungar. 138 (2013) 85–101.

[18] L.F. Marques, Zeros de Polinômios Perturbados, Master Thesis (Presidente Prudente, Brazil, 2013).

[19] G.V. Milovanović, D.S. Mitrinović, Th.M. Rassias, Topics in Polynomials: Extremal Problems, Inequalities, Zeros, World Scientific, Singapore, 1994.

[20] T.J. Rivlin, Chebyshev Polynomials, Wiley–Interscience Publ., 1990.

[21] J. Shohat, On mechanical quadratures, in particular, with positive coefficients, Trans. Amer. Math. Soc. 42 (1937) 461–496.

[22] A.P. da Silva, A. Sri Ranga, Polynomials generated by a three term recurrence relation: bounds for complex zeros, Linear Algebra Appl. 397 (2005) 299–324.

[23] A. Zhedanov, On some classes of polynomials orthogonal on arcs of the unit circle connected with symmetric orthogonal polynomials on an interval, J. Approx. Theory 94 (1998) 73–106.