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Global estimates for non-uniformly nonlinear elliptic equations in a convex domain

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A B S T R A C T

In this paper we obtain the following global a priori Calderón–Zygmund estimates in a convex domain Ω

$$|\mathbf{f}|^{p_1}+|\mathbf{f}|^{p_2}\in L^{\gamma}(\Omega)\Rightarrow|\nabla u|^{p_1}+|\nabla u|^{p_2}\in L^{\gamma}(\Omega)\quad\text{for any }\gamma\geq 1$$

of weak solutions for a class of non-uniformly nonlinear elliptic equations with vanishing Dirichlet data

$$\sum_{i=1}^{2}\operatorname{div}\left((A_i\nabla u\cdot\nabla u)^{\frac{p_i-2}{2}}A_i\nabla u\right)=\sum_{i=1}^{2}\operatorname{div}\left(|\mathbf{f}|^{p_i-2}\mathbf{f}\right),$$

where $1<p_1<p_2<\infty$.

1. Introduction

Recently, Banerjee & Lewis [1] proved that

$$|\nabla u|^p\,(y)\leq Cr^{-n}\int\limits_{\Omega\cap B_{4r}(x)}|\nabla u|^p\,dx\quad\text{for any }y\in\Omega\cap B_r(x),\tag{1.1}$$

where $x\in\partial\Omega$ and Ω is a convex domain, for the weak solution of

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)=0\quad\text{in }\Omega\cap B_{4r}(x)$$

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with $u = 0$ on $\partial\Omega \cap B_{4r}(x)$. In the subsection "Remark (A Few Open Problems)" of [1] they conjectured that "It also would be interesting to look at the problem of obtaining higher integrability or BMO estimates up to the boundary for equations/systems with non-homogeneous term in convex domains along the lines of [10]". Actually, Kinnunen and Zhou [9,10] obtained $W^{1,q}$ $(q \geq p)$ estimates for the weak solution of

$$\operatorname{div}\left((A\nabla u \cdot \nabla u)^{\frac{p-2}{2}} A\nabla u\right) = \operatorname{div}\left(|\mathbf{f}|^{p-2}\mathbf{f}\right) \quad \text{in } \Omega \tag{1.2}$$

with $C^{1,\alpha}$-boundary $\partial\Omega$ and VMO coefficients. Moreover, Colombo and Mingione [6] proved that

$$|\mathbf{f}|^p + a(x)|\mathbf{f}|^q \in L^\gamma_{loc}(\Omega) \Rightarrow |\nabla u|^p + a(x)|\nabla u|^q \in L^\gamma_{loc}(\Omega) \quad \text{for any } \gamma \geq 1,$$

for

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\right) = \operatorname{div}\left(|\mathbf{f}|^{p-2}\mathbf{f} + a(x)|\mathbf{f}|^{q-2}\mathbf{f}\right)$$

and the general case. In this work we consider the following nonlinear elliptic boundary value problem of

$$\sum_{i=1}^{2} \operatorname{div}\left((A_i\nabla u \cdot \nabla u)^{\frac{p_i-2}{2}} A_i\nabla u\right) = \sum_{i=1}^{2} \operatorname{div}\left(|\mathbf{f}|^{p_i-2}\mathbf{f}\right) \quad \text{in } \Omega \tag{1.3}$$

with the vanishing Dirichlet data

$$u = 0 \qquad \text{on } \partial\Omega, \tag{1.4}$$

where $1 < p_1 < p_2 < \infty$, Ω is a bounded convex domain in $\mathbb{R}^n$, $\mathbf{f} = (f^1, \cdots, f^n) \in L^{p_2}(\Omega)$, and $A_i = \{a^i_{lj}(x)\}_{n\times n}$ is a symmetric matrix with discontinuous coefficients satisfying the uniform ellipticity condition; namely,

$$\Lambda_i^{-1}|\xi|^2 \leq A_i(x)\xi \cdot \xi \leq \Lambda_i|\xi|^2 \qquad i = 1, 2 \tag{1.5}$$

for all $\xi \in \mathbb{R}^n$, for almost every $x \in \mathbb{R}^n$ and for some positive constant Λ_i for $i = 1, 2$.

As usual, the solutions of (1.3)–(1.4) are taken in a weak sense. We now state the definition of weak solutions.

Definition 1.1. A function $u \in W^{1,p_2}_0(\Omega)$ is a weak solution of (1.3)–(1.4) if for any $\varphi \in W^{1,p_2}_0(\Omega)$, we have

$$\sum_{i=1}^{2} \int_\Omega (A_i\nabla u \cdot \nabla u)^{\frac{p_i-2}{2}} A_i\nabla u \cdot \nabla\varphi dx = \sum_{i=1}^{2} \int_\Omega |\mathbf{f}|^{p_i-2}\mathbf{f} \cdot \nabla\varphi dx.$$

There have been a wide research activities [2–5,7–12] on the study on the local/global L^q estimates of the gradient for the weak solution of (1.3) and the general case. Our approach is very much influenced by a series of works [1,3,4,6,9,10]. For simplicity, we define

$$H(z) =: |z|^{p_1} + |z|^{p_2}. \tag{1.6}$$

The purpose of this paper is to obtain the global a priori Calderón–Zygmund estimates in a convex domain Ω for the weak solutions of (1.3)–(1.4). In particular, we shall prove that

$$H(\mathbf{f}) \in L^\gamma(\Omega) \Rightarrow H(\nabla u) \in L^\gamma(\Omega) \quad \text{for any } \gamma \geq 1$$

for the weak solution of (1.3)–(1.4) with the estimate

$$\int_{\Omega} [H(\nabla u)]^{\gamma}\, dx \le C \int_{\Omega} [H(\mathbf{f}])^{\gamma}\, dx, \tag{1.7}$$

where C is a constant independent from u and $\mathbf{f}$.

Definition 1.2 *(Small BMO condition).* We say that the matrix A_i with coefficients is (δ, R_i)-vanishing for $i = 1, 2$ if

$$\sup_{0<r\le R_i} \sup_{x\in\mathbb{R}^n} \fint_{B_r(x)} \left|A_i(y) - \overline{A_i}_{B_r(x)}\right| dy \le \delta,$$

where

$$\overline{A_i}_{B_r(x)} = \fint_{B_r(x)} A_i(y)dy.$$

Now we are set to state the main result.

Theorem 1.3. *Assume that Ω is a bounded convex domain in $\mathbb{R}^n$, A_i is uniformly elliptic and (δ, R_i)-vanishing for $i = 1, 2$, $H(\mathbf{f}) \in L^{\gamma}(\Omega)$ for any $\gamma \ge 1$ and u is the weak solution of (1.3)–(1.4). Then there exists a small $\delta = \delta(n, p_1, p_2, \gamma, R_1, R_2, \Lambda_1, \Lambda_2)$ such that*

$$H(\nabla u) \in L^{\gamma}(\Omega)$$

with the estimate

$$\int_{\Omega} [H(\nabla u)]^{\gamma}\, dx \le C \int_{\Omega} [H(\mathbf{f})]^{\gamma}\, dx,$$

where the constant C is independent of u and $\mathbf{f}$.

2. Proof of Theorem 1.3

2.1. Preliminary tools

In this work we shall use the Hardy–Littlewood maximal function which controls the local behavior of a function.

Definition 2.1. Let g be a locally integrable function. The Hardy–Littlewood maximal function $\mathcal{M}g(x)$ is defined as

$$\mathcal{M}g(x) = \sup_{r>0} \fint_{B_r(x)} |g(y)|dy.$$

If g is not defined outside Ω, then

$$\mathcal{M}g(x) = \mathcal{M}(g\,\chi_{\Omega})(x).$$

Lemma 2.2. *(See [13].) Assume that $g \in L^q(\Omega)$ for some $q > 1$. Then we have*

(1)
$$\|\mathcal{M}g\|_{L^q(\Omega)} \leq C\|g\|_{L^q(\Omega)}.$$

(2)
$$|\{x \in \Omega : \mathcal{M}g(x) > \mu\}| \leq \frac{C}{\mu} \int_{\Omega} |g|\, dx.$$

(3)
$$\int_{\Omega} |g|^q dx = q \int_0^{\infty} \mu^{q-1} |\{x \in \Omega : |g| > \mu\}|\, d\mu.$$

We will use the following modified Vitali covering lemma.

Lemma 2.3. *Assume that E and F are measurable sets with $E \subset F \subset \Omega$, and that there exists an $\epsilon \in (0, 1/2^n)$ such that*

$$|E| < \epsilon |B_1|, \tag{2.1}$$

and for all $x \in B_1$ and for all $r \in (0, 1]$ with $|E \cap B_r(x)| \geq \epsilon |B_r(x)|$,

$$B_r(x) \cap \Omega \subset F. \tag{2.2}$$

Then we have

$$|E| \leq C_0 \epsilon |F|,$$

where the constant C_0 depends only on n, Ω.

Proof. In view of (2.1) and the fact that $\epsilon \leq 1/2^n$, for a.e. $x \in E$ there exists a small $r_x \in (0, 1)$ such that

$$|E \cap B_{r_x}(x)| = \epsilon\, |B_{r_x}(x)| \text{ and } |E \cap B_r(x)| < \epsilon\, |B_r(x)| \text{ for any } r > r_x. \tag{2.3}$$

From the Vitali covering lemma there exists a disjoint $\{B_{r_i}(x_i)\}_{j=1}^{\infty}$ for $r_i = r_{x_i}$ such that

$$E \subset \bigcup_i B_{5r_i}(x_i) \quad \text{and} \quad |E| < 5^n \sum |B_{r_i}|\,. \tag{2.4}$$

Then from (2.3) we find that

$$|E \cap B_{5r_i}(x_i)| < \epsilon\, |B_{5r_i}(x_i)| = 5^n \epsilon\, |B_{r_i}(x_i)|\,. \tag{2.5}$$

Fix any $x \in \Omega$ and $r \in (0, 1]$. Since Ω is a bounded convex domain, we know that there exists a ball in Ω. Without loss of generality we assume that

$$B_{r_0} \subset \Omega \subset B_{r_1} \quad \text{for some } r_0, r_1 > 0.$$

Then from the geometry knowledge we have

$$|\Omega \cap B_r(x)| \geq \frac{1}{\pi} \arcsin \frac{r_0}{r_1}\, |B_r(x)|\,,$$

which implies that

$$\sup_{0<r\leq 1}\sup_{x\in\Omega}\frac{|B_r(x)|}{|\Omega\cap B_r(x)|}\leq \frac{\pi}{\arcsin\frac{r_0}{r_1}}. \tag{2.6}$$

Finally, from (2.3)–(2.6) we conclude that

$$|E|\leq \sum_i |B_{5r_i}(x_i)\cap E| < 5^n\epsilon\sum_i |B_{r_i}(x_i)| \leq 5^n\frac{\pi}{\arcsin\frac{r_0}{r_1}}\epsilon\sum_i |\Omega\cap B_{r_i}(x_i)|,$$

which implies that

$$|E|\leq 5^n\frac{\pi}{\arcsin\frac{r_0}{r_1}}\epsilon\,|F|,$$

since $\{B_{r_i}(x_i)\}_{j=1}^{\infty}$ is disjoint and $\Omega\cap B_{r_i}(x_i)\subset F$ in view of (2.2). □

2.2. Final proof

We first prove the following boundary $W^{1,\gamma}$ estimates for the special case that $\gamma=1$. Actually, when $B_2\subset\Omega$, this result can be reduced to the local estimate.

Lemma 2.4. *Assume that A_i is uniformly elliptic for $i=1,2$, $H(\mathbf{f})\in L^\gamma(\Omega)$ for any $\gamma\geq 1$ and u is the weak solution of (1.3)–(1.4). Then we have*

$$\int_{\Omega_1} H(\nabla u)dx \leq C\int_{\Omega_2} H(u)+H(\mathbf{f})\,dx, \tag{2.7}$$

where $\Omega_r=\Omega\cap B_r$, $H(z)$ is defined in (1.6) and C only depends on $n,p_1,p_2,\Lambda_1,\Lambda_2$.

Proof. We may as well select the test function $\varphi=\zeta^{p_2}u\in W_0^{1,p_2}(\Omega)$, where $\zeta\in C_0^\infty(\mathbb{R}^n)$ is a cut-off function satisfying

$$0\leq\zeta\leq 1,\quad |\nabla\zeta|\leq C,\quad \zeta\equiv 1\ \text{ in } B_1\ \text{ and }\ \zeta\equiv 0\ \text{ in } \mathbb{R}^n/B_2.$$

Then by Definition 1.1, we have

$$\sum_{i=1}^{2}\int_{\Omega_2}(A_i\nabla u\cdot\nabla u)^{(p_i-2)/2}A_i\nabla u\cdot\nabla(\zeta^{p_2}u)dx=\sum_{i=1}^{2}\int_{\Omega_2}|\mathbf{f}|^{p_i-2}\mathbf{f}\cdot\nabla(\zeta^{p_2}u)dx$$

and write the resulting expression as

$$I_1=I_2+I_3+I_4+I_5,$$

where

$$I_1=\sum_{i=1}^{2}\int_{\Omega_2}\zeta^{p_2}(A_i\nabla u\cdot\nabla u)^{p_i/2}dx,$$

$$I_2=-\sum_{i=1}^{2}\int_{\Omega_2}p_2\zeta^{p_2-1}u(A_i\nabla u\cdot\nabla u)^{(p_i-2)/2}A_i\nabla u\cdot\nabla\zeta dx,$$

$$I_3 = \sum_{i=1}^{2} \int_{\Omega_2} \zeta^{p_2} |\mathbf{f}|^{p_i-2}\mathbf{f} \cdot \nabla u dx,$$

$$I_4 = \sum_{i=1}^{2} \int_{\Omega_2} p_2 \zeta^{p_2-1} u |\mathbf{f}|^{p_i-2}\mathbf{f} \cdot \nabla \zeta dx.$$

Estimate of I_1. It follows from the uniformly elliptic condition (1.5) that

$$I_1 \geq \frac{1}{\Lambda} \sum_{i=1}^{2} \int_{\Omega_2} \zeta^{p_2} |\nabla u|^{p_i} dx = \frac{1}{\Lambda} \int_{\Omega_2} \zeta^{p_2} H(\nabla u)\, dx \quad \text{for } \Lambda = \max\{\Lambda_1^{\frac{p_1}{2}}, \Lambda_2^{\frac{p_2}{2}}\}.$$

Estimate of I_2. From the uniformly elliptic condition (1.5) and Young's inequality with τ we have

$$\begin{aligned} I_2 &\leq C \int_{\Omega_2} \zeta^{p_2-p_1} \zeta^{p_1-1} |\nabla u|^{p_1-1} |u| dx + C \int_{\Omega_2} \zeta^{p_2-1} |\nabla u|^{p_2-1} |u| dx \\ &\leq \tau \int_{\Omega_2} \zeta^{p_2-p_1} \zeta^{p_1} |\nabla u|^{p_1} + \zeta^{p_2} |\nabla u|^{p_2} dx + C(\tau) \int_{\Omega_2} |u|^{p_1} + |u|^{p_2} dx \\ &= \tau \int_{\Omega_2} \zeta^{p_2} H(\nabla u)\, dx + C(\tau) \int_{\Omega_2} H(u)\, dx. \end{aligned}$$

Estimate of I_3 *and* I_4. Similarly to I_2, we have

$$I_3 \leq \tau \int_{\Omega_2} \zeta^{p_2} H(\nabla u)\, dx + C(\tau) \int_{\Omega_2} H(\mathbf{f})\, dx,$$

$$I_4 \leq C \int_{\Omega_2} H(\mathbf{f}) + H(u)\, dx.$$

Combining all the estimates of I_i $(1 \leq i \leq 4)$, we deduce that

$$\frac{1}{\Lambda} \int_{\Omega_2} \zeta^{p_2} H(\nabla u)\, dx \leq 2\tau \int_{\Omega_2} \zeta^{p_2} H(\nabla u)\, dx + C(\tau) \int_{\Omega_2} H(\mathbf{f}) + H(u)\, dx.$$

Selecting $\tau = 1/(4\Lambda)$ and recalling the definition of ζ, we complete the proof. $\square$

Let us introduce the following reference equation

$$\begin{cases} \sum_{i=1}^{2} \operatorname{div} \left(\left(\widetilde{A}_i^0 \nabla v \cdot \nabla v \right)^{\frac{p_i-2}{2}} \widetilde{A}_i^0 \nabla v \right) = 0 & \text{in } \Omega_R, \\ v = 0 & \text{on } \partial\Omega \cap B_R, \end{cases} \tag{2.8}$$

where $\widetilde{A}_i^0$ is a constant matrix with $\|(A_i)_{\Omega_R} - \widetilde{A}_i^0\|_\infty$ small for $i = 1, 2$.

Similarly to the proof of Theorem 1.1 in [1], we can obtain the following result.

Lemma 2.5. *Assume that Ω is a bounded convex domain, $x \in \partial\Omega$ and v is a weak solution of*

$$\begin{cases} div\left(|\nabla v|^{p_1-2}\nabla v + |\nabla v|^{p_2-2}\nabla v\right) = 0 & in\ \Omega_R, \\ v = 0 & on\ \partial\Omega \cap B_R. \end{cases} \tag{2.9}$$

Then we have

$$H(|\nabla v|)(y) \le CR^{-n} \int_{\Omega_R} H(|\nabla v|)dx \quad for\ any\ y \in \Omega_{R/4}, \tag{2.10}$$

where C depends only on n, p_1, p_2, Ω.

Proof. We only give the skeleton of the proof since the proof is similar to that of Theorem 1.1 in [1]. Similarly to (2.8) in [1] we consider the approximation problem

$$\operatorname{div}\left(\left(\epsilon + |\nabla v|^2\right)^{p_1/2-1}\nabla v + \left(\epsilon + |\nabla v|^2\right)^{p_2/2-1}\nabla v\right) = 0 \quad \text{in } \Omega_R. \tag{2.11}$$

Differentiating (2.11) with respect to x_k $(k = 1, 2, \ldots, n)$, we have

$$\sum_{l,j=1}^{n} \left(b_{lj}(x) v_{x_j x_k}\right)_{x_l} = 0 \quad \text{in } \Omega_R, \tag{2.12}$$

where

$$b_{lj}(x) = \sum_{i=1}^{2} \left(\epsilon + |\nabla v|^2\right)^{p_i/2-2} \left(\left(\epsilon + |\nabla v|^2\right)\delta_{lj} + (p_i - 2)v_{x_l}v_{x_j}\right).$$

Let $h = \left(\epsilon + |\nabla v|^2\right)^{1/2}$. Then we have

$$\sum_{i=1}^{2} \min\{p_i - 1, 1\} h^{p_i-2}|\xi|^2 \le \sum_{l,j=1}^{n} b_{lj}(x)\xi_l\xi_j \le \sum_{i=1}^{2} \max\{p_i - 1, 1\} h^{p_i-2}|\xi|^2. \tag{2.13}$$

Let

$$c_{lj}(x) = \frac{b_{ij}(x)}{p_1\left(\epsilon + |\nabla v|^2\right)^{p_1/2-1} + p_2\left(\epsilon + |\nabla v|^2\right)^{p_2/2-1}} = \frac{b_{ij}(x)}{p_1 h^{p_1-2} + p_2 h^{p_2-2}}.$$

We observe that

$$\begin{aligned} \sum_{l,j=1}^{n} c_{lj}(x)\xi_l\xi_j &\ge \frac{\min\{p_1 - 1, 1\}h^{p_1-2} + \min\{p_2 - 1, 1\}h^{p_2-2}}{p_1 h^{p_1-2} + p_2 h^{p_2-2}}|\xi|^2 \\ &\ge \min\{p_1 - 1, 1\}\frac{|\xi|^2}{p_1 + p_2} \end{aligned}$$

and

$$\sum_{i,j=1}^{n} c_{ij}(x)\xi_i\xi_j \le \frac{\max\{p_1-1,1\}h^{p_1-2}+\max\{p_2-1,1\}h^{p_2-2}}{p_1h^{p_1-2}+p_2h^{p_2-2}}|\xi|^2$$
$$\le \frac{1}{p_1}\big(\max\{p_1-1,1\}+\max\{p_2-1,1\}\big)|\xi|^2.$$

We define the differential operator

$$L =: \sum_{l,j=1}^{n} \frac{\partial}{\partial x_l}\left(c_{lj}(x)\frac{\partial}{\partial x_j}\right).$$

From (2.12) and (2.13) we conclude that

$$L(h^{p_1}+h^{p_2}) = \sum_{l,j=1}^{n} \big(b_{lj}(x)v_{x_k}v_{x_kx_j}\big)_{x_l} = \sum_{l,j=1}^{n} b_{lj}(x)v_{x_kx_l}v_{x_kx_j} \ge 0.$$

The rest of the proof is totally similar to that of Theorem 1.1 in [1]. □

Remark 2.6. Similarly to the above lemma, we can prove (2.10) is still true for the weak solution of (2.8).

Lemma 2.7. *For any $\epsilon > 0$, there exists a small $\delta = \delta(\epsilon) > 0$ such that if $u \in W^{1,p_2}(\Omega)$ is the weak solution of (1.3)–(1.4) in a convex domain Ω with*

$$\fint_{\Omega_4} H(\nabla u)\,dx \le 1 \quad \text{and} \quad \fint_{\Omega_4} H(\mathbf{f}) + \left|A_i - \overline{A}_{i\Omega_4}\right| dx \le \delta \quad \text{for } i=1,2, \tag{2.14}$$

then there exist constant matrixes $\widetilde{A}_i^0$ for $i = 1,2$ with $\|\overline{A}_{i\Omega_4} - \widetilde{A}_i^0\|_\infty \le \epsilon$ and a corresponding weak solution v of (2.8) in Ω_4 such that

$$\fint_{\Omega_4} H(u-v)\,dx \le \epsilon. \tag{2.15}$$

Moreover, there exists a constant $N_0 > 1$ such that

$$H(\nabla v) \le N_0 \quad \text{for any } x \in \Omega_3. \tag{2.16}$$

Proof. We first set out to prove (2.15) by contradiction. If not, there exists $\epsilon_0 > 0$, $\{A_1^k\}_{k=1}^\infty$, $\{A_2^k\}_{k=1}^\infty$, $\{u_k\}_{k=1}^\infty$ and $\{\mathbf{f}_k\}_{k=1}^\infty$ such that u_k is a weak solution of the following problem

$$\sum_{i=1}^{2} \operatorname{div}\left((A_i^k\nabla u_k\cdot\nabla u_k)^{\frac{p_1-2}{2}}A_i^k\nabla u_k\right) = \sum_{i=1}^{2}\operatorname{div}\left(|\mathbf{f}_k|^{p_i-2}\mathbf{f}_k\right) \quad \text{in } \Omega_4 \tag{2.17}$$

and

$$u_k = 0 \quad \text{on} \quad \partial\Omega\cap B_4, \tag{2.18}$$

with

$$\fint_{\Omega_4} H(\nabla u_k)\,dx \le 1 \quad \text{and} \quad \fint_{\Omega_4} H(\mathbf{f}_k) + \left|A_i^k - \overline{A_i^k}_{\Omega_4}\right| dx \le \frac{1}{k} \quad \text{for } i=1,2. \tag{2.19}$$

But we have

$$\fint_{\Omega_4} H\left(u_k - v_k\right) dx > \epsilon_0 \quad \text{for some } \epsilon_0 > 0 \tag{2.20}$$

for any constant matrixes $\widetilde{A}_i^0$ for $i = 1, 2$ with $\|\overline{A_i}_{\Omega_4} - \widetilde{A}_i^0\|_\infty \leq \epsilon_0$ and a corresponding weak solution v of (2.8) in Ω_4. From (2.18) and (2.19) we observe that $\{u_k\}_{k=1}^\infty$ is bounded in $W^{1,p_2}(\Omega_4)$. Consequently there exists a subsequence, which we still denote by $\{u_k\}_{k=1}^\infty$, and $u_0 \in W^{1,p_2}(\Omega_4)$ such that

$$u_k \rightharpoonup u_0 \quad \text{in } W^{1,p_2}(\Omega_4) \quad \text{and} \quad u_k \to u_0 \quad \text{in } L^{p_2}(\Omega_4). \tag{2.21}$$

As $\{\overline{A_i^k}_{\Omega_4}\}_{k=1}^\infty$ is bounded in l^∞, there exists a subsequence, which we denote by $\{\overline{A_i^k}\}$, such that

$$\|\overline{A_i^k} - A_i^0\|_\infty \to 0 \quad \text{as } k \to \infty \tag{2.22}$$

for some constant coefficients matrix A_i^0 and $i = 1, 2$. It follows from (2.19) and $A_i \in L^\infty$ that

$$A_i^k \to A_0^i \qquad \text{in} \quad L^{\frac{p_1}{p_1-1}}(\Omega_4) \quad \text{for} \quad i = 1, 2. \tag{2.23}$$

Next, we will show that u_0 is a weak solution of

$$\sum_{i=1}^{2} \operatorname{div}\left((A_i^0 \nabla u_0 \cdot \nabla u_0)^{\frac{p-2}{2}} A_i^0 \nabla u_0\right) = 0 \quad \text{in} \quad \Omega_4 \tag{2.24}$$

and

$$u_0 = 0 \quad \text{on} \quad \partial\Omega \cap B_4. \tag{2.25}$$

The (2.25) is trivial from (2.18). Indeed, if $\varphi \in C_0^\infty(B_4)$, it follows from (2.17) and Definition 1.1 that

$$\sum_{i=1}^{2} \int_{\Omega_4} (A_i^k \nabla u_k \cdot \nabla u_k)^{\frac{p_i-2}{2}} A_i^k \nabla u_k \cdot \nabla\varphi dx = \sum_{i=1}^{2} \int_{\Omega_4} |\mathbf{f}_k|^{p_i-2} \mathbf{f}_k \cdot \nabla\varphi dx. \tag{2.26}$$

Using (2.19), (2.21), (2.22) and (2.23), and letting $k \to \infty$ in (2.26), we have

$$\sum_{i=1}^{2} \int_{\Omega_4} (A_i^0 \nabla u_0 \cdot \nabla u_0)^{\frac{p_i-2}{2}} A_i^0 \nabla u_0 \cdot \nabla\varphi dx = 0,$$

which implies (2.24). Taking $v = u_0$ and sending $k \to \infty$, we reach a contradiction to (2.20). Moreover, from Lemma 2.4, the fact that $u = 0$ on $\partial\Omega \cap B_4$, (2.14) and (2.15) we conclude that

$$\begin{aligned} &\int_{\Omega_{7/2}} H(\nabla v) dx \\ &\leq C \int_{\Omega_4} H(v) dx \leq C \int_{\Omega_4} H(u - v) + H(u) dx \leq C + C \int_{\Omega_4} H(\nabla u) dx \leq C, \end{aligned}$$

which implies that the conclusion (2.16) is true by Lemma 2.5 and Remark 2.6. This completes our proof. □

Corollary 2.8. *Under the same assumptions on $u, \mathbf{f}, A_i, v$ as those in Lemma 2.7. Then we have*

$$\fint_{\Omega_2} H\left(\nabla u - \nabla v\right) dx \leq \epsilon. \tag{2.27}$$

Proof. From Lemma 2.7, for any $\eta > 0$ there exists a small $\delta = \delta(\eta)$ such that

$$\fint_{\Omega_4} H\left(u - v\right) dx \leq \eta \quad \text{and} \quad H\left(\nabla v\right) \leq N_0 \quad \text{for any } x \in \Omega_3. \tag{2.28}$$

Let $\zeta \in C_0^\infty(\mathbb{R}^n)$ be a cut-off function satisfying

$$0 \leq \zeta \leq 1, \quad |\nabla \zeta| \leq C, \quad \zeta \equiv 1 \ \text{ in } B_2 \ \text{ and } \ \zeta \equiv 0 \ \text{ in } \mathbb{R}^n/B_3. \tag{2.29}$$

We may choose the test function $\varphi = \zeta^{p_2}(u - v) \in W_0^{1,p_2}(\Omega_4)$ for u and v, and then by Definition 1.1, a direct calculation shows the resulting expression as

$$I_1 = I_2 + I_3 + I_4 + I_5,$$

where

$$I_1 = \sum_{i=1}^{2} \int_{\Omega_4} \zeta^{p_2} \left[\left(A_i \nabla u \cdot \nabla u\right)^{\frac{p_i-2}{2}} A_i \nabla u - \left(A_i \nabla v \cdot \nabla v\right)^{\frac{p_i-2}{2}} A_i \nabla v \right] \cdot \nabla\left(u - v\right) dx,$$

$$I_2 = \sum_{i=1}^{2} \int_{\Omega_4} \zeta^{p_2} \left[\left(A_i^0 \nabla v \cdot \nabla v\right)^{\frac{p_i-2}{2}} A_i^0 \nabla v - \left(A_i \nabla v \cdot \nabla v\right)^{\frac{p_i-2}{2}} A_i \nabla v \right] \cdot \nabla\left(u - v\right) dx,$$

$$I_3 = -\sum_{i=1}^{2} p_2 \int_{\Omega_4} \zeta^{p_2-1}(u - v)(A_i \nabla u \cdot \nabla u)^{\frac{p_i-2}{2}} A_i \nabla u \cdot \nabla \zeta dx,$$

$$I_4 = \sum_{i=1}^{2} p_2 \int_{\Omega_4} \zeta^{p_2-1}(u - v)|\mathbf{f}|^{p_i-2}\mathbf{f} \cdot \nabla \zeta dx + \sum_{i=1}^{2} \int_{\Omega_4} \zeta^{p_2} |\mathbf{f}|^{p_i-2}\mathbf{f} \cdot \nabla(u - v) dx,$$

$$I_5 = \sum_{i=1}^{2} p_2 \int_{\Omega_4} \zeta^{p_2-1}(u - v) \left(A_i^0 \nabla v \cdot \nabla v\right)^{\frac{p_i-2}{2}} A_i^0 \nabla v \cdot \nabla \zeta dx.$$

Estimate of I_1. We divide it into three cases.

Case 1. $p_2 > p_1 \geq 2$. Using the elementary inequality

$$\left[\left(A\xi \cdot \xi\right)^{\frac{p-2}{2}} A\xi - \left(A\eta \cdot \eta\right)^{\frac{p-2}{2}} A\eta \right] \cdot (\xi - \eta) \geq C \left|\xi - \eta\right|^p$$

for every $\xi, \eta \in \mathbb{R}^n$, we have

$$I_1 \geq C \sum_{i=1}^{2} \int_{\Omega_4} \zeta^{p_2} \left|\nabla u - \nabla v\right|^{p_i} dx.$$

Case 2. $1 < p_1 < p_2 < 2$. Using the elementary inequality

$$C(p)\tau^{\frac{p-2}{p}}\left[(A\xi\cdot\xi)^{\frac{p-2}{2}}A\xi-(A\eta\cdot\eta)^{\frac{p-2}{2}}A\eta\right]\cdot(\xi-\eta)+\tau|\eta|^p\geq C|\xi-\eta|^p$$

for every $\xi,\eta\in\mathbb{R}^n$ and every $\tau\in(0,1]$, we have

$$I_1+\tau\sum_{i=1}^{2}\int\limits_{\Omega_4}\zeta^{p_2}|\nabla v|^{p_i}\,dx\geq C(\tau)\sum_{i=1}^{2}\int\limits_{\Omega_4}\zeta^{p_2}|\nabla u-\nabla v|^{p_i}\,dx. \tag{2.30}$$

Case 3. $1<p_1\leq 2\leq p_2$. Similarly to Case 1 and Case 2, (2.30) is still true.

Estimate of I_2. Using the elementary inequality

$$\left|(A^0\xi\cdot\xi)^{\frac{p-2}{2}}A^0\xi-(A\xi\cdot\xi)^{\frac{p-2}{2}}A\xi\right|\leq C\left|A^0-A\right||\xi|^{p-1}$$

for every $\xi\in\mathbb{R}^n$, (1.5), (2.14), (2.28), (2.29) and Young's inequality with τ, we have

$$\begin{aligned}I_2&\leq C\sum_{i=1}^{2}\int\limits_{\Omega_4}\zeta^{p_2}\left|A_i^0-A_i\right||\nabla v|^{p_i-1}|\nabla(u-v)|\,dx\\&\leq C\sum_{i=1}^{2}\int\limits_{\Omega_4}\zeta^{p_2}\left|A_i^0-A_i\right|^{\frac{p_i-1}{p_i}}|\nabla(u-v)|\,dx\\&\leq\tau\sum_{i=1}^{2}\int\limits_{\Omega_4}\zeta^{p_2}|\nabla(u-v)|^{p_i}\,dx+C(\tau)\delta+C(\tau)\eta.\end{aligned}$$

Estimate of I_3. From (1.5), (2.14), (2.28) and Young's inequality with τ, we have

$$\begin{aligned}I_3&\leq C\sum_{i=1}^{2}\int\limits_{\Omega_4}|u-v||\nabla u|^{p_i-1}\,dx\\&\leq\tau\sum_{i=1}^{2}\int\limits_{\Omega_4}|\nabla u|^{p_i}\,dx+C(\tau)\sum_{i=1}^{2}\int\limits_{\Omega_4}|u-v|^{p_i}\,dx\\&\leq C\tau+C(\tau)\eta.\end{aligned}$$

Estimate of I_4. Using Young's inequality with τ and (2.28), we have

$$\begin{aligned}I_4&\leq C\sum_{i=1}^{2}\int\limits_{\Omega_4}|u-v||\mathbf{f}|^{p_i-1}dx+\sum_{i=1}^{2}\int\limits_{\Omega_4}\zeta^{p_2}|\mathbf{f}|^{p_i-1}|\nabla(u-v)|\,dx\\&\leq C\sum_{i=1}^{2}\int\limits_{\Omega_4}|u-v|^{p_i}\,dx+\tau\sum_{i=1}^{2}\int\limits_{\Omega_4}\zeta^{p_2}|\nabla(u-v)|^{p_i}\,dx+C(\tau)\sum_{i=1}^{2}\int\limits_{\Omega_4}|\mathbf{f}|^{p_i}dx\\&\leq C\eta+\tau\sum_{i=1}^{2}\int\limits_{\Omega_4}\zeta^{p_2}|\nabla(u-v)|^{p_i}\,dx+C(\tau)\delta.\end{aligned}$$

Estimate of I_5. Using (2.28), (2.29) and Young's inequality with τ, we have

$$I_5 \leq C \sum_{i=1}^{2} \int_{\Omega_4} |u-v| \zeta^{p_2} |\nabla v|^{p_i - 1} dx$$
$$\leq C \sum_{i=1}^{2} \int_{\Omega_4} |u-v| dx \leq C \sum_{i=1}^{2} \left(\int_{\Omega_4} |u-v|^{p_i} dx \right)^{1/p_i} \leq C \sum_{i=1}^{2} \eta^{1/p_i} \leq C\eta^{1/p_2}.$$

Combining all the estimates of I_i $(1 \leq i \leq 5)$, and selecting $0 < \tau << \eta < 1$ small enough, we obtain

$$\sum_{i=1}^{2} \int_{\Omega_4} \zeta^{p_2} |\nabla u - \nabla v|^{p_i} dx \leq C\delta + \tau \sum_{i=1}^{2} \int_{\Omega_4} \zeta^{p_2} |\nabla v|^{p_i} dx + C\tau + C\eta^{1/p_2} \leq C\delta + C\eta^{1/p_2} = \epsilon$$

by taking η and δ satisfying the last identity above. Finally, we can finish the proof of (2.27) by the above inequality and (2.29). □

Lemma 2.9. *There is a constant $N_1 > 1$ so that for any $\epsilon > 0$, there exists a small $\delta = \delta(\epsilon)$ with A_i uniformly elliptic and (δ, R_i)-vanishing for $i = 1, 2$, and if u is the weak solution of (1.3)–(1.4) in the convex domain Ω with*

$$\{x \in \Omega_1 : \mathcal{M}(H(|\nabla u|)) \leq 1\} \cap \{x \in \Omega_1 : \mathcal{M}(H(|\mathbf{f}|)) \leq \delta\} \neq \phi, \tag{2.31}$$

then

$$|\{x \in \Omega_1 : \mathcal{M}(H(|\nabla u|)) > N_1\}| < \epsilon |B_1|.$$

Proof. From (2.31) there exists a point $x_0 \in \Omega_1$ such that

$$\mathcal{M}(H(|\nabla u|))(x_0) \leq 1 \quad \text{and} \quad \mathcal{M}(H(|\mathbf{f}|))(x_0) \leq \delta,$$

which implies that

$$\frac{1}{|B_r|} \int_{\Omega_r(x_0)} H(|\nabla u|) dx \leq 1 \quad \text{and} \quad \frac{1}{|B_r|} \int_{\Omega_r(x_0)} H(|\mathbf{f}|) dx \leq \delta$$

for any $r > 0$. Since $\Omega_4 \subset \Omega_5(x_0)$, we have

$$\frac{1}{|B_4|} \int_{\Omega_4} H(|\nabla u|) dx \leq \left(\frac{5}{4}\right)^n \frac{1}{|B_5|} \int_{\Omega_5(x_0)} H(|\nabla u|) dx \leq \left(\frac{5}{4}\right)^n$$

and

$$\frac{1}{|B_4|} \int_{\Omega_4} H(|\mathbf{f}|) dx \leq \left(\frac{5}{4}\right)^n \delta.$$

Using Corollary 2.8, we find that for any $\eta > 0$, there exist a small $\delta(\eta)$ and a corresponding weak solution v of (2.8) in Ω_4 such that

$$\int_{\Omega_2} H(|\nabla u - \nabla v|) dx \leq \eta \quad \text{and} \quad H(|\nabla v|) \leq N_0 \quad \text{for any } x \in \Omega_3. \tag{2.32}$$

Next, we shall claim that

$$\{x \in \Omega_1 : \mathcal{M}(H(|\nabla u|)) > N_1\} \subset \{x \in \Omega_1 : \mathcal{M}(H(|\nabla u - \nabla v|)) > N_0\} \tag{2.33}$$

for $N_1 = \max\{2^{p_2} N_0, 2^n\}$. To prove this, suppose that

$$x_1 \in \{x \in \Omega_1 : \mathcal{M}(H(|\nabla u - \nabla v|)) \leq N_0\}.$$

Case 1: $r \leq 2$. Then $\Omega_r(x_1) \subset \Omega_3$. From (2.32) we have

$$\frac{1}{|B_r|} \int\limits_{\Omega_r(x_1)} H(|\nabla u|)\, dx \leq \frac{2^{p_2-1}}{|B_r|} \int\limits_{\Omega_r(x_1)} H(|\nabla u - \nabla v|) + H(|\nabla v|)\, dx \leq 2^{p_2} N_0,$$

since $(a+b)^p \leq 2^{p-1}(a^p + b^p)$ for $p > 1$ and $a, b > 0$.
Case 2: $r > 2$. Then $x_0 \in \Omega_r(x_1) \subset \Omega_{2r}(x_0)$. From (2.32) we have

$$\frac{1}{|B_r|} \int\limits_{\Omega_r(x_1)} H(|\nabla u|)\, dx \leq \frac{1}{|B_r|} \int\limits_{\Omega_{2r}(x_0)} H(|\nabla u|)\, dx \leq 2^n.$$

Consequently, we have $x_1 \in \{x \in \Omega_1 : \mathcal{M}(H(|\nabla u|)) \leq N_1\}$, which implies that (2.33) is true. Finally, from (2.32), (2.33) and Lemma 2.2 (2) we have

$$\begin{aligned}
&|\{x \in \Omega_1 : \mathcal{M}(H(|\nabla u|)) > N_1\}| \\
&\leq |\{x \in \Omega_1 : \mathcal{M}(H(|\nabla u - \nabla v|)) > N_0\}| \leq C \int\limits_{\Omega_1} H(|\nabla u - \nabla v|)\, dx < C\eta = \epsilon,
\end{aligned}$$

by choosing η small enough satisfying the last inequality. Thus we complete the proof. □

The following results can follow from the above lemma and a scaling argument.

Corollary 2.10. *Assume that u is the weak solution of (1.3)–(1.4) in the convex domain Ω with A_i uniformly elliptic and (δ, R_i)-vanishing for $i = 1, 2$. For any $\epsilon > 0$ and $r \in (0, 1]$, there exists a small $\delta = \delta(\epsilon)$ such that*

(1) *If $\{x \in \Omega_r : \mathcal{M}(H(|\nabla u|)) \leq 1\} \cap \{x \in \Omega_r : \mathcal{M}(H(|\mathbf{f}|)) \leq \delta\} \neq \phi$, then*

$$|\{x \in \Omega_r : \mathcal{M}(H(|\nabla u|)) > N_1\}| < \epsilon |B_r|.$$

(2) *If $|\{x \in \Omega : \mathcal{M}(H(|\nabla u|)) > N_1\} \cap B_r| \geq \epsilon |B_r|$, then*

$$\Omega_r \subset \{x \in \Omega : \mathcal{M}(H(|\nabla u|)) > 1\} \cup \{x \in \Omega : \mathcal{M}(H(|\mathbf{f}|)) > \delta\}.$$

Furthermore, we can obtain the following result.

Lemma 2.11. *Assume that u is the weak solution of (1.3)–(1.4) in the convex domain Ω with A_i uniformly elliptic and (δ, R_i)-vanishing for $i = 1, 2$. If*

$$|\{x \in \Omega : \mathcal{M}(H(|\nabla u|)) > N_1\}| < \epsilon |B_1| \quad \text{for any } \epsilon \in (0, 1/2^n), \tag{2.34}$$

then for any $\lambda \geq 1$ *we have*

$$\begin{aligned}&|\{x \in \Omega : \mathcal{M}(H(|\nabla u|))(x) > \lambda N_1\}| \\ &\quad \leq C_0\epsilon\big(|\{x \in \Omega : \mathcal{M}(H(|\nabla u|))(x) > \lambda\}| + |\{x \in \Omega : \mathcal{M}(H(|\mathbf{f}|))(x) > \lambda\delta\}|\big).\end{aligned}$$

Proof. We prove this lemma by two steps.
Step 1. $\lambda = 1$. We denote

$$E = \{x \in \Omega : \mathcal{M}(H(|\nabla u|))(x) > N_1\}$$

and

$$F = \{x \in \Omega : \mathcal{M}(H(|\nabla u|))(x) > 1\} \cup \{x \in \Omega : \mathcal{M}(H(|\mathbf{f}|))(x) > \delta\}.$$

Then $E \subset F \subset \Omega$ and $|E| < \epsilon |B_1|$. Furthermore, from Lemma 2.3 and Corollary 2.10 we find that

$$|E| < C_0\epsilon |F|,$$

which implies that the result is true for $\lambda = 1$.
Step 2. $\lambda > 1$. Then from (2.34) we find that

$$\begin{aligned}&\left|\left\{x \in \Omega : \frac{1}{\lambda}\mathcal{M}(H(|\nabla u|)) > N_1\right\}\right| \\ &= |\{x \in \Omega : \mathcal{M}(H(|\nabla u|)) > \lambda N_1\}| \leq |\{x \in \Omega : \mathcal{M}(H(|\nabla u|)) > N_1\}| < \epsilon |B_1|.\end{aligned}$$

Therefore, if we replace $H(|\nabla u|), H(|\mathbf{f}|)$ by $\frac{H(|\nabla u|)}{\lambda}, \frac{H(|\mathbf{f}|)}{\lambda}$, similarly to Step 1 we can complete the proof. □

Finally, we are set to prove the main result of this paper, Theorem 1.3.

Proof. Let

$$\lambda_0 = \frac{1}{\delta}\left[\int_\Omega (H(|\mathbf{f}|))^\gamma \, dx\right]^{1/\gamma} \quad \text{for some small } \delta \in (0,1). \tag{2.35}$$

Choosing the test function $\varphi = u \in W_0^{1,p_2}(\Omega)$ for (1.3)–(1.4), from Young's inequality with τ we can prove that

$$\int_\Omega H(\nabla u)\, dx \leq C \int_\Omega H(\mathbf{f})\, dx, \tag{2.36}$$

which implies that

$$\int_\Omega \frac{1}{\lambda_0} H(|\nabla u|)\, dx \leq C \int_\Omega \frac{1}{\lambda_0} H(|\mathbf{f}|) dx \leq C\delta \leq \epsilon |B_1| \tag{2.37}$$

in view of Hölder's inequality and (2.35), by taking δ sufficiently small in order to get the last inequality.

Furthermore, from Lemma 2.2 (2) and (2.37) we find that

$$\left|\left\{x \in \Omega : \frac{1}{\lambda_0}\mathcal{M}\left(H\left(|\nabla u|\right)\right) > N_1\right\}\right| \le \frac{1}{N_1}\int_\Omega \frac{1}{\lambda_0} H\left(|\nabla u|\right) dx < \epsilon \left|B_1\right|.$$

Therefore, if we replace $H\left(|\nabla u|\right)$, $H\left(|\mathbf{f}|\right)$ by $\frac{H(|\nabla u|)}{\lambda_0}$, $\frac{H(|\mathbf{f}|)}{\lambda_0}$, similarly to Lemma 2.9 we deduce that

$$\begin{aligned}&\left|\left\{x \in \Omega : \frac{1}{\lambda_0}\mathcal{M}\left(H\left(|\nabla u|\right)\right)(x) > \lambda N_1\right\}\right|\\ &\le C_0\epsilon\left|\left\{x \in \Omega : \frac{1}{\lambda_0}\mathcal{M}\left(H\left(|\nabla u|\right)\right)(x) > \lambda\right\}\right|\\ &\quad + C_0\epsilon\left|\left\{x \in \Omega : \frac{1}{\lambda_0}\mathcal{M}\left(H\left(|\mathbf{f}|\right)\right)(x) > \lambda\delta\right\}\right| \quad \text{for any } \lambda \ge 1. \end{aligned} \tag{2.38}$$

By (2.36) we may as well assume that $\gamma > 1$. Moreover, recalling Lemma 2.2 and (2.38), for any $\gamma > 1$ we compute

$$\begin{aligned}&\int_\Omega \left(\frac{1}{\lambda_0}\mathcal{M}\left(H\left(|\nabla u|\right)\right)\right)^\gamma dx\\ &= \gamma N_1^\gamma \left\{\int_0^1 + \int_1^\infty\right\} \lambda^{\gamma-1}\left|\left\{x \in \Omega : \frac{1}{\lambda_0}\mathcal{M}\left(H\left(|\nabla u|\right)\right)(x) > \lambda N_1\right\}\right| d\lambda\\ &\le \gamma N_1^\gamma |\Omega| \int_0^1 \lambda^{\gamma-1} d\lambda + C_0\epsilon\gamma N_1^\gamma \int_0^\infty \lambda^{\gamma-1}\left|\left\{x \in \Omega : \frac{1}{\lambda_0}\mathcal{M}\left(H\left(|\nabla u|\right)\right)(x) > \lambda\right\}\right| d\lambda\\ &\quad + C_0\epsilon\gamma N_1^\gamma \int_0^\infty \lambda^{\gamma-1}\left|\left\{x \in \Omega : \frac{1}{\lambda_0}\mathcal{M}\left(H\left(|\mathbf{f}|\right)\right)(x) > \lambda\delta\right\}\right| d\lambda\\ &\le C_2 + C_3\epsilon \int_\Omega \left(\frac{1}{\lambda_0}\mathcal{M}\left(H\left(|\nabla u|\right)\right)\right)^\gamma dx + C_4 \int_\Omega \left(\frac{1}{\lambda_0}\mathcal{M}\left(H\left(|\mathbf{f}|\right)\right)\right)^\gamma dx\\ &\le C_2 + C_3\epsilon \int_\Omega \left(\frac{1}{\lambda_0}\mathcal{M}\left(H\left(|\nabla u|\right)\right)\right)^\gamma dx + C_5 \int_\Omega \left(\frac{1}{\lambda_0}H\left(|\mathbf{f}|\right)\right)^\gamma dx,\end{aligned}$$

where $C_2 = C_2(n, \Omega, \gamma, N_1)$, $C_3 = C_3(n, \gamma, N_1)$ and $C_5 = C_5(n, \gamma, \epsilon, N_1)$. Then choosing a suitable ϵ such that $C_3\epsilon < \frac{1}{2}$, thereby determining δ with $0 < \delta < 1$, from (2.35) we obtain

$$\int_\Omega \left(\frac{1}{\lambda_0}\mathcal{M}\left(H\left(|\nabla u|\right)\right)\right)^\gamma dx \le C,$$

which implies that

$$\int_\Omega \left(\frac{1}{\lambda_0}H\left(|\nabla u|\right)\right)^\gamma dx \le C,$$

by using the fact that $H\left(|\nabla u|\right)(x) \le \mathcal{M}\left(H\left(|\nabla u|\right)\right)(x)$. Finally, from (2.35) we obtain

$$\int_{\Omega} \left(H\left(|\nabla u|\right)\right)^{\gamma} dx \le C \int_{\Omega} \left(H\left(|\mathbf{f}|\right)\right)^{\gamma} dx,$$

which finishes our proof. □

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