J. Math. Anal. Appl. 434 (2016) 1106–1122

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Spectra and variance of quantum random variables

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ARTICLE INFO

Article history:
Received 10 February 2015
Available online 30 September 2015
Submitted by J.A. Ball

Keywords:
Positive operator-valued measure
Quantum probability measure
Quantum random variable
Variance
Quantum noise
Smearing

ABSTRACT

We study essentially bounded quantum random variables and show that the Gelfand spectrum of such a quantum random variable ψ coincides with the hypoconvex hull of the essential range of ψ. Moreover, a notion of operator-valued variance is introduced, leading to a formulation of the moment problem in the context of quantum probability spaces in terms of operator-theoretic properties involving semi-invariant subspaces and spectral theory. As an application of quantum variance, new measures of random and inherent quantum noise are introduced for measurements of quantum systems, modifying some recent ideas of Polterovich [17].

1. Introduction

Some of the most basic and useful properties of classical random variables are altered when passing from real- or complex-valued measurable functions to operator-valued measurable functions (that is, from classical to quantum random variables). In earlier works [8,9,12], a certain operator-valued formulation of the notion of expectation of a quantum random variable was considered. In the present paper, we consider a similar formulation for the variance of a quantum random variable. As in these earlier investigations, the noncommutativity of operator algebra will lead to some structure that simply does not appear in the classical setting.

It is a basic fact of functional analysis that the essential range of an essentially bounded random variable coincides with the spectrum of a certain element in an abelian von Neumann algebra. Specifically, if $\psi : X \to \mathbb{C}$ is an essentially bounded function on a probability space $(X, \mathcal{F}(X), \mu)$, then the essential range of ψ is precisely the spectrum of ψ, where one considers ψ as an element of the von Neumann algebra

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http://dx.doi.org/10.1016/j.jmaa.2015.09.055
0022-247X/

$L^\infty(X, \mathcal{F}(X), \mu)$. We will arrive at a similar result for essentially bounded quantum random variables on quantum probability spaces using higher dimensional spectra. However, it will turn out that our investigation of quantum variance will also involve notions from spectral theory. In particular, the quantum moment problem admits a characterisation entirely within spectral terms.

As an application of our operator-valued variance, we consider some recent work of Polterovich [17] on random and inherent quantum noise in which the variance has a role. In Polterovich's work, a somewhat hybrid context is at play: while the measures are operator-valued, the random variables are classical. In modifying Polterovich's ideas to account for operator-valued measures and operator-valued random variables, we formulate new measures of quantum noise. One of the main consequences of our results in this direction is that if an experimental apparatus is free of random quantum noise, then it is classical, not quantum mechanical. Our work on quantum noise involves another idea that may be of value in other settings, namely that of quantum randomisation (or smearing), which is in contrast to the hybrid notion of smearing studied in early works such as [4,11]. By way of quantum randomisation, we also modify another concept of Polterovich to obtain a measure of the intrinsic quantum noise of the apparatus represented by ν.

If $(X, \mathcal{F}(X))$ denotes an arbitrary measurable space, and if M is a von Neumann algebra with predual M_* and positive cone M_+, then a function $\nu : \mathcal{F}(X) \to M$ is a positive operator-valued measure (POVM) if

1. $\nu(E) \in M_+$ for every $E \in \mathcal{F}(X)$,
2. $\nu(X) \neq 0$, and
3. $\omega \circ \nu : \mathcal{F}(X) \to \mathbb{C}$ is a complex measure for every $\omega \in M_*$.

Note that the third condition above asserts that, for every countable collection $\{E_k\}_{k\in\mathbb{N}} \subseteq \mathcal{F}(X)$ with $E_j \cap E_k = \emptyset$ for $j \neq k$,

$$\nu\left(\bigcup_{k\in\mathbb{N}} E_k\right) = \sum_{k\in\mathbb{N}} \nu(E_k), \tag{1}$$

where the convergence is with respect to the ultraweak topology of M.

If a POVM ν also satisfies $\nu(E \cap F) = \nu(E)\nu(F)$ for all $E, F \in \mathcal{F}(X)$, then ν is called a projective POVM. An important theorem of M.A. Neumark [14], [15, Theorem 4.6] states that every POVM admits a dilation to a projective POVM. Lastly, if a POVM ν has the property that $\nu(X) = 1$, the identity element of M, then ν is called a quantum probability measure.

A function $\psi : X \to M$ is said to be measurable if the complex-valued function $\omega \circ f$ on X is measurable for every $\omega \in M_*$. Furthermore, if ν is a quantum probability measure, then a measurable function $\psi : X \to M$ is called a quantum random variable.

Suppose that $\omega \in M_*$ is a faithful state on M and that ν is a quantum probability measure. Then $\omega \circ \nu$ is a (classical) probability measure and, because ω is faithful, ν and $\omega \circ \nu$ are mutually absolutely continuous. The predual of the von Neumann algebra $L^\infty(X, \omega\circ\nu)\overline{\otimes}M$ is given by $L^1_{M_*}(X, \omega\circ\nu)$ [20, Theorem IV.7.17]. By way of this duality isomorphism, if $\Psi \in L^\infty(X, \omega\circ\nu)\overline{\otimes}M$, then there is a bounded measurable function $\psi : X \to M$ such that, for each $f \in L^1_{M_*}(X, \omega\circ\nu)$, the complex number $\Psi(f)$ is given by

$$\Psi(f) = \int_X \omega\left(f(x)\psi(x)\right)\, \mathrm{d}(\omega\circ\nu)(x).$$

Although ψ is not unique, it is unique up to a set of $\omega\circ\nu$-measure zero. We therefore identify Ψ and ψ and consider the elements of $L^\infty(X, \omega\circ\nu)\overline{\otimes}M$, in the case where ν is a quantum probability measure, to be bounded quantum random variables $\psi : X \to M$.

The general context described above for operator-valued measures and functions is considered in this paper only in the setting a finite factor M of type I_d; that is, $M = \mathcal{B}(\mathcal{H})$ for some d-dimensional Hilbert

space $\mathcal{H}$ and $d \in \mathbb{N}$. The predual M_* is denoted by $\mathcal{T}(\mathcal{H})$ (the Banach space of trace-class operators on $\mathcal{H}$). Owing to the finite-dimensionality of $\mathcal{H}$, the Banach spaces $\mathcal{B}(\mathcal{H})$ and $\mathcal{T}(\mathcal{H})$ are equal as sets, but as Banach spaces any one of these spaces is isometrically isomorphic to the dual of the other. In this setting, the faithful normal state $\omega \in M_*$ is chosen to be the normalised trace and, for a fixed quantum probability measure ν, we denote by μ the classical probability measure $\mu = \frac{1}{d}\operatorname{Tr} \circ \nu$, where Tr is the canonical trace on $\mathcal{B}(\mathcal{H})$. Because $\mathcal{H}$ has finite dimension d, we adopt the following notation:

$$L^\infty_{\mathcal{H}}(X, \nu) = L^\infty(X, \mu) \,\overline{\otimes}\, \mathcal{B}(\mathcal{H}) \cong L^\infty(X, \mu) \otimes M_d(\mathbb{C}),$$

where $M_d(\mathbb{C})$ is the space of $d \times d$ matrices over $\mathbb{C}$.

(The restriction to factors of type I_d is made for two reasons. The first reason is that the notion of quantum measurement most often in practice entails a POVM ν with values in $\mathcal{B}(\mathcal{H})$ for a finite-dimensional Hilbert space $\mathcal{H}$. The second reason is that certain results, when formulated for infinite-dimensional factors, become far less interesting than is the case with finite-dimensional factors. As an example of this particular situation, compare [9, Theorem 5.1] on the affine structure of the set of all quantum probability measures (type I_d case) with the analogous result in [10] (type I_∞ case). Specifically, in the type I_d case extremal quantum probability measures are certain linear combinations of point-mass measures, whereas in the type I_∞ case the set of projective quantum probability measures is dense (with need of forming convex combinations of such projective measures) in the space of all quantum probability measures.)

Lastly, all homomorphisms and isomorphisms of C*-algebras are assumed, without saying so each time, to be unital and $*$-preserving. If Z is a compact Hausdorff space, then $C(Z)$ is the unital abelian C*-algebra of all continuous functions $f : Z \to \mathbb{C}$.

2. Basic properties of measurability and quantum expectation

Some elementary but useful facts concerning measurable functions are noted in this section.

Theorem 2.1. *The following two statements are equivalent for a function $\psi : X \to \mathcal{B}(\mathcal{H})$.*

1. *ψ is measurable.*
2. *$\psi^{-1}(U)$ is a measurable set, for every open set $U \subseteq \mathcal{B}(\mathcal{H})$.*

Proof. Fix an orthonormal basis $\{e_1, \ldots, e_d\}$ of $\mathcal{H}$. Because ψ is measurable if and only if each coordinate function $\psi_{ij}(x) = \langle \psi(x)e_j, e_i\rangle$ is measurable [9, Section III], and because $\mathcal{B}(\mathcal{H})$ is topologically equivalent to $\mathbb{C}^{d^2}$ in the product topology, we may assume without loss of generality that $\psi : X \to \mathbb{C}^{d^2}$ and $\psi(x) = (\psi_{11}(x), \psi_{12}(x), \ldots, \psi_{dd}(x))$ for $x \in X$. Furthermore, every open set $U \subseteq \mathbb{C}^{d^2}$ will be viewed as a product $U = \prod_{i,j=1}^{d} U_{ij}$ of open sets $U_{ij} \subseteq \mathbb{C}$. Suppose now that ψ is measurable. As each ψ_{ij} is therefore measurable, we have that $\psi_{ij}(U_{ij}) \in \mathcal{F}(X)$ for every open set $U_{ij} \subseteq \mathbb{C}^{d^2}$. Thus, if $U = \prod_{i,j=1}^{d} U_{ij}$ is open in $\mathbb{C}^{d^2}$, then $\psi^{-1}(U) = \bigcap_{i,j=1}^{d} \psi_{ij}^{-1}(U_{ij})$ is a measurable set. Conversely, if $\psi^{-1}(U)$ is a measurable set, then for a fixed ordered pair (k, ℓ) and any open set $U_{k\ell} \subseteq \mathbb{C}$, we have $\psi_{k\ell}^{-1}(U_{k\ell}) = \psi^{-1}(U)$, where $U = \prod_{i,j=1}^{d} U_{ij}$ is the open set for which $U_{ij} = \mathbb{C}$ for all $(i, j) \neq (k, \ell)$. Thus, $\psi_{k\ell}$ is a measurable function. □

Mimicking the classical definition of a regular probability measure, we have the following.

Definition 2.2. A quantum probability measure $\nu : \mathcal{F}(X) \to \mathcal{B}(\mathcal{H})$ is *regular* if for every $E \in \mathcal{F}(X)$,

$$\begin{aligned}\nu(E) &= \inf\{\nu(U) \,|\, U \subseteq X \text{ is open, and } E \subseteq U\} \\ &= \sup\{\nu(K) \,|\, K \text{ is compact, and } K \subseteq E\}.\end{aligned}$$

We note that because $\mathcal{B}(\mathcal{H})$ is a von Neumann algebra, the infimum and supremum in the definition of regular measure above exist. Furthermore, because the normalised trace τ on $\mathcal{B}(\mathcal{H})$ is a normal linear functional, the induced classical probability measure $\mu = \tau \circ \nu$ on $(X, \mathcal{F}(X))$ is regular if the quantum probability measure ν is. This leads to the next result which is the quantum analogue of the classical Lusin theorem.

Theorem 2.3. *If $\psi : X \to \mathcal{B}(\mathcal{H})$ is a quantum random variable and if ν is a regular quantum probability measure on $(X, \mathcal{F}(X))$, where $\mathcal{F}(X)$ is the σ-algebra of Borel sets of a locally compact Hausdorff space X, then for every $\varepsilon > 0$ there is a continuous function $\vartheta : X \to \mathcal{B}(\mathcal{H})$ with compact support such that $\mu(\{x \in X \,|\, \psi(x) \neq \vartheta(x)\}) < \epsilon$.*

Proof. Let $\{\psi_{ij}\}_{i,j=1}^{d}$ be the set of coordinate functions defined in the proof of Theorem 2.1. Because ν is a regular measure and τ is a normal state, the induced measure $\mu = \tau \circ \nu$ is also regular. Hence, the classical Lusin theorem may be invoked to obtain, for each i and j, a continuous function $\vartheta_{ij} : X \to \mathbb{C}$ with compact support and such that $\mu(D_{ij}) < \varepsilon/d^2$, where $D_{ij} = \{x \in X \,|\, \psi_{ij}(x) \neq \vartheta_{ij}(x)\}$. Let $\vartheta : X \to \mathcal{B}(\mathcal{H})$ be the continuous map induced by the coordinate functions ϑ_{ij}, and define D to be the set $D = \{x \in X \,|\, \psi(x) \neq \vartheta(x)\}$ which is measurable by Theorem 2.1. Because $D \subseteq \bigcup_{i,j=1}^{d} D_{ij}$, we deduce that $\mu(D) \leq \sum_{i,j=1}^{d} \mu(D_{ij}) < \varepsilon$. □

The following theorem and two definitions summarise the results of [9, Section III] relevant for our purposes; see also [8,12] for additional details.

Theorem 2.4. *If ν is a quantum probability measure, then ν is absolutely continuous with respect to the induced classical measure μ, and there exists a quantum random variable denoted by $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ such that*

$$\int_E \mathrm{Tr}\left(\rho \, \frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right) \mathrm{d}\mu(x) \;=\; \mathrm{Tr}\left(\rho\, \nu(E)\right), \tag{2}$$

for all $E \in \mathcal{F}(X)$ and every density operator ρ.

The Borel function $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ is called the *principal Radon–Nikodým derivative of ν* and is a positive operator for μ-almost all $x \in X$.

Definition 2.5.

1. A quantum random variable ψ is *ν-integrable* if for every density operator ρ the complex-valued function

$$\psi_\rho(x) \;=\; \mathrm{Tr}\left(\rho \left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2} \psi(x) \left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2}\right), \; x \in X,$$

is μ-integrable.

2. The *integral* of a ν-integrable function $\psi : X \to \mathcal{B}(\mathcal{H})$ is defined to be the unique operator acting on $\mathcal{H}$ having the property that

$$\mathrm{Tr}\left(\rho \int_X \psi \, \mathrm{d}\nu\right) = \int_X \psi_\rho \, \mathrm{d}\mu \,,$$

for every density operator ρ.

Definition 2.6. If $\nu : \mathcal{F}(X) \to \mathcal{B}(\mathcal{H})$ is a quantum probability measure, then the map $\mathbb{E}_\nu : L^\infty_{\mathcal{H}}(X, \nu) \to \mathcal{B}(\mathcal{H})$ defined by

$$\mathbb{E}_\nu[\psi] = \int_X \psi \, \mathrm{d}\nu$$

is called the *quantum expectation* of ψ with respect to ν.

A version of the following example first appeared in [9, Example 3.4]; see also [8, Theorem 2.3(4)].

Example 2.7. Let $X = \{x_1, \ldots, x_n\}$ and let $\mathcal{F}(X)$ be the power set of X. If $h_1, \ldots, h_n \in \mathcal{B}(\mathcal{H})_+$ are such that $h_1 + \cdots + h_n = 1 \in \mathcal{B}(\mathcal{H})$, and ν satisfies $\nu(\{x_j\}) = h_j$ for $j = 1, \ldots, n$, then for every $\psi : X \to \mathcal{B}(\mathcal{H})$,

$$\mathbb{E}_\nu[\psi] = \int_X \psi \, \mathrm{d}\nu = \sum_{j=1}^n h_j^{1/2} \psi(x_j) h_j^{1/2}.$$

Thus one can view $\mathbb{E}_\nu[\psi]$ as a quantum averaging of ψ.

Recall [15, Chapter 3] that a linear map $\varphi : \mathcal{A} \to \mathcal{B}$ of unital C*-algebras is a unital completely positive (ucp) map if $\varphi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$ and the induced linear maps

$$\varphi \otimes \mathrm{id}_n : \mathcal{A} \otimes M_n(\mathbb{C}) \to \mathcal{B} \otimes M_n(\mathbb{C})$$

are positive for every $n \in \mathbb{N}$.

Theorem 2.8. *Quantum expectation is a completely positive operation. That is, the linear map* $\mathbb{E}_\nu : L^\infty_{\mathcal{H}}(X, \nu) \to \mathcal{B}(\mathcal{H})$ *is a unital completely positive map, for every quantum probability measure* ν.

Proof. The linearity of the map $\mathbb{E}_\nu$ follows readily by definition. Because the algebra $L^\infty(X, \mu)$ is a unital abelian C*-algebra, where μ is induced by a quantum probability measure ν, the Gelfand transform $\Gamma : L^\infty(X, \mu) \to C(Z_\nu)$ is a unital C*-algebra isomorphism, where Z_ν is the maximal ideal space of $L^\infty(X, \mu)$. The topological space Z_ν is necessarily compact, Hausdorff, and totally disconnected. Hence, if $\mathcal{H}$ has finite dimension d, then $L^\infty_{\mathcal{H}}(X, \nu)$ and $C(Z_\nu) \otimes \mathcal{B}(\mathcal{H})$ are isomorphic C*-algebras via the unital $*$-isomorphism $\Gamma_{\mathcal{H}} = \Gamma \otimes \mathrm{id}_{\mathcal{B}(\mathcal{H})} : L^\infty_{\mathcal{H}}(X, \nu) \to C(Z_\nu) \otimes \mathcal{B}(\mathcal{H})$. Because the map $\mathbb{E}_\nu \circ \Gamma^{-1}_{\mathcal{H}} : C(Z_\nu) \otimes \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is unital and completely positive [9, Theorem 3.5] and because the homomorphism $\Gamma^{-1}_{\mathcal{H}}$ is completely positive, the linear map $\mathbb{E}_\nu : L^\infty_{\mathcal{H}}(X, \nu) \to \mathcal{B}(\mathcal{H})$ is necessarily completely positive. □

Theorem 2.8 gives rise to the following operator inequality.

Corollary 2.9 *(Schwarz inequality). If the operators $h_1, \ldots, h_n \in \mathcal{B}(\mathcal{H})_+$ satisfy $h_1^2 + \cdots + h_n^2 = 1$, then for all $z_1, \ldots, z_n \in \mathcal{B}(\mathcal{H})$,*

$$\left(\sum_{j=1}^{n} h_j z_j h_j\right)^* \left(\sum_{j=1}^{n} h_j z_j h_j\right) \leq \sum_{j=1}^{n} h_j z_j^* z_j h_j. \tag{3}$$

Proof. Let $X = \{x_1, \ldots, x_n\}$, and let $\mathcal{F}(X)$ be the power set of X. If ν is the quantum probability measure for which $\nu(\{x_j\}) = h_j^2$, and if the quantum random variable $\psi : X \to \mathcal{B}(\mathcal{H})$ is defined by $\psi(x_j) = z_j$, for each $j = 1, \ldots, n$, then $\psi \in L^\infty_{\mathcal{H}}(X, \nu)$ and $\mathbb{E}_\nu[\psi] = h_1 z_1 h_1 + \cdots + h_n z_n h_n$ as in Example 2.7. By the Schwarz inequality for completely positive linear maps [15, Proposition 3.3], we have $\mathbb{E}_\nu[\psi]^* \mathbb{E}_\nu[\psi] \leq \mathbb{E}_\nu[\psi^*\psi]$, which is precisely inequality (3). □

3. The essential range of quantum random variables

Definition 3.1. Let $\psi : X \to \mathcal{B}(\mathcal{H})$ be a quantum random variable. The *essential range* of ψ is the set $\operatorname{ess-ran}\psi$ of all operators $\lambda \in \mathcal{B}(\mathcal{H})$ for which $\mu\left(\psi^{-1}(U)\right) > 0$, for every neighbourhood U of λ.

The essential range of $\psi : X \to \mathcal{B}(\mathcal{H})$ is closed and the μ-measure of the set $\{x \in X \mid \psi(x) \notin \operatorname{ess-ran}\psi\}$ is zero. Thus, if ψ_1 and ψ_2 determine the same element $\psi \in L^\infty_{\mathcal{H}}(X, \nu)$, then ψ_1 and ψ_2 have the same essential range. Our aim is to identify the essential range with certain spectral elements of ψ.

Definition 3.2. If $\mathcal{A}$ is a unital C*-algebra and $a \in \mathcal{A}$, let $\mathrm{C}^*(a)$ be the unital C*-subalgebra generated by a. For $d \in \mathbb{N}$, the set

$$\operatorname{Spec}^d(a) = \{\varrho(a) \mid \varrho : \mathrm{C}^*(a) \to M_d(\mathbb{C}) \text{ is a homomorphism}\}$$

is called the *Gelfand spectrum* of a.

Of course, for many elements a, it will be the case that $\operatorname{Spec}^d(a)$ is empty. A notable exception occurs with (essentially) bounded measurable functions $\psi : X \to \mathbb{C}$, in which case $\operatorname{Spec}^1(\psi)$, where ψ is considered to be an element of the abelian von Neumann algebra $L^\infty(X, \mu)$, coincides with the essential range of ψ [13]. However, the case of quantum random variables (see Theorem 3.4 below) requires the notion of hypoconvexity [18, Definition 1.6].

Definition 3.3. A nonempty compact subset $Q \subset \mathcal{B}(\mathcal{H})$ is *hypoconvex* if

1. $u^*\lambda u \in Q$ for every unitary $u \in \mathcal{B}(\mathcal{H})$ and $\lambda \in Q$, and
2. $\sum_{j=1}^{m} p_j \lambda_j \in Q$ for all $\lambda_1, \ldots, \lambda_m \in Q$ and projections $p_1, \ldots, p_m \in \mathcal{B}(\mathcal{H})$ satisfying $p_1 + \cdots + p_m = 1$ and $p_j \lambda_j = \lambda_j p_j$ for each j.

If $Q \subset \mathcal{B}(\mathcal{H})$ is an arbitrary nonempty compact set, then $Q^\sim$ denotes the *hypoconvex hull* of Q, namely, the smallest hypoconvex set that contains Q.

If $\mathcal{H} = \mathbb{C}$, then the two conditions above for the hypoconvexity of a compact set Q are trivially satisfied. Hence, the notion of hypoconvex set is distinguished from compactness only at dimension $d = 2$ and higher.

Theorem 3.4. *The following two statements are equivalent for a Hilbert space $\mathcal{H}$ of dimension d and a quantum random variable $\psi \in L^\infty_{\mathcal{H}}(X, \nu)$.*

1. $\lambda \in \mathrm{Spec}^d(\psi)$.
2. *There exists a unitary* $v : \mathbb{C}^d \to \mathcal{H}$ *such that* $v\lambda v^{-1} \in (\text{ess-ran}\,\psi)^\sim$.

Proof. Consider the isomorphism $\Gamma_{\mathcal{H}} : L^\infty_{\mathcal{H}}(X,\nu) \to C(Z_\nu) \otimes \mathcal{B}(\mathcal{H})$ defined in the proof of Theorem 2.8, and fix a computational basis $\mathfrak{C} = \{e_1, \ldots, e_d\}$ of $\mathcal{H}$. Let $\pi_{\mathfrak{C}} : \mathcal{B}(\mathcal{H}) \to M_d(\mathbb{C})$ be the isomorphism that sends each rank-1 operator $e_i \otimes e_j \in \mathcal{B}(\mathcal{H})$ to the canonical matrix unit E_{ij} of $M_d(\mathbb{C})$. Thus, $\rho_{\mathfrak{C}} = \mathrm{id}_{C(Z_\nu)} \otimes \pi_{\mathfrak{C}}$ is an isomorphism of $C(Z_\nu) \otimes \mathcal{B}(\mathcal{H})$ and $C(Z_\nu) \otimes M_d(\mathbb{C})$ that maps each $f \in C(Z_\nu) \otimes \mathcal{B}(\mathcal{H})$ to a $d \times d$ matrix $\rho_{\mathfrak{C}}(f) = [f_{ij}]_{i,j}$ of continuous maps $f_{ij} : Z_\nu \to \mathbb{C}$.

Suppose now that $\psi \in L^\infty_{\mathcal{H}}(X,\nu)$ and let $f = \Gamma_{\mathcal{H}}(\psi)$ and $[f_{ij}]_{i,j} = \rho_{\mathfrak{C}}(f) = \rho_{\mathfrak{C}} \circ \Gamma_{\mathcal{H}}(\psi)$. Define a subset $\Omega_{\psi,\mathfrak{C}} \subset M_d(\mathbb{C})$ by setting

$$\Omega_{\psi,\mathfrak{C}} \;=\; \left\{ [\omega(f_{ij})]_{i,j=1}^d \in M_d(\mathbb{C}) \,|\, \omega : \mathrm{C}^*\left(\{f_{ij}\}_{i,j}\right) \to \mathbb{C} \text{ is a homomorphism} \right\}.$$

Note that by restricting the domain of a homomorphism $\omega : C(Z_\nu) \to \mathbb{C}$ to the unital C*-subalgebra $\mathrm{C}^*\left(\{f_{ij}\}_{i,j}\right)$ of $C(Z_\nu)$ we obtain an inclusion $\Delta_{\psi,\mathfrak{C}} \subseteq \Omega_{\psi,\mathfrak{C}}$, where

$$\Delta_{\psi,\mathfrak{C}} \;=\; \left\{ [\omega(f_{ij})]_{i,j=1}^d \in M_d(\mathbb{C}) \,|\, \omega : C(Z_\nu) \to \mathbb{C} \text{ is a homomorphism} \right\}.$$

On the other hand, because every homomorphism $\omega_0 : \mathrm{C}^*\left(\{f_{ij}\}_{i,j}\right) \to \mathbb{C}$ is, by the fact that $\mathbb{C}$ is 1-dimensional, irreducible, there is a homomorphism $\omega : C(Z_\nu) \to \mathbb{C}$ such that $\omega_{|\mathrm{C}^*(\{f_{ij}\}_{i,j})} = \omega_0$. Hence, $\Delta_{\psi,\mathfrak{C}} = \Omega_{\psi,\mathfrak{C}}$. Because homomorphisms $C(Z_\nu) \to \mathbb{C}$ are point evaluations $g \mapsto g(z_0)$ for $z_0 \in Z_\nu$, the set $\Omega_{\psi,\mathfrak{C}}$ is the range of the matrix-valued function $z \mapsto [f_{ij}(z)]_{i,j=1}^d$, which we have denoted by $\rho_{\mathfrak{C}} \circ \Gamma_{\mathcal{H}}(\psi)$. Thus, $\Omega^\sim_{\psi,\mathfrak{C}} = \mathrm{Spec}^d\left(\rho_{\mathfrak{C}} \circ \Gamma_{\mathcal{H}}(\psi)\right)$ [18, Lemma 2.3]. The Gelfand spectrum is an isomorphism invariant; hence, $\mathrm{Spec}^d(\psi) = \Omega^\sim_{\psi,\mathfrak{C}}$.

Now let $v_{\mathfrak{C}} : \mathbb{C}^d \to \mathcal{H}$ be the unitary operator that sends the j-th coordinate vector of $\mathbb{C}^d$ to the unit vector $e_j \in \mathcal{H}$ in the computational basis $\mathfrak{C} = \{e_1, \ldots, e_d\}$ of $\mathcal{H}$. If $\lambda = [\lambda_{ij}]_{i,j} \in \Omega_{\psi,\mathfrak{C}}$, then there is a $z_0 \in Z_\nu$ such that $\lambda_{ij} = f_{ij}(z_0) = \langle f(z_0)e_j, e_i\rangle$. Hence, $v_{\mathfrak{C}}\lambda v_{\mathfrak{C}}^{-1}$ is an element of the range of $f = \Gamma_{\mathcal{H}}(\nu)$, which coincides with the essential range of ψ. As the map $s \mapsto v_{\mathfrak{C}} s v_{\mathfrak{C}}^{-1}$ is an automorphism of $\mathcal{B}(\mathcal{H})$, it is also true that if $\lambda \in \Omega^\sim_{\psi,\mathfrak{C}}$, then $v_{\mathfrak{C}}\lambda v_{\mathfrak{C}}^{-1}$ is an element of the hypoconvex hull of the range of f. This completes the proof that $\lambda \in \mathrm{Spec}^d(\psi)$ only if there exists a unitary operator $v : \mathbb{C}^d \to \mathcal{H}$ such that $v\lambda v^{-1} \in (\text{ess-ran}\,\psi)^\sim$.

Conversely, for each choice of computational basis $\mathfrak{C} = \{e_1, \ldots, e_d\}$ of $\mathcal{H}$ there is an isometry $v_{\mathfrak{C}} : \mathbb{C}^d \to \mathcal{H}$ that sends the j-th coordinate vector of $\mathbb{C}^d$ to the unit vector $e_j \in \mathcal{H}$. Hence, if $v_{\mathfrak{C}}\lambda v_{\mathfrak{C}}^{-1} \in (\text{ess-ran}\,\psi)^\sim$, then $\lambda \in \mathrm{Spec}^d(\psi)$. □

The expectation $\mathbb{E}_\nu[\psi]$ of ψ is just one of many operators $\varphi(\psi) \in \mathcal{B}(\mathcal{H})$ obtained by evaluating ψ at a ucp map $\varphi : L^\infty_{\mathcal{H}}(X,\nu) \to \mathcal{B}(\mathcal{H})$; that is,

$$\mathbb{E}_\nu[\psi] \in \{\Phi(\psi) \,|\, \Phi : L^\infty_{\mathcal{H}}(X,\nu) \to \mathcal{B}(\mathcal{H}) \text{ is a ucp map}\}.$$

Theorem 3.5 below clarifies the relationship between operators of this type and the essential range of ψ.

Theorem 3.5. *If* $\varphi : L^\infty_{\mathcal{H}}(X,\nu) \to \mathcal{B}(\mathcal{H})$ *is a ucp map, then for every* $\psi \in L^\infty_{\mathcal{H}}(X,\nu)$ *there exist* $x_1, \ldots, x_m \in X$ *(not necessarily distinct) and* $t_1, \ldots, t_m \in \mathcal{B}(\mathcal{H})$ *such that*

$$\varphi(\psi) \;=\; \sum_{j=1}^m t_j^* \psi(x_j) t_j \qquad \textit{and} \qquad \sum_{j=1}^m t_j^* t_j = 1 \in \mathcal{B}(\mathcal{H}).$$

Proof. Because $L^\infty_{\mathcal{H}}(X,\nu)$ is isomorphic to $L^\infty(X,\mu) \otimes M_d(\mathbb{C})$, every $\psi \in L^\infty_{\mathcal{H}}(X,\nu)$ may be represented as a $d \times d$ matrix whose entries are taken from an abelian C*-algebra; that is, ψ is d-normal. Select an

orthonormal basis $\{e_1, \ldots, e_d\}$ of $\mathcal{H}$ and consider the $d \times d$ matrix $\omega = [\langle \varphi(\psi)e_j, e_i\rangle]_{i,j=1}^d$. Thus, ω is an element of the matricial range $W^d(\psi)$, namely the set of all $d \times d$ matrices of the form $\Phi(\psi)$ for some ucp map $\Phi : L^\infty_{\mathcal{H}}(X, \nu) \to M_d(\mathbb{C})$. By [3, Theorem 3.9], there are $\lambda_1, \ldots, \lambda_q \in \mathrm{Spec}^d(\psi)$ (not necessarily distinct) and $s_1, \ldots, s_q \in M_d(\mathbb{C})$ such that $\omega = \sum_j s_j^* \lambda_j s_j$ and $\sum_j s_j^* s_j = 1$. By Theorem 3.4, there exist unitaries $v_j : \mathbb{C}^d \to \mathcal{H}$ such that $v_j \lambda_j v_j^* \in (\text{ess-ran}\, \psi)^\sim$. For each $j = 1, \ldots, q$ there are $x_1^{(j)}, \ldots, x_{n_j}^{(j)} \in X$ and pairwise-orthogonal projections $p_1^{(j)}, \ldots, p_{n_j}^{(j)} \in \mathcal{B}(\mathcal{H})$ such that each $p_i^{(j)}$ commutes with $\psi(x_i^{(j)})$ and $v_j \lambda_j v_j^* = \sum_{i=1}^{n_j} \psi(x_i^{(j)}) p_i^{(j)}$. Let $u : \mathbb{C}^d \to \mathcal{H}$ be the unitary for which $u^* z u = [\langle z e_j, e_i\rangle]_{i,j}$ for all $z \in \mathcal{B}(\mathcal{H})$. In particular, $\omega = u^* \varphi(\psi) u$ and each $s_j = u^* r_j u$ for a unique $r_j \in \mathcal{B}(\mathcal{H})$. Thus,

$$\begin{aligned} \varphi(\psi) = \sum_{j=1}^{q} \sum_{i=1}^{n_j} u s_j^* \lambda_j s_j u^* &= \sum_{j=1}^{q} \sum_{i=1}^{n_j} (p_i^{(j)} u^* r_j)^* \psi(x_i^{(j)}) (p_i^{(j)} u^* r_j) \\ &= \sum_{\ell=1}^{m} t_\ell^* \psi(x_\ell) t_\ell, \end{aligned}$$

where $\{t_\ell\}_\ell = \{p_i^{(j)} u^* r_j\}_{i,j}$ and $\{x_\ell\}_\ell = \{x_i^{(j)}\}_{i,j}$ are relabellings and renumberings of the operators and points in the decomposition of $\varphi(\psi)$ above. □

Recall that a subset $K \subseteq \mathcal{B}(\mathcal{H})$ is *C*-convex* if $\sum_{j=1}^m t_j^* z_j t_j \in K$, for every $z_1, \ldots, z_m \in K$ and $t_1, \ldots, t_m \in \mathcal{B}(\mathcal{H})$ with $\sum_{j=1}^m t_j^* t_j = 1 \in \mathcal{B}(\mathcal{H})$. If $S \subset \mathcal{B}(\mathcal{H})$ is a nonempty set, then $\mathrm{C}^*\mathrm{conv}(S)$ is the smallest C*-convex set that contains S. Thus, Theorem 3.5 leads immediately to the following corollary which is a generalisation of [8, Theorem 2.3(8)].

Corollary 3.6. *If $\psi \in L^\infty_{\mathcal{H}}(X, \nu)$, then*

$$\{\Phi(\psi) \mid \Phi : L^\infty_{\mathcal{H}}(X, \nu) \to \mathcal{B}(\mathcal{H}) \text{ is a ucp map}\} = \mathrm{C}^*\mathrm{conv}(\text{ess-ran}\, \psi).$$

In particular, $\mathbb{E}_\nu[\psi] \in \mathrm{C}^*\mathrm{conv}(\text{ess-ran}\, \psi)$.

4. Quantum variance

Definition 4.1. If $\psi \in L^\infty_{\mathcal{H}}(X, \nu)$ is a quantum random variable, then

1. the *left variance* of ψ with respect to ν is the operator
$$\mathrm{Var}_{\ell\nu}[\psi] = \mathbb{E}_\nu[\psi^* \psi] - \mathbb{E}_\nu[\psi^*]\, \mathbb{E}_\nu[\psi],$$
2. the *right variance* of ψ with respect to ν is the operator
$$\mathrm{Var}_{r\nu}[\psi] = \mathbb{E}_\nu[\psi \psi^*] - \mathbb{E}_\nu[\psi]\, \mathbb{E}_\nu[\psi^*],$$

and

3. the *variance* of ψ with respect to ν is the operator
$$\mathrm{Var}_\nu[\psi] = \frac{1}{2}\left(\mathrm{Var}_{\ell\nu}[\psi] + \mathrm{Var}_{r\nu}[\psi]\right).$$

The Schwarz inequality ensures that all three of the variances defined above are positive operators. However, this occurrence of positivity is a consequence of the fact that ψ is essentially bounded and that $L^\infty_{\mathcal{H}}(X,\nu)$ is a von Neumann algebra. In contrast, variance is defined classically for square-integrable random variables rather than essentially bounded random variables, as it is a result of Chebyshev's inequality that square-integrable random variables are necessarily integrable (i.e., $L^2 \subset L^1$). To similarly define the variance of an arbitrary quantum random variable ψ, it is necessary to fulfil the *second moment condition* that $\psi^*\psi$ be ν-integrable. The obvious question is whether or not the second moment condition implies that ψ is itself ν-integrable; the theorem below answers this question affirmatively.

Theorem 4.2. *Suppose that $\psi : X \to \mathcal{B}(\mathcal{H})$ is a quantum random variable. If $\psi^*\psi$ is ν-integrable, then ψ is ν-integrable.*

Proof. Let ρ be a density operator and consider the functions $(\psi^*\psi)_\rho$ and ψ_ρ on X defined by

$$(\psi^*\psi)_\rho(x) \;=\; \operatorname{Tr}\left(\rho^{1/2}\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2}[\psi^*\psi(x)]\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2}\rho^{1/2}\right), \text{ and}$$

$$\psi_\rho(x) \;=\; \operatorname{Tr}\left(\rho^{1/2}\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2}\psi(x)\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2}\rho^{1/2}\right),$$

which coincide with Definition 2.5 using elementary properties of the trace. Let us also define, using the constant function $\iota(x) = 1 \in \mathcal{B}(\mathcal{H})$, the scalar-valued function

$$\iota_\rho(x) \;=\; \operatorname{Tr}\left(\rho\,\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right).$$

Note that ι_ρ is μ-integrable. To complete the proof we shall require the following two tracial inequalities for arbitrary $y, z \in \mathcal{B}(\mathcal{H})$:

1. ([21, Theorem 1]) $|\operatorname{Tr}(y)| \leq \operatorname{Tr}(|y|)$, and
2. ([1, Theorem 1]) $\operatorname{Tr}(|yz^*|) \leq \frac{1}{2}\operatorname{Tr}(|y|^2) + \frac{1}{2}\operatorname{Tr}(|z|^2)$.

Consider the function $w : X \to \mathcal{B}(\mathcal{H})$ defined by $w(x) = \left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2}\rho^{1/2}$. The two tracial inequalities above imply that

$$\begin{aligned}
|\psi_\rho(x)| &= |\operatorname{Tr}(w(x)^*\psi(x)w(x))| = |\operatorname{Tr}([\psi(x)w(x)]w^*(x))| \\
&\leq \operatorname{Tr}(|[\psi(x)w(x)]w^*(x)|) \leq \frac{1}{2}\operatorname{Tr}(|\psi(x)w(x)|^2) + \frac{1}{2}\operatorname{Tr}(|w(x)|^2) \\
&= \frac{1}{2}\operatorname{Tr}(w(x)^*\psi(x)^*\psi(x)w(x)) + \frac{1}{2}\operatorname{Tr}(w^*(x)w(x)) = \frac{1}{2}\psi^*\psi_\rho(x) + \frac{1}{2}\iota_\rho(x).
\end{aligned}$$

Thus, $|\psi_\rho|$ is bounded above by the average of the two nonnegative μ-integrable functions $(\psi^*\psi)_\rho$ and ι_ρ. Hence, $\psi_\rho \in L^1(X,\mu)$. As this is true for every density operator ρ, we deduce that ψ is ν-integrable. □

Corollary 4.3. *The three variances in Definition 4.1 can be defined for quantum random variables ψ for which $\psi^*\psi$ is ν-integrable.*

Notwithstanding the extension of the variance domains as indicated in Corollary 4.3, it is not necessarily true that the left or right variance is positive. In other words, there is no natural analogue of the Schwarz inequality from essentially bounded quantum random variables to square-integrable quantum random variables.

We now turn our attention to essentially bounded quantum random variables having variance zero. Although random variables having variance zero are trivially constant in the classical case, we will show that a much richer structure exists for quantum random variables having variance zero.

One family of quantum random variables that have variance zero is the following. For $z \in \mathcal{B}(\mathcal{H})$, let $\psi_z : X \to \mathcal{B}(\mathcal{H})$ denote the constant function defined by $\psi_z(x) = z$ for every $x \in X$. Because of the noncommutativity of operator algebra, quantum averaging of z, by way of $z \mapsto \mathbb{E}_\nu[\psi_z]$, may in fact alter z. This phenomenon was observed in [8, Theorem 2.3(8)], where it was shown that, in general, one only has $\mathbb{E}_\nu[\psi_z] \in \mathrm{C}^*\mathrm{conv}(\{z\})$. However, if $\mathbb{E}_\nu[\psi_z] = z$, namely if quantum averaging does not disturb z, then the variance of ψ_z is zero; this is the immediate analogue of the fact that scalars (i.e., constant random variables) have variance zero.

Proposition 4.4. *If* $\mathbb{E}_\nu[\psi_z] = z$, *then* $\mathrm{Var}_\nu[\psi_z] = 0$.

Proof. By [8, Theorem 2.3(8)], the set of all $y \in \mathcal{B}(\mathcal{H})$ for which $\mathbb{E}_\nu[\psi_y] = y$ is a unital C*-subalgebra of $\mathcal{B}(\mathcal{H})$. Hence, $\mathbb{E}_\nu[\psi_z] = z$ implies that $\mathrm{Var}_{\ell\nu}[\psi] = \mathrm{Var}_{r\nu}[\psi] = 0$. □

The following result is a concise spectral characterisation of variance zero in the case of essentially bounded quantum random variables.

Theorem 4.5. *The following two statements are equivalent for the quantum random variable* $\psi \in L^\infty_{\mathcal{H}}(X, \nu)$.

1. $\mathrm{Var}_\nu[\psi] = 0$.
2. *There exist a unitary* $u : \mathcal{H} \to \mathbb{C}^d$ *and a* $\lambda \in \mathrm{Spec}^d(\psi)$ *such that* $u^*\lambda u = \mathbb{E}_\nu[\psi]$.

Proof. The condition that $\mathrm{Var}_\nu[\psi] = 0$ is equivalent to the two equations $\mathbb{E}_\nu[\psi^*\psi] = \mathbb{E}_\nu[\psi]^*\,\mathbb{E}_\nu[\psi]$ and $\mathbb{E}_\nu[\psi\psi^*] = \mathbb{E}_\nu[\psi]\,\mathbb{E}_\nu[\psi]^*$ holding simultaneously, which in turn is equivalent to ψ belonging to the multiplicative domain of the ucp map $\mathbb{E}_\nu$ [15, Theorem 3.18]. Because the multiplicative domain of $\mathbb{E}_\nu$ is a unital C*-subalgebra of $L^\infty_{\mathcal{H}}(X, \nu)$ and contains ψ, the restriction of $\mathbb{E}_\nu$ to $\mathrm{C}^*(\psi)$ is a homomorphism. Thus, by selecting an orthonormal basis $\{\phi_1, \ldots, \phi_d\}$ of $\mathcal{H}$ and in letting $u : \mathcal{H} \to \mathbb{C}^d$ be the unitary operator that sends each ϕ_j to e_j, we have that $\lambda = u^*\mathbb{E}_\nu[\psi]\,u$ is an element of $\mathrm{Spec}^d(\psi)$. Conversely, if there exist $\lambda \in \mathrm{Spec}^d(\psi)$ and a unitary $u : \mathcal{H} \to \mathbb{C}^d$ such that $u^*\lambda u = \mathbb{E}_\nu[\psi]$, then the restriction of $\mathbb{E}_\nu$ to $\mathrm{C}^*(\psi)$ is a homomorphism and so $\mathrm{Var}_\nu[\psi] = 0$. □

5. The quantum moment problem

The classical Hamburger moment problem, named after the German mathematician Hans Ludwig Hamburger, is as follows. Suppose that $\{g_k\}_{k\in\mathbb{N}}$ is a sequence of real numbers. Does there exist a positive Borel measure μ on the real line such that

$$g_k = \int_{-\infty}^{\infty} x^k \,\mathrm{d}\mu \,? \tag{4}$$

This problem, as well as many variants of it, has been extensively studied for almost a century. If $\{g_k\}_{k\in\mathbb{N}}$ is given, then we say that μ is a *solution* to the moment problem for $\{g_k\}_{k\in\mathbb{N}}$ if (4) holds.

A condition for a unique solution to one variant of the moment problem that is very well known in classical probability is found in [2, Theorem 30.1], namely if $g = \{g_k\}_{k\in\mathbb{N}}$ is given and satisfies

$$P_g(t) = \sum_{k=0}^{\infty} \frac{g_k t^k}{k!} < \infty \tag{5}$$

for all $t \in \mathbb{R}$, then there is a unique probability measure $\mathbf{P}$ on $\mathbb{R}$ that is the solution to the moment problem for $\{g_k\}_{k\in\mathbb{N}}$. In the case that $\{g_k\}_{k\in\mathbb{N}}$ is a *multiplicative moment sequence*, meaning that $g_k = (g_1)^k$ for all k for some $g_1 \in \mathbb{R}$, the unique solution to the moment problem is trivial. That is, (5) implies the solution to the moment problem is unique since $P_g(t) = e^{g_1 t} < \infty$ for all $t \in \mathbb{R}$. If $\mathbf{P}$ denotes the probability measure on $\mathbb{R}$ supported on g_1 and Y is a random variable with $\mathbf{P}\{Y = g_1\} = 1$, then $\int_{\mathbb{R}} Y^k \,\mathrm{d}\mathbf{P} = (g_1)^k$. Thus, only constant random variables have multiplicative moment sequences.

Suppose now that $\{g_k\}_{k\in\mathbb{N}}$ is a sequence in $\mathcal{B}(\mathcal{H})$. We say that a quantum probability measure ν on the Borel sets of $\mathcal{B}(\mathcal{H})$ is a solution to the quantum moment problem for $\{g_k\}_{k\in\mathbb{N}}$ if there exists a Borel subset $X \subseteq \mathcal{B}(\mathcal{H})$ and a quantum random variable $\psi : X \to \mathcal{B}(\mathcal{H})$ such that

$$\mathbb{E}_\nu\left[\psi^k\right] = \int_X \psi^k \,\mathrm{d}\nu = g_k$$

for all $k = 0, 1, 2, \ldots$. A natural question to ask is if we can develop an operator-theoretic criterion to determine when the quantum moment problem for a multiplicative moment sequence has only a trivial solution.

By Stinespring's dilation theorem for unital completely positive linear maps [15, Theorem 4.1], we deduce that for every quantum probability measure ν, there exist a Hilbert space $\mathcal{K}_\nu$, an isometry $v : \mathcal{H} \to \mathcal{K}_\nu$, and a homomorphism $\Delta_\nu : L^\infty_{\mathcal{H}}(X, \nu) \to \mathcal{B}(\mathcal{K}_\nu)$ such that

1. $\mathbb{E}_\nu[\psi] = v^*\Delta_\nu(\psi)v$ for every $\psi \in L^\infty_{\mathcal{H}}(X, \nu)$, and
2. $\mathrm{Span}\,\{\Delta_\nu(\psi)v\xi \,|\, \psi \in L^\infty_{\mathcal{H}}(X, \nu),\ \xi \in \mathcal{H}\}$ is dense in $\mathcal{K}_\nu$.

The two conditions above determine the triple $(\mathcal{K}_\nu, \Delta_\nu, v)$ up to unitary equivalence [15, Proposition 4.2], and so we may refer unambiguously to the triple $(\mathcal{K}_\nu, \Delta_\nu, v)$ as the minimal Stinespring dilation of the quantum expectation $\mathbb{E}_\nu$. (Here, "minimal" is in reference to the second condition, which is to say that the Hilbert space $\mathcal{K}_\nu$ is no larger than it needs to be.)

A second operator-theoretic concept that we will employ is that of a semi-invariant subspace. A subspace $\mathcal{M}$ of a Hilbert space $\mathcal{K}$ is said to be semi-invariant for an operator $z \in \mathcal{B}(\mathcal{K})$ if $\mathcal{M} = \mathcal{L}_0^\perp \cap \mathcal{L}_1$ for some z-invariant subspaces $\mathcal{L}_0$ and $\mathcal{L}_1$.

Theorem 5.1. *Let ν be a quantum probability measure, and let $\psi \in L^\infty_{\mathcal{H}}(X, \nu)$. Assume that $\{g_k\}_{k\in\mathbb{N}} \subset \mathcal{B}(\mathcal{H})$ is a sequence of operators with $g_k = \mathbb{E}_\nu\left[\psi^k\right]$, and let $(\mathcal{K}_\nu, \Delta_\nu, v)$ be a minimal Stinespring dilation of the quantum expectation $\mathbb{E}_\nu$. Then the following two statements are equivalent.*

1. *$g_k = (g_1)^k$ for every $k \in \mathbb{N}$.*
2. *$v(\mathcal{H})$ is a semi-invariant subspace for $\Delta_\nu(\psi)$.*

Proof. By a result of Sarason [19, Lemma 0], a subspace $\mathcal{M}$ of $\mathcal{K}_\nu$ is semi-invariant for $\Delta_\nu(\psi)$ if and only if $p\Delta_\nu(\psi)^k{}_{|\mathcal{M}} = \left(p\Delta_\nu(\psi)_{|\mathcal{M}}\right)^k$ for every $k \in \mathbb{N}$, where $p \in \mathcal{B}(\mathcal{K}_\nu)$ is the projection with range $\mathcal{M}$. In the case at hand, the dimension of $\mathcal{M}$ is necessarily d. Further, any projection $p \in \mathcal{B}(\mathcal{K}_\nu)$ of rank d can be factored as $p = vv^*$ for some isometry $v : \mathcal{H} \to \mathcal{H}_\nu$ and, conversely, for every isometry $v : \mathcal{H} \to \mathcal{H}_\nu$ the operator vv^* is

a projection of rank d. Because Δ_ν is a homomorphism, Sarason's criterion is, for an isometry $v : \mathcal{H} \to \mathcal{H}_\nu$, equivalent to: $v(\mathcal{H})$ is semi-invariant for $\Delta_\nu(\psi)$ if and only if $v^*\Delta_\nu(\psi^k)v = (v^*\Delta_\nu(\psi)v)^k$ for every $k \in \mathbb{N}$. Thus, because $g_k = v^*\Delta_\nu(\psi^k)v$ for all $k \in \mathbb{N}$, the proof of the theorem is complete. □

In order to state our final result, we recall the following definition [3, Definition 5.1].

Definition 5.2. Assume that $\mathcal{A}$ is a unital C*-algebra and that $k \in \mathbb{N}$. The $k \times k$ *matricial spectrum* of $a \in \mathcal{A}$ is the subset $\sigma^k(a) \subset M_k(\mathbb{C})$ defined by

$$\sigma^k(a) \;=\; \big\{\varphi(a) \,|\, \varphi : \mathcal{A} \to M_k(\mathbb{C}) \text{ is ucp and } \varphi_{|\mathcal{R}(a)} \text{ is a homomorphism}\big\}\,,$$

where $\mathcal{R}(a)$ is the rational Banach subalgebra of $\mathcal{A}$ generated by a.

To be more precise, the algebra $\mathcal{R}(a)$ is the norm-closure of the abelian algebra of all elements of the form $p(a)q(a)^{-1}$, where p and q are complex polynomials such that q has no roots in the spectrum $\sigma(a)$ of a.

If one considers the classical $d = 1$ case, then every point λ in the spectrum of ψ gives rise to a measure μ for which $\lambda^k = \int_X \psi^k \, \mathrm{d}\mu$ for every $k \in \mathbb{N}$. Indeed, this measure μ is a point-mass measure corresponding to a point evaluation of ψ that yields the complex number λ (which is basically the situation described at the end of the second paragraph of this section). The matter is simplified somewhat by the fact that $\sigma^1(\psi) = \mathrm{Spec}^1(\psi)$ if ψ is a classical random variable. However, in higher dimensions, the matricial spectrum $\sigma^d(\psi)$ is generally much larger than the Gelfand spectrum $\mathrm{Spec}^d(\psi)$ and, consequently, the quantum moment problem for multiplicative moment sequences entails certain obstructions not seen at the classical level. Our final result, Theorem 5.3, illustrates the obstruction in that it demonstrates that a correction by a unitary quantum random variable is necessary prior to integration. The underlying complicating factor is that an analogue of the Riesz Representation Theorem holds only for a certain subset of ucp maps $L^\infty_{\mathcal{H}}(X, \nu)$ [9, Corollary 4.5].

If $(X, \mathcal{F}(X))$ is the Borel space of a compact metric space X, and if $\psi : X \to M_d(\mathbb{C})$ is continuous, then below we consider ψ as an element of the unital C*-algebra $C(X) \otimes M_d(\mathbb{C})$, and the matricial spectra of ψ are defined relative to this choice of C*-algebra.

Theorem 5.3. *If $\psi : X \to M_d(\mathbb{C})$ is a continuous quantum random variable on a compact metric space X, and if $\lambda \in \sigma^d(\psi)$, then there exist sequences $\{\nu_n\}_{n\in\mathbb{N}}$ and $\{w_n\}_{n\in\mathbb{N}}$ of quantum probability measures and quantum random variables, respectively, such that*

1. *$w_n(x)$ is unitary for all $x \in X$ and every $n \in \mathbb{N}$, and*
2. *$\lim_{n\to\infty} \big\|\lambda^k - \mathbb{E}_{\nu_n}\left[w_n^*\psi^k w_n\right]\big\| = 0$ for every $k \in \mathbb{N}$.*

Proof. By hypothesis there is a ucp map $\vartheta : C(X) \otimes M_d(\mathbb{C}) \to M_d(\mathbb{C})$ such that the restriction of ϑ to the rational algebra $\mathcal{R}(\psi)$ is a homomorphism. Let $v^*\Delta v$ be a minimal Stinespring representation of ϑ. Because X is a metric space, the C*-algebra $C(X) \otimes M_d(\mathbb{C})$ is separable; hence, the minimal Stinespring dilation Δ of ϑ takes place on a representing Hilbert space $\mathcal{H}_\Delta$ that is separable. Hence, by Voiculescu's Theorem [7, Corollary II.5.9], Δ is approximately unitarily equivalent to a direct sum $\tilde{\Delta}$ of a countable family of irreducible representations of $C(X) \otimes M_d(\mathbb{C})$. Because $C(X) \otimes M_d(\mathbb{C})$ is homogeneous, every irreducible representation of it is a point evaluation. Thus, there are a countable subset $X_1 \subseteq X$, a separable Hilbert

space $\mathcal{H}_{\tilde{\Delta}} = \bigoplus_{x\in X_1} \mathbb{C}^d_x$, where $\mathbb{C}^d_x = \mathbb{C}^d$ for each $x \in X_1$, and a sequence $\{u_n\}_{n\in\mathbb{N}}$ of unitary operators $u_n : \mathcal{H}_\Delta \to \mathcal{H}_{\tilde{\Delta}}$ such that, for every $f \in C(X) \otimes M_d(\mathbb{C})$,

$$\tilde{\Delta}(f) = \bigoplus_{x\in X_1} f(x) \quad \text{and} \quad \lim_{n\to\infty} \|\Delta(f) - u_n^* \tilde{\Delta}(f) u_n\| = 0.$$

Hence,

$$\lim_{n\to\infty} \|\vartheta(f) - (u_n v)^* \tilde{\Delta}(f)(u_n v)\| = 0$$

for every $f \in C(X) \otimes M_d(\mathbb{C})$.

The isometry $u_n v : \mathbb{C}^d \to \bigoplus_{x\in X_1} \mathbb{C}^d_x$ acts as $u_n v\xi = \bigoplus_{x\in X_1} v_{n,x}\xi$ for $\xi \in \mathbb{C}^d$, for some $v_{n,x} \in M_d(\mathbb{C})$ and has the property that $\sum_{x\in X_1} v_{n,x}^* v_{n,x} = 1$ for each n. By the Polar Decomposition, there is a unitary operator $w_{n,x} \in \mathcal{B}(\mathbb{C}^d_x) = M_d(\mathbb{C})$ such that $v_{n,x} = w_{n,x}|v_{n,x}|$. Now define $w_n : X \to M_d(\mathbb{C})$ by $w_n(x) = w_{n,x}$, if $x \in X_1$, and $w_n(x) = 1$ if $x \notin X_1$. Because point sets are closed in a Hausdorff space, each w_n is a measurable function. By defining $\nu_n : \mathcal{F}(X) \to M_d(\mathbb{C})$ by

$$\nu_n = \sum_{x\in X_1} \delta_{\{x\}} v_{n,x}^* v_{n,x},$$

where $\delta_{\{x\}}$ is a classical point-mass measure concentrated at $\{x\}$, we see that ν_n is a quantum probability measure such that $(u_n v)^* \tilde{\Delta}(f)(u_n v) = \mathbb{E}_{\nu_n}[w_n^* f w_n]$ for every $f \in C(X) \otimes M_d(\mathbb{C})$. Thus, using the fact that ϑ is a homomorphism on $\mathcal{R}(\psi)$, we have that

$$\lambda^k = \vartheta(\psi)^k = \vartheta(\psi^k) = \lim_{n\to\infty} \mathbb{E}_{\nu_n}\left[w_n^* \psi^k w_n\right]$$

for every $k \in \mathbb{N}$. □

In the following special case, one can dispense with the sequences $\{\nu_n\}_n$ and $\{w_n\}_n$, and the quantum moment problem for multiplicative moment sequences is solved exactly rather than asymptotically.

Corollary 5.4. *If $\psi : X \to M_d(\mathbb{C})$ is a quantum random variable on $X = \{x_1, \ldots, x_n\}$ and if $\lambda \in \sigma^d(\psi)$, then there exist a quantum probability measure ν on $(X, \mathcal{F}(X))$, where $\mathcal{F}(X)$ is the power set of X, and a unitary-valued quantum random variable $w : X \to M_d(\mathbb{C})$ such that $\lambda^k = \mathbb{E}_\nu\left[w^* \psi^k w\right]$ for every $k \in \mathbb{N}$.*

Proof. The C*-algebra $C(X) \otimes M_d(\mathbb{C})$ has, in this case, finite dimension. Therefore, the minimal Stinespring dilation of every ucp map on $C(X) \otimes M_d(\mathbb{C})$ has a representing Hilbert space of finite dimension. Hence, in the proof of Theorem 5.3, the representations Δ and $\tilde{\Delta}$ may be assumed to be equal. □

6. Measures of quantum noise

Our focus to this point has been with purely mathematical issues. However, the probability measures that we have studied herein feature prominently in the theory of quantum measurement. In this regard, the variance of a quantum random variable has a particularly crucial role.

To explain briefly the physical context, assume that a d-dimensional Hilbert $\mathcal{H}$ is used to model (the states of) some physical quantum system. The states of the quantum system are represented by density operators ρ acting on $\mathcal{H}$. The system will have various physical properties; those properties of the system that can actually be measured using some experimental apparatus or device are called observable properties.

In the mathematical formulation of quantum theory, an observable property is represented by a hermitian operator, while an experimental apparatus is represented by a quantum probability measure ν on $(X, \mathcal{F}(X))$, where X is the sample space of possible outcomes of the measurement and $\mathcal{F}(X)$ is a σ-algebra of events. Therefore, in practice, X is a finite set and $\mathcal{F}(X)$ is the power set of X. Our assumptions here about X are a little more general: namely, that X is a compact Hausdorff space and that $\mathcal{F}(X)$ contains the Borel sets of X. The statistical element of quantum measurement is realised by the following axiom: if, at the moment of the measurement, the system is in state ρ, then the probability that event $E \in \mathcal{F}(X)$ will be measured is $\mathrm{Tr}\,(\rho\nu(E))$.

The observable properties of a system associated with a particular quantum measurement ν will, in general, intermingle information about the system with random disturbances coming from the measuring apparatus. These random disturbances are called quantum noise of ν. (The physics of quantum noise is treated in [6], for example.) To quantify the amount of quantum noise present in a quantum mechanical measurement, various numerical measures of quantum noise have been introduced (see, for example, [5,16, 17]). Two forms of quantum noise—random noise and inherent noise—have been investigated recently by Polterovich [17]. In Polterovich's approach, the "quantum" aspect is captured by a certain scalar-valued measurable function (specifically, a Markov kernel), which is integrated with respect to a POVM to produce a Hilbert space operator whose norm is used to determine a numerical indicator of the amount of quantum noise present. We outline below how a similar process is carried out using operator-valued measurable functions; we adopt, as much as possible, the notation of Polterovich.

Suppose that ν and $\mathcal{H}$ are fixed, and consider $K(\nu)$, the closed unit ball of $L^\infty_{\mathcal{H}}(X, \nu)$. The *random quantum noise* of ν is the quantity $N(\nu)$ defined by

$$N(\nu) = \sup_{\psi \in K(\nu)} \|\mathrm{Var}_\nu\,[\psi]\,\|.$$

Because quantum expectation is a contractive completely positive map, the operator inequality

$$\mathrm{Var}_{\ell\nu}\,[\psi] = \mathbb{E}_\nu\,[\psi^*\psi] - \mathbb{E}_\nu\,[\psi^*]\,\mathbb{E}_\nu\,[\psi] \leq \mathbb{E}_\nu\,[\psi^*\psi]$$

yields the norm inequality

$$\|\mathrm{Var}_{\ell\nu}\,[\psi]\| \leq \|\mathbb{E}_\nu\,[\psi^*\psi]\| \leq \|\psi^*\psi\| = \|\psi\|^2.$$

Likewise, $\|\mathrm{Var}_{r\nu}\,[\psi]\| \leq \|\psi\|^2$ for all ψ. Hence,

$$0 \leq \sup_{\psi \in K(\nu)} \|\mathrm{Var}_\nu\,[\psi]\,\| \leq 1.$$

Our definition above of the random quantum noise of ν differs from that of Polterovich (see [17, p. 489]), although we have used the same notation. The difference lies in the fact that we are using a larger class of functions ψ in defining $K(\nu)$—that is, we use operator-valued ψ, not just scalar-valued ψ.

Proposition 6.1. *$N(\nu) = 0$ if and only if the mass of ν is concentrated at a point x_0 of X.*

Proof. If $N(\nu) = 0$, then $\mathrm{Var}_\nu\,[\psi] = 0$ for every $\psi \in K(\nu)$. Hence, by Theorem 4.5, the quantum expectation map $\mathbb{E}_\nu$ is a unital homomorphism of $L^\infty(X, \mu) \otimes M_d(\mathbb{C})$ onto $M_d(\mathbb{C})$. If Z is the maximal ideal space of $L^\infty(X, \mu)$, then the unital homomorphisms of the homogeneous C*-algebra $C(Z) \otimes M_d(\mathbb{C})$ onto $M_d(\mathbb{C})$ are point evaluations $f \mapsto f(x_0)$. Hence, there is an $x_0 \in X$ such that $\nu = \delta_{\{x_0\}} 1$.

Conversely, if ν has its mass concentrated at a point, then $\mathbb{E}_\nu$ is a unital homomorphism and so $\mathrm{Var}_\nu\,[\psi] = 0$ for every $\psi \in K(\nu)$. □

Returning to the postulate that quantum probability measures are associated with measurements of quantum systems, Proposition 6.1 has the following consequence:

Corollary 6.2. *Every apparatus that performs measurements of a physical quantum system admits random quantum noise.*

Proof. Suppose that ν is the quantum probability measure associated with the measurement apparatus of some physical quantum system represented by a finite-dimensional Hilbert space. If $N(\nu) = 0$, then $\nu = \delta_{\{x_0\}}1$ for some $x_0 \in X$; in other words, the probability is exactly 1 that an event E containing outcome x_0 is measured, regardless of the state of the system; however, this contravenes the axioms of quantum mechanics. Hence, it must be that $N(\nu) > 0$. □

A subtler and potentially more descriptive notion of quantum noise is that of *inherent quantum noise*. To discuss inherent quantum noise, we first extend the concept of "smearing or randomisation of a measurement" [4,11,17] to one which involves quantum random variables rather than classical random variables.

Assume that $(X, \mathcal{F}(X))$ and $(Y, \mathcal{F}(Y))$ are Borel spaces for compact Hausdorff spaces X and Y, and suppose that ν and ν' are quantum probability measures on $(X, \mathcal{F}(X))$ and $(Y, \mathcal{F}(Y))$, respectively, with values in $\mathcal{B}(\mathcal{H})$ for some d-dimensional Hilbert space $\mathcal{H}$. The measure ν is said to be a *quantum randomisation* of ν' if there exists a function γ (sending y to γ_y) of Y into the space of quantum probability measures on $(X, \mathcal{F}(X))$ with values in $\mathcal{B}(\mathcal{H})$ such that

1. for every $E \in \mathcal{F}(X)$, the map $f_E^\gamma : Y \to \mathcal{B}(\mathcal{H})$ defined by $f_E^\gamma(y) = \gamma_y(E)$, for $y \in Y$, is a measurable function on $(Y, \mathcal{F}(Y))$, and
2. $\nu(E) = \int_Y f_E^\gamma \, \mathrm{d}\nu'$, for every $E \in \mathcal{F}(X)$.

Furthermore, the linear transformation $\Gamma_{\nu'} : L^\infty_{\mathcal{H}}(X, \nu) \to L^\infty_{\mathcal{H}}(Y, \nu')$ defined by

$$\Gamma_{\nu'}\psi(y) = \int_X \psi \, \mathrm{d}\gamma_y,$$

is called a *quantum randomisation operator*.

Proposition 6.3. *The quantum randomisation operator $\Gamma_{\nu'}$ is a unital completely positive linear map.*

Proof. The linearity of $\Gamma_{\nu'}$ has already been noted, and it is clear that $\Gamma_{\nu'}(1) = 1$ because each POVM γ_y satisfies $\gamma_y(X) = 1$. To show that $\Gamma_{\nu'}$ is completely positive, let $n \in \mathbb{N}$ be given and suppose that $\Psi = [\psi_{ij}]_{i,j=1}^n \in M_n\left(L^\infty_{\mathcal{H}}(X, \nu)\right)$ is positive. Because $L^\infty_{\mathcal{H}}(X, \nu)$ is a homogeneous C*-algebra, so is $M_n\left(L^\infty_{\mathcal{H}}(X, \nu)\right)$. Indeed,

$$M_n\left(L^\infty_{\mathcal{H}}(X, \nu)\right) \simeq C(Z) \otimes M_{nd}(\mathbb{C}),$$

where Z is the maximal ideal space of the abelian von Neumann algebra $L^\infty(X, \mu)$. Thus, to say that the matrix Ψ is positive in $M_n\left(L^\infty_{\mathcal{H}}(X, \nu)\right)$ is to say that the operator matrix $\Psi(x) = [\psi_{ij}(x)]_{i,j=1}^n$ acting on $\bigoplus_1^n \mathcal{H}$ is a positive operator for μ-almost all $x \in X$. Likewise, $\Gamma_{\nu'}^{(n)}(\Psi) = [\Gamma_{\nu'}(\psi_{ij})]_{i,j=1}^n$ is positive in $M_n\left(L^\infty_{\mathcal{H}}(Y, \nu')\right)$ if $\Gamma_{\nu'}^{(n)}(\Psi)(y)$ is a positive operator matrix for μ'-almost all $y \in Y$. Now if $y \in Y$, then

$$\Gamma_{\nu'}^{(n)}(\Psi)(y) = [\Gamma_{\nu'}(\psi_{ij})(y)]_{i,j=1}^n = \Gamma_{\nu'}^{(n)}(\Psi) = \left[\mathbb{E}_{\gamma_y}[\psi_{ij}]\right]_{i,j=1}^n = \mathbb{E}_{\gamma_y}^{(n)}[\Psi].$$

Because γ_y is a quantum probability measure, the expectation $\mathbb{E}_{\gamma_y}$ is completely positive. Thus, $\mathbb{E}^{(n)}_{\gamma_y}[\Psi] = \Gamma^{(n)}_{\nu'}(\Psi)(y)$ is a positive operator on $\bigoplus_1^n \mathcal{H}$, which proves that $\Gamma_{\nu'}$ is completely positive. □

Another useful property of the quantum randomisation operator $\Gamma_{\nu'}$ is:

Proposition 6.4. $\mathbb{E}_{\nu'} \circ \Gamma_{\nu'} = \mathbb{E}_{\nu}$.

Proof. Select $E \in \mathcal{F}(X)$ and consider the quantum random variable χ_E (the characteristic function of E). If $y \in Y$, then

$$\Gamma_{\nu'}\chi_E(y) = \int_X \chi_E \, \mathrm{d}\gamma_y = \gamma_y(E) = f^{\gamma}_E(y).$$

Hence,

$$\mathbb{E}_{\nu'}\left[\Gamma_{\nu'}\chi_E\right] = \int_Y f^{\gamma}_E \, \mathrm{d}\nu' = \nu(E) = \mathbb{E}_{\nu}\left[\chi_E\right].$$

Because the span of the characteristic functions is norm dense in $L^{\infty}_H(X,\nu)$, the linearity and continuity of Γ and of the expectations $\mathbb{E}_{\nu'}$ and $\mathbb{E}_{\nu}$ yield $\mathbb{E}_{\nu'}\left[\Gamma_{\nu'}\psi\right] = \mathbb{E}_{\nu}\left[\psi\right]$, for every $\psi \in L^{\infty}_{\mathcal{H}}(X,\nu)$. □

The main result of this section is the following theorem, which states that to every quantum random variable ψ on $(X, \mathcal{F}(X), \nu)$ there corresponds a quantum random variable $\Gamma_{\nu'}(\psi)$ on $(Y, \mathcal{F}(Y), \nu')$ such that $\Gamma_{\nu'}(\psi)$ and ψ have the same quantum expectation and the quantum variance of $\Gamma_{\nu'}(\psi)$ is bounded above in the Loewner ordering of $\mathcal{B}(\mathcal{H})_{\mathrm{sa}}$ by the quantum variance of ψ.

Theorem 6.5. $\mathrm{Var}_{\nu'}\left[\Gamma_{\nu'}\psi\right] \leq \mathrm{Var}_{\nu}\left[\psi\right]$.

Proof. Consider first the left variance. Given $\psi \in L^{\infty}_{\mathcal{H}}(X,\nu)$, we have that $\Gamma_{\nu'}(\psi^*\psi) \geq \Gamma_{\nu'}(\psi)^*\Gamma_{\nu'}(\psi)$ because every unital completely positive linear map satisfies the Schwarz inequality. Hence,

$$\begin{aligned}
\mathrm{Var}_{\ell\nu}\left[\psi\right] &= \mathbb{E}_{\nu}\left[\psi^*\psi\right] - \mathbb{E}_{\nu}\left[\psi\right]^* \mathbb{E}_{\nu}\left[\psi\right] \\
&= \mathbb{E}_{\nu'}\left[\Gamma_{\nu'}(\psi^*\psi)\right] - \mathbb{E}_{\nu'}\left[\Gamma_{\nu'}\psi\right]^* \mathbb{E}_{\nu'}\left[\Gamma_{\nu'}\psi\right] \\
&\geq \mathbb{E}_{\nu'}\left[\Gamma_{\nu'}(\psi)^*\Gamma_{\nu'}(\psi)\right] - \mathbb{E}_{\nu'}\left[\Gamma_{\nu'}\psi\right]^* \mathbb{E}_{\nu'}\left[\Gamma_{\nu'}\psi\right] \\
&= \mathrm{Var}_{\ell\nu'}\left[\Gamma_{\nu'}\psi\right].
\end{aligned}$$

A similar inequality holds for the right variance. Therefore, the inequality holds for the average of the left and right variances; hence, $\mathrm{Var}_{\nu'}\left[\Gamma_{\nu'}\psi\right] \leq \mathrm{Var}_{\nu}\left[\psi\right]$. □

The *intrinsic quantum noise* of a quantum probability measure ν on $(X, \mathcal{F}(X))$ is the quantity $N_{\mathrm{in}}(\nu)$ defined by

$$N_{\mathrm{in}}(\nu) = \inf_{(\Gamma_{\nu'},\nu')} \sup_{\psi \in K(\nu)} \|\mathrm{Var}_{\nu'}\left[\Gamma_{\nu'}\psi\right]\|.$$

An immediate consequence of Theorem 6.5 is a fundamental inequality that extends a similar inequality of Polterovich [17, Proposition 2.2].

Theorem 6.6. $0 \leq N_{\mathrm{in}}(\nu) \leq N(\nu) \leq 1$.

Acknowledgments

The work of all three authors is supported, in part, by the Natural Sciences and Engineering Research Council of Canada. The second author thanks the Isaac Newton Institute for Mathematical Sciences, Cambridge, for its hospitality during the Random Geometry programme in Spring 2015 where part of this paper was completed.

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