J. Math. Anal. Appl. 444 (2016) 1000–1014

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Multiple-angle formulas of generalized trigonometric functions with two parameters ☆

Shingo Takeuchi

Department of Mathematical Sciences, Shibaura Institute of Technology, Japan [1]

A R T I C L E   I N F O

Article history:
Received 19 April 2016
Available online 11 July 2016
Submitted by S. Cooper

Dedicated to Professor Eiji Yanagida on the occasion of his 60th birthday

Keywords:
Multiple-angle formulas
Generalized trigonometric functions
p-Laplacian
Eigenvalue problems
Pendulum equation
Lemniscate

A B S T R A C T

Generalized trigonometric functions with two parameters were introduced by Drábek and Manásevich to study an inhomogeneous eigenvalue problem of the p-Laplacian. Concerning these functions, no multiple-angle formula has been known except for the classical cases and a special case discovered by Edmunds, Gurka and Lang, not to mention addition theorems. In this paper, we will present new multiple-angle formulas which are established between two kinds of the generalized trigonometric functions, and apply the formulas to generalize classical topics related to the trigonometric functions and the lemniscate function.

1. Introduction

Let $p,\ q \in (1, \infty)$ be any constants. We define $\sin_{p,q} x$ by the inverse function of

$$\sin_{p,q}^{-1} x := \int_0^x \frac{dt}{(1-t^q)^{1/p}}, \quad 0 \le x \le 1,$$

and

$$\pi_{p,q} := 2\sin_{p,q}^{-1} 1 = 2\int_0^1 \frac{dt}{(1-t^q)^{1/p}} = \frac{2}{q} B\left(\frac{1}{p^*}, \frac{1}{q}\right), \tag{1.1}$$

☆ This work was supported by JSPS KAKENHI Grant Number 24540218.
E-mail address: shingo@shibaura-it.ac.jp.

[1] 307 Fukasaku, Minuma-ku, Saitama-shi, Saitama 337-8570, Japan.

http://dx.doi.org/10.1016/j.jmaa.2016.06.074
0022-247X/

where $p^* := p/(p-1)$ and B denotes the beta function. The function $\sin_{p,q} x$ is increasing in $[0, \pi_{p,q}/2]$ onto $[0, 1]$. We extend it to $(\pi_{p,q}/2, \pi_{p,q}]$ by $\sin_{p,q}(\pi_{p,q} - x)$ and to the whole real line $\mathbb{R}$ as the odd $2\pi_{p,q}$-periodic continuation of the function. Since $\sin_{p,q} x \in C^1(\mathbb{R})$, we also define $\cos_{p,q} x$ by $\cos_{p,q} x := (\sin_{p,q} x)'$. Then, it follows that

$$|\cos_{p,q} x|^p + |\sin_{p,q} x|^q = 1.$$

In case $p = q = 2$, it is obvious that $\sin_{p,q} x$, $\cos_{p,q} x$ and $\pi_{p,q}$ are reduced to the ordinary $\sin x$, $\cos x$ and π, respectively. This is a reason why these functions and the constant are called *generalized trigonometric functions* (with parameter (p, q)) and *the generalized* π, respectively.

Drábek and Manásevich [5] introduced the generalized trigonometric functions with two parameters to study an inhomogeneous eigenvalue problem of p-Laplacian. They gave a closed form of solutions (λ, u) of the eigenvalue problem

$$-(|u'|^{p-2}u')' = \lambda|u|^{q-2}u, \quad u(0) = u(L) = 0.$$

Indeed, for any $n = 1, 2, \ldots$, there exists a curve of solutions $(\lambda_{n,R}, u_{n,R})$ with a parameter $R \in \mathbb{R} \setminus \{0\}$ such that

$$\lambda_{n,R} = \frac{q}{p^*}\left(\frac{n\pi_{p,q}}{L}\right)^p |R|^{p-q}, \tag{1.2}$$

$$u_{n,R}(x) = R\sin_{p,q}\left(\frac{n\pi_{p,q}}{L}x\right) \tag{1.3}$$

(see also [12]). Conversely, there exists no other solution of the eigenvalue problem. Thus, the generalized trigonometric functions play important roles to study problems of the p-Laplacian.

It is of interest to know whether the generalized trigonometric functions have multiple-angle formulas unless $p = q = 2$. A few multiple-angle formulas seem to be known. Actually, in case $2p = q = 4$, the function $\sin_{p,q} x = \sin_{2,4} x$ coincides with the lemniscate sine function $\mathrm{sl}\, x$, whose inverse function is defined as

$$\mathrm{sl}^{-1} x := \int_0^x \frac{dt}{\sqrt{1-t^4}}.$$

Furthermore, $\pi_{2,4}$ is equal to the lemniscate constant $\varpi := 2\,\mathrm{sl}^{-1} 1 = 2.6220\cdots$. Concerning $\mathrm{sl}\, x$ and ϖ, we refer to the reader to [11, p. 81], [15] and [16, §22.8]. Since $\mathrm{sl}\, x$ has the multiple-angle formula

$$\mathrm{sl}\,(2x) = \frac{2\,\mathrm{sl}\, x\sqrt{1-\mathrm{sl}^4 x}}{1+\mathrm{sl}^4 x}, \quad 0 \le x \le \frac{\varpi}{2}, \tag{1.4}$$

we see that

$$\sin_{2,4}(2x) = \frac{2\sin_{2,4} x \cos_{2,4} x}{1+\sin_{2,4}^4 x}, \quad 0 \le x \le \frac{\pi_{2,4}}{2}.$$

Also in case $p^* = q = 4$, it is possible to show that $\sin_{p,q} x = \sin_{4/3,4} x$ can be expressed in terms of the Jacobian elliptic function, whose multiple-angle formula yields

$$\sin_{4/3,4}(2x) = \frac{2\sin_{4/3,4} x \cos_{4/3,4}^{1/3} x}{\sqrt{1+4\sin_{4/3,4}^4 x \cos_{4/3,4}^{4/3} x}} \quad 0 \le x < \frac{\pi_{4/3,4}}{4}. \tag{1.5}$$

The formula (1.5) was investigated by Edmunds, Gurka and Lang [6, Proposition 3.4]. They also proved an addition theorem for $\sin_{4/3,4} x$ involving (1.5). Such reductions to the elliptic functions have previously been used by Cayley [4] and Lindqvist and Peetre [8].

In this paper, we will present multiple-angle formulas which are established between two kinds of the generalized trigonometric functions with parameters $(2, p)$ and (p^*, p).

Theorem 1.1. *For $p \in (1, \infty)$ and $x \in [0, \pi_{2,p}/2^{2/p}] = [0, \pi_{p^*,p}/2]$, we have*

$$\sin_{2,p}(2^{2/p}x) = 2^{2/p} \sin_{p^*,p} x \cos_{p^*,p}^{p^*-1} x \tag{1.6}$$

and

$$\begin{aligned} \cos_{2,p}(2^{2/p}x) &= \cos_{p^*,p}^{p^*} x - \sin_{p^*,p}^{p} x \\ &= 1 - 2\sin_{p^*,p}^{p} x = 2\cos_{p^*,p}^{p^*} x - 1. \end{aligned} \tag{1.7}$$

Moreover, for $x \in \mathbb{R}$, we have

$$\sin_{2,p}(2^{2/p}x) = 2^{2/p} \sin_{p^*,p} x |\cos_{p^*,p} x|^{p^*-2} \cos_{p^*,p} x \tag{1.8}$$

and

$$\begin{aligned} \cos_{2,p}(2^{2/p}x) &= |\cos_{p^*,p} x|^{p^*} - |\sin_{p^*,p} x|^{p} \\ &= 1 - 2|\sin_{p^*,p} x|^{p} = 2|\cos_{p^*,p} x|^{p^*} - 1. \end{aligned} \tag{1.9}$$

In Theorem 1.1, the fact

$$\frac{\pi_{2,p}}{2^{2/p}} = \frac{\pi_{p^*,p}}{2} \tag{1.10}$$

is the special case $n = 2$ of the following identity.

Theorem 1.2. *Let $2 \le n < p + 1$. Then*

$$\pi_{\frac{p}{p-1},p}\pi_{\frac{p}{p-2},p}\cdots\pi_{\frac{p}{p-n+1},p} = n^{1-n/p}\pi_{\frac{n}{n-1},p}\pi_{\frac{n}{n-2},p}\cdots\pi_{\frac{n}{1},p}.$$

We give a series expansion of $\pi_{p^*,p}$ as a counterpart of the Gregory–Leibniz series for π. It is worth pointing out that $\pi_{p^*,p}$ is the area enclosed by the p-circle $|x|^p + |y|^p = 1$ (see [7,9]).

Theorem 1.3.

$$\begin{aligned} \frac{\pi_{p^*,p}}{4} &= \sum_{n=0}^{\infty} \frac{(2/p)_n}{n!} \frac{(-1)^n}{pn+1} \\ &= 1 - \frac{2}{p(p+1)} + \frac{2+p}{p^2(2p+1)} - \frac{(2+p)(2+2p)}{3p^3(3p+1)} + \cdots, \end{aligned}$$

where $(a)_n := \Gamma(a+n)/\Gamma(a) = a(a+1)(a+2)\cdots(a+n-1)$ and Γ denotes the gamma function.

We will apply Theorems 1.1–1.3 to the following problems (I)–(V).

(I) *An alternative proof of* (1.5). It should be noted that the multiple-angle formula (1.6) in Theorem 1.1 allows (1.5) to be rewritten in terms of the lemniscate function $\mathrm{sl}\, x = \sin_{2,4} x$:

$$\sin_{4/3,4}(2x) = \frac{\sqrt{2}\,\mathrm{sl}\,(\sqrt{2}x)}{\sqrt{1+\mathrm{sl}^4\,(\sqrt{2}x)}}, \quad 0 \le x < \frac{\pi_{4/3,4}}{4} = \frac{\varpi}{2\sqrt{2}},$$

where the last equality above follows from (1.10) with $\pi_{2,4} = \varpi$. This indicates that it is possible to obtain (1.5) from the multiple-angle formula (1.4) for the lemniscate function.

(II) *A relation between eigenvalue problems of the p-Laplacian and that of the Laplacian.* Let u be a function with $(n-1)$-zeros in $(0, L)$ satisfying

$$-(|u'|^{p-2}u')' = \lambda |u|^{p^*-2}u, \quad u(0) = u(L) = 0$$

for some $\lambda > 0$. Similarly, let v be a function with n-zeros in $(0, L)$ satisfying

$$-(|v'|^{p-2}v')' = \mu |v|^{p^*-2}v, \quad v'(0) = v'(L) = 0$$

for some $\mu > 0$. Then, by Theorem 1.1, we can show that the product $w = uv$ is a function with $(2n-1)$-zeros in $(0, L)$ satisfying

$$-w'' = 2p^*(\lambda\mu)^{1/p}|w|^{p^*-2}w, \quad w(0) = w(L) = 0.$$

The curious fact is the consequence of a straightforward calculation with (1.2), (1.3), (1.8) and (1.10). Such a relation between the eigenvalue problems of the p-Laplacian and that of the Laplacian may be known. However, we can not find a literature proving it, while the assertion in case $p = 2$ is trivial because

$$w = \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right) = \frac{1}{2}\sin\left(\frac{2n\pi}{L}x\right).$$

(III) *A pendulum-type equation with the p-Laplacian.* We give a closed form of solutions of the pendulum-type equation

$$-(|\theta'|^{p-2}\theta')' = \lambda^p |\sin_{2,p}\theta|^{p-2}\sin_{2,p}\theta.$$

In case $p = 2$, this equation is the ordinary pendulum equation $-\theta'' = \lambda^2 \sin\theta$ and it is well known that the solutions can be expressed in terms of the Jacobian elliptic function. We will obtain an expression of the solution for the pendulum-type equation above by using our special functions involving a generalization of the Jacobian elliptic function in [12,13]. There are studies of other (forced) pendulum-type equations with p-Laplacian versus $\sin\theta$ in [10]; versus $\sin_{p,p}\theta$ in [1], for the purpose of finding periodic solutions.

(IV) *Catalan-type constants.* Catalan's constant, which occasionally appears in estimates in combinatorics, is defined by

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 0.9159\cdots.$$

We can find a lot of representation of G in [2]; for a typical example,

$$\frac{1}{2}\int_0^{\pi/2} \frac{x}{\sin x}\,dx = G. \tag{1.11}$$

The multiple-angle formula (1.6) gives a generalization of (1.11) as

$$\frac{1}{2^{2/p}}\int_0^{\pi_{2,p}/2}\frac{x}{\sin_{2,p}x}\,dx=\sum_{n=0}^{\infty}\frac{(2/p)_n}{n!}\frac{(-1)^n}{(pn+1)^2}. \tag{1.12}$$

In case $p=2$, the formula (1.12) coincides with (1.11). Moreover, for $p=4$ we obtain the interesting formula

$$\frac{1}{\sqrt{2}}\int_0^{\varpi/2}\frac{x}{\operatorname{sl}x}\,dx=\sum_{n=0}^{\infty}\frac{(1/2)_n}{n!}\frac{(-1)^n}{(4n+1)^2}.$$

(V) *Series expansions of the lemniscate constant* ϖ. The lemniscate constant ϖ has the formula [15, Theorem 5]:

$$\frac{\varpi}{2}=1+\frac{1}{10}+\frac{1}{24}+\frac{5}{208}+\cdots+\frac{(2n-1)!!}{(2n)!!}\frac{1}{4n+1}+\cdots,$$

where $(-1)!!:=1$. For this, using Theorem 1.3 with (1.10), we will obtain

$$\frac{\varpi}{2\sqrt{2}}=1-\frac{1}{10}+\frac{1}{24}-\frac{5}{208}+\cdots+\frac{(2n-1)!!}{(2n)!!}\frac{(-1)^n}{4n+1}+\cdots,$$

which does not appear in Todd [15] and seems to be unfamiliar. We will also produce some other formulas of ϖ.

This paper is organized as follows. Section 2 is devoted to the proofs of Theorems 1.1–1.3. In Section 3, we deal with the above-mentioned problems (I)–(V).

2. The multiple-angle formulas

Let $p,\ q\in(1,\infty)$ and $x\in(0,\pi_{p,q}/2)$. It is easy to see that

$$\cos_{p,q}^p x+\sin_{p,q}^q x=1,$$
$$(\sin_{p,q}x)'=\cos_{p,q}x,\quad (\cos_{p,q}x)'=-\frac{q}{p}\sin_{p,q}^{q-1}x\cos_{p,q}^{2-p}x,$$
$$(\cos_{p,q}^{p-1}x)'=-\frac{q}{p^*}\sin_{p,q}^{q-1}x.$$

If we extend to these formulas for any $x\in\mathbb{R}$, then the last one, for example, corresponds to

$$(|\cos_{p,q}x|^{p-2}\cos_{p,q}x)'=-\frac{q}{p^*}|\sin_{p,q}x|^{q-2}\sin_{p,q}x. \tag{2.1}$$

In a particular case,

$$\cos_{p^*,p}^{p^*}x+\sin_{p^*,p}^{p}x=1, \tag{2.2}$$
$$(\sin_{p^*,p}x)'=\cos_{p^*,p}x,\quad (\cos_{p^*,p}x)'=-(p-1)\sin_{p^*,p}^{p-1}x\cos_{p^*,p}^{2-p^*}x,$$
$$(\cos_{p^*,p}^{p^*-1}x)'=-\sin_{p^*,p}^{p-1}x.$$

From the last one and the differentiation of inverse functions,

$$(\cos_{p^*,p}^{p^*-1})^{-1}(y) = \int_y^1 \frac{dt}{(1-t^p)^{1/p^*}}, \quad 0 \le y \le 1,$$

hence

$$\sin_{p^*,p}^{-1} y + (\cos_{p^*,p}^{p^*-1})^{-1}(y) = \frac{\pi_{p^*,p}}{2}.$$

Therefore, for $x \in [0, \pi_{p^*,p}/2]$

$$\sin_{p^*,p}\left(\frac{\pi_{p^*,p}}{2} - x\right) = \cos_{p^*,p}^{p^*-1} x, \tag{2.3}$$

$$\cos_{p^*,p}^{p^*-1}\left(\frac{\pi_{p^*,p}}{2} - x\right) = \sin_{p^*,p} x. \tag{2.4}$$

Throughout this paper, the following function is useful:

$$\tau_{p,q}(x) := \frac{\sin_{p,q} x}{|\cos_{p,q} x|^{p/q-1} \cos_{p,q} x}, \quad x \neq \frac{2n+1}{2}\pi_{p,q},\ n \in \mathbb{Z}.$$

Then, it follows immediately from (2.3) and (2.2) that

Lemma 2.1. *For $x \in (0, \pi_{p^*,p}/2)$, $\tau_{p^*,p}(x) = 1$ implies $x = \pi_{p^*,p}/4$. Moreover, $\sin_{p^*,p}^{-1}(2^{-1/p}) = \cos_{p^*,p}^{-1}(2^{-1/p^*}) = \pi_{p^*,p}/4$.*

Let us prove the multiple-angle formulas in Theorem 1.1.

Proof of Theorem 1.1. Let $x \in [0, \pi_{p^*,p}/4]$. Then, $y = \sin_{p^*,p} x \in [0, 2^{-1/p}]$ by Lemma 2.1. Setting $t^p = (1-(1-s^p)^{1/2})/2$ in

$$\sin_{p^*,p}^{-1} y = \int_0^y \frac{dt}{(1-t^p)^{1/p^*}},$$

we have

$$\sin_{p^*,p}^{-1} y = \int_0^{y(4(1-y^p))^{1/p}} \frac{\dfrac{2^{-1-1/p} s^{p-1}}{(1-s^p)^{1/2}(1-(1-s^p)^{1/2})^{1-1/p}}\, ds}{2^{-1+1/p}(1+(1-s^p)^{1/2})^{1-1/p}}$$

$$= 2^{-2/p} \int_0^{y(4(1-y^p))^{1/p}} \frac{ds}{(1-s^p)^{1/2}};$$

that is,

$$\sin_{p^*,p}^{-1} y = 2^{-2/p} \sin_{2,p}^{-1}(y(4(1-y^p))^{1/p}). \tag{2.5}$$

Hence we obtain

$$\sin_{2,p}(2^{2/p} x) = 2^{2/p} \sin_{p^*,p} x \cos_{p^*,p}^{p^*-1} x,$$

and (1.6) is proved. In particular, letting $y = 2^{-1/p}$ in (2.5) and using Lemma 2.1, we get

$$\frac{\pi_{p^*,p}}{4} = 2^{-2/p} \sin_{2,p}^{-1} 1 = \frac{\pi_{2,p}}{2^{1+2/p}},$$

which implies (1.10).

Next, let $x \in (\pi_{p^*,p}/4, \pi_{p^*,p}/2]$ and $y := \pi_{p^*,p}/2 - x \in [0, \pi_{p^*,p}/4)$. By the symmetry properties (2.3) and (2.4), we obtain

$$2^{2/p} \sin_{p^*,p} x \cos_{p^*,p}^{p^*-1} x = 2^{2/p} \cos_{p^*,p}^{p^*-1} y \sin_{p^*,p} y.$$

According to the argument above, the right-hand side is identical to $\sin_{2,p}(2^{2/p}y)$. Moreover, (1.10) gives

$$\sin_{2,p}(2^{2/p}y) = \sin_{2,p}(\pi_{2,p} - 2^{2/p}x) = \sin_{2,p}(2^{2/p}x).$$

The formula (1.7) is deduced from differentiating both sides of (1.6). Moreover, (1.8) and (1.9) come from the periodicities of the functions. □

Proof of Theorem 1.2. Setting $x = 1/n$ and $y = 1/p$ in the formula of the beta function (see [16, §12.15, Example])

$$B(nx, ny) = \frac{1}{n^{ny}} \frac{\prod_{k=0}^{n-1} B(x + k/n, y)}{\prod_{k=1}^{n-1} B(ky, y)}, \quad n \geq 2, \ x, \ y > 0,$$

we have

$$\frac{p}{n} = \frac{1}{n^{n/p}} \frac{\prod_{k=1}^{n} B(k/n, 1/p)}{\prod_{k=1}^{n-1} B(k/p, 1/p)}.$$

Hence,

$$\prod_{k=1}^{n-1} B\left(\frac{k}{p}, \frac{1}{p}\right) = n^{1-n/p} \prod_{k=1}^{n-1} B\left(\frac{k}{n}, \frac{1}{p}\right), \quad n \geq 2.$$

From (1.1), this is rewritten as

$$\prod_{k=1}^{n-1} \pi_{(p/k)^*,p} = n^{1-n/p} \prod_{k=1}^{n-1} \pi_{(n/k)^*,p}, \quad 2 \leq n < p+1,$$

which is precisely the assertion of the theorem. □

Remark 2.2. Taking $n = 2$ in Theorem 1.2, we have the relation (1.10) between $\pi_{p^*,p}$ and $\pi_{2,p}$. In fact, (1.10) is equivalent to the duplication formula of the gamma function (see [16, §12.15, Corollary])

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}} \Gamma(x) \Gamma\left(x + \frac{1}{2}\right).$$

Proof of Theorem 1.3. Let $x \in (0, 1)$. Differentiating the inverse function of $\tau_{p^*,p}$, we have

$$\tau_{p^*,p}^{-1}(x) = \int_0^x \frac{dt}{(1+t^p)^{2/p}}.$$

Hence

$$\tau_{p^*,p}^{-1}(x) = \int_0^x \sum_{n=0}^{\infty} \binom{-2/p}{n} t^{pn}\,dt = x\sum_{n=0}^{\infty} \frac{(2/p)_n}{n!}\frac{(-x^p)^n}{pn+1}. \tag{2.6}$$

By Abel's continuity theorem [16, §3.71], it is sufficient to show that the right-hand side of (2.6) converges at $x = 1$; i.e., the series

$$\sum_{n=0}^{\infty} \frac{(2/p)_n}{n!}\frac{(-1)^n}{pn+1} =: \sum_{n=0}^{\infty}(-1)^n a_n$$

converges. This is an alternating series and $\{a_n\}$ is decreasing because

$$0 \le \frac{a_{n+1}}{a_n} = \frac{(2/p+n)(pn+1)}{(n+1)(pn+p+1)} = \frac{(n+1/p)(pn+2)}{(n+1)(pn+p+1)} < 1.$$

Moreover, $\{a_n\}$ converges to 0 as $n \to \infty$. Indeed, Euler's formula for the gamma function [16, §12.11, Example] gives

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{2/p(2/p+1)(2/p+2)\cdots(2/p+n-1)}{(n-1)!n^{2/p}}\frac{n^{2/p-1}}{pn+1}\\ &= \frac{1}{\Gamma(2/p)}\cdot 0 = 0.\end{aligned}$$

Therefore, the series above converges to $\tau_{p^*,p}^{-1}(1)$ (see for instance [16, §2.31, Corollary (ii)]). From Lemma 2.1, we conclude the theorem. $\square$

Remark 2.3. Combining (1.8) and (1.9), we can assert that $\tau_{2,p}$ and $\tau_{p^*,p}$ satisfy the multiple-angle formula

$$\tau_{2,p}(2^{2/p}x) = \frac{2^{2/p}\tau_{p^*,p}(x)}{|1-|\tau_{p^*,p}(x)|^p|^{2/p-1}(1-|\tau_{p^*,p}(x)|^p)},$$

which coincides with that of the tangent function if $p = 2$.

3. Applications

3.1. An alternative proof of (1.5)

Let us give an alternative proof of the multiple-angle formula of $\sin_{4/3,4} x$:

$$\sin_{4/3,4}(2x) = \frac{2\sin_{4/3,4} x \cos_{4/3,4}^{1/3} x}{\sqrt{1+4\sin_{4/3,4}^4 x \cos_{4/3,4}^{4/3} x}}, \quad 0 \le x < \frac{\pi_{4/3,4}}{4}, \tag{1.5}$$

which was discovered by Edmunds, Gurka and Lang [6].

Recall that $\sin_{2,4} x$ is equal to the lemniscate function $\mathrm{sl}\, x$. Applying (1.6) in case $p = 4$ with x replaced by $2x \in [0, \pi_{4/3,4}/2)$, we get

$$\mathrm{sl}\,(2\sqrt{2}x) = \sqrt{2}\sin_{4/3,4}(2x)(1-\sin_{4/3,4}^4(2x))^{1/4}. \tag{3.1}$$

First, we consider the case

$$0 \le x < \frac{\pi_{4/3,4}}{8} = \frac{\varpi}{4\sqrt{2}}.$$

Then, since $0 \le 2\sin^4_{4/3,4}(2x) < 1$ by Lemma 2.1, the equation (3.1) gives

$$2\sin^4_{4/3,4}(2x) = 1 - \sqrt{1 - \mathrm{sl}^4(2\sqrt{2}x)}.$$

Set $S = S(x) := \mathrm{sl}(\sqrt{2}x)$. Using the multiple-angle formula (1.4) for the lemniscate function, we have

$$\begin{aligned} 2\sin^4_{4/3,4}(2x) &= 1 - \sqrt{1 - \left(\frac{2S\sqrt{1-S^4}}{1+S^4}\right)^4} \\ &= 1 - \frac{\sqrt{1 - 12S^4 + 38S^8 - 12S^{12} + S^{16}}}{(1+S^4)^2} \\ &= 1 - \frac{|1 - 6S^4 + S^8|}{(1+S^4)^2}. \end{aligned}$$

Since $0 \le S < \mathrm{sl}(\varpi/4) = (3 - 2\sqrt{2})^{1/4}$, evaluated by (1.4), we see that $1 - 6S^4 + S^8 \ge 0$. Thus,

$$2\sin^4_{4/3,4}(2x) = 1 - \frac{1 - 6S^4 + S^8}{(1+S^4)^2} = \frac{8S^4}{(1+S^4)^2}. \tag{3.2}$$

Therefore, by (1.6),

$$\sin_{4/3,4}(2x) = \frac{\sqrt{2}S}{\sqrt{1+S^4}} = \frac{2\sin_{4/3,4} x \cos^{1/3}_{4/3,4} x}{\sqrt{1 + 4\sin^4_{4/3,4} x \cos^{4/3}_{4/3,4} x}}.$$

In the remaining case

$$\frac{\varpi}{4\sqrt{2}} = \frac{\pi_{4/3,4}}{8} \le x < \frac{\pi_{4/3,4}}{4} = \frac{\varpi}{2\sqrt{2}},$$

it follows easily that $1 \le 2\sin^4_{4/3,4}(2x) < 2$ and $1 - 6S^4 + S^8 \le 0$, hence we obtain (3.2) again. The proof of (1.5) is complete.

3.2. A relation between eigenvalue problems of the p-Laplacian and that of the Laplacian

Theorem 3.1. *Let $n \in \mathbb{N}$ and $p \in (1, \infty)$. Let u be an eigenfunction with $(n-1)$-zeros in $(0, L)$ for an eigenvalue $\lambda > 0$ of the eigenvalue problem*

$$-(|u'|^{p-2}u')' = \lambda |u|^{p^*-2}u, \quad u(0) = u(L) = 0, \tag{3.3}$$

and v an eigenfunction with n-zeros in $(0, L)$ for an eigenvalue $\mu > 0$ of the eigenvalue problem

$$-(|v'|^{p-2}v')' = \mu |v|^{p^*-2}v, \quad v'(0) = v'(L) = 0. \tag{3.4}$$

Then, the product $w = uv$ is an eigenfunction for the eigenvalue $\xi = 2p^*(\lambda\mu)^{1/p}$ with $(2n-1)$-zeros in $(0, L)$ of the eigenvalue problem

$$-w'' = \xi |w|^{p^*-2}w, \quad w(0) = w(L) = 0. \tag{3.5}$$

Proof. By (1.2) and (1.3), the solution (λ, u) of (3.3) can be expressed as follows:

$$\lambda = \left(\frac{n\pi_{p,p^*}}{L}\right)^p |R|^{p-p^*},$$
$$u(x) = R\sin_{p,p^*}\left(\frac{n\pi_{p,p^*}}{L}x\right), \quad R \neq 0.$$

Similarly, by the symmetry (2.3), the solution (μ, v) of (3.4) is represented as

$$\mu = \left(\frac{n\pi_{p,p^*}}{L}\right)^p |Q|^{p-p^*},$$
$$v(x) = Q\left|\cos_{p,p^*}\left(\frac{n\pi_{p,p^*}}{L}x\right)\right|^{p-2}\cos_{p,p^*}\left(\frac{n\pi_{p,p^*}}{L}x\right), \quad Q \neq 0.$$

Applying (1.8) in Theorem 1.1 and (1.10) to the product $w = uv$, we have

$$\begin{aligned} w(x) &= RQ\sin_{p,p^*}\left(\frac{n\pi_{p,p^*}}{L}x\right)\left|\cos_{p,p^*}\left(\frac{n\pi_{p,p^*}}{L}x\right)\right|^{p-2}\cos_{p,p^*}\left(\frac{n\pi_{p,p^*}}{L}x\right) \\ &= 2^{-2/p^*}RQ\sin_{2,p^*}\left(\frac{2n\pi_{2,p^*}}{L}x\right), \end{aligned}$$

which belongs to $C^2(\mathbb{R})$ and has $(2n-1)$-zeros in $(0, L)$. Therefore, by (2.1) with $p = 2$, a direct calculation shows

$$\begin{aligned} w'' &= -p^*2^{1-2/p^*}\left(\frac{n\pi_{2,p^*}}{L}\right)^2 RQ\left|\sin_{2,p^*}\left(\frac{2n\pi_{2,p^*}}{L}x\right)\right|^{p^*-2}\sin_{2,p^*}\left(\frac{2n\pi_{2,p^*}}{L}x\right) \\ &= -p^*2^{3-4/p^*}\left(\frac{n\pi_{2,p^*}}{L}\right)^2 |RQ|^{2-p^*}|w|^{p^*-2}w. \end{aligned} \tag{3.6}$$

On the other hand, (1.10) gives

$$(\lambda\mu)^{1/p} = 2^{2-4/p^*}\left(\frac{n\pi_{2,p^*}}{L}\right)^2 |RQ|^{2-p^*}. \tag{3.7}$$

Combining (3.6) and (3.7), we obtain (3.5). □

3.3. A pendulum-type equation

We give an expression of the solution of the following initial value problem:

$$-(|\theta'|^{p-2}\theta')' = \lambda^p|\sin_{2,p}\theta|^{p-2}\sin_{2,p}\theta, \quad \theta(0) = 0, \ \theta'(0) = \omega_0. \tag{3.8}$$

For $p, \ q \in (1, \infty)$ and $k \in [0, 1)$ we define $\operatorname{am}_{p,q}(x, k)$ by the inverse function of

$$\operatorname{am}_{p,q}^{-1}(x, k) := \int_0^x \frac{d\theta}{(1 - k^q|\sin_{p,q}\theta|^q)^{1/p^*}}, \quad -\infty < x < \infty,$$

in particular,

$$K_{p,q}(k) := \operatorname{am}_{p,q}^{-1}\left(\frac{\pi_{p,q}}{2}, k\right) = \int_0^{\pi_{p,q}/2} \frac{d\theta}{(1 - k^q\sin_{p,q}^q\theta)^{1/p^*}}. \tag{3.9}$$

Then, the function

$$\operatorname{sn}_{p,q}(x,k) := \sin_{p,q}(\operatorname{am}_{p,q}(x,k))$$

is a $4K_{p,q}(k)$-periodic odd function in $\mathbb{R}$. In case $p = q = 2$, the functions $\operatorname{am}_{p,q}$, $K_{p,q}$ and $\operatorname{sn}_{p,q}$ coincide with the amplitude function, the complete elliptic integral of the first kind and the Jacobian elliptic function, respectively (see [11, §2.8] for them). It is easy to see that $K_{p,q}(0) = \pi_{p,q}/2$, $\operatorname{sn}_{p,q}(x,0) = \sin_{p,q} x$, $\lim_{k\to 1-0} K_{p,q}(k) = \infty$ and $\lim_{k\to 1-0} \operatorname{sn}_{p,q}(x,k) = \tanh_q x$, defined as the inverse function of

$$\tanh_q^{-1} x = \int_0^x \frac{dt}{1-|t|^q}, \quad -1 < x < 1.$$

See [13,14] for details of $\operatorname{sn}_{p,q}$ and $K_{p,q}$. Another type of generalization of the Jacobian elliptic function also appears in a bifurcation problem of the p-Laplacian; see [12].

Theorem 3.2. *Let $p \in (1,\infty)$ and $\lambda,\ \omega_0 \in (0,\infty)$, and set $k := \omega_0/(2^{2/p}\lambda)$. Then, the solution of (3.8) is as follows:*

(i) *Case $k < 1$.*

$$\theta(t) = 2^{2/p} \sin_{p^*,p}^{-1}(k \operatorname{sn}_{p,p}(\lambda t, k)),$$

which is a $4K_{p,p}(k)/\lambda$-periodic function and

$$\max_{0\le t<\infty} |\theta(t)| = 2^{2/p} \sin_{p^*,p}^{-1} k.$$

(ii) *Case $k = 1$.*

$$\theta(t) = 2^{2/p} \sin_{p^*,p}^{-1}(\tanh_p(\lambda t)),$$

which is a strictly monotone increasing function and

$$\lim_{t\to\infty} \theta(t) = 2^{2/p-1}\pi_{p^*,p} = \pi_{2,p}.$$

(iii) *Case $k > 1$.*

$$\theta(t) = 2^{2/p} \operatorname{am}_{p^*,p}\left(k\lambda t, \frac{1}{k}\right),$$

which is a strictly monotone increasing function and

$$\lim_{t\to\infty} \theta(t) = \infty.$$

Proof. By the standard argument (for example, [5, Proposition 2.1] or [7, Theorem 3.1]), we can show that there exists a unique global solution of (3.8).

Let θ be the solution of (3.8) and $T := \inf\{t > 0 : \theta'(t) = 0\}$. On the interval $(0,T)$, θ satisfies $\theta(t) > 0$ and $\theta'(t) > 0$. Then, using (2.1) with $p = 2$, we obtain

$$\frac{1}{p}\theta'(t)^p - \frac{1}{p}\omega_0^p = \frac{2\lambda^p}{p}\cos_{2,p}\theta(t) - \frac{2\lambda^p}{p}. \tag{3.10}$$

From (1.9) in Theorem 1.1 we have

$$\cos_{2,p}\theta(t) = 1 - 2|\sin_{p^*,p}(2^{-2/p}\theta(t))|^p. \tag{3.11}$$

Combining (3.10) and (3.11) we obtain

$$\theta'(t)^p = 4\lambda^p(k^p - |\sin_{p^*,p}(2^{-2/p}\theta(t))|^p),$$

where $k = \omega_0/(2^{2/p}\lambda)$, hence, for $t \in [0, T]$,

$$t = \frac{1}{2^{2/p}\lambda}\int_0^{\theta(t)} \frac{d\theta}{(k^p - |\sin_{p^*,p}(2^{-2/p}\theta)|^p)^{1/p}}. \tag{3.12}$$

(i) Case $k < 1$. We can find $\alpha \in (0, 2^{2/p-1}\pi_{p^*,p})$ such that $k = \sin_{p^*,p}(2^{-2/p}\alpha)$ and $T = \theta^{-1}(\alpha)$. Letting $\sin_{p^*,p}(2^{-2/p}\theta) = k\sin_{p,p}\varphi$ in (3.12), we obtain

$$t = \frac{1}{\lambda}\int_0^{\varphi(t)} \frac{d\varphi}{(1 - k^p|\sin_{p,p}\varphi|^p)^{1/p^*}},$$

which implies

$$\varphi(t) = \mathrm{am}_{p,p}(\lambda t, k).$$

Therefore,

$$\theta(t) = 2^{2/p}\sin_{p^*,p}^{-1}(k\,\mathrm{sn}_{p,p}(\lambda t, k)).$$

We have thus found the unique solution θ of (3.8) in $[0, T]$. However, in view of the periodicity properties of $\mathrm{sn}_{p,p}$, this function θ is actually the unique global solution of (3.8), which is periodic of $4T = 4K_{p,p}(k)/\lambda$ and whose maximum value is $\alpha = 2^{2/p}\sin_{p^*,p}^{-1}k$.

(ii) Case $k = 1$. In this case, letting $\sin_{p^*,p}(2^{-2/p}\theta) = x$ in (3.12), we obtain

$$t = \frac{1}{\lambda}\int_0^{x(t)} \frac{dx}{1 - |x|^p} = \frac{1}{\lambda}\tanh_p^{-1} x(t).$$

Therefore,

$$\theta(t) = 2^{2/p}\sin_{p^*,p}^{-1}(\tanh_p(\lambda t))$$

and $T = \infty$. Moreover, by (1.10)

$$\lim_{t\to\infty}\theta(t) = 2^{2/p}\sin_{p^*,p}^{-1}1 = 2^{2/p-1}\pi_{p^*,p} = \pi_{2,p}.$$

(iii) Case $k > 1$. In this case, (3.12) becomes

$$t = \frac{1}{2^{2/p}k\lambda}\int_0^{\theta(t)} \frac{d\theta}{(1 - k^{-p}|\sin_{p^*,p}(2^{-2/p}\theta)|^p)^{1/p}}$$

$$= \frac{1}{k\lambda} \int_0^{\varphi(t)} \frac{d\varphi}{(1-k^{-p}|\sin_{p^*,p}\varphi|^p)^{1/p}}$$

$$= \frac{1}{k\lambda} \mathrm{am}_{p^*,p}^{-1}\left(\varphi(t), \frac{1}{k}\right).$$

Therefore,

$$\theta(t) = 2^{2/p} \mathrm{am}_{p^*,p}\left(k\lambda t, \frac{1}{k}\right)$$

and $T = \infty$. It is obvious that $\lim_{t\to\infty} \theta(t) = \infty$. □

Remark 3.3. The solution $\theta(t)$ in (ii) does not attain $\pi_{2,p}$ for any finite t, while equations with p-Laplacian sometimes have flat-core solutions (cf. [12]).

3.4. Catalan-type constants

We define

$$G_p := \sum_{n=0}^{\infty} \frac{(2/p)_n}{n!} \frac{(-1)^n}{(pn+1)^2}.$$

It is clear that $G_2 = G$, i.e. Catalan's constant described in Introduction.

Theorem 3.4. *Let $p \in (1,\infty)$, then*

$$G_p = \frac{1}{2^{2/p}} \int_0^{\pi_{2,p}/2} \frac{x}{\sin_{2,p} x}\, dx = \frac{1}{2^{2/p}} \int_0^1 K_{2,p}(k)\, dk, \tag{3.13}$$

where $K_{2,p}(k)$ is defined by (3.9).

Proof. By (2.6),

$$\int_0^1 \frac{\tau_{p^*,p}^{-1}(x)}{x}\, dx = \sum_{n=0}^{\infty} \frac{(2/p)_n(-1)^n}{n!} \int_0^1 \frac{x^{pn}}{pn+1}\, dx = G_p.$$

On the other hand, letting $\tau_{p^*,p}^{-1}(x) = 2^{-2/p}y$, we obtain

$$\int_0^1 \frac{\tau_{p^*,p}^{-1}(x)}{x}\, dx = \frac{1}{2^{4/p}} \int_0^{2^{2/p-2}\pi_{p^*,p}} \frac{y}{\sin_{p^*,p}(2^{-2/p}y)\cos_{p^*,p}^{p^*-1}(2^{-2/p}y)}\, dy$$

$$= \frac{1}{2^{2/p}} \int_0^{\pi_{2,p}/2} \frac{y}{\sin_{2,p} y}\, dy.$$

Here, we have used (1.10) and (1.6). This shows the first equality in (3.13).

The second equality in (3.13) follows from Fubini's theorem:

$$\begin{aligned}\int_0^1 K_{2,p}(k)\,dk &= \int_0^{\pi_{2,p}/2}\int_0^1 \frac{1}{(1-k^p\sin_{2,p}^p x)^{1/2}}\,dk\,dx\\ &= \int_0^{\pi_{2,p}/2} \frac{1}{\sin_{2,p} x}\int_0^{\sin_{2,p} x}\frac{1}{(1-t^p)^{1/2}}\,dt\,dx\\ &= \int_0^{\pi_{2,p}/2}\frac{\sin_{2,p}^{-1}(\sin_{2,p} x)}{\sin_{2,p} x}\,dx\\ &= \int_0^{\pi_{2,p}/2}\frac{x}{\sin_{2,p} x}\,dx.\end{aligned}$$

The proof is accomplished. □

By Theorem 3.4 with $p=4$, we obtain

Corollary 3.5.

$$\frac{1}{\sqrt{2}}\int_0^{\varpi/2}\frac{x}{\mathrm{sl}\,x}\,dx = \sum_{n=0}^{\infty}\frac{(1/2)_n}{n!}\frac{(-1)^n}{(4n+1)^2}.$$

Remark 3.6. In a similar way to [3, §3.2] or the last paragraph of [7, §2.1], we can also obtain the formula

$$\int_0^{\pi_{2,p}/2}\frac{x}{\sin_{2,p} x}\,dx = \frac{\pi_{2,p}}{2}\sum_{n=0}^{\infty}\frac{(1/2)_n(1/p)_n}{(1/2+1/p)_n n!}\frac{1}{pn+1} =: \frac{\pi_{2,p}}{2}C_p. \tag{3.14}$$

Therefore, from (3.13), (3.14) and (1.10),

$$\frac{\pi_{p^*,p}}{4} = \frac{\pi_{2,p}}{2^{2/p+1}} = \frac{G_p}{C_p} = \frac{\sum_{n=0}^{\infty}\frac{(2/p)_n}{n!}\frac{(-1)^n}{(pn+1)^2}}{\sum_{n=0}^{\infty}\frac{(1/2)_n(1/p)_n}{(1/2+1/p)_n n!}\frac{1}{pn+1}}, \tag{3.15}$$

particularly,

$$\frac{\pi}{4} = \frac{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}}{\sum_{n=0}^{\infty}\left(\frac{(1/2)_n}{n!}\right)^2\frac{1}{2n+1}}.$$

3.5. Series expansions of the lemniscate constant ϖ

The series of Proposition 1.3 for $p=2$ is nothing but the Gregory–Leibniz series. Letting $p=4$ and using (1.10), we have the expansion series for the lemniscate constant ϖ:

$$\frac{\varpi}{2\sqrt{2}} = 1 - \frac{1}{10} + \frac{1}{24} - \frac{5}{208} + \cdots + \frac{(2n-1)!!}{(2n)!!}\frac{(-1)^n}{4n+1} + \cdots. \tag{3.16}$$

On the other hand, there is the similar series to this in [15, Theorem 5]:

$$\frac{\varpi}{2} = 1 + \frac{1}{10} + \frac{1}{24} + \frac{5}{208} + \cdots + \frac{(2n-1)!!}{(2n)!!}\frac{1}{4n+1} + \cdots . \tag{3.17}$$

Combining (3.16) and (3.17), we obtain

$$\frac{2+\sqrt{2}}{8}\varpi = 1 + \frac{1}{24} + \cdots + \frac{(4n-1)!!}{(4n)!!}\frac{1}{8n+1} + \cdots ,$$
$$\frac{2-\sqrt{2}}{8}\varpi = \frac{1}{10} + \frac{5}{208} + \cdots + \frac{(4n+1)!!}{(4n+2)!!}\frac{1}{8n+5} + \cdots .$$

Finally, letting $p = 4$ in (3.15) we obtain

$$\frac{\varpi}{2\sqrt{2}} = \frac{\sum_{n=0}^{\infty} \frac{(1/2)_n}{n!} \frac{(-1)^n}{(4n+1)^2}}{\sum_{n=0}^{\infty} \frac{(1/2)_n (1/4)_n}{(3/4)_n n!} \frac{1}{4n+1}}.$$

References

[1] P. Amster, M.C. Mariani, P. De Nápoli, Boundary nonlinearities for a one-dimensional p-Laplacian like equation, Rev. Un. Mat. Argentina 45 (2) (2004) 1–10.
[2] David M. Bradley, Representations of Catalan's constant, CiteSeerX, http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.26.1879, 2001.
[3] P.J. Bushell, D.E. Edmunds, Remarks on generalized trigonometric functions, Rocky Mountain J. Math. 42 (1) (2012) 25–57.
[4] A. Cayley, Reduction of $\int \frac{dx}{(1-x^3)^{2/3}}$ to elliptic integrals, Messenger Math. 11 (1882) 142–143;
Also in: The Collected Mathematical Papers of Arthur Cayley, vol. XI, no. 760, Cambridge University Press, Cambridge, 1896.
[5] P. Drábek, R. Manásevich, On the closed solution to some nonhomogeneous eigenvalue problems with p-Laplacian, Differential Integral Equations 12 (1999) 773–788.
[6] D.E. Edmunds, P. Gurka, J. Lang, Properties of generalized trigonometric functions, J. Approx. Theory 164 (1) (2012) 47–56.
[7] J. Lang, D.E. Edmunds, Eigenvalues, Embeddings and Generalised Trigonometric Functions, Lecture Notes in Math., vol. 2016, Springer, Heidelberg, 2011.
[8] P. Lindqvist, J. Peetre, Two remarkable identities, called twos, for inverses to some Abelian integrals, Amer. Math. Monthly 108 (5) (2001) 403–410.
[9] P. Lindqvist, J. Peetre, p-arclength of the q-circle, Math. Student 72 (2003) 139–145.
[10] J. Mawhin, Periodic solutions of the forced pendulum: classical vs relativistic, Matematiche (Catania) 65 (2) (2010) 97–107.
[11] V. Prasolov, Y. Solovyev, Elliptic Functions and Elliptic Integrals, translated from the Russian manuscript by D. Leites, Transl. Math. Monogr., vol. 170, American Mathematical Society, Providence, RI, 1997.
[12] S. Takeuchi, Generalized Jacobian elliptic functions and their application to bifurcation problems associated with p-Laplacian, J. Math. Anal. Appl. 385 (1) (2012) 24–35.
[13] S. Takeuchi, The basis property of generalized Jacobian elliptic functions, Commun. Pure Appl. Anal. 13 (6) (2014) 2675–2692.
[14] S. Takeuchi, A new form of the generalized complete elliptic integrals, Kodai Math. J. 39 (2016) 202–226.
[15] J. Todd, The lemniscate constants, in: Collection of articles honoring Alston S. Householder, Commun. ACM 18 (1975) 14–19;
J. Todd, Corrigendum, Commun. ACM 18 (8) (1975) 462.
[16] E.T. Whittaker, G.N. Watson, A Course of Modern Analysis. An Introduction to the General Theory of Infinite Processes and of Analytic Functions; With an Account of the Principal Transcendental Functions, reprint of the fourth (1927) edition, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1996.