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On the Pólya–Wiman properties of differential operators [☆]

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ABSTRACT

Let $\phi(x) = \sum \alpha_n x^n$ be a formal power series with real coefficients, and let D denote differentiation. It is shown that "for every real polynomial f there is a positive integer m_0 such that $\phi(D)^m f$ has only real zeros whenever $m \geq m_0$" if and only if "$\alpha_0 = 0$ or $2\alpha_0\alpha_2 - \alpha_1^2 < 0$", and that if ϕ does not represent a Laguerre–Pólya function, then there is a Laguerre–Pólya function f of genus 0 such that for every positive integer m, $\phi(D)^m f$ represents a real entire function having infinitely many nonreal zeros.

1. Introduction

A real entire function is an entire function which takes real values on the real axis. If f is a real entire function, we denote the number of nonreal zeros (counting multiplicities) of f by $Z_C(f)$. (If f is identically equal to 0, we set $Z_C(f) = 0$.) A real entire function f is said to be of *genus* 1^* if it can be expressed in the form

$$f(x) = e^{-\gamma x^2} g(x),$$

where $\gamma \geq 0$ and g is a real entire function of genus at most 1. (For the definition of the order and genus of an entire function, see [2,14].) If f is a real entire function of genus 1^* and $Z_C(f) = 0$, then f is called a *Laguerre–Pólya function* and we write $f \in \mathcal{LP}$. We denote by $\mathcal{LP}^*$ the class of real entire functions f of genus 1^* such that $Z_C(f) < \infty$. It is well known that $f \in \mathcal{LP}$ if and only if there is a sequence $\langle f_n \rangle$ of real polynomials such that $Z_C(f_n) = 0$ for all n and $f_n \to f$ uniformly

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on compact sets in the complex plane. (See Chapter 8 of [14] and [15,17,21].) From this and an elementary argument based on Rolle's theorem, it follows that the classes $\mathcal{LP}$ and $\mathcal{LP}^*$ are closed under differentiation, and that $Z_C(f) \geq Z_C(f')$ for all $f \in \mathcal{LP}^*$. The *Pólya–Wiman theorem* states that for every $f \in \mathcal{LP}^*$ there is a positive integer m_0 such that $f^{(m)} \in \mathcal{LP}$ for all $m \geq m_0$ [6,7,10,12,20]. On the other hand, it follows from recent results of W. Bergweiler, A. Eremenko and J. Langley that if f is a real entire function, $Z_C(f) < \infty$ and $f \notin \mathcal{LP}^*$, then $Z_C(f^{(m)}) \to \infty$ as $m \to \infty$ [1,13].

Let ϕ be a formal power series given by

$$\phi(x) = \sum_{n=0}^{\infty} \alpha_n x^n.$$

For convenience we express the n-th coefficient α_n of ϕ as $\phi^{(n)}(0)/n!$ even when the radius of convergence is equal to 0. If f is an entire function and the series

$$\sum_{n=0}^{\infty} \alpha_n f^{(n)}$$

converges uniformly on compact sets in the complex plane, so that it represents an entire function, we write $f \in \operatorname{dom}\phi(D)$ and denote the entire function by $\phi(D)f$. For $m \geq 2$ we denote by $\operatorname{dom}\phi(D)^m$ the class of entire functions f such that $f, \phi(D)f, \ldots, \phi(D)^{m-1}f \in \operatorname{dom}\phi(D)$. It is obvious that if f is a polynomial, then $f \in \operatorname{dom}\phi(D)^m$ for all m. For more general restrictions on the growth of ϕ and f under which $f \in \operatorname{dom}\phi(D)^m$ for all m, see [3,5].

The following version of the Pólya–Wiman theorem for the operator $\phi(D)$ was established by T. Craven and G. Csordas [5, Theorem 2.4].

Theorem A. *Suppose that ϕ is a formal power series with real coefficients, $\phi'(0) = 0$ and $\phi''(0)\phi(0) < 0$. Then for every real polynomial f there is a positive integer m_0 such that all the zeros of $\phi(D)^m f$ are real and simple whenever $m \geq m_0$.*

Remark. The assumption implies that $\phi(0) \neq 0$. On the other hand, if $\phi(0) = 0$ and f is a real polynomial, then it is trivial to see that $Z_C(\phi(D)^m f) \to 0$ as $m \to \infty$. (Recall that we have set $Z_C(f) = 0$ if f is identically equal to 0.)

The following version of the Pólya–Wiman theorem is a consequence of the results in Section 3 of [5].

Theorem B. *Suppose that $\phi \in \mathcal{LP}$, $f \in \mathcal{LP}^*$, and that f is of order less than 2. Then $f \in \operatorname{dom}\phi(D)^m$, $\phi(D)^m f \in \mathcal{LP}^*$ and $Z_C(\phi(D)^m f) \geq Z_C(\phi(D)^{m+1} f)$ for all m. Furthermore, if ϕ is not of the form $\phi(x) = ce^{\gamma x}$ with $c \neq 0$, then $Z_C(\phi(D)^m f) \to 0$ as $m \to \infty$.*

Remarks. (1) If $\phi(x) = ce^{\gamma x}$, then

$$\phi(D)f(x) = \sum_{n=0}^{\infty} \frac{c\gamma^n}{n!} f^{(n)}(x) = cf(x+\gamma)$$

for every entire function f. Hence $Z_C(\phi(D)^m f) = Z_C(f)$ for all m whenever $c, \gamma \in \mathbb{R}$, $c \neq 0$ and f is a real entire function. We also remark that $\phi(x) = ce^{\gamma x}$ with $c \neq 0$ if and only if $\phi(0) \neq 0$ and $\phi^{(n)}(0)\phi(0)^{n-1} = \phi'(0)^n$ for all n.

(2) From [5, Lemma 3.2], [11, Theorem 2.3] and the arguments given in [3], it follows that the restriction "f is of order less than 2" can be weakened to "ϕ or f is of genus at most 1". See also [5, Theorem 3.3].

(3) In the case where ϕ is of genus 2, so that ϕ is of the form $\phi(x) = e^{-\gamma x^2}\psi(x)$ where $\gamma > 0$ and $\psi \in \mathcal{LP}$ is of genus at most 1, we have the following stronger result: If f is a real entire function of genus at most 1, and if the imaginary parts of the zeros of f are uniformly bounded, then $f \in \operatorname{dom}\phi(D)^m$ and $Z_C(\phi(D)^m f) \geq Z_C(\phi(D)^{m+1} f)$ for all m, and $Z_C(\phi(D)^m f) \to 0$ as $m \to \infty$, even when f has infinitely many nonreal zeros. See [3], [5, Lemma 3.2], [8, Theorems 9a, 13 and 14] and [11, Theorem 2.3].

In this paper, we complement Theorems A and B above. Let ϕ be a formal power series with real coefficients and f be a real entire function. If $f \in \operatorname{dom}\phi(D)^m$ for all m and $Z_C(\phi(D)^m f) \to 0$ as $m \to \infty$, then we will say that ϕ (or the corresponding operator $\phi(D)$) has the *Pólya–Wiman property* with respect to f. For instance, if f is a real entire function and $Z_C(f) < \infty$, then the operator D $(= d/dx)$ has the Pólya–Wiman property with respect to f if and only if $f \in \mathcal{LP}^*$. Theorem A gives a sufficient condition for ϕ to have the Pólya–Wiman property with respect to arbitrary real polynomials. The following two theorems imply that this is the case if and only if $\phi(0) = 0$ or $\phi''(0)\phi(0) - \phi'(0)^2 < 0$.

Theorem 1.1. *Suppose that ϕ is a formal power series with real coefficients, $\phi(0) \neq 0$ and $\phi''(0)\phi(0) - \phi'(0)^2 < 0$. Then for every real polynomial f there is a positive integer m_0 such that all the zeros of $\phi(D)^m f$ are real and simple whenever $m \geq m_0$.*

Theorem 1.2. *Suppose that ϕ is a formal power series with real coefficients, $\phi(0) \neq 0$, $\phi''(0)\phi(0) - \phi'(0)^2 \geq 0$, ϕ is not of the form $\phi(x) = ce^{\gamma x}$ with $c \neq 0$, f is a real polynomial, and that*

$$\deg f \geq \min\{n \geq 2 : \phi^{(n)}(0)\phi(0)^{n-1} \neq \phi'(0)^n\}.$$

Then there is a positive integer m_0 such that $Z_C(\phi(D)^m f) > 0$ for all $m \geq m_0$.

If $\phi \in \mathcal{LP}$ is not of the form $\phi(x) = ce^{\gamma x}$ with $c \neq 0$, then it is easy to see that $\phi(0) = 0$ or $\phi''(0)\phi(0) - \phi'(0)^2 < 0$ (for a proof, see [4,9]); hence Theorem B as well as Theorem 1.1 implies that ϕ has the Pólya–Wiman property with respect to arbitrary real polynomials. On the other hand, there are plenty of formal power series ϕ with real coefficients which satisfy $\phi(0) = 0$ or $\phi''(0)\phi(0) - \phi'(0)^2 < 0$, but do not represent Laguerre–Pólya functions. The following theorem, which is a strong version of the converse of Theorem B, implies that if ϕ is one such formal power series, then ϕ does not have the Pólya–Wiman property with respect to some (transcendental) Laguerre–Pólya function of genus 0, although it has the property with respect to arbitrary real polynomials.

Theorem 1.3. *Suppose that ϕ is a formal power series with real coefficients and ϕ does not represent a Laguerre–Pólya function. Then there is a Laguerre–Pólya function f of genus 0 such that $f \in \operatorname{dom}\phi(D)^m$ and $Z_C(\phi(D)^m f) = \infty$ for all positive integers m.*

As we shall see in the next section, Theorems 1.1 and 1.2 are almost immediate consequences of Theorems 2.1 and 2.2 below, which are proved in the same section by refining the arguments of Craven and Csordas given in Section 2 of [5]. Theorem 1.3 is a consequence of Pólya's characterization of the class $\mathcal{LP}$ given in [18,21] and a diagonal argument. It is proved in Section 3. Finally, in Section 4, we conclude the paper with some consequences of Theorems 2.1 and 2.2 on the asymptotic behavior of the distribution of zeros of $\phi(D)^m f$ as $m \to \infty$, in the general case where ϕ is a formal power series with complex coefficients and f is an arbitrary complex polynomial.

2. Proofs of Theorems 1.1 and 1.2

For notational clarity, we denote the monic monomial of degree d by M^d; that is, $M^d(x) = x^d$. With this notation, we have

$$\left(\exp\left(\beta D^p\right) M^d\right)(x) = \sum_{k=0}^{\lfloor d/p \rfloor} \frac{d!\beta^k}{k!(d-pk)!} x^{d-pk} \qquad (\beta \in \mathbb{C};\ d, p = 1, 2, \dots).$$

The following two theorems will be proved after the proofs of Theorems 1.1 and 1.2.

Theorem 2.1. *Suppose that ϕ is a formal power series with complex coefficients, $\phi(0) = 1$, ϕ is not of the form $\phi(x) = e^{\gamma x}$,*

$$p = \min\{n : n \geq 2 \text{ and } \phi^{(n)}(0) \neq \phi'(0)^n\},$$

$\alpha = \phi'(0)$ and $\beta = \left(\phi^{(p)}(0) - \phi'(0)^p\right)/p!$. Suppose also that f is a monic complex polynomial of degree d, and $f_1, f_2, \dots$ are given by

$$f_m(x) = m^{-d/p} \left(\phi(D)^m f\right) \left(m^{1/p} x - m\alpha\right). \tag{2.1}$$

Then $f_m \to \exp\left(\beta D^p\right) M^d$ uniformly on compact sets in the complex plane.

Theorem 2.2. *Suppose that d and p are positive integers, $p \geq 2$, $q = \lfloor d/p \rfloor$ and $r = d - pq$.*

(1) *If $q = 0$ ($d < p$), then $\exp\left(-D^p\right) M^d = M^d$.*
(2) *If $q \geq 1$, then $\exp\left(-D^p\right) M^d$ has exactly q distinct positive zeros; and if we denote them by $\rho_1, \dots, \rho_q$, then*

$$\left(\exp\left(-D^p\right) M^d\right)(x) = x^r \prod_{j=1}^{q} \prod_{k=0}^{p-1} \left(x - e^{2k\pi i/p} \rho_j\right).$$

Remark. The d-th Hermite polynomial H_d is given by

$$H_d(x) = \sum_{k=0}^{\lfloor d/2 \rfloor} \frac{(-1)^k d!}{k!(d-2k)!} (2x)^{d-2k}.$$

Thus we have $\left(\exp\left(-D^2\right) M^d\right)(x) = H_d(x/2)$ for all d, and Theorem 2.2 implies the well known fact that all the zeros of the Hermite polynomials are real and simple.

Corollary. *If $\beta > 0$, then all the zeros of $\exp\left(-\beta D^2\right) M^d$ are real and simple, and $\exp\left(\beta D^2\right) M^d$ has exactly $2\lfloor d/2 \rfloor$ distinct purely imaginary zeros; and if $\beta \neq 0$ and $3 \leq p \leq d$, then $\exp\left(\beta D^p\right) M^d$ has nonreal zeros.*

This corollary is an immediate consequence of Theorem 2.2 and the following relations which are trivially proved: If $\beta > 0$ and $\rho^p = -1$, then

$$\left(\exp\left(-\beta D^p\right) M^d\right)(x) = \beta^{d/p} \left(\exp\left(-D^p\right) M^d\right) \left(\frac{x}{\beta^{1/p}}\right)$$

and

$$\left(\exp\left(\beta D^p\right) M^d\right)(x) = \left(\rho\beta^{1/p}\right)^d \left(\exp\left(-D^p\right) M^d\right)\left(\frac{x}{\rho\beta^{1/p}}\right).$$

Proof of Theorem 1.1. Let f be a (nonconstant) real polynomial. Since multiplication by a nonzero constant does not change the zeros of a polynomial, we may assume that f is monic and $\phi(0) = 1$. Let $d = \deg f$, $\alpha = \phi'(0)$, $\beta = \left(\phi''(0) - \phi'(0)^2\right)/2$, and $f_1, f_2, \ldots$ be given by

$$f_m(x) = m^{-d/2}\left(\phi(D)^m f\right)\left(m^{1/2}x - m\alpha\right). \tag{2.2}$$

Then $\beta < 0$, and Theorem 2.1 implies that $f_m \to \exp\left(\beta D^2\right) M^d$ uniformly on compact sets in the complex plane. We have $\deg f_m = d = \deg(\exp\left(\beta D^2\right) M^d)$ for all m; and since $\beta < 0$, the corollary to Theorem 2.2 implies that all the zeros of $\exp\left(\beta D^2\right) M^d$ are real and simple. Hence the intermediate value theorem implies that there is a positive integer m_0 such that all the zeros of f_m are real and simple whenever $m \geq m_0$, and (2.2) shows that the same holds for $\phi(D)^m f$. $\square$

Proof of Theorem 1.2. Again, we may assume that f is monic and $\phi(0) = 1$. Let $d = \deg f$, and let p, α, β and the polynomials $f_1, f_2, \ldots$ be as in Theorem 2.1. We have $\beta \neq 0$; and in the case where $p = 2$, we must have $\beta > 0$, because we are assuming that $\phi''(0) - \phi'(0)^2 \geq 0$. Hence the corollary to Theorem 2.2 implies that $Z_C\left(\exp\left(\beta D^p\right) M^d\right) > 0$. By Theorem 2.1, $f_m \to \exp\left(\beta D^p\right) M^d$ uniformly on compact sets in the complex plane. Hence Rouche's theorem implies that there is a positive integer m_0 such that $Z_C(f_m) > 0$ whenever $m \geq m_0$, and (2.1) shows that the same holds for $\phi(D)^m f$. $\square$

In order to prove Theorem 2.1, we need some preliminaries. Let $\mathbb{C}[x]$ denote the (complex) vector space of complex polynomials, let $\mathbb{C}[x]^d$ denote the $(d+1)$-dimensional subspace of $\mathbb{C}[x]$ whose members are complex polynomials of degree $\leq d$, and let $\|\ \|_\infty$ denote the norm on $\mathbb{C}[x]$ defined by

$$\|f\|_\infty = \max\{|f^{(k)}(0)/k!| : 0 \leq k \leq \deg f\}.$$

Note that if $\langle f_m \rangle$ is a sequence of polynomials in $\mathbb{C}[x]^d$, then $\|f_m\|_\infty \to 0$ if and only if $f_m \to 0$ uniformly on compact sets in the complex plane. When ϕ is a formal power series (with complex coefficients) and d is a nonnegative integer, we denote the *operator norm* of $\phi(D)|_{\mathbb{C}[x]^d}$ with respect to $\|\ \|_\infty$ by $\|\phi(D)\|_d$; that is,

$$\|\phi(D)\|_d = \sup\{\|\phi(D)f\|_\infty : f \in \mathbb{C}[x]^d \text{ and } \|f\|_\infty \leq 1\}.$$

Let us denote the d-th partial sum of ϕ by $\phi|_d$:

$$\phi|_d(x) = \sum_{k=0}^{d} \frac{\phi^{(k)}(0)}{k!} x^k.$$

It is then clear that the restriction of $\phi(D)$ to $\mathbb{C}[x]^d$ is completely determined by the polynomial $\phi|_d$. Hence there are positive constants A_d and B_d such that

$$A_d\|\phi(D)\|_d \leq \|\phi|_d\|_\infty \leq B_d\|\phi(D)\|_d$$

for all formal power series ϕ.

Proof of Theorem 2.1. Let $r = \max\{p, d\}$. If $\tilde{\phi}$ is a formal power series and $\tilde{\phi}|_r = \phi|_r$, then $\tilde{\phi}$ satisfies the identical assumptions in the theorem that are satisfied by ϕ, and we have $\tilde{\phi}(D)^m f = \phi(D)^m f$ for all m. In other words, the theorem is about the first $r+1$ coefficients of ϕ only, and the coefficients $\phi^{(n)}(0)/n!$, $n > r$, are irrelevant to the theorem. For this reason, we may assume that $\phi^{(n)}(0) = 0$ for all $n > r$. Then ϕ is a polynomial and the assumptions on ϕ imply that

$$\phi(x) = e^{\alpha x} + \beta x^p + O\left(x^{p+1}\right) \qquad (x \to 0).$$

Let $\Phi(x) = e^{-\alpha x}\phi(x)$. Since $e^x = 1 + O(x)$ as $x \to 0$, it follows that

$$\Phi(x) = 1 + \beta x^p + O\left(x^{p+1}\right) \qquad (x \to 0);$$

hence there is a neighborhood U of 0 in the complex plane and there is an analytic function ψ in U such that

$$\Phi(x) = \exp\left(\beta x^p + x^{p+1}\psi(x)\right) \qquad (x \in U).$$

For $m = 1, 2, \ldots$ we have

$$\Phi\left(m^{-1/p}x\right)^m = \exp\left(\beta x^p + m^{-1/p}x^{p+1}\psi\left(m^{-1/p}x\right)\right) \qquad (x \in m^{1/p}U),$$

and this implies that

$$\sup_{|x|\le 1}\left|\Phi\left(m^{-1/p}x\right)^m - \exp\left(\beta x^p\right)\right| = O\left(m^{-1/p}\right) \qquad (m \to \infty).$$

Hence

$$\left\|\Phi\left(m^{-1/p}D\right)^m - \exp\left(\beta D^p\right)\right\|_d = O\left(m^{-1/p}\right) \qquad (m \to \infty). \tag{2.3}$$

For $m = 1, 2, \ldots$ define F_m by

$$F_m(x) = m^{-d/p} f\left(m^{1/p}x\right).$$

Since f is monic and of degree d, it follows that

$$\left\|F_m - M^d\right\|_\infty = O\left(m^{-1/p}\right) \qquad (m \to \infty). \tag{2.4}$$

For each m, it is trivial to see that

$$\left(\left(m^{-1/p}D\right)^k F_m\right)(x) = m^{-d/p}\left(D^k f\right)\left(m^{1/p}x\right) \qquad (k = 0, 1, 2, \ldots).$$

Form this, it follows that

$$\left(\Phi\left(m^{-1/p}D\right)^m F_m\right)(x) = m^{-d/p}\left(\Phi(D)^m f\right)\left(m^{1/p}x\right).$$

Since

$$\begin{aligned} m^{-d/p}\left(\Phi(D)^m f\right)\left(m^{1/p}x\right) &= m^{-d/p}\left(e^{-m\alpha D}\phi(D)^m f\right)\left(m^{1/p}x\right) \\ &= m^{-d/p}\left(\phi(D)^m f\right)\left(m^{1/p}x - m\alpha\right), \end{aligned}$$

we see that $f_m = \Phi\left(m^{-1/p}D\right)^m F_m$. Therefore we have

$$\begin{aligned}
&\left\|f_m - \exp\left(\beta D^p\right) M^d\right\|_\infty \\
&= \left\|\Phi\left(m^{-1/p}D\right)^m F_m - \exp\left(\beta D^p\right) M^d\right\|_\infty \\
&\le \left\|\Phi\left(m^{-1/p}D\right)^m - \exp\left(\beta D^p\right)\right\|_d \|F_m\|_\infty + \left\|\exp\left(\beta D^p\right)\right\|_d \left\|F_m - M^d\right\|_\infty \\
&= O\left(m^{-1/p}\right) \qquad (m \to \infty),
\end{aligned}$$

by (2.3) and (2.4). This proves the theorem. □

As we shall see soon, Theorem 2.2 is a consequence of known results on Jensen polynomials and Mittag–Leffler functions. The following is a simplified version of [5, Proposition 4.1].

Proposition 2.3. *Suppose that $\phi \in \mathcal{LP}$, q is a positive integer and f is given by*

$$f(x) = \sum_{k=0}^{q} \binom{q}{k} \phi^{(k)}(0) x^k.$$

Suppose also that $\phi(0) \neq 0$ and ϕ is not of the form $\phi(x) = p(x)e^{\alpha x}$, where p is a polynomial and $\alpha \neq 0$. Then all the zeros of f are real and simple.

Remark. The polynomial f is called the q-th *Jensen polynomial* associated with ϕ.

The *Mittag–Leffler functions* $E_1, E_2, \ldots$ are given by

$$E_p(x) = \sum_{k=0}^{\infty} \frac{x^k}{(pk)!}.$$

It is known that E_p is of order $1/p$ and $E_p \in \mathcal{LP}$ for all p (p. 5 of [14] and [16,19,22]). Let $J_{(p,q)}$ denote the q-th Jensen polynomial associated with E_p:

$$J_{(p,q)}(x) = \sum_{k=0}^{q} \binom{q}{k} E_p^{(k)}(0) x^k = \sum_{k=0}^{q} \frac{q! x^k}{(q-k)!(pk)!}.$$

Proposition 2.4. *The zeros of $J_{(p,q)}$ are all negative and simple for $p = 2, 3, \ldots$ and for $q = 1, 2, \ldots$.*

Proof. Suppose that $p \ge 2$ and $q \ge 1$. Then E_p is of order $\le 1/2$, hence it is not of the form $E_p(x) = p(x)e^{\alpha x}$ where p is a polynomial and $\alpha \neq 0$; and we have $E_p(0) = 1 \neq 0$. Since $E_p \in \mathcal{LP}$, Proposition 2.3 implies that all the zeros of $J_{(p,q)}$ are real and simple. Finally, they are all negative, because the coefficients of $J_{(p,q)}$ are all positive. □

Proof of Theorem 2.2. We have $d = pq + r$, $0 \le r \le p - 1$ and

$$\left(\exp\left(-D^p\right) M^d\right)(x) = x^r \sum_{k=0}^{q} \frac{(-1)^k d!}{k!(d-pk)!} x^{p(q-k)}.$$

The right hand side is of the form $x^r f(x^p)$, where f is a monic polynomial of degree q and $f(0) = (-1)^q d!/(q!r!) \neq 0$. From this, we see that part (1) of Theorem 2.2 is trivial, $\exp\left(-D^p\right) M^d$ has exactly r

zeros at the origin, and that the second assertion of part (2) follows from the first one. If $a \neq 0$ is a zero of $\exp(-D^p) M^d$, then so are $e^{2k\pi i/p}a$, $k = 0, 1, \dots, p-1$, and they are distinct. Since $\exp(-D^p) M^d$ has exactly $d = pq + r$ zeros in the whole plane and has exactly r zeros at the origin, it follows that $\exp(-D^p) M^d$ has at most q distinct positive zeros. Hence it is enough to show that if $q \geq 1$, then $\exp(-D^p) M^d$ has (at least) q distinct positive zeros.

Suppose that $q \geq 1$. We first consider the case where d is a multiple of p. In this case, we have $d = pq$, $r = 0$ and

$$\begin{aligned}
\left(\exp(-D^p) M^d\right)(x) &= \sum_{k=0}^{q} \frac{(-1)^k (pq)!}{k!(p(q-k))!} x^{p(q-k)} \\
&= \sum_{k=0}^{q} \frac{(-1)^{q-k} (pq)!}{(q-k)!(pk)!} x^{pk} \\
&= (-1)^q \frac{(pq)!}{q!} \sum_{k=0}^{q} \frac{q!}{(q-k)!(pk)!} (-x^p)^k \\
&= (-1)^q \frac{(pq)!}{q!} J_{(p,q)}(-x^p).
\end{aligned}$$

Since $p \geq 2$, Proposition 2.4 implies that all the zeros of $J_{(p,q)}$ are negative and simple. Hence $\exp(-D^p) M^d$ has exactly q ($= \deg J_{(p,q)}$) distinct positive zeros.

Finally, the result for the remaining case follows from an inductive argument based on Rolle's theorem, because $(\exp(-D^p) M^{pq+r})(0) = 0$ for $1 \leq r \leq p-1$,

$$\exp(-D^p) M^d = \frac{1}{d+1} D\left(\exp(-D^p) M^{d+1}\right),$$

and $\exp(-D^p) M^{p(q+1)}$ has exactly $q+1$ distinct positive zeros. □

3. Proof of Theorem 1.3

Let ϕ be a formal power series. First of all, we need to find a sufficient condition for an entire function f to be such that $f \in \operatorname{dom} \phi(D)^m$ and $\phi(D)^m f$ is not identically equal to 0 for all positive integers m. Let $\langle C_n \rangle$ be a sequence of positive numbers. If $|\phi^{(n)}(0)| < C_n$ for all n, we write $\phi \ll \langle C_n \rangle$. More generally, if there are constants c and d such that $c > 0$, $d \geq 0$ and $\phi \ll \langle c(1+n)^d C_n \rangle$, then we will write $\phi \prec \langle C_n \rangle$.

Lemma 3.1. *Suppose that $\langle B_n \rangle$ is an increasing sequence of positive numbers,*

$$B_m B_n \leq B_0 B_{m+n} \qquad (m, n = 0, 1, 2, \dots), \tag{3.1}$$

ϕ and ψ are formal power series, $\phi, \psi \prec \langle n! B_n \rangle$, f is an entire function, and that $f \prec \langle (n! B_n)^{-1} \rangle$. Then $\phi\psi \prec \langle n! B_n \rangle$, $f \in \operatorname{dom} \phi(D)$, $\phi(D) f \prec \langle (n! B_n)^{-1} \rangle$ and $\psi(D)(\phi(D) f) = (\phi\psi)(D) f$.

Proof. Suppose that a, b are nonnegative constants, $\phi \ll \langle (1+n)^a n! B_n \rangle$ and $\psi \ll \langle (1+n)^b n! B_n \rangle$. Then

$$\begin{aligned}
\left|(\phi\psi)^{(n)}(0)\right| &\leq \sum_{k=0}^{n} \binom{n}{k} \left|\phi^{(k)}(0)\right| \left|\psi^{(n-k)}(0)\right| \\
&< B_0 (1+n)^{a+b+1} n! B_n \qquad (n = 0, 1, 2, \dots),
\end{aligned}$$

hence $\phi\psi \prec \langle n! B_n \rangle$.

Now suppose that c is a nonnegative constant, $f \ll \langle (1+n)^c (n!B_n)^{-1} \rangle$, $R > 0$, and $|x| \leq R$. Then

$$\left| \frac{\phi^{(n)}(0) f^{(n+k)}(0) x^k}{n!k!} \right| \leq \frac{B_0 (1+n)^{a+c} (1+k)^c R^k}{n!(k!)^2 B_k}, \tag{3.2}$$

and we have

$$\sum_{n,k \geq 0} \frac{B_0 (1+n)^{a+c} (1+k)^c R^k}{n!(k!)^2 B_k} \leq \sum_{n=0}^{\infty} \frac{(1+n)^{a+c}}{n!} \sum_{k=0}^{\infty} \frac{(1+k)^c R^k}{(k!)^2} < \infty.$$

Hence the double series

$$\sum_{n,k \geq 0} \frac{\phi^{(n)}(0) f^{(n+k)}(0) x^k}{n!k!}$$

converges absolutely and uniformly on compact sets in the complex plane. As a consequence, the series

$$\sum_{n=0}^{\infty} \frac{\phi^{(n)}(0)}{n!} f^{(n)}$$

converges uniformly on compact sets in the complex plane; that is, $f \in \operatorname{dom} \phi(D)$. Furthermore, the absolute convergence of the double series implies that

$$\phi(D) f(x) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{\phi^{(n)}(0) f^{(n+k)}(0)}{n!} \frac{x^k}{k!} \qquad (x \in \mathbb{C}),$$

from which we obtain

$$(\phi(D) f)^{(k)}(0) = \sum_{n=0}^{\infty} \frac{\phi^{(n)}(0) f^{(n+k)}(0)}{n!} \qquad (k = 0, 1, 2, \ldots),$$

and the assumptions imply that

$$\left| (\phi(D) f)^{(k)}(0) \right| < \frac{B_0 (1+k)^c}{k! B_k} \sum_{n=0}^{\infty} \frac{(1+n)^{a+c}}{n!} \qquad (k = 0, 1, 2, \ldots),$$

hence we have $\phi(D) f \prec \langle (n!B_n)^{-1} \rangle$.

Finally, an estimate which is similar to (3.2) shows that the triple series

$$\sum_{m,n,k \geq 0} \frac{\psi^{(m)}(0) \phi^{(n)}(0) f^{(m+n+k)}(0) x^k}{m!n!k!}$$

converges absolutely for every $x \in \mathbb{C}$, hence the last assertion follows. □

Corollary. *Suppose that ϕ, ψ, f and $\langle B_n \rangle$ are as in Lemma 3.1, μ is a nonnegative integer, $\phi(x)\psi(x) = x^\mu$, and that f is transcendental. Then $f \in \operatorname{dom} \phi(D)^m$ and $\phi(D)^m f$ is not identically equal to 0 for all positive integers m.*

Proof. An inductive argument shows that $f \in \operatorname{dom} \phi(D)^m$, $\phi(D)^m f \in \operatorname{dom} \psi(D)^m$, and that $\psi(D)^m (\phi(D)^m f) = f^{(m\mu)}$ for all m. Since f is transcendental, $f^{(m\mu)}$ is not identically equal to 0 for all m, hence the same is true for $\phi(D)^m f$. □

Lemma 3.2. *Suppose that $\langle B_n \rangle$ and ϕ are as in Lemma 3.1, f is an entire function, $\langle f_N \rangle$ is a sequence of entire functions, $f_N \ll \langle (n!B_n)^{-1} \rangle$ for all N, and that $f_N \to f$ as $N \to \infty$ uniformly on compact sets in the complex plane. Then $f_1, f_2, \dots, f \in \operatorname{dom} \phi(D)$ and $\phi(D)f_N \to \phi(D)f$ as $N \to \infty$ uniformly on compact sets in the complex plane.*

Proof. First of all, Lemma 3.1 implies that $f_N \in \operatorname{dom} \phi(D)$ for all N. Since $f_N \to f$ uniformly on compact sets in the complex plane, and since

$$|f_N^{(n)}(0)| < (n!B_n)^{-1} \qquad (N = 1, 2, \dots;\ n = 0, 1, 2, \dots),$$

it follows that

$$|f^{(n)}(0)| \le (n!B_n)^{-1} \qquad (n = 0, 1, 2, \dots),$$

hence $f \in \operatorname{dom} \phi(D)$, by Lemma 3.1.

To prove the uniform convergence on compact sets in the complex plane, let $R > 0$ and $\epsilon > 0$ be arbitrary. Suppose that a is a nonnegative constant and $\phi \ll \langle (1+n)^a n!B_n \rangle$. If we put

$$b = \sum_{k=0}^{\infty} \frac{B_0 R^k}{(k!)^2 B_k},$$

then it is easy to see that

$$|f_N^{(n)}(x)| < \frac{b}{n!B_n} \qquad (|x| \le R;\ N = 1, 2, \dots;\ n = 0, 1, 2, \dots),$$

and that

$$|f^{(n)}(x)| \le \frac{b}{n!B_n} \qquad (|x| \le R;\ n = 0, 1, 2, \dots).$$

Let ν be a positive integer such that

$$b \sum_{n=\nu+1}^{\infty} \frac{(1+n)^a}{n!} < \epsilon.$$

Then there is a positive integer N_0 such that

$$\left| \sum_{n=0}^{\nu} \frac{\phi^{(n)}(0)}{n!} \left(f_N^{(n)}(x) - f^{(n)}(x) \right) \right| < \epsilon \qquad (|x| \le R;\ N \ge N_0),$$

because $f_N \to f$ uniformly on compact sets in the complex plane.

Now, suppose that $|x| \le R$ and $N \ge N_0$. Then we have

$$|\phi(D)f_N(x) - \phi(D)f(x)| \le \\ \left| \sum_{n=0}^{\nu} \frac{\phi^{(n)}(0)}{n!} \left(f_N^{(n)}(x) - f^{(n)}(x) \right) \right| + 2 \sum_{n=\nu+1}^{\infty} \frac{|\phi^{(n)}(0)|}{n!} \frac{b}{n!B_n} < 3\epsilon.$$

This completes the proof. $\square$

Corollary. *Under the same assumptions as in Lemma 3.2, $\phi(D)^m f_N \to \phi(D)^m f$ as $N \to \infty$ uniformly on compact sets in the complex plane for every positive integer m.*

Proof. Lemma 3.1 implies that $\phi^m \prec \langle n!B_n \rangle$ for all positive integers m. $\square$

We denote the open disk with center at a and radius r by $D(a;r)$, and its closure by $\bar{D}(a;r)$. For a complex constant c we define the *translation operator* T^c by $(T^c f)(x) = f(x+c)$. It is clear that if f is a monic polynomial of degree d, then $c^{-d}T^c f \to 1$ as $|c| \to \infty$ uniformly on compact sets in the complex plane. If m is a nonnegative integer, ϕ is a formal power series given by

$$\phi(x) = \sum_{n=m}^{\infty} \alpha_n x^n$$

with $\alpha_m \neq 0$, and f is a polynomial, then it is clear that $\phi(D)f$ is not identically equal to 0 if and only if f is not identically equal to 0 and $m \leq \deg f$. These observations lead to the following:

Lemma 3.3. *Suppose that ϕ is a formal power series, f and g are polynomials, $a_1, \ldots, a_N$ are zeros of $\phi(D)f$, b is a zero of $\phi(D)g$, and that neither $\phi(D)f$ nor $\phi(D)g$ is identically equal to 0. Then for every $c \in \mathbb{C}$ the polynomial $\phi(D)(fT^c g)$ is not identically equal to 0, and for every $\epsilon > 0$ there is an $R > 0$ such that if $|c| > R$, then $\phi(D)(fT^c g)$ has a zero in each of the disks $D(a_1;\epsilon), \ldots, D(a_N;\epsilon)$ and $D(b-c;\epsilon)$.*

Proof. The assumptions imply that neither f nor g is identically equal to 0. In particular, we have $\deg(fT^c g) \geq \deg f$, hence the first assertion follows, because $\phi(D)f$ is not identically equal to zero.

Let $\epsilon > 0$. We first note that if c is a constant, then $\phi(D)(fT^c g)$ has a zero in $D(b-c;\epsilon)$ if and only if $\phi(D)(gT^{-c}f)$ has a zero in $D(b;\epsilon)$. Since neither f nor g is identically equal to 0, we may assume that f and g are monic. Then $c^{-\deg g} fT^c g \to f$ and $(-c)^{-\deg f} gT^{-c}f \to g$ as $|c| \to \infty$ uniformly on compact sets in the complex plane. Hence there is an $R > 0$ such that if $|c| > R$, then $\phi(D)(fT^c g)$ has a zero in each of the disks $D(a_1;\epsilon), \ldots, D(a_N;\epsilon)$ and $\phi(D)(gT^{-c}f)$ has a zero in $D(b;\epsilon)$. $\square$

The following characterization of the class $\mathcal{LP}$ given in [18,21] will play a crucial role in the proof of Theorem 1.3.

Theorem (Pólya). *Let ϕ be a formal power series with real coefficients. Then $\phi \in \mathcal{LP}$ if and only if $Z_C(\phi(D)M^d) = 0$ for all positive integers d.*

Corollary. *Suppose that ϕ is a formal power series with real coefficients and ϕ does not represent a Laguerre–Pólya function. Then there is a positive integer d_0 such that $Z_C(\phi(D)M^d) > 0$ for all $d \geq d_0$.*

Proof. By Pólya's theorem, there is a positive integer d_0 such that $Z_C(\phi(D)M^{d_0}) > 0$, and Rolle's theorem implies that if $Z_C(\phi(D)M^{d+1}) = 0$, then $Z_C(\phi(D)M^d) = 0$. $\square$

Proof of Theorem 1.3. We will construct a sequence $\langle d(k) \rangle$ of positive integers and a sequence $\langle \gamma(k) \rangle$ of positive numbers such that $\sum_{k=1}^{\infty} d(k)\gamma(k) < \infty$ and the entire function f represented by

$$f(x) = \prod_{k=1}^{\infty} (1+\gamma(k)x)^{d(k)}$$

has the desired property.

Since ϕ does not represent a Laguerre–Pólya function, it follows that neither does the formal power series ϕ^m for every positive integer m. Hence the corollary to Pólya's theorem implies that there is an increasing sequence $\langle d(m)\rangle$ of positive integers such that $Z_C(\phi(D)^m M^{d(m)}) > 0$ for all positive integers m. Since $\langle d(m)\rangle$ is increasing, we have $Z_C(\phi(D)^m M^{d(k)}) > 0$ whenever $m \le k$. For each pair (m, k) of positive integers with $m \le k$ choose a nonreal zero of $\phi(D)^m M^{d(k)}$ in the upper half plane, denote it by $a(m, k)$ and set $r(m, k) = \operatorname{Im} a(m, k)/2$. It is obvious that $r(m, k) > 0$, and that $\bar{D}(a(m, k) - \gamma; r(m, k)) \cap \mathbb{R} = \emptyset$ for all $\gamma \in \mathbb{R}$. The assumption also implies that $\phi^{(n)}(0) \neq 0$ for some n, hence there is a nonnegative integer μ and there is a formal power series ψ such that $\phi(x)\psi(x) = x^\mu$. Choose an increasing sequence $\langle A_n\rangle$ of positive numbers such that $\phi, \psi \ll \langle A_n\rangle$, and define $\langle B_n\rangle$ by $B_0 = A_0$, $B_1 = A_1$ and

$$B_{n+1} = \max\left[\{A_{n+1}\} \cup \{B_0^{-1} B_k B_{n+1-k} : k = 1, \dots, n\}\right] \qquad (n = 1, 2, \dots).$$

It is clear that $\langle B_n\rangle$ is an increasing sequence of positive numbers, $\langle B_n\rangle$ satisfies (3.1), and that $\phi, \psi \prec \langle n! B_n\rangle$.

For $k = 1, 2, \dots$ and for $\gamma > 0$ define $g_{k,\gamma}$ by

$$g_{k,\gamma}(x) = (1 + \gamma x)^{d(k)};$$

that is, $g_{k,\gamma} = \gamma^{d(k)} T^{1/\gamma} M^{d(k)}$. From the definition, it follows that $g_{k,\gamma}$ is a real polynomial of degree $d(k)$, $g_{k,\gamma}(0) = 1$, $\phi(D)^m g_{k,\gamma}$ is not identically equal to 0 for $1 \le m \le k$,

$$\left(\phi(D)^m g_{k,\gamma}\right)\left(a(m, k) - \gamma^{-1}\right) = 0 \qquad (1 \le m \le k), \tag{3.3}$$

and that $g_{k,\gamma} \to 1$ as $\gamma \to 0$ uniformly on compact sets in the complex plane.

Since $g_{1,\gamma}(0) = 1 < 2$ and $g_{1,\gamma}$ is a polynomial of degree $d(1)$ for every $\gamma > 0$, and since $g_{1,\gamma} \to 1$ as $\gamma \to 0$ uniformly on compact sets in the complex plane, there is a positive number $\gamma(1)$ such that $g_{1,\gamma(1)} \ll \langle 2B_0(n! B_n)^{-1}\rangle$. From the definition, the polynomial $\phi(D) g_{1,\gamma(1)}$ is not identically equal to 0, and from (3.3) we have $\left(\phi(D) g_{1,\gamma(1)}\right)\left(a(1, 1) - \gamma(1)^{-1}\right) = 0$. Now suppose that $\gamma(1), \dots, \gamma(N)$ are positive numbers,

$$\prod_{k=1}^{N} g_{k,\gamma(k)} \ll \langle 2B_0(n! B_n)^{-1}\rangle, \tag{3.4}$$

and that for each $m \in \{1, \dots, N\}$, the closures of the disks

$$D\left(a(m, k) - \gamma(k)^{-1}; r(m, k)\right) \qquad (m \le k \le N) \tag{3.5}$$

are mutually disjoint and the polynomial $\phi(D)^m \left(\prod_{k=1}^{N} g_{k,\gamma(k)}\right)$ has a zero in each of these disks. Suppose also that the polynomials $\phi(D)^m \left(\prod_{k=1}^{N} g_{k,\gamma(k)}\right)$, $m = 1, \dots, N$ are not identically equal to 0. Since $\prod_{k=1}^{N} g_{k,\gamma(k)}$ is a polynomial, $g_{N+1,\gamma}$ is a polynomial of degree $d(N+1)$ for every $\gamma > 0$, and since $g_{N+1,\gamma} \to 1$ as $\gamma \to 0$ uniformly on compact sets in the complex plane, (3.4) implies that there is a $\delta > 0$ such that

$$\left(\prod_{k=1}^{N} g_{k,\gamma(k)}\right) g_{N+1,\gamma} \ll \langle 2B_0(n! B_n)^{-1}\rangle \qquad (0 < \gamma < \delta). \tag{3.6}$$

From Lemma 3.3, it follows that for each $m \in \{1, \dots, N\}$ there is an $R_m > 0$ such that if $|c| > R_m$, then $\phi(D)^m \left(\left(\prod_{k=1}^{N} g_{k,\gamma(k)}\right) T^c M^{d(N+1)}\right)$ has a zero in each of the disks given in (3.5) and also has a zero in the disk $D\left(a(m,N+1)-c;r(m,N+1)\right)$, because $\phi(D)^m\left(\prod_{k=1}^N g_{k,\gamma(k)}\right)$ has a zero in each of the disks given in (3.5) and $\left(\phi(D)^m M^{d(N+1)}\right)(a(m,N+1))=0$. By taking R_m sufficiently large, we may assume that

$$\bar{D}\left(a(m,k)-\gamma(k)^{-1};r(m,k)\right)\cap\bar{D}\left(a(m,N+1)-c;r(m,N+1)\right)=\emptyset$$

for $|c|>R_m$ and for $m\le k\le N$. Since $\phi(D)^{N+1}M^{d(N+1)}$ has a zero at $a(N+1,N+1)$ and $r(N+1,N+1)>0$, Lemma 3.3 implies that there is an $R_{N+1}>0$ such that if $|c|>R_{N+1}$, then $\phi(D)^{N+1}\left(\left(\prod_{k=1}^N g_{k,\gamma(k)}\right)T^c M^{d(N+1)}\right)$ has a zero in $D\left(a(N+1,N+1)-c;r(N+1,N+1)\right)$. Let $\gamma(N+1)$ be such that $0<\gamma(N+1)<\min\{\delta,R_1^{-1},\dots,R_N^{-1},R_{N+1}^{-1}\}$. Then (3.6) implies that

$$\prod_{k=1}^{N+1} g_{k,\gamma(k)}\ll\langle 2B_0(n!B_n)^{-1}\rangle.$$

The construction shows that for each $m\in\{1,\dots,N+1\}$ the closures of the disks

$$D\left(a(m,k)-\gamma(k)^{-1};r(m,k)\right)\qquad(m\le k\le N+1)$$

are mutually disjoint and the polynomial $\phi(D)^m\left(\prod_{k=1}^{N+1} g_{k,\gamma(k)}\right)$ has a zero in each of these disks. Finally, the polynomials $\phi(D)^m\left(\prod_{k=1}^{N+1} g_{k,\gamma(k)}\right)$, $m=1,\dots,N+1$, are not identically equal to 0, by Lemma 3.3.

By induction, this process produces a sequence $\langle\gamma(k)\rangle$ of positive numbers which has the following properties:

(1) For each positive integer N we have

$$\prod_{k=1}^{N} g_{k,\gamma(k)}\ll\langle 2B_0(n!B_n)^{-1}\rangle.$$

(2) For each positive integer m, the closed disks

$$\bar{D}\left(a(m,k)-\gamma(k)^{-1};r(m,k)\right)\qquad(k=m,m+1,m+2,\dots)\tag{3.7}$$

are mutually disjoint.

(3) For each positive integer m the polynomial $\phi(D)^m\left(\prod_{k=1}^N g_{k,\gamma(k)}\right)$ has a zero in each of the disks given in (3.5), whenever $N\ge m$.

For $N=1,2,\dots$ we set $f_N=\prod_{k=1}^N g_{k,\gamma(k)}$; that is,

$$f_N(x)=\prod_{k=1}^N\left(1+\gamma(k)x\right)^{d(k)}.$$

From (1), it follows that

$$0\le f_N^{(n)}(0)<2B_0(n!B_n)^{-1}\qquad(N=1,2,\dots;\ n=0,1,2,\dots).\tag{3.8}$$

In particular, we have

$$\sum_{k=1}^N d(k)\gamma(k)=f_N'(0)<2B_0/B_1\qquad(N=1,2,\dots),$$

hence the infinite product $\prod_{k=1}^{\infty}(1+\gamma(k)x)^{d(k)}$ represents an entire function of genus 0. Let f denote the entire function. It is then obvious that f is transcendental, $f \in \mathcal{LP}$, $f_N \to f$ uniformly on compact sets in the complex plane, and that

$$0 < f^{(n)}(0) \leq 2B_0(n!B_n)^{-1} \qquad (n = 0, 1, 2, \dots).$$

To complete the proof, let m be a positive integer. From the corollary to Lemma 3.1, it follows that $f \in \operatorname{dom}\phi(D)^m$ and $\phi(D)^m f$ is not identically equal to 0; and from (3.8) and the corollary to Lemma 3.2, we see that $\phi(D)^m f_N \to \phi(D)^m f$ as $N \to \infty$ uniformly on compact sets in the complex plane. Furthermore, $\phi(D)^m f_N$ has a zero in each of the disks given in (3.5) whenever $N \geq m$. Hence $\phi(D)^m f$ has a zero in each of the closed disks given in (3.7) which are mutually disjoint and do not intersect the real axis. Therefore $Z_C(\phi(D)^m f) = \infty$. $\square$

4. Some consequences of Theorems 2.1 and 2.2

In this brief section, we will be concerned with the asymptotic behavior of the distribution of zeros of $\phi(D)^m f$ as $m \to \infty$, in the case where the coefficients of ϕ are complex numbers and f is a complex polynomial. When f is an entire function, we denote its zero set by $\mathcal{Z}(f)$; that is, $\mathcal{Z}(f) = \{z \in \mathbb{C} : f(z) = 0\}$. For $a \in \mathcal{Z}(f)$ the multiplicity is denoted by $\mathrm{m}(a, f)$.

Let ϕ, p, α, β, f, d and $f_1, f_2, \dots$ be as in Theorem 2.1. Then $\beta \neq 0$ and $f_m \to \exp(\beta D^p)M^d$ uniformly on compact sets in the complex plane. We also have

$$\mathcal{Z}(\phi(D)^m f) = -m\alpha + m^{1/p}\mathcal{Z}(f_m) \tag{4.1}$$

and

$$\mathrm{m}(a, \phi(D)^m f) = \mathrm{m}(m^{-1/p}(a + m\alpha), f_m) \tag{4.2}$$

for all $a \in \mathcal{Z}(\phi(D)^m f)$. Let $\epsilon > 0$ be so small that the disks $D(b;\epsilon)$, $b \in \mathcal{Z}\left(\exp(\beta D^p)M^d\right)$, are mutually disjoint. Then Rouche's theorem implies that there is a positive integer m_0 such that

$$\sum_{c \in D(b;\epsilon)\cap\mathcal{Z}(f_m)} \mathrm{m}(c, f_m) = \mathrm{m}\left(b, \exp(\beta D^p)M^d\right) \tag{4.3}$$

holds for all $b \in \mathcal{Z}\left(\exp(\beta D^p)M^d\right)$ and for all $m \geq m_0$. As a consequence, we have

$$\mathcal{Z}(f_m) \subset D(0;\epsilon) + \mathcal{Z}\left(\exp(\beta D^p)M^d\right) \tag{4.4}$$

for all $m \geq m_0$. Let γ be a complex number such that $\gamma^p = -\beta$. Then $\gamma \neq 0$ and we have

$$\mathcal{Z}\left(\exp(\beta D^p)M^d\right) = \gamma\mathcal{Z}\left(\exp(-D^p)M^d\right),$$

because

$$\left(\exp(\beta D^p)M^d\right)(x) = \gamma^d\left(\exp(-D^p)M^d\right)(x/\gamma).$$

Now, (4.1) and (4.4) imply that

$$\mathcal{Z}(\phi(D)^m f) \subset -m\alpha + m^{1/p}\left(D(0;\epsilon) + \gamma\mathcal{Z}\left(\exp(-D^p)M^d\right)\right) \tag{4.5}$$

holds for all $m \geq m_0$.

With the aid of Theorem 2.2, the above results give us some information on the zeros of $\phi(D)^m f$ for large values of m. From Theorem 2.2, it follows that

$$\mathcal{Z}\left(\exp(-D^p)M^d\right) \subset S_p,$$

where

$$S_p = \bigcup_{k=0}^{p-1} \left\{ re^{2k\pi i/p} : r \geq 0 \right\}.$$

It also follows from Theorem 2.2 that if $d \equiv 0$ or $1 \bmod p$, then all the zeros of $\exp(-D^p)M^d$ are simple. Hence (4.5) implies that for every $\epsilon > 0$ there is a positive integer m_0 such that

$$\mathcal{Z}(\phi(D)^m f) \subset -m\alpha + N(0, m^{1/p}\epsilon) + \gamma S_p$$

for all $m \geq m_0$, and (4.1) through (4.3) imply that if $d \equiv 0$ or $1 \bmod p$, then all the zeros of $\phi(D)^m f$ are simple whenever m becomes sufficiently large.

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