



On the Cauchy problem for a class of shallow water wave equations with $(k + 1)$ -order nonlinearities [☆]



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ABSTRACT

This paper considers the Cauchy problem for a class of shallow water wave equations with $(k + 1)$ -order nonlinearities in the Besov spaces

$$\partial_t u - \partial_t \partial_x^2 u = u^k \partial_x^3 u + bu^{k-1} \partial_x u \partial_x^2 u - (b + 1)u^k \partial_x u,$$

which involves the Camassa–Holm, the Degasperis–Procesi and the Novikov equations as special cases. Firstly, by means of the transport equation and the Littlewood–Paley theory, we obtain the local well-posedness of the equations in the nonhomogeneous Besov space $B_{p,r}^s$ ($s > \max\{1 + \frac{1}{p}, \frac{3}{2}\}$ and $p, r \in [1, +\infty]$). Secondly, we consider the local well-posedness in $B_{2,r}^s$ with the critical index $s = \frac{3}{2}$, and show that the solutions continuously depend on the initial data. Thirdly, the blow-up criteria and the conservative property for the strong solutions are derived. Finally, with the help of a new Ovsyannikov theorem, we investigate the Gevrey regularity and analyticity of the solutions. Moreover, we get a lower bound of the lifespan and the continuity of the data-to-solution mapping.

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1. Introduction

In this paper, we consider the Cauchy problem for a class of shallow water wave equations with $(k + 1)$ -order nonlinearities,

$$\partial_t u - \partial_t \partial_x^2 u = u^k \partial_x^3 u + bu^{k-1} \partial_x u \partial_x^2 u - (b + 1)u^k \partial_x u, \quad x \in \mathbb{R}, \quad t > 0, \quad (1.1)$$

which was recently proposed by Himonas and Holliman in [29]. Here, k is a given positive integer number, and the parameter b is assumed to range over the real line \mathbb{R} . As special cases, the Camassa–Holm, the

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Degasperis–Procesi and the Novikov equations are integrable members of this family of equations. By means of a Galerkin type approximation method, the local well-posedness of (1.1) in the Sobolev spaces $C([0, T]; H^s(\mathbb{R})) \cap C^1([0, T]; H^{s-1}(\mathbb{R}))$ is established [29], and the data-to-solution mapping is proved to be continuous but not uniformly continuous on any bounded subset of $H^s(\mathbb{R})$ ($s > \frac{3}{2}$). Furthermore, Holmes [32] proved that the data-to-solution mapping is Hölder continuous in $H^s(\mathbb{R})$ ($s > \frac{3}{2}$) endowed with the H^r -topology for $0 \leq r < s$, and the Hölder exponent is expressed in terms of s and r . On the other hand, if we write the Eq. (1.1) in the nonlocal form ((1.7) below), it can be regarded as a weakly dispersive perturbation of the generalized Burgers equation $\partial_t u + u^k \partial_x u = 0$. One of the important properties that makes the Eq. (1.1) an interesting evolution equation is that the Burgers equation has no peakon traveling wave solutions (called peakons) while the Eq. (1.1) does. More precisely, on the line \mathbb{R} , the peakons are given by

$$u_c(x, t) = c^{\frac{1}{k}} e^{|x-ct|}, \quad (1.2)$$

and on the circle \mathbb{S} , the peakons are given by

$$u_c(x, t) = \frac{c^{\frac{1}{k}}}{\cosh(\pi)} \cosh([x - ct]_p - \pi), \quad (1.3)$$

where c is any positive constant and $[x - ct]_p := x - ct - 2\pi[\frac{x-ct}{2\pi}]$ in [27]. By using these peakon solutions, one can construct two sequences of solutions whose distance at the initial time goes to zero while at any later time their distance goes to infinitely in the Sobolev spaces $H^s(\mathbb{R})$ with $s \leq \frac{3}{2}$ [27]. Consequently, the data-to-solution mapping for the equation (1.1) is not uniformly continuous in Sobolev spaces with exponent less than $\frac{3}{2}$.

For $k = 2$ and $b = 3$, the Eq. (1.1) becomes the Novikov equation with cubic nonlinearity

$$\partial_t u - \partial_t \partial_x^2 u + 4u^2 \partial_x u = 3u \partial_x u \partial_x^2 u + u^2 \partial_x^3 u, \quad (1.4)$$

which was discovered by V. Novikov in a symmetry classification of nonlocal partial differential equations with quadratic or cubic nonlinearity [42]. In [28], Himonas and Holliman considered the Cauchy problem of (1.4) in the Sobolev space $H^s(\mathbb{R})$ on the circle \mathbb{T} and the line \mathbb{R} for $s > \frac{3}{2}$. In [26], Grayshan investigated the non-periodic and the periodic low regularity solutions of the Eq. (1.4) in the Sobolev space with the exponent less than $\frac{3}{2}$. It is shown that the Eq. (1.4) admits an analytic solution if the initial data is analytic [47]. If the initial data satisfies a sign condition, Lai et al. [33] proved that the Eq. (1.4) admits a unique global weak solution in the Sobolev space $H^s(\mathbb{R})$ with $1 \leq s \leq \frac{3}{2}$. In [39], Ni and Zhou considered the local well-posedness for the Eq. (1.4) in the Besov spaces $B_{2,r}^s(\mathbb{R})$ with the critical index $s = \frac{3}{2}$, and they also studied the well-posedness in $H^s(\mathbb{R})$ with $s > \frac{3}{2}$ by applying the Kato's semigroup theory. For the other works related to the Eq. (1.4), we refer the readers to [14,36,38,37,49] and the references therein.

The most celebrated integrable member of Eq. (1.1) is the following Camassa–Holm (CH) equation (for $k = 1, b = 2$) with quadratic nonlinearity,

$$\partial_t u - \partial_x^2 \partial_t u + 3u \partial_x u = 2\partial_x u \partial_x^2 u + u \partial_x^3 u. \quad (1.5)$$

The CH equation is completely integrable, and it has a bi-Hamilton structure and infinite conservation laws as well as global dissipative solutions [4,8,31]. The CH equation has peakons which describe a fundamental characteristic of the traveling waves of largest amplitude, and these solutions can be formulated in the form of $ce^{-|x-ct|}$ with $c > 0$ [13,16,18,9]. It is worth to mention that the CH equation leads to the geodesic flow of a certain invariant metric on the Bott–Virasoro group [15], which implies that the Least Action Principle remains to be true. The local well-posedness and blow-up phenomena of the CH equation in the Sobolev

and the Besov spaces are investigated in [10–12,7,19,46]. For the global existence of the weak and strong solutions, we refer the readers to [10,11,17,7,48]. The global conservative and dissipative solutions of the CH equation were studied in [2,3]. The analyticity of the Cauchy problem for the CH equation with the analytic data was considered in [30].

By taking $k = 1, b = 3$ in Eq. (1.1), we obtain the well-known quadratic Degasperis–Procesi (DP) equation [23],

$$\partial_t u - \partial_x^2 \partial_t u = 4u \partial_x u - 3 \partial_x u \partial_x^2 u - u \partial_x^3 u. \quad (1.6)$$

The DP equation is similar to the CH equation in several aspects, such as its asymptotic accuracy is the same as the ones for the CH equation [24], it is completely integrable and also has a bi-Hamiltonian structure [22]. However, these two equations are truly different. One of the novel features of the DP equation different from the CH equation is that it not only has peakon solutions [22] and periodic peakon solutions [51], but also the periodic shock waves [25] and shock peakons [34].

Moreover, for the special case of $b = k + 1$, the local well-posedness and blow-up criteria of solutions are investigated in the recent works [6] and [50].

Noting that $G(x) = \frac{1}{2}e^{-|x|}$ and $G(x) * f = (I - \partial_x^2)^{-1}f$ for all $f \in L^2(\mathbb{R})$. Denote $P(D) = (I - \partial_x^2)^{-1}$, then the Eq. (1.1) can be expressed as the following hyperbolic type:

$$\begin{aligned} \partial_t u + u^k \partial_x u + P(D) \left[\frac{(k-1)(b-k)}{2} u^{k-2} (\partial_x u)^3 \right] \\ + \partial_x P(D) \left[\frac{3k-b}{2} u^{k-1} (\partial_x u)^2 + \frac{b}{k+1} u^{k+1} \right] = 0, \quad x \in \mathbb{R}, \quad t > 0, \end{aligned} \quad (1.7)$$

and we supplement (1.7) with the initial data

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}. \quad (1.8)$$

To the best known of our knowledge, it seems that the Cauchy problem (1.7)–(1.8) in the Besov spaces has not been studied yet. In this paper, by using the transport equation theory and the Littlewood–Paley theory, we firstly show that the Cauchy problem (1.7)–(1.8) is locally well-posed in $B_{p,r}^s(\mathbb{R})$ with $s > \max\{1 + \frac{1}{p}, \frac{3}{2}\}$ and $p, r \in [1, +\infty]$. The problematic issue here is that we need to deal with the Eq. (1.1) with $(k+1)$ -order nonlinearities which makes the proof of several required nonlinear estimates somewhat delicate. Unfortunately, in the case of $s = \frac{3}{2}$, one can prove that the local well-posedness in the Besov space $B_{2,\infty}^{\frac{3}{2}}(\mathbb{R})$ fails (see Lemma 4.1). However, we can establish the local well-posedness of the Cauchy problem (1.7)–(1.8) in the critical Besov space $B_{2,1}^{\frac{3}{2}}(\mathbb{R})$. Compared with the cases of $s > \frac{3}{2}$, in order to overcome the problems caused by the low regularity of $B_{2,\infty}^{-\frac{1}{2}}(\mathbb{R})$ and the high-order nonlinearities, the tools of logarithmic interpolation inequality and the Osgood Lemma are essential in our argument. After that, the blow-up criteria and the conservative law for the solutions are also derived. Finally, based on a generalized Ovsyannikov theorem (generalization of the classical Cauchy–Kovalevsky theorem) [35] and the basic properties of Sobolev–Gevrey spaces, we investigate the Gevrey regularity and analyticity of the solutions in $G_{\sigma,s}^1(\mathbb{R})$ ($\sigma \geq 1, s > \frac{3}{2}$). As a special case, the Eq. (1.1) admits a unique analytic solution with $G_{1,s}^1(\mathbb{R})$ data, which is analytic in both variables, locally in time and globally in space. Furthermore, we obtain an estimate for the analytic lifespan and the continuity of the data-to-solution mapping.

We sketch the outline of this paper. In section 2, we recall the Littlewood–Paley theory and the transport theory. In the section 3 and section 4, we establish the local well-posedness of the Cauchy problem (1.7)–(1.8) in the Besov spaces and the critical Besov space, respectively. Section 5 is devoted to the blow-up criteria

and conservative law property of the strong solutions. In section 6, we investigate the Gevrey regularity and analyticity of the solutions.

Notation. All function spaces are considered in \mathbb{R} , and we shall drop them in our notation if there is no ambiguity. We denote by C the estimates that hold up to some universal constant which may change from line to line but whose meaning is clear throughout the context.

2. Preliminaries

Our results mostly rely on the Littlewood–Paley decomposition. Unless otherwise specified, all the results which are presented in this section are proved in [1,5,21].

To define a Littlewood–Paley decomposition, fix a function $\psi(x) \in C_0^\infty(\mathcal{B}_{4/3})$ with $\mathcal{B}_{4/3} = \{x \in \mathbb{R}; |x| \leq \frac{4}{3}\}$ and a function $\varphi(x) \in C_0^\infty(\mathcal{C})$ with $\mathcal{C} = \{x \in \mathbb{R}; \frac{3}{4} \leq |x| \leq \frac{8}{3}\}$ such that

$$\begin{aligned}\psi(x) + \sum_{j \geq 0} \varphi(2^{-j}x) &= 1, \quad \forall x \in \mathbb{R}, \\ |i-j| \geq 2 &\implies \text{supp } \varphi(2^{-j}\cdot) \cap \text{supp } \varphi(2^{-i}\cdot) = \emptyset, \\ j \geq 1 &\implies \text{supp } \psi(\cdot) \cap \text{supp } \varphi(2^{-j}\cdot) = \emptyset.\end{aligned}$$

The nonhomogeneous dyadic blocks $\{\Delta_q\}_{q \in \mathbb{N}^+}$ are defined as follows:

$$\Delta_q u = 0, \quad \text{if } q < -1; \quad \Delta_{-1} u := \psi(D)u; \quad \Delta_q u := \varphi(2^{-q}D)u, \quad \text{if } q \geq 0,$$

where $f(D)$ stands for the pseudo-differential operator $u \rightarrow \mathcal{F}^{-1}(f\mathcal{F}u)$. Since $\varphi(\xi) = \psi(\frac{\xi}{2}) - \psi(\xi)$, we also introduce the following low frequency cut-off S_q :

$$S_q u := \varphi(2^{-q}D)u = \sum_{1 \leq p \leq q-1} \Delta_p u.$$

Definition 2.1. Let $1 \leq p, r \leq \infty$ and $s \in \mathbb{R}$, the nonhomogeneous Besov space $B_{p,r}^s$ ($r < \infty$) in one dimension is defined by

$$B_{p,r}^s := \left\{ u \in \mathcal{S}'; \quad \|u\|_{B_{p,r}^s} = \|(2^{sj} \|\Delta_j u\|_{L^p})_{j \in \mathbb{N}}\|_{l^r} = \left(\sum_{j \in \mathbb{N}} 2^{rsj} \|\Delta_j u\|_{L^p}^r \right)^{\frac{1}{r}} < \infty \right\}.$$

Moreover, if $s = \infty$, we define $B_{p,r}^\infty := \bigcap_{s > 0} B_{p,r}^s$.

Remark 2.2. The following classical properties will be used freely in the paper:

- Density: \mathcal{S} is dense in $B_{p,r}^s$ if $1 \leq p, r < \infty$, where \mathcal{S} denotes the Schwartz space.
- Algebraic properties: If $s > 0$, $B_{p,r}^s \cap L^\infty$ is an algebra. Furthermore, $B_{p,r}^s$ is an algebra provided that $s > 1/p$ or $s = 1/p$ and $r = 1$.
- Embedding: If $p_1 \leq p_2$ and $r_1 \leq r_2$, then $B_{p_1,r_1}^s \hookrightarrow B_{p_2,r_2}^{s-(1/p_1-1/p_2)}$. If $s_1 < s_2$, the embedding $B_{p,r_2}^{s_2} \hookrightarrow B_{p,r_1}^{s_1}$ is locally compact.
- Fatou’s lemma: If $\{u_n\}_{n \in \mathbb{N}^+}$ is bounded in $B_{p,r}^s$ and $u_n \rightarrow u$ in \mathcal{S}' , then $u \in B_{p,r}^s$ and

$$\|u\|_{B_{p,r}^s} \leq \liminf_{n \rightarrow \infty} \|u_n\|_{B_{p,r}^s}.$$

- If $s_1 \leq \frac{1}{p} < s_2$ ($s_2 \geq \frac{1}{p}$ if $r = 1$) and $s_1 + s_2 > 0$, then we have

$$\|uv\|_{B_{p,r}^{s_1}} \leq C\|u\|_{B_{p,r}^{s_1}}\|v\|_{B_{p,r}^{s_2}}.$$

– A smooth function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is said to be an S^m -multiplier if $\forall \alpha \in \mathbb{N}^n$, there exists a constant $C_\alpha > 0$ such that $|\partial^\alpha f(\xi)| \leq C_\alpha(1 + |\xi|)^{m-|\alpha|}$, for all $\xi \in \mathbb{R}^d$. The operator $f(D)$ is continuous from $B_{p,r}^s$ to $B_{p,r}^{s-m}$, for all $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$.

Lemma 2.3. *If $u \in B_{p,r}^{s_1} \cap B_{p,r}^{s_2}$, then $u \in B_{p,r}^{\theta s_1 + (1-\theta)s_2}$ for any $\theta \in [0, 1]$, and we have*

$$\|u\|_{B_{p,r}^{\theta s_1 + (1-\theta)s_2}} \leq \|u\|_{B_{p,r}^{s_1}}^\theta \|u\|_{B_{p,r}^{s_2}}^{1-\theta}.$$

For $\forall s \in \mathbb{R}$, $\epsilon > 0$ and $1 \leq p \leq \infty$, there exists a constant $C > 0$ such that

$$\|u\|_{B_{p,1}^s} \leq C \frac{\epsilon + 1}{\epsilon} \|u\|_{B_{p,\infty}^s} \left(1 + \log \frac{\|u\|_{B_{p,\infty}^{s+\epsilon}}}{\|u\|_{B_{p,\infty}^s}} \right).$$

Lemma 2.4 (Commutator estimates). *Let $\sigma > 0$, $1 \leq r \leq \infty$ and $1 \leq p \leq p_1 \leq \infty$. Let v be a vector field over \mathbb{R} . Then the following estimate holds:*

$$\|(2^{j\sigma} \|[v\partial_x, \Delta_j]f\|_{L^p})_{j \in \mathbb{N}}\|_{l^r} \leq C(\|v_x\|_{L^\infty} \|f\|_{B_{p,r}^\sigma} + \|f_x\|_{L^{p_2}} \|v\|_{B_{p_1,r}^{\sigma-1}}),$$

where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. In addition, if $\sigma < 1$, we have

$$\|(2^{j\sigma} \|[v\partial_x, \Delta_j]f\|_{L^p})_{j \in \mathbb{N}}\|_{l^r} \leq C\|v_x\|_{L^\infty} \|f\|_{B_{p,r}^\sigma}.$$

The following result is in some sense a generalization of the Gronwall's Lemma.

Lemma 2.5 (Osgood's Lemma). *Let $\rho \geq 0$ be a measurable function, let $\gamma > 0$ be a locally integrable function and let μ be a continuous and increasing function. For some $c \geq 0$, if*

$$\rho(t) \leq c + \int_{t_0}^t \gamma(s) \mu(\rho(s)) ds.$$

- If $c > 0$, then $-\mathcal{M}(\rho(t)) + \mathcal{M}(c) \leq \int_{t_0}^t \gamma(s) ds$, where $\mathcal{M}(x) := \int_x^1 \frac{1}{\mu(r)} dr$.
- If $c = 0$ and μ satisfies the condition $\int_0^1 \frac{dr}{\mu(r)} dr = +\infty$, then $\rho \equiv 0$.

Our results concerning Eq. (1.1) rely strongly on a priori estimates in Besov spaces for the transport equation:

$$\begin{cases} \partial_t f + v \partial_x f = g, \\ f(x, 0) = f_0(x). \end{cases} \quad (2.1)$$

Lemma 2.6 (A priori estimates). *Let $1 \leq p, r \leq \infty$ and $s \geq -\min(\frac{1}{p}, 1 - \frac{1}{p})$. Assume that $f_0 \in B_{p,r}^s$ and $g \in L^1([0, T]; B_{p,r}^s)$. For any solution $f \in L^\infty([0, T]; B_{p,r}^s)$ of (2.1) with $v_x \in L^1([0, T]; B_{p,r}^{s-1})$ if $s > 1 + \frac{1}{p}$ or $v_x \in L^1([0, T]; B_{p,r}^{\frac{1}{p}} \cap L^\infty)$ otherwise.*

- If $r = 1$ or $s \neq 1 + \frac{1}{p}$, there exists $C > 0$ depending only on s, p and r such that

$$\|f\|_{B_{p,r}^s} \leq \exp\{CV_p(t)\} \|f_0\|_{B_{p,r}^s} + \int_0^t \exp\{CV_p(t) - CV_p(s)\} \|g(s)\|_{B_{p,r}^s} ds, \quad (2.2)$$

with

$$V_p(t) := \begin{cases} \int_0^t \|v_x(s)\|_{B_{p,\infty}^{\frac{1}{p}} \cap L^\infty} ds, & \text{if } s < 1 + \frac{1}{p}; \\ \int_0^t \|v_x(s)\|_{B_{p,r}^{s-1}} ds, & \text{if } s > 1 + \frac{1}{p} \text{ or } s = 1 + \frac{1}{p}, r = 1. \end{cases} \quad (2.3)$$

- If $r < \infty$, then $f \in C([0, T]; B_{p,r}^s)$. If $r = \infty$, then $f \in C([0, T]; B_{p,1}^{s'})$, for all $s' < s$.
- If $v = f$ and $s > 0$, the inequality (2.2) holds true with $V_p(t) := \int_0^t \|v_x(s)\|_{L^\infty} ds$.

Lemma 2.7. Let p, r, s, f_0 and g be as in Lemma 2.6. Suppose that $v \in L^\rho([0, T]; B_{\infty,\infty}^{-M})$ for some $\rho > 1$, $M > 0$ and $v_x \in L^1([0, T]; B_{p,r}^{s-1})$ if $s > 1 + \frac{1}{p}$ or $s = 1 + \frac{1}{p}$ and $r = 1$, and $v_x \in L^1([0, T]; B_{p,\infty}^{\frac{1}{p}} \cap L^\infty)$ if $s < 1 + \frac{1}{p}$. Then the transport equation (2.1) admits a unique solution u in

- the space $C([0, T]; B_{p,r}^s)$ if $r < \infty$;
- the space $L^\infty([0, T]; B_{p,r}^s) \cap (\bigcap_{s' < s} C([0, T]; B_{p,1}^{s'}))$ if $r < \infty$.

Moreover, the inequalities of Lemma 2.6 hold true.

3. Local well-posedness in $B_{p,r}^s$, $p, r \in [1, +\infty]$ and $s > \max\{1 + \frac{1}{p}, \frac{3}{2}\}$

In this section, we prove the local well-posedness of the Cauchy problem in the Besov spaces by using the Littlewood–Paley theory and the transport equation theory.

Definition 3.1. For $T > 0$, $s \in \mathbb{R}$ and $1 \leq p \leq +\infty$, we set

$$\begin{aligned} E_{p,r}^s(T) &:= C([0, T]; B_{p,r}^s) \cap C^1([0, T]; B_{p,r}^{s-1}), \quad \text{if } r < \infty, \\ E_{p,\infty}^s(T) &:= L^\infty([0, T]; B_{p,\infty}^s) \cap Lip([0, T]; B_{p,\infty}^{s-1}), \end{aligned}$$

and $E_{p,r}^s := \bigcap_{T>0} E_{p,r}^s(T)$.

Our first main result can be stated as follows.

Theorem 3.2. Assume that $u_0 \in B_{p,r}^s$, where $1 \leq p, r \leq +\infty$ and $s > \max\{1 + \frac{1}{p}, \frac{3}{2}\}$. Then there exists a $T := T(\|u_0\|_{B_{p,r}^s}) > 0$ such that the initial-value problem (1.7)–(1.8) admits a unique solution $u \in E_{p,r}^s(T)$. Moreover, the solution mapping $u_0 \mapsto u$ is continuous from $B_{p,r}^s$ into

$$C([0, T]; B_{p,r}^{s'}) \cap C^1([0, T]; B_{p,r}^{s'-1})$$

for all $s' < s$ if $r = \infty$, and $s' = s$ if $1 \leq r < \infty$.

Remark 3.3. The Theorem 3.2 extends the result obtained in [29]. Indeed, since $B_{2,2}^s = H^s$ for every $s \in \mathbb{R}$, we get the local well-posedness of (1.7)–(1.8) in the Sobolev spaces $C([0, T]; H^s) \cap C^1([0, T]; H^{s-1})$ with $s > \frac{3}{2}$, which can be proved by Galerkin type approximation scheme.

The uniqueness and continuity of the solution with respect to the initial data can be achieved by the following a priori estimates.

Lemma 3.4. Assume that $1 \leq p, r \leq \infty$ and $s > \max\{\frac{3}{2}, 1 + \frac{1}{p}\}$. Let u and $v \in L^\infty([0, T]; B_{p,r}^s) \cap C([0, T]; \mathcal{S}')$ be two solutions of the Cauchy problem (1.7)–(1.8) with the initial data u_0 and $v_0 \in B_{p,r}^s$, respectively. Then for $\forall t \in [0, T]$, we have that

- If $s > \max\{\frac{3}{2}, 1 + \frac{1}{p}\}$ but $s \neq 2 + \frac{1}{p}$, the following estimate holds:

$$\begin{aligned} & \|u(t) - v(t)\|_{B_{p,r}^{s-1}} \\ & \leq C \|u_0 - v_0\|_{B_{p,r}^{s-1}} \exp \left\{ C \sum_{i=0}^{2k} \int_0^t (\|u(\tau)\|_{B_{p,r}^s}^i + \|v(\tau)\|_{B_{p,r}^s}^i) d\tau \right\}. \end{aligned} \quad (3.1)$$

- If $s = 2 + \frac{1}{p}$, for $\forall \theta \in (0, 1)$, the following estimate holds:

$$\begin{aligned} \|u(t) - v(t)\|_{B_{p,r}^{s-1}} & \leq C \|u_0 - v_0\|_{B_{p,r}^{s-1}}^\theta \left(\|u(t)\|_{B_{p,r}^s}^{1-\theta} + \|v(t)\|_{B_{p,r}^s}^{1-\theta} \right) \\ & \quad \times \exp \left\{ C \theta \sum_{i=0}^{2k} \int_0^t (\|u(\tau)\|_{B_{p,r}^s}^i + \|v(\tau)\|_{B_{p,r}^s}^i) d\tau \right\}. \end{aligned} \quad (3.2)$$

Proof. Let u, v be two solutions of the problem (1.7)–(1.8) with the initial data $u_0, v_0 \in B_{p,r}^s$, and denote $w = v - u$ and $w_0 = v_0 - u_0$. It is obvious that w satisfies the following equation:

$$\partial_t w + u^k \partial_x w = - \sum_{i=0}^{k-1} u^i v^{k-1-i} (\partial_x v) w + \tilde{\mathcal{F}}(w, u, v), \quad (3.3)$$

with $w(x, 0) = w_0(x) = v_0(x) - u_0(x)$, where

$$\begin{aligned} \tilde{\mathcal{F}}(w, u, v) &= \frac{(k-1)(b-k)}{2} P(D) \left[\sum_{i=0}^{k-3} v^i u^{k-3-i} (\partial_x v)^3 w + \sum_{i=0}^2 (\partial_x v)^i (\partial_x u)^{2-i} u^{k-2} \partial_x w \right] \\ &+ \frac{3k-b}{2} \partial_x P(D) \left[\sum_{i=0}^{k-2} v^i u^{k-2-i} (\partial_x v)^2 w + u^{k-1} (\partial_x u + \partial_x v) \partial_x w \right] \\ &+ \frac{b}{k+1} \partial_x P(D) \left[\sum_{i=0}^k v^i u^{k-i} w \right]. \end{aligned} \quad (3.4)$$

- Let $s > \max\{\frac{3}{2}, 1 + \frac{1}{p}\}$ but $s \neq 2 + \frac{1}{p}$, it follows from the Lemma 2.6 that

$$\begin{aligned} e^{-C \int_0^t \|\partial_x(u^k)(\tau')\|_{B_{p,r}^{s-2}} d\tau'} \|w\|_{B_{p,r}^{s-1}} & \leq \|w_0\|_{B_{p,r}^{s-1}} + \int_0^t e^{-C \int_0^\tau \|\partial_x(u^k)(\tau')\|_{B_{p,r}^{s-2}} d\tau'} \\ & \quad \times \left\| - \sum_{i=0}^{k-1} u^i v^{k-1-i} (\partial_x v) w + \tilde{\mathcal{F}}(w, u, v) \right\|_{B_{p,r}^{s-1}} d\tau. \end{aligned} \quad (3.5)$$

Noting that $s-1 > \max\{\frac{1}{2}, \frac{1}{p}\}$, then $B_{p,r}^{s-1} \hookrightarrow L^\infty$ and $B_{p,r}^{s-1}$ is a Banach algebra. Therefore, one can deduce that

$$\int_0^t \|\partial_x u^k(\tau')\|_{B_{p,r}^{s-2}} d\tau' \leq C \int_0^t \|u(\tau')\|_{B_{p,r}^{s-1}}^k d\tau', \quad (3.6)$$

and

$$\begin{aligned}
\left\| \sum_{i=0}^{k-1} u^i v^{k-1-i} (\partial_x v) w \right\|_{B_{p,r}^{s-1}} &\leq C \sum_{i=0}^{k-1} \|u\|_{B_{p,r}^{s-1}}^i \|v\|_{B_{p,r}^{s-1}}^{k-1-i} \|v\|_{B_{p,r}^s} \|w\|_{B_{p,r}^{s-1}} \\
&\leq C \sum_{i=0}^{k-1} \|u\|_{B_{p,r}^s}^i \|v\|_{B_{p,r}^s}^{k-i} \|w\|_{B_{p,r}^{s-1}} \\
&\leq C \sum_{i=0}^{2k} \left(\|u\|_{B_{p,r}^s}^i + \|v\|_{B_{p,r}^s}^i \right) \|w\|_{B_{p,r}^{s-1}}. \tag{3.7}
\end{aligned}$$

If $s > 2 + \frac{1}{p}$, the Besov space $B_{p,r}^{s-2}$ is a Banach algebra. By virtue of the property that $P(D)$ is an S^{-2} -multiplier and the Cauchy–Schwarz inequality, we get that

$$\begin{aligned}
\|\tilde{\mathcal{F}}(w, u, v)\|_{B_{p,r}^{s-1}} &\leq \frac{|b|}{k+1} \left\| \sum_{i=0}^k v^i u^{k-i} w \right\|_{B_{p,r}^{s-2}} \\
&\quad + \frac{|(k-1)(b-k)|}{2} \left\| \sum_{i=0}^{k-3} v^i u^{k-3-i} (\partial_x v)^3 w + \sum_{i=0}^2 (\partial_x v)^i (\partial_x u)^{2-i} u^{k-2} \partial_x w \right\|_{B_{p,r}^{s-2}} \\
&\quad + \frac{|3k-b|}{2} \left\| \sum_{i=0}^{k-2} v^i u^{k-2-i} (\partial_x v)^2 w + u^{k-1} (\partial_x u + \partial_x v) \partial_x w \right\|_{B_{p,r}^{s-2}} \\
&\leq \frac{C|(k-1)(b-k)|}{2} \left(\sum_{i=0}^{k-3} \|u\|_{B_{p,r}^s}^i \|v\|_{B_{p,r}^s}^{k-i} \|w\|_{B_{p,r}^{s-1}} + \sum_{i=0}^2 \|v\|_{B_{p,r}^s}^i \|u\|_{B_{p,r}^s}^{k-i} \|w\|_{B_{p,r}^{s-1}} \right) \\
&\quad + \frac{C|3k-b|}{2} \left(\sum_{i=0}^{k-2} \|v\|_{B_{p,r}^s}^{i+2} \|u\|_{B_{p,r}^s}^{k-2-i} \|w\|_{B_{p,r}^{s-1}} + \|u\|_{B_{p,r}^s}^{k-1} (\|u\|_{B_{p,r}^s} + \|v\|_{B_{p,r}^s}) \|w\|_{B_{p,r}^{s-1}} \right) \\
&\quad + \frac{C|b|}{k+1} \sum_{i=0}^k \|v\|_{B_{p,r}^s}^i \|u\|_{B_{p,r}^s}^{k-i} \|w\|_{B_{p,r}^{s-1}} \\
&\leq C \sum_{i=0}^{2k} \left(\|u\|_{B_{p,r}^s}^i + \|v\|_{B_{p,r}^s}^i \right) \|w\|_{B_{p,r}^{s-1}}. \tag{3.8}
\end{aligned}$$

If $\max\{\frac{3}{2}, 1 + \frac{1}{p}\} < s < 2 + \frac{1}{p}$, we observe that $s-2 < \frac{1}{p} < s-1$ and $(s-2) + (s-1) > 0$. By means of the [Remark 2.2](#), we have that

$$\begin{aligned}
&\left\| P(D) \left[\sum_{i=0}^{k-3} v^i u^{k-3-i} (\partial_x v)^3 w + \sum_{i=0}^2 (\partial_x v)^i (\partial_x u)^{2-i} u^{k-2} \partial_x w \right] \right\|_{B_{p,r}^{s-1}} \\
&\leq C \left\| \sum_{i=0}^{k-3} v^i u^{k-3-i} (\partial_x v)^3 w + \sum_{i=0}^2 (\partial_x v)^i (\partial_x u)^{2-i} u^{k-2} \partial_x w \right\|_{B_{p,r}^{s-2}} \\
&\leq C \left(\sum_{i=0}^{k-3} \|v^i u^{k-3-i} (\partial_x v)^3\|_{B_{p,r}^{s-1}} \|w\|_{B_{p,r}^{s-2}} + \sum_{i=0}^2 \|(\partial_x v)^i (\partial_x u)^{2-i} u^{k-2}\|_{B_{p,r}^{s-1}} \|\partial_x w\|_{B_{p,r}^{s-2}} \right) \\
&\leq C \left(\sum_{i=0}^{k-3} \|u\|_{B_{p,r}^s}^i \|v\|_{B_{p,r}^s}^{k-i} + \sum_{i=0}^2 \|v\|_{B_{p,r}^s}^i \|u\|_{B_{p,r}^s}^{k-i} \right) \|w\|_{B_{p,r}^{s-1}}. \tag{3.9}
\end{aligned}$$

In the last inequality, we used the fact that $B_{p,r}^{s-1}$ is a Banach algebra since $s-1 > \frac{1}{p}$. Similarly, it follows from the [Remark 2.2](#) and the embedding $B_{p,r}^{s-1} \hookrightarrow B_{p,r}^{s-2}$ that

$$\begin{aligned}
& \left\| \partial_x P(D) \left[\sum_{i=0}^{k-2} v^i u^{k-2-i} (\partial_x v)^2 w + u^{k-1} (\partial_x u + \partial_x v) \partial_x w \right] \right\|_{B_{p,r}^{s-1}} \\
& \leq C \left(\sum_{i=0}^{k-2} \|v^i u^{k-2-i} (\partial_x v)^2 w\|_{B_{p,r}^{s-2}} + \|u^{k-1} (\partial_x u + \partial_x v) \partial_x w\|_{B_{p,r}^{s-2}} \right) \\
& \leq C \left(\sum_{i=0}^{k-2} \|v^i u^{k-2-i} (\partial_x v)^2\|_{B_{p,r}^{s-1}} \|w\|_{B_{p,r}^{s-2}} + \|u^{k-1} (\partial_x u + \partial_x v)\|_{B_{p,r}^{s-1}} \|\partial_x w\|_{B_{p,r}^{s-2}} \right) \\
& \leq C \left(\sum_{i=0}^{k-2} \|v\|_{B_{p,r}^s}^{i+2} \|u\|_{B_{p,r}^s}^{k-2-i} + \|u\|_{B_{p,r}^s}^k + \|u\|_{B_{p,r}^s}^{k-1} \|v\|_{B_{p,r}^s} \right) \|w\|_{B_{p,r}^{s-1}}, \tag{3.10}
\end{aligned}$$

and

$$\left\| \partial_x P(D) \left[\sum_{i=0}^k v^i u^{k-i} w \right] \right\|_{B_{p,r}^{s-1}} \leq C \sum_{i=0}^k \|v\|_{B_{p,r}^s}^i \|u\|_{B_{p,r}^s}^{k-i} \|w\|_{B_{p,r}^{s-1}}. \tag{3.11}$$

Combining the estimates of (3.9)–(3.11) and the Cauchy–Schwarz inequality, we deduce that

$$\begin{aligned}
\|\tilde{\mathcal{F}}(w, u, v)\|_{B_{p,r}^{s-1}} & \leq \frac{C|(k-1)(b-k)|}{2} \left(\sum_{i=0}^{k-3} \|u\|_{B_{p,r}^s}^i \|v\|_{B_{p,r}^s}^{k-i} + \sum_{i=0}^2 \|v\|_{B_{p,r}^s}^i \|u\|_{B_{p,r}^s}^{k-i} \right) \|w\|_{B_{p,r}^{s-1}} \\
& \quad + \frac{C|3k-b|}{2} \left(\sum_{i=0}^{k-2} \|v\|_{B_{p,r}^s}^{i+2} \|u\|_{B_{p,r}^s}^{k-2-i} + \|u\|_{B_{p,r}^s}^k + \|u\|_{B_{p,r}^s}^{k-1} \|v\|_{B_{p,r}^s} \right) \|w\|_{B_{p,r}^{s-1}} \\
& \quad + \frac{C|b|}{k+1} \sum_{i=0}^k \|v\|_{B_{p,r}^s}^i \|u\|_{B_{p,r}^s}^{k-i} \|w\|_{B_{p,r}^{s-1}} \\
& \leq C \sum_{i=0}^{2k} \left(\|u\|_{B_{p,r}^s}^i + \|v\|_{B_{p,r}^s}^i \right) \|w\|_{B_{p,r}^{s-1}}. \tag{3.12}
\end{aligned}$$

Plugging the inequalities (3.6)–(3.8) and (3.12) into (3.5), we get

$$\begin{aligned}
& e^{-C \int_0^t \|\partial_x(u^k)(\tau')\|_{B_{p,r}^{s-2}} d\tau'} \|w\|_{B_{p,r}^{s-1}} \leq \|w_0\|_{B_{p,r}^{s-1}} \\
& \quad + C \int_0^t \sum_{i=0}^{2k} (\|u\|_{B_{p,r}^s}^i + \|v\|_{B_{p,r}^s}^i) e^{-C \int_0^\tau \|\partial_x(u^k)(\tau')\|_{B_{p,r}^{s-2}} d\tau'} \|w\|_{B_{p,r}^{s-1}} d\tau. \tag{3.13}
\end{aligned}$$

By applying the Gronwall inequality, we can get (3.1).

• In the case of $s = 2 + \frac{1}{p}$, we shall deal with it by utilizing the interpolation argument. To this end, let us choose s_1, s_2 satisfying $(1-\theta)s_1 + \theta s_2 = s-1$, for $\forall \theta \in (0, 1)$, where

$$s-1 < s_1 < s, \quad \max \left\{ \frac{1}{2}, \frac{1}{p} \right\} < s_2 < s-1.$$

By using the complex interpolation inequality (see Lemma 2.3) and the consequence obtained in the previous case, we get

$$\begin{aligned}
\|u(t) - v(t)\|_{B_{p,r}^{s-1}} &\leq \|u(t) - v(t)\|_{B_{p,r}^{s_1}}^{1-\theta} \|u(t) - v(t)\|_{B_{p,r}^{s_2}}^{\theta} \\
&\leq \left(\|u(t)\|_{B_{p,r}^{s_1}} + \|v(t)\|_{B_{p,r}^{s_1}} \right)^{1-\theta} \|u_0 - v_0\|_{B_{p,r}^{s_2}}^{\theta} \\
&\quad \times \exp \left\{ C\theta \sum_{i=0}^{2k} \int_0^t (\|u(\tau)\|_{B_{p,r}^s}^i + \|v(\tau)\|_{B_{p,r}^s}^i) d\tau \right\} \\
&\leq \|u_0 - v_0\|_{B_{p,r}^{s-1}}^{\theta} \left(\|u(t)\|_{B_{p,r}^s}^{1-\theta} + \|v(t)\|_{B_{p,r}^s}^{1-\theta} \right) \\
&\quad \times \exp \left\{ C\theta \sum_{i=0}^{2k} \int_0^t (\|u(\tau)\|_{B_{p,r}^s}^i + \|v(\tau)\|_{B_{p,r}^s}^i) d\tau \right\}. \tag{3.14}
\end{aligned}$$

This completes the proof of [Lemma 3.4](#). \square

Next, we focus on the smooth approximation of the solutions to the problem (1.7)–(1.8).

Lemma 3.5. *Let p, r be the same as in the statement of [Lemma 3.4](#). Assume that $s > \max\{\frac{3}{2}, 1 + \frac{1}{p}\}$ and $u_0 \in B_{p,r}^s$ is the initial data.*

(1) *There exists a sequence of smooth functions $\{u^{(n)}\}_{n \in \mathbb{N}^+} \in C([0, \infty); B_{p,r}^{\infty})$, which solves the following iterative linear transport equation: $u^{(0)} = 0$ and*

$$\begin{cases} \partial_t u^{(n+1)} + (u^{(n)})^k \partial_x u^{(n+1)} + \sum_{i=1}^3 \mathcal{F}_i^{(n)}(x, t) = 0, \\ u^{(n+1)}(x, 0) = u_0^{(n+1)}(x) = S_{n+1} u_0, \end{cases} \tag{3.15}$$

where $S_{n+1} u_0$ is the low frequency cut-off of u_0 given by $S_{n+1} u_0 := \sum_{p=-1}^n \Delta_p u_0$. $\mathcal{F}_1^{(n)}(x, t) = \frac{(k-1)(b-k)}{2} P(D)[(u^{(n)})^{k-2} (\partial_x u^{(n)})^3]$, $\mathcal{F}_2^{(n)}(x, t) = \frac{3k-b}{2} \partial_x P(D)[(u^{(n)})^{k-1} (\partial_x u^{(n)})^2]$ and $\mathcal{F}_3^{(n)}(x, t) = \frac{b}{k+1} \times \partial_x P(D)(u^{(n)})^{k+1}$.

(2) *There exists a time $T := T(\|u_0\|_{B_{p,r}^s}) > 0$ such that the approximation solutions $\{u^{(n)}\}_{n \in \mathbb{N}^+}$ are uniformly bounded in $E_{p,r}^s(T)$. Moreover, $\{u^{(n)}\}_{n \in \mathbb{N}^+}$ is a Cauchy sequence in $C([0, T]; B_{p,r}^{s-1})$ and it converges to a function u in $C([0, T]; B_{p,r}^{s-1})$.*

Proof. (1) Noting that all the datum $S_{n+1} u_0$ belong to $B_{p,r}^{\infty}$, by applying the [Lemma 2.7](#) and the induction argument with respect to n , we conclude that Cauchy problem (1.7)–(1.8) admits a global solution $u^{(n)}$ in $C([0, \infty); B_{p,r}^{\infty})$.

(2) We first prove the uniformly boundedness of $\{u^{(n)}\}_{n \in \mathbb{N}^+}$ with some $T > 0$. By applying the [Lemma 2.6](#) to the first equation of (3.15), we obtain

$$\begin{aligned}
\|u^{(n+1)}\|_{B_{p,r}^s} &\leq \exp\{CV(t)\} \|S_{n+1} u_0\|_{B_{p,r}^s} \\
&\quad + \int_0^t \exp\{CV(t) - CV(s)\} \sum_{i=1}^3 \|\mathcal{F}_i^{(n)}(x, s)\|_{B_{p,r}^s} ds, \tag{3.16}
\end{aligned}$$

where

$$V(t) := \int_0^t \|\partial_x (u^{(n)})^k(s)\|_{B_{p,r}^{s-1}} ds \leq C \int_0^t \|(u^{(n)})(s)\|_{B_{p,r}^s}^k ds. \tag{3.17}$$

Since $s > \max\{\frac{3}{2}, 1 + \frac{1}{p}\}$, it follows from the [Remark 2.2](#) that

$$\begin{aligned}
\|\mathcal{F}_1^{(n)}(x, t)\|_{B_{p,r}^s} &\leq \left| \frac{(k-1)(b-k)}{2} \right| \|P(D)[(u^{(n)})^{k-2}(\partial_x u^{(n)})^3]\|_{B_{p,r}^s} \\
&\leq C\|(u^{(n)})^{k-2}(\partial_x u^{(n)})^3\|_{B_{p,r}^{s-2}} \\
&\leq C\|(u^{(n)})^{k-2}\|_{B_{p,r}^{s-2}} \|(\partial_x u^{(n)})^3\|_{B_{p,r}^{s-1}} \leq C\|u^{(n)}\|_{B_{p,r}^s}^{k+1}.
\end{aligned} \tag{3.18}$$

Using the facts that $B_{p,r}^{s-1}$ is a Banach algebra and the operator $\partial_x P(D)$ is an S^{-1} -multiplier, we can deduce that

$$\begin{aligned}
\|\mathcal{F}_2^{(n)}(x, t)\|_{B_{p,r}^s} &\leq \left| \frac{3k-b}{2} \right| \|\partial_x P(D)[(u^{(n)})^{k-1}(\partial_x u^{(n)})^2]\|_{B_{p,r}^s} \\
&\leq C\|(u^{(n)})^{k-1}\|_{B_{p,r}^{s-1}} \|\partial_x u^{(n)}\|_{B_{p,r}^{s-1}}^2 \leq C\|u^{(n)}\|_{B_{p,r}^s}^{k+1}.
\end{aligned} \tag{3.19}$$

Similarly, we also have

$$\|\mathcal{F}_3^{(n)}(x, t)\|_{B_{p,r}^s} \leq C\|u^{(n)}\|_{B_{p,r}^s}^{k+1}. \tag{3.20}$$

Plugging the estimates (3.17)–(3.20) into (3.16), we get that

$$\|u^{(n+1)}(t)\|_{B_{p,r}^s} \leq \exp\{CV(t)\}\|u_0\|_{B_{p,r}^s} + C \int_0^t \exp\{CV(t) - CV(s)\} \|u^{(n)}(s)\|_{B_{p,r}^s}^{k+1} ds, \tag{3.21}$$

since $\|S_{n+1}u_0\|_{B_{p,r}^s} \leq C\|u_0\|_{B_{p,r}^s}$, where $C > 0$ is a constant independent of n .

Let us choose a $T > 0$ satisfying $2kCT\|u_0\|_{B_{p,r}^s}^k < 1$, and

$$\|u^{(n)}(t)\|_{B_{p,r}^s} \leq \frac{\sqrt[k]{2}\|u_0\|_{B_{p,r}^s}}{\sqrt[k]{1 - 4kC\|u_0\|_{B_{p,r}^s}^k t}}, \quad \text{for } \forall t \in (0, T). \tag{3.22}$$

In view of the definition of $V(t)$, we can deduce from (3.22) that

$$\begin{aligned}
\exp\{CV(t) - CV(s)\} &\leq \exp\left\{ \int_s^t \frac{2C\|u_0\|_{B_{p,r}^s}^k}{1 - 4kC\|u_0\|_{B_{p,r}^s}^k \tau} d\tau \right\} \\
&= \exp\left\{ -\frac{1}{2k} \int_s^t \frac{d(1 - 4kC\|u_0\|_{B_{p,r}^s}^k \tau)}{1 - 4kC\|u_0\|_{B_{p,r}^s}^k \tau} \right\} \\
&= \frac{\sqrt[2k]{1 - 4kC\|u_0\|_{B_{p,r}^s}^k s}}{\sqrt[2k]{1 - 4kC\|u_0\|_{B_{p,r}^s}^k t}}.
\end{aligned} \tag{3.23}$$

Since $V(0) = 0$, by taking $s = 0$, it follows from (3.23) that

$$\exp\{CV(t)\} \leq \frac{1}{\sqrt[2k]{1 - 4kC\|u_0\|_{B_{p,r}^s}^k t}}. \tag{3.24}$$

Combing (3.23)–(3.24) and (3.21), we get that

$$\begin{aligned}
& \|u^{(n+1)}(t)\|_{B_{p,r}^s} \\
& \leq \frac{\|u_0\|_{B_{p,r}^s}}{2^k \sqrt{1-4kC}\|u_0\|_{B_{p,r}^s}^k t} + \frac{C\|u_0\|_{B_{p,r}^s}}{2^k \sqrt{1-4kC}\|u_0\|_{B_{p,r}^s}^k t} \int_0^t \frac{2^{1+\frac{1}{k}}\|u_0\|_{B_{p,r}^s}^k}{(1-4kC\|u_0\|_{B_{p,r}^s}^k s)^{1+\frac{1}{2k}}} ds \\
& \leq \frac{\|u_0\|_{B_{p,r}^s}}{2^k \sqrt{1-4kC}\|u_0\|_{B_{p,r}^s}^k t} - \frac{\sqrt[2k]{2}\|u_0\|_{B_{p,r}^s}}{2k \sqrt[2k]{1-4kC}\|u_0\|_{B_{p,r}^s}^k t} \int_0^t \frac{d(1-4kC\|u_0\|_{B_{p,r}^s}^k s)}{(1-4kC\|u_0\|_{B_{p,r}^s}^k s)^{1+\frac{1}{2k}}} \\
& = \frac{(1-\sqrt[2k]{2})\|u_0\|_{B_{p,r}^s}}{2^k \sqrt[2k]{1-4kC}\|u_0\|_{B_{p,r}^s}^k t} + \frac{\sqrt[2k]{2}\|u_0\|_{B_{p,r}^s}}{\sqrt[2k]{1-4kC}\|u_0\|_{B_{p,r}^s}^k t} \\
& \leq \frac{\sqrt[2k]{2}\|u_0\|_{B_{p,r}^s}}{\sqrt[2k]{1-4kC}\|u_0\|_{B_{p,r}^s}^k t}, \quad \text{for } t \in [0, T].
\end{aligned} \tag{3.25}$$

Hence, the sequence $\{u^{(n)}\}_{n \in \mathbb{N}^+}$ is uniformly bounded in $C([0, T]; B_{p,r}^s)$. By using the Equ. (3.15) and applying the similar argument, we can prove that $\{\partial_t u^{(n)}\}_{n \in \mathbb{N}^+}$ is uniformly bounded in $C([0, T]; B_{p,r}^{s-1})$. Thus, the sequence $\{u^{(n)}\}_{n \in \mathbb{N}^+}$ is uniformly bounded in $E_{p,r}^s$.

Next, we are going to show that

$$\{u^{(n)}\}_{n \in \mathbb{N}^+} \text{ is a Cauchy sequence in } C([0, T]; B_{p,r}^{s-1}).$$

To this end, for arbitrary $m, n \in \mathbb{N}^+$, consider the following equation

$$\begin{cases} \partial_t(u^{(n+m+1)} - u^{(n+1)}) + (u^{(m+n)})^k \partial_x(u^{(n+m+1)} - u^{(n+1)}) \\ \quad = ((u^{(n)})^k - (u^{(m+n)})^k) \partial_x u^{(n+1)} + \mathcal{F}(u^{(n)}, u^{(m+n)}), \\ (u^{(n+m+1)} - u^{(n+1)})(x, 0) = S_{m+n+1}u_0 - S_{n+1}u_0, \end{cases} \tag{3.26}$$

where

$$\begin{aligned}
\mathcal{F}(u^{(n)}, u^{(m+n)}) &= \frac{(k-1)(b-k)}{2} P(D) \left[(u^{(n)})^{k-2} (\partial_x u^{(n)})^3 - (u^{(m+n)})^{k-2} (\partial_x u^{(m+n)})^3 \right] \\
&+ \frac{3k-b}{2} \partial_x P(D) \left[(u^{(n)})^{k-1} (\partial_x u^{(n)})^2 - (u^{(m+n)})^{k-1} (\partial_x u^{(m+n)})^2 \right] \\
&+ \frac{b}{k+1} \partial_x P(D) \left[(u^{(n)})^{k+1} - (u^{(m+n)})^{k+1} \right].
\end{aligned} \tag{3.27}$$

Since $B_{p,r}^{s-1}$ is a Banach algebra, we have

$$\begin{aligned}
& \|((u^{(n)})^k - (u^{(m+n)})^k) \partial_x u^{(n+1)}\|_{B_{p,r}^{s-1}} \\
& \leq C \|((u^{(n)})^k - (u^{(m+n)})^k)\|_{B_{p,r}^{s-1}} \|\partial_x u^{(n+1)}\|_{B_{p,r}^{s-1}} \\
& \leq C \sum_{i=0}^k \|u^{(n)}\|_{B_{p,r}^s}^i \|u^{(m+n)}\|_{B_{p,r}^s}^{k-i} \|u^{(n)} - u^{(m+n)}\|_{B_{p,r}^{s-1}} \|u^{(n+1)}\|_{B_{p,r}^s} \\
& \leq C \sum_{i=0}^{2k} \left(\|u^{(n)}\|_{B_{p,r}^s}^i + \|u^{(m+n)}\|_{B_{p,r}^s}^i \right) \|u^{(n)} - u^{(m+n)}\|_{B_{p,r}^{s-1}} \|u^{(n+1)}\|_{B_{p,r}^s}.
\end{aligned} \tag{3.28}$$

Similar to the proof of Lemma 3.4, the term $\mathcal{F}(u^{(n)}, u^{(m+n)})$ can be estimated by

$$\|\mathcal{F}(u^{(n)}, u^{(m+n)})\|_{B_{p,r}^{s-1}} \leq C \|u^{(n)} - u^{(m+n)}\|_{B_{p,r}^{s-1}} \sum_{i=0}^{2k} \left(\|u^{(n)}\|_{B_{p,r}^s}^i + \|u^{(m+n)}\|_{B_{p,r}^s}^i \right). \quad (3.29)$$

Thanks to the facts that $\Delta_i \Delta_j u = 0$ if $|i - j| \geq 2$ and $\|\Delta_j u\|_p \leq C \|u\|_p$ for $j \geq -1$, we have

$$\begin{aligned} \|u_0^{(m+n)} - u_0^{(n)}\|_{B_{p,r}^{s-1}} &= \|S_{n+m+1} u_0 - S_{n+1} u_0\|_{B_{p,r}^{s-1}} \\ &= \left[\sum_{k \geq -1} 2^{-kr} 2^{krs} \left\| \Delta_k \left(\sum_{n+1 \leq j \leq n+m} \Delta_j u_0 \right) \right\|_{L^p}^r \right]^{1/r} \\ &\leq 2^{-n} \|u_0\|_{B_{p,r}^s}. \end{aligned} \quad (3.30)$$

On the one hand, in the case of $s > \{\frac{3}{2}, 1 + \frac{1}{p}\}$ but $s \neq 2 + \frac{1}{p}$, by applying the [Lemma 2.6](#) to the Equ. (3.15), it follows from (3.28)–(3.30) that

$$\begin{aligned} &\|u^{(m+n)}(t) - u^{(n)}(t)\|_{B_{p,r}^{s-1}} \\ &\leq \|u_0^{(m+n)} - u_0^{(n)}\|_{B_{p,r}^{s-1}} e^{C \int_0^t \|\partial_x(u^k)(\tau')\|_{B_{p,r}^{s-2}} d\tau'} + \int_0^t e^{-C \int_\tau^t \|\partial_x(u^k)(\tau')\|_{B_{p,r}^{s-2}} d\tau'} \\ &\quad \times \left(\|((u^{(n)})^k - (u^{(m+n)})^k) \partial_x u^{(n+1)}\|_{B_{p,r}^{s-1}} + \|\mathcal{F}(u^{(n)}, u^{(m+n)})\|_{B_{p,r}^{s-1}} \right) d\tau \\ &\leq C \|u_0^{(m+n)} - u_0^{(n)}\|_{B_{p,r}^{s-1}} + C \int_0^t \|u^{(n)} - u^{(m+n)}\|_{B_{p,r}^{s-1}} (1 + \|u^{(n+1)}\|_{B_{p,r}^s}) \\ &\quad \times \sum_{i=0}^{2k} \left(\|u^{(n)}\|_{B_{p,r}^s}^i + \|u^{(m+n)}\|_{B_{p,r}^s}^i \right) ds \\ &\leq C 2^{-n} + C \int_0^t \|u^{(n)}(s) - u^{(m+n)}(s)\|_{B_{p,r}^{s-1}} ds. \end{aligned} \quad (3.31)$$

In the last inequality, we used the uniformly boundedness of the $\{u^{(n)}\}_{n \in \mathbb{N}^+}$ in $C([0, T]; B_{p,r}^s)$.
Set

$$\Xi_{n,m}(t) = \|u^{(m+n)}(t) - u^{(n)}(t)\|_{B_{p,r}^{s-1}}.$$

By (3.31) and using the induction with respect to n , we get that

$$\begin{aligned} \sup_{t \in [0, T]} \Xi_{n+1,m}(t) &\leq C \left(2^{-n} + \frac{2^{2-n} t^2}{2!} + \cdots + \frac{t^n}{n!} \right) + C \int_0^t \frac{(t-\tau)^n}{n!} \Xi_{0,m}(\tau) d\tau \\ &\leq C 2^{-n} \left(1 + 2T + \cdots + \frac{(2T)^n}{n!} \right) + \frac{CT^{n+1}}{(n+1)!} \\ &\leq C 2^{-n} e^{2T} + \frac{CT^{n+1}}{(n+1)!} \longrightarrow 0, \quad \text{for } \forall m \in \mathbb{N}^+, \text{ as } n \rightarrow \infty. \end{aligned} \quad (3.32)$$

Therefore, we get that the $\{u^{(n)}\}_{n \in \mathbb{N}^+}$ is a Cauchy sequence in $C([0, T]; B_{p,r}^{s-1})$.

If $s = 2 + \frac{1}{p}$, by using the uniformly boundedness of $\{u^{(n)}\}_{n \in \mathbb{N}^+}$ and the interpolation argument as we did in the proof of [Lemma 3.4](#), one can conclude that $\{u^{(n)}\}_{n \in \mathbb{N}^+}$ is also a Cauchy sequence in $C([0, T]; B_{p,r}^{s-1})$. This completes the proof of [Lemma 3.5](#). \square

Proof of Theorem 3.2. By [Lemma 3.5](#), we obtain that $\{u^{(n)}\}_{n \in \mathbb{N}^+}$ is a Cauchy sequence in $C([0, T]; B_{p,r}^{s-1})$, so there exists a function $u \in C([0, T]; B_{p,r}^{s-1})$ such that $u^{(n)} \rightarrow u$ in $C([0, T]; B_{p,r}^{s-1})$ as $n \rightarrow \infty$. Since $\{u^{(n)}\}_{n \in \mathbb{N}^+}$ is uniformly bounded in $C([0, T]; B_{p,r}^s)$, it follows from the [Remark 2.2](#) that u is actually in $C([0, T]; B_{p,r}^s)$, and

$$\|u\|_{B_{p,r}^s} \leq C \liminf_{n \rightarrow \infty} \|u^{(n)}\|_{B_{p,r}^s}.$$

As $\{u^{(n)}\}_{n \in \mathbb{N}^+}$ converges to u in $C([0, T]; B_{p,r}^{s-1})$, an interpolation argument ensures that the convergence holds true in $C([0, T]; B_{p,r}^{s'})$, $s' < s$. Then, by passing to the limit in [\(3.15\)](#), we conclude that u is a solution to the problem [\(1.7\)–\(1.8\)](#). Since $u \in C([0, T]; B_{p,r}^s)$, the right hand side of the equation

$$\begin{aligned} \partial_t u + u^k \partial_x u = & -P(D) \left[\frac{(k-1)(b-k)}{2} u^{k-2} (\partial_x u)^3 \right] \\ & - \partial_x P(D) \left[\frac{3k-b}{2} u^{k-1} (\partial_x u)^2 + \frac{b}{k+1} u^{k+1} \right] \end{aligned} \quad (3.33)$$

belongs to $C([0, T]; B_{p,r}^s)$. Particularly, in the case of $r < \infty$, [Lemma 2.7](#) enables us to conclude that $u \in C([0, T]; B_{p,r}^{s'})$ for $\forall s' < s$. Finally, by means of the equation again, we get that $\partial_t u \in C([0, T]; B_{p,r}^{s'})$ when $r < \infty$, and $\partial_t u \in L^\infty([0, T]; B_{p,r}^{s'})$ when $r = \infty$. Moreover, the continuity of the solution in $C([0, T]; B_{p,r}^s) \cap C^1([0, T]; B_{p,r}^{s-1})$ can be proved by a standard use of a sequence of viscosity approximate solutions $\{u_\epsilon\}_{\epsilon > 0}$, which uniformly converges in $C([0, T]; B_{p,r}^s) \cap C^1([0, T]; B_{p,r}^{s-1})$. Thus the proof of [Theorem 3.2](#) is completed. \square

4. Local well-posedness in critical space $B_{2,1}^{\frac{3}{2}}$

In the previous section, the local well-posedness in $B_{p,r}^s$ with $s > \max\{\frac{3}{2}, 1 + \frac{1}{p}\}$ is established. However, the following result shows that the local well-posedness in $B_{2,\infty}^{\frac{3}{2}}$ fails. Namely,

Lemma 4.1. *There exists solution $u \in L^\infty(0, \infty; B_{2,\infty}^{\frac{3}{2}})$ to the Cauchy problem [\(1.7\)–\(1.8\)](#) such that for any $T, \epsilon > 0$, there exists a solution $v \in L^\infty(0, \infty; B_{2,\infty}^{\frac{3}{2}})$ with*

$$\|u(0) - v(0)\|_{B_{2,\infty}^{\frac{3}{2}}} \leq \epsilon, \quad \|u - v\|_{L^\infty(0,T; B_{2,\infty}^{\frac{3}{2}})} \geq 1. \quad (4.1)$$

Proof. The Equ. [\(1.7\)](#) possesses a peaked solitary wave (see [\[27\]](#)), which takes the form of

$$u_c(x, t) = c^{\frac{1}{k}} e^{-|x-ct|}, \quad x \in \mathbb{R}$$

where c is a positive constant. By using the above solitary wave solution and the similar argument as that in [\[20, Proposition 4\]](#), one can establish the [Lemma 4.1](#); we leave the details of the proof to the readers. \square

Based on the [Lemma 4.1](#), according to the following embedding properties

$$H^s \hookrightarrow B_{2,1}^{\frac{3}{2}} \hookrightarrow H^{\frac{3}{2}} \hookrightarrow B_{2,\infty}^{\frac{3}{2}} \hookrightarrow H^{s'}, \quad \text{for } \forall s' < \frac{3}{2} < s,$$

one can see that $s = \frac{3}{2}$ is the index critical for the local well-posedness. Nevertheless, it is proved that the Cauchy problem (1.7)–(1.8) is locally well-posed in the critical space $B_{2,1}^{\frac{3}{2}}$ even though the space $B_{2,1}^{\frac{3}{2}}$ is very close to $B_{2,\infty}^{\frac{3}{2}}$. More precisely, we have

Theorem 4.2. *Let $u_0 \in B_{2,1}^{\frac{3}{2}}$, then there exists a $T := T(\|u_0\|_{B_{2,1}^{\frac{3}{2}}}) > 0$ such that the Cauchy problem (1.7)–(1.8) admits a unique solution $u \in C([0, T]; B_{2,1}^{\frac{3}{2}}) \cap C^1([0, T]; B_{2,1}^{\frac{1}{2}})$.*

Proof. For the sake of simplicity, let us assume that $B_{2,\infty}^{\frac{1}{2}} \hookrightarrow Lip$, and we shall use a standard iterative process to build a solution. To this end, let $\rho(x) \in C_0^\infty(\mathbb{R})$ be a nonnegative mollifier such that $\int_{\mathbb{R}} \rho(x) dx = 1$ and denote $\rho^{(n)}(x) = n^d \rho(nx)$. Let $u^0 = 0$ and define a sequence of smooth functions $\{u^{(n)}\}_{n \in \mathbb{N}^+}$ satisfying the following iterative equation:

$$\begin{cases} \partial_t u^{(n+1)} + (u^{(n)})^k \partial_x u^{(n+1)} + \sum_{i=1}^3 \mathcal{F}_i^{(n)}(x, t) = 0, \\ u^{(n+1)}(x, 0) = u_0^{(n+1)}(x) = \rho^{(n+1)} * u_0, \end{cases} \quad (4.2)$$

where the nonlinearities $\mathcal{F}_i^{(n)}(x, t)$, $i = 1, 2, 3$, are the same as those provided in (3.15).

Since $u_0 \in B_{2,1}^{\frac{5}{2}}$, the transport equation theory in the Besov space can be applied. Similar to the argument for the case of $B_{p,r}^s$ with $s > \{\frac{3}{2}, 1 + \frac{1}{p}\}$, one can find a time $T > 0$ which depends on the initial data such that for all $n \in \mathbb{N}^+$,

$$\|u^{(n)}(t)\|_{B_{2,1}^{\frac{3}{2}}} \leq \frac{\sqrt[k]{2} \|u_0\|_{B_{2,1}^{\frac{3}{2}}}}{\sqrt[k]{1 - 4kC \|u_0\|_{B_{2,1}^{\frac{3}{2}}}^k t}}, \quad \text{for } \forall t \in [0, T].$$

Therefore, $\{u^{(n)}\}_{n \in \mathbb{N}^+}$ is uniformly bounded in $C([0, T], B_{2,1}^{\frac{3}{2}})$. By the Equ. (4.2), one can easily verify that $\{u^{(n)}\}_{n \in \mathbb{N}^+}$ is uniformly bounded in $C([0, T], B_{2,1}^{\frac{3}{2}}) \cap C^1([0, T], B_{2,1}^{\frac{1}{2}})$. The sequence $\{u^{(n)}\}_{n \in \mathbb{N}^+}$ tends to a limit $u \in C([0, T], B_{2,1}^{\frac{3}{2}}) \cap C^1([0, T], B_{2,1}^{\frac{1}{2}})$, which enables us to finish the proof of the existence of the solution. \square

The uniqueness of the solution is a direct consequence of the following Lemma 4.3.

Lemma 4.3. *Assume that u, v are two solutions of the Cauchy problem (1.7)–(1.8), and $u_0, v_0 \in B_{2,\infty}^{\frac{3}{2}} \cap Lip$ are the corresponding initial data, respectively. Let $u, v \in L^\infty([0, T]; B_{2,\infty}^{\frac{3}{2}} \cap Lip) \cap C([0, T]; B_{2,\infty}^{\frac{1}{2}})$. There exists a constant C such that for some $T_0 \leq T$,*

$$\sup_{t \in [0, T]} \left(e^{-C \int_0^t \|\partial_x(u^k)(\tau')\|_{B_{2,\infty}^{1/2} \cap L^\infty} d\tau'} \|u(t) - v(t)\|_{B_{2,\infty}^{\frac{1}{2}}} \right) < 1, \quad (4.3)$$

then the following inequality holds true for $t \in [0, T_0]$,

$$\begin{aligned} & \frac{\|u(t) - v(t)\|_{B_{2,\infty}^{1/2}}}{e} \\ & \leq e^{C \int_0^t \|\partial_x(u^k)(\tau')\|_{B_{2,\infty}^{1/2} \cap L^\infty} d\tau'} \left(\frac{\|u_0 - v_0\|_{B_{2,\infty}^{1/2}}}{e} \right)^{\exp\{-C \int_0^t \mathcal{R}(\tau) \ln(e + \mathcal{R}(\tau)) d\tau\}}, \end{aligned} \quad (4.4)$$

where $\mathcal{R}(t) = \sum_{i=0}^{2k} (\|u(t)\|_{B_{2,\infty}^{3/2} \cap Lip}^i + \|v(t)\|_{B_{2,\infty}^{3/2} \cap Lip}^i)$. In particular, the inequality (4.3) holds true on $[0, T]$ provided that

$$\|u_0 - v_0\|_{B_{2,\infty}^{1/2}} \leq \exp \left\{ 1 - e^{\int_0^t \mathcal{R}(\tau) \ln(e + \mathcal{R}(\tau)) d\tau} \right\}. \quad (4.5)$$

Proof. Setting $w = u - v$ and $w_0 = u_0 - v_0$, it is clear that w satisfies the following equation:

$$\partial_t w + u^k \partial_x w = - \sum_{i=0}^{k-1} u^i v^{k-1-i} (\partial_x v) w + \sum_{i=1}^5 \mathcal{G}_i(u, v), \quad (4.6)$$

with $w(x, 0) = w_0(x) = v_0(x) - u_0(x)$, where

$$\begin{aligned} \mathcal{G}_1(u, v) &= P(D) \left[\frac{(k-1)(b-k)}{2} \sum_{i=0}^{k-3} v^i u^{k-3-i} (\partial_x v)^3 w \right], \\ \mathcal{G}_2(u, v) &= P(D) \left[\frac{(k-1)(b-k)}{2} \sum_{i=0}^2 (\partial_x v)^i (\partial_x u)^{2-i} u^{k-2} \partial_x w \right], \\ \mathcal{G}_3(u, v) &= \partial_x P(D) \left[\frac{3k-b}{2} \sum_{i=0}^{k-2} v^i u^{k-2-i} (\partial_x v)^2 w \right], \\ \mathcal{G}_4(u, v) &= \partial_x P(D) \left[\frac{3k-b}{2} u^{k-1} (\partial_x u + \partial_x v) \partial_x w \right], \\ \mathcal{G}_5(u, v) &= \partial_x P(D) \left[\frac{b}{k+1} \sum_{i=0}^k v^i u^{k-i} w \right]. \end{aligned}$$

The following inequality is obvious,

$$\|uv\|_{B_{2,\infty}^{\frac{1}{2}}} \leq C \|u\|_{B_{2,1}^{\frac{1}{2}}} \|v\|_{L^\infty} \leq C \|u\|_{B_{2,1}^{\frac{1}{2}}} \|v\|_{B_{2,\infty}^{\frac{1}{2}} \cap L^\infty}. \quad (4.7)$$

Thanks to the facts that $B_{2,\infty}^{\frac{1}{2}} \cap L^\infty$ is a Banach algebra, $B_{2,\infty}^{\frac{3}{2}} \cap Lip \hookrightarrow B_{2,1}^{\frac{1}{2}} \hookrightarrow B_{2,\infty}^{\frac{1}{2}} \cap L^\infty$ and the inequality (4.7), one can obtain that

$$\begin{aligned} \left\| \sum_{i=0}^{k-1} u^i v^{k-1-i} (\partial_x v) w \right\|_{B_{2,\infty}^{\frac{1}{2}}} &\leq C \sum_{i=0}^{k-1} \|u^i v^{k-1-i} (\partial_x v)\|_{B_{2,\infty}^{\frac{1}{2}} \cap L^\infty} \|w\|_{B_{2,1}^{\frac{1}{2}}} \\ &\leq C \sum_{i=0}^{k-1} \|u\|_{B_{2,\infty}^{\frac{1}{2}} \cap L^\infty}^i \|v\|_{B_{2,\infty}^{\frac{1}{2}} \cap L^\infty}^{k-1-i} \|v\|_{B_{2,\infty}^{\frac{3}{2}} \cap Lip} \|w\|_{B_{2,1}^{\frac{1}{2}}} \\ &\leq C \sum_{i=0}^{2k} \left(\|u\|_{B_{2,\infty}^{\frac{3}{2}} \cap Lip}^i + \|v\|_{B_{2,\infty}^{\frac{3}{2}} \cap Lip}^i \right) \|w\|_{B_{2,1}^{\frac{1}{2}}}. \end{aligned} \quad (4.8)$$

Using the embedding $B_{2,\infty}^{-\frac{1}{2}} \hookrightarrow B_{2,\infty}^{-\frac{3}{2}}$, and the estimate $\|uv\|_{B_{2,\infty}^{-\frac{1}{2}}} \leq C \|u\|_{B_{2,\infty}^{-\frac{1}{2}}} \|v\|_{B_{2,1}^{\frac{1}{2}}}$ for $\forall u \in B_{2,\infty}^{-\frac{1}{2}}$ and $v \in B_{2,1}^{\frac{1}{2}}$, one can estimate that

$$\begin{aligned}
\|\mathcal{G}_1(u, v)\|_{B_{2,\infty}^{\frac{1}{2}}} &\leq \frac{|(k-1)(b-k)|}{2} \sum_{i=0}^{k-3} \|v^i u^{k-3-i} (\partial_x v)^3 w\|_{B_{2,\infty}^{-\frac{1}{2}}} \\
&\leq \frac{C|(k-1)(b-k)|}{2} \sum_{i=0}^{k-3} \|v^i u^{k-3-i} (\partial_x v)^3\|_{B_{2,\infty}^{-\frac{1}{2}}} \|w\|_{B_{2,1}^{\frac{1}{2}}} \\
&\leq C \sum_{i=0}^{k-3} \|u\|_{B_{2,\infty}^{\frac{3}{2}} \cap Lip}^i \|v\|_{B_{2,\infty}^{\frac{3}{2}} \cap Lip}^{k-i} \|w\|_{B_{2,1}^{\frac{1}{2}}}.
\end{aligned} \tag{4.9}$$

Since the operator $\partial_x P(D)$ is an S^{-1} -multiplier, by means of the Moser-type estimates, one can obtain the estimates for $\mathcal{G}_i(u, v)$ ($i = 2, 3, 4, 5$) as follows,

$$\|\mathcal{G}_2(u, v)\|_{B_{2,\infty}^{\frac{1}{2}}} \leq C \sum_{i=0}^2 \|u\|_{B_{2,\infty}^{\frac{3}{2}} \cap Lip}^i \|v\|_{B_{2,\infty}^{\frac{3}{2}} \cap Lip}^{k-i} \|w\|_{B_{2,1}^{\frac{1}{2}}}, \tag{4.10}$$

$$\|\mathcal{G}_3(u, v)\|_{B_{2,\infty}^{\frac{1}{2}}} \leq C \sum_{i=0}^{k-2} \|u\|_{B_{2,\infty}^{\frac{3}{2}} \cap Lip}^i \|v\|_{B_{2,\infty}^{\frac{3}{2}} \cap Lip}^{k-i} \|w\|_{B_{2,1}^{\frac{1}{2}}}, \tag{4.11}$$

$$\|\mathcal{G}_4(u, v)\|_{B_{2,\infty}^{\frac{1}{2}}} \leq C \|u\|_{B_{2,\infty}^{\frac{3}{2}} \cap Lip}^k \left(\|u\|_{B_{2,\infty}^{\frac{3}{2}} \cap Lip} + \|v\|_{B_{2,\infty}^{\frac{3}{2}} \cap Lip} \right) \|w\|_{B_{2,1}^{\frac{1}{2}}}, \tag{4.12}$$

$$\|\mathcal{G}_5(u, v)\|_{B_{2,\infty}^{\frac{1}{2}}} \leq C \sum_{i=0}^k \|v\|_{B_{2,\infty}^{\frac{3}{2}} \cap Lip}^i \|u\|_{B_{2,\infty}^{\frac{3}{2}} \cap Lip}^{k-i} \|w\|_{B_{2,1}^{\frac{1}{2}}}. \tag{4.13}$$

By applying the a priori estimate for the transform equation ($r = 1$ in this case) to the Equ. (4.6) and using the estimates (4.8)–(4.13), we get that

$$\begin{aligned}
&e^{-C \int_0^t \|\partial_x(u^k)(\tau')\|_{B_{2,\infty}^{1/2} \cap L^\infty} d\tau'} \|w(t)\|_{B_{2,\infty}^{\frac{1}{2}}} \\
&\leq \|w_0\|_{B_{2,\infty}^{\frac{1}{2}}} + C \int_0^t \sum_{i=0}^{2k} \left(\|u\|_{B_{2,\infty}^{\frac{3}{2}} \cap Lip}^i + \|v\|_{B_{2,\infty}^{\frac{3}{2}} \cap Lip}^i \right) \\
&\quad \times e^{-C \int_0^\tau \|\partial_x(u^k)(\tau')\|_{B_{2,\infty}^{1/2} \cap L^\infty} d\tau'} \|w\|_{B_{2,1}^{\frac{1}{2}}} d\tau \\
&\leq \|w_0\|_{B_{2,\infty}^{\frac{1}{2}}} + C \int_0^t \sum_{i=0}^{2k} \left(\|u\|_{B_{2,\infty}^{\frac{3}{2}} \cap Lip}^i + \|v\|_{B_{2,\infty}^{\frac{3}{2}} \cap Lip}^i \right) \\
&\quad \times e^{-C \int_0^\tau \|\partial_x(u^k)(\tau')\|_{B_{2,\infty}^{1/2} \cap L^\infty} d\tau'} \|w\|_{B_{2,\infty}^{\frac{1}{2}}} \ln \left(e + \frac{\|w\|_{B_{2,\infty}^{\frac{3}{2}}}}{\|w\|_{B_{2,\infty}^{\frac{1}{2}}}} \right) d\tau.
\end{aligned} \tag{4.14}$$

Here, the logarithmic interpolation inequality is applied in the last inequality of (4.14), which is essential for the following discussion. For convenience, set

$$\mathcal{H}(t) = e^{-C \int_0^t \|\partial_x(u^k)(\tau')\|_{B_{2,\infty}^{1/2} \cap L^\infty} d\tau'} \|w(t)\|_{B_{2,\infty}^{\frac{1}{2}}},$$

and

$$\mathcal{R}(t) = \sum_{i=0}^{2k} \left(\|u(t)\|_{B_{2,\infty}^{\frac{3}{2}} \cap Lip}^i + \|v(t)\|_{B_{2,\infty}^{\frac{3}{2}} \cap Lip}^i \right).$$

It is easy to prove that $f(x) = x(1 - \ln x)$ is a monotonic increasing function and

$$\ln \left(e + \frac{C}{x} \right) \leq \ln(e + C)(1 - \ln x), \quad \forall x \in (0, 1] \text{ and } \forall C > 0.$$

By the hypothesis, we see that $\mathcal{H}(t) \leq 1$ on $[0, T]$. It follows from (4.14) that

$$\begin{aligned} \mathcal{H}(t) &\leq \mathcal{H}(0) + C \int_0^t \mathcal{R}(\tau) \mathcal{H}(\tau) \|w\|_{B_{2,\infty}^{\frac{1}{2}}} \ln \left(e + \frac{\|u\|_{B_{2,\infty}^{3/2} \cap Lip} + \|v\|_{B_{2,\infty}^{3/2} \cap Lip}}{\|w\|_{B_{2,\infty}^{1/2}}} \right) d\tau \\ &\leq \mathcal{H}(0) + C \int_0^t \mathcal{R}(\tau) \ln(e + \mathcal{R}(\tau)) \mathcal{H}(\tau) (1 - \ln \mathcal{H}(\tau)) d\tau. \end{aligned} \quad (4.15)$$

By applying the Osgood Lemma to the previous inequality with $\mu(r) = r(1 - \ln r)$ and $\gamma(t) = \mathcal{R}(t) \ln(e + \mathcal{R}(t))$, we get that

$$- \int_{\mathcal{H}(0)}^{\mathcal{H}(t)} \frac{dr}{r(1 - \ln r)} \leq \int_0^t \mathcal{R}(s) \ln(e + \mathcal{R}(s)) ds. \quad (4.16)$$

After some direct computations, we finally obtain that

$$\frac{\mathcal{H}(t)}{e} \leq \left(\frac{\mathcal{H}(0)}{e} \right)^{\exp\{-C \int_0^t \mathcal{R}(\tau) \ln(e + \mathcal{R}(\tau)) d\tau\}},$$

which yields the desired result. Since (4.5) implies (4.3) with $T_0 = T$, we obtain the same estimate.

Finally, in view of the uniform boundedness of $\{u^{(n)}\}_{n \in \mathbb{N}^+}$ in $C([0, T]; B_{2,1}^{\frac{3}{2}})$, by using an interpolation argument, for $\forall \theta \in (0, 1)$, we have that

$$\begin{aligned} \|u - v\|_{B_{2,1}^{1/2}} &\leq C(\theta) \|u - v\|_{B_{2,\infty}^{\frac{1}{2}}}^\theta \|u - v\|_{B_{2,\infty}^{\frac{3}{2}}}^{1-\theta} \\ &\leq C e^{C \int_0^t \|u(\tau')\|_{B_{2,1}^{3/2}}^k d\tau'} \left(\frac{\|u_0 - v_0\|_{B_{2,\infty}^{1/2}}}{e} \right)^{G(t)} \left(\|u\|_{B_{2,1}^{3/2}}^{1-\theta} + \|v\|_{B_{2,1}^{3/2}}^{1-\theta} \right) \\ &\leq C \left(\frac{\|u_0 - v_0\|_{B_{2,\infty}^{1/2}}}{e} \right)^{G(t)}, \end{aligned} \quad (4.17)$$

where $G(t) = \exp\{-\theta C \int_0^t \mathcal{R}(\tau) \ln(e + \mathcal{R}(\tau)) d\tau\}$. It is obvious that (4.17) implies the uniqueness of the solution. Thus the proof of Lemma 4.3 is completed. \square

In the reminder of this section, we discuss the continuity of the solution with respect to the initial data. The following lemma plays a crucial role in the argument.

Lemma 4.4. [20] Denote $\bar{\mathbb{N}} = \mathbb{N} \cup \infty$. Let $\{v^{(n)}\}_{n \in \bar{\mathbb{N}}}$ be a sequence of functions in $C([0, T]; B_{2,1}^{\frac{1}{2}})$. Assume that $\{v^{(n)}\}_{n \in \bar{\mathbb{N}}}$ is the solution to

$$\begin{cases} \partial_t v^{(n)} + a^{(n)} \partial_x v^{(n)} = f, \\ v^{(n)}(0) = v_0 \end{cases}$$

with $v_0 \in B_{2,1}^{\frac{1}{2}}$, $f \in L^1(0, T; B_{2,1}^{\frac{1}{2}})$ and for some $\beta \in L^1(0, T)$,

$$\sup_{n \in \mathbb{N}} \|\partial_x a^{(n)}(t)\|_{B_{2,1}^{\frac{1}{2}}} \leq \beta(t).$$

In addition, if $a^{(n)} \rightarrow a^{(\infty)}$ in $L^1(0, T; B_{2,1}^{\frac{1}{2}})$, then we have

$$v^{(n)} \rightarrow v^{(\infty)} \quad \text{in } C([0, T]; B_{2,1}^{\frac{1}{2}}).$$

Theorem 4.5. Let $u_0 \in B_{2,1}^{\frac{3}{2}}$ and let u be a solution to the Cauchy problem (1.7)–(1.8). Then there exist a time $T > 0$ and a neighborhood V of u_0 in $B_{2,1}^{\frac{3}{2}}$ such that the mapping $\Theta : v_0 \mapsto v$, where u is the solution to (1.7) with initial data u_0 in $B_{2,1}^{\frac{3}{2}}$, is continuous from V into $C([0, T]; B_{2,1}^{\frac{3}{2}})$.

Proof. Step 1: The Hölder continuity in $C([0, T]; B_{2,1}^{\frac{1}{2}})$.

For given $u_0 \in B_{2,1}^{\frac{3}{2}}$ and $\sigma > 0$, we claim that there exist a $T > 0$ and $\overline{M} > 0$ such that for any $\bar{u}_0 \in B_{2,1}^{\frac{3}{2}}$ with $\|\bar{u}_0 - u_0\|_{B_{2,1}^{\frac{3}{2}}} \leq \sigma$, the solution $\bar{u} = \Theta(\bar{u}_0)$ of the Cauchy problem (1.7)–(1.8) associated to \bar{u}_0 belongs to $C([0, T]; B_{2,1}^{\frac{3}{2}})$ and $\|\bar{u}\|_{L^\infty(0, T; B_{2,1}^{\frac{3}{2}})} \leq \overline{M}$. To this end, one can choose

$$T = \frac{1}{8kC[(\|u_0\|_{B_{2,1}^{3/2}} + \sigma)^k + \sigma^k]}. \quad (4.18)$$

It is clear that $T \leq 1/8kC[\|\bar{u}_0\|_{B_{2,1}^{3/2}}^k + \sigma^k]$. On the other hand, from the proof of the local well-posedness one can find that for such a time $T < 1/2kC\|\bar{u}_0\|_{B_{p,r}^s}^k$, we have that

$$\|\bar{u}(t)\|_{B_{2,1}^{3/2}} \leq \frac{\sqrt[k]{2}\|\bar{u}_0\|_{B_{2,1}^{3/2}}}{\sqrt[k]{1 - 4kC\|\bar{u}_0\|_{B_{2,1}^{3/2}}^k t}}, \quad \text{for } \forall t \in [0, T]. \quad (4.19)$$

Hence, it follows from (4.18) and (4.19) that

$$\|\bar{u}(t)\|_{B_{2,1}^{3/2}} \leq \frac{\sqrt[k]{2}\|\bar{u}_0\|_{B_{2,1}^{3/2}}}{\sqrt[k]{1 - \frac{4kC\|\bar{u}_0\|_{B_{2,1}^{3/2}}^k}{8kC[\|\bar{u}_0\|_{B_{2,1}^{3/2}}^k + \sigma^k]}}} \leq 4\|\bar{u}_0\|_{B_{2,1}^{3/2}} \leq 4(\|u_0\|_{B_{2,1}^{3/2}} + \sigma) := \overline{M}. \quad (4.20)$$

Combing the above uniform bounds and the Lemma 4.3, we infer that

$$\begin{aligned} & \frac{\|\Theta(\bar{u}_0) - \Theta(u_0)\|_{L^\infty(0, T; B_{2,\infty}^{1/2})}}{e} \\ &= \frac{\|\bar{u} - u\|_{L^\infty(0, T; B_{2,\infty}^{1/2})}}{e} \leq e^{C\overline{M}T} \left(\frac{\|\bar{u}_0 - u_0\|_{B_{2,\infty}^{1/2}}}{e} \right)^{\exp\{-C\overline{M}T\}}. \end{aligned} \quad (4.21)$$

Note that for $\forall \theta \in (0, 1)$, the real interpolation inequality implies that $\|u\|_{B_{2,1}^{1/2}} \leq \|u\|_{B_{2,1}^{3/2-\theta}} \leq C\|u\|_{B_{2,\infty}^{1/2}}^\theta \|u\|_{B_{2,\infty}^{3/2}}^{1-\theta}$. In view of the uniform bounds and (4.21), we have

$$\begin{aligned} \|\Theta(\bar{u}_0) - \Theta(u_0)\|_{B_{2,1}^{1/2}} &\leq \|\bar{u} - u\|_{B_{2,\infty}^{1/2}}^\theta \|\bar{u} - u\|_{B_{2,\infty}^{3/2}}^{1-\theta} \\ &\leq C\bar{M}^{1-\theta} e^{\theta C\bar{M}T} \left(\frac{\|\bar{u}_0 - u_0\|_{B_{2,1}^{3/2}}}{e} \right)^{\theta \exp\{-C\bar{M}T\}}, \end{aligned} \quad (4.22)$$

which implies that the solution mapping is Hölder continuous from V to $C([0, T]; B_{2,1}^{\frac{1}{2}})$.

Step 2: The continuity in $C([0, T]; B_{2,1}^{\frac{3}{2}})$.

Let $u_0^{(\infty)} \in B_{2,\infty}^{\frac{3}{2}}$ and let $u_0^{(n)}$ tend to $u_0^{(\infty)}$ in $C([0, T]; B_{2,1}^{\frac{3}{2}})$. From the argument in Step 1, there exist positive constants T and \bar{M} such that for all $n \in \mathbb{N}^+$, the sequence $u^{(n)}$ which is the solution with respect to the datum $u_0^{(n)}$ is defined on $[0, T]$, and satisfies

$$\sup_{n \in \mathbb{N}^+} \|u^{(n)}\|_{L^\infty(0, T; B_{2,1}^{\frac{3}{2}})} \leq \bar{M}. \quad (4.23)$$

As a consequence of the Step 1, to prove $u^{(n)}$ tends to $u^{(\infty)}$ in $C([0, T]; B_{2,1}^{3/2})$ is equivalent to proving that $v^{(n)} = \partial_x u^{(n)}$ tends to $v^{(\infty)} = \partial_x u^{(\infty)}$ in $C([0, T]; B_{2,1}^{1/2})$. Note that $v^{(n)}$ satisfies the following equation:

$$\begin{cases} \partial_t v^{(n)} + (u^{(n)})^k \partial_x v^{(n)} = \mathcal{P}^{(n)}, \\ v^{(n)}(x, 0) = \partial_x u_0^{(n)}(x), \end{cases} \quad (4.24)$$

where

$$\begin{aligned} \mathcal{P}^{(n)} &= -k(u^{(n)})^{k-1}(\partial_x u^{(n)})^2 - \partial_x P(D) \left[\frac{(k-1)(b-k)}{2} (u^{(n)})^{k-2} (\partial_x u^{(n)})^3 \right] \\ &\quad - P(D) \left[\frac{3k-b}{2} (u^{(n)})^{k-1} (\partial_x u^{(n)})^2 + \frac{b}{k+1} (u^{(n)})^{k+1} \right] \\ &\quad + \left[\frac{3k-b}{2} (u^{(n)})^{k-1} (\partial_x u^{(n)})^2 + \frac{b}{k+1} (u^{(n)})^{k+1} \right]. \end{aligned}$$

According to the Kato's theory, we can decompose $v^{(n)}$ by $v^{(n)} = w^{(n)} + z^{(n)}$ with

$$\begin{cases} \partial_t w^{(n)} + (u^{(n)})^k \partial_x w^{(n)} = \mathcal{P}^{(n)} - \mathcal{P}^{(\infty)}, \\ w^{(n)}(x, 0) = \partial_x u_0^{(n)}(x) - \partial_x u_0^{(\infty)}(x), \end{cases} \quad (4.25)$$

and

$$\begin{cases} \partial_t z^{(n)} + (u^{(n)})^k \partial_x z^{(n)} = \mathcal{P}^{(\infty)}, \\ z^{(n)}(x, 0) = \partial_x u_0^{(\infty)}(x). \end{cases} \quad (4.26)$$

Using the properties of Besov spaces exhibited in Section 2 and the fact that $B_{2,1}^{\frac{1}{2}}$ is a Banach algebra, it is easy to verify that $\{\mathcal{P}^{(n)}\}_{n \in \mathbb{N}^+}$ is uniformly bounded in $C([0, T]; B_{2,1}^{\frac{1}{2}})$. Moreover,

$$\begin{aligned} \mathcal{P}^{(n)} - \mathcal{P}^{(\infty)} &= k \left((u^{(\infty)})^{k-1} (\partial_x u^{(\infty)})^2 - (u^{(n)})^{k-1} (\partial_x u^{(n)})^2 \right) + \frac{(k-1)(b-k)}{2} \\ &\quad \partial_x P(D) \left((u^{(\infty)})^{k-2} (\partial_x u^{(\infty)})^3 - (u^{(n)})^{k-2} (\partial_x u^{(n)})^3 \right) + P(D) \\ &\quad \left[\frac{3k-b}{2} \left((u^{(\infty)})^{k-1} (\partial_x u^{(\infty)})^2 - (u^{(n)})^{k-1} (\partial_x u^{(n)})^2 \right) + \frac{b}{k+1} \right] \end{aligned}$$

$$\begin{aligned} & \left[(u^{(\infty)})^{k+1} - (u^{(n)})^{k+1} \right] - \frac{3k-b}{2} \left((u^{(\infty)})^{k-1} (\partial_x u^{(\infty)})^2 \right. \\ & \left. - (u^{(n)})^{k-1} (\partial_x u^{(n)})^2 \right) + \frac{b}{k+1} \left((u^{(\infty)})^{k+1} - (u^{(n)})^{k+1} \right). \end{aligned} \quad (4.27)$$

By means of the [Remark 2.2](#), there exists a constant C depending only on k, b such that

$$\begin{aligned} \|\mathcal{P}^{(n)} - \mathcal{P}^{(\infty)}\|_{B_{2,1}^{\frac{1}{2}}} & \leq C \sum_{i=0}^{2k} \left(\|u^{(n)}\|_{B_{2,1}^{\frac{3}{2}}}^i + \|u^{(\infty)}\|_{B_{2,1}^{\frac{3}{2}}}^i \right) \\ & \times \left(\|u^{(n)} - u^{(\infty)}\|_{B_{2,1}^{1/2}} + \|\partial_x u^{(n)} - \partial_x u^{(\infty)}\|_{B_{2,1}^{1/2}} \right). \end{aligned} \quad (4.28)$$

Therefore, applying the transport theory to the Equ. (4.25), we get that

$$\begin{aligned} \|w^{(n)}\|_{B_{2,1}^{1/2}} & \leq e^{C \int_0^t \|\partial_x (u^{(n)})^k(s)\|_{B_{2,1}^{1/2}} ds} \left(\|\partial_x u_0^{(n)} - \partial_x u_0^{(\infty)}\|_{B_{2,1}^{1/2}} \right. \\ & \quad \left. + \int_0^t e^{-C \int_0^\tau \|\partial_x (u^{(n)})^k(s)\|_{B_{2,1}^{1/2}} d\tau} \|\mathcal{P}^{(n)} - \mathcal{P}^{(\infty)}\|_{B_{2,1}^{1/2}} d\tau \right) \\ & \leq e^{C \int_0^t \|u^{(n)}(s)\|_{B_{2,1}^{3/2}}^k ds} \left[\|\partial_x u_0^{(n)} - \partial_x u_0^{(\infty)}\|_{B_{2,1}^{1/2}} \right. \\ & \quad \left. + C \int_0^t \sum_{i=0}^{2k} \left(\|u^{(n)}\|_{B_{2,1}^{\frac{3}{2}}}^i + \|u^{(\infty)}\|_{B_{2,1}^{\frac{3}{2}}}^i \right) \right. \\ & \quad \left. \times \left(\|u^{(n)} - u^{(\infty)}\|_{B_{2,1}^{1/2}} + \|\partial_x u^{(n)} - \partial_x u^{(\infty)}\|_{B_{2,1}^{1/2}} \right) d\tau \right]. \end{aligned} \quad (4.29)$$

On the other hand, since the sequence $\{z^{(n)}\}_{n \in \mathbb{N}^+}$ is uniformly bounded in $C([0, T]; B_{2,1}^{\frac{1}{2}})$ and converges to $u^{(\infty)}$ in $C([0, T]; B_{2,1}^{\frac{1}{2}})$, [Lemma 4.4](#) tells us that $z^{(n)} \rightarrow z^{(\infty)}$ in $C([0, T]; B_{2,1}^{1/2})$. Moreover, since $u^{(n)}$ tends to $u^{(\infty)}$ in $C([0, T]; B_{2,1}^{1/2})$, for $\forall \epsilon > 0$, there exists $N > 0$ such that

$$\|z^{(n)} - z^{(\infty)}\|_{B_{2,1}^{1/2}} + \|u^{(n)} - u^{(\infty)}\|_{B_{2,1}^{1/2}} \leq \epsilon, \quad \forall n > N. \quad (4.30)$$

Hence, it follows from (4.23), (4.29) and (4.30) that

$$\begin{aligned} \|\partial_x u^{(n)} - \partial_x u^{(\infty)}\|_{B_{2,1}^{1/2}} & \leq \epsilon + C \left(\epsilon + \|\partial_x u_0^{(n)} - \partial_x u_0^{(\infty)}\|_{B_{2,1}^{1/2}} \right. \\ & \quad \left. + \int_0^t \|\partial_x u^{(n)} - \partial_x u^{(\infty)}\|_{B_{2,1}^{1/2}} ds \right), \end{aligned} \quad (4.31)$$

which combined with the Gronwall inequality yields that

$$\|\partial_x u^{(n)} - \partial_x u^{(\infty)}\|_{L^\infty(0, T; B_{2,1}^{1/2})} \leq C e^{CT} \left(\epsilon + \|\partial_x u_0^{(n)} - \partial_x u_0^{(\infty)}\|_{B_{2,1}^{1/2}} \right). \quad (4.32)$$

This implies the desired results, and the proof of [Theorem 4.5](#) is completed. \square

5. Blow-up criteria and conservative property

In this section, we derive the precise blow-up criteria of the strong solution to the equation (1.7), and provide a conservative law property for the solutions by using the particle method. Firstly, we have the following blow-up result in the critical Besov space.

Theorem 5.1. *Let u be the solution of the Cauchy problem (1.7)–(1.8) with the initial data $u_0 \in B_{2,1}^{\frac{3}{2}}$, and let T^* denote the maximum existence time of the solution. If $T^* < \infty$, then*

$$\int_0^{T^*} \|u(s)\|_{L^\infty}^{k-2} \|\partial_x u(s)\|_{L^\infty}^2 ds = \infty \quad \text{or} \quad \int_0^{T^*} \|u(s)\|_{L^\infty}^k ds = \infty.$$

Proof. Applying the nonhomogeneous dyadic blocks Δ_q to the Equ. (1.7), we get that

$$\partial_t \Delta_q u + u^k \partial_x \Delta_q u = u^k \Delta_q \partial_x u - \Delta_q [u^k \partial_x u] + \mathcal{F}_q(u, \partial_x u), \quad (5.1)$$

where

$$\begin{aligned} \mathcal{F}_q(u, \partial_x u) &= \frac{(k-1)(b-k)}{2} \Delta_q P(D) u^{k-2} (\partial_x u)^3 + \frac{b}{k+1} \Delta_q \partial_x P(D) u^{k+1} \\ &\quad + \frac{3k-b}{2} \Delta_q \partial_x P(D) u^{k-1} (\partial_x u)^2. \end{aligned}$$

Multiplying both sides of the Equ. (5.1) by $\Delta_q u$, and then integrating on \mathbb{R} with respect to x , it follows from the Hölder inequality that

$$\begin{aligned} \frac{d}{dt} \|\Delta_q u\|_{L^2} &\leq k \|u^{k-1}\|_{L^\infty} \|\partial_x u\|_{L^\infty} \|\Delta_q u\|_{L^2} \\ &\quad + \|u^k \Delta_q u \partial_x u - \Delta_q [u^k \partial_x u]\|_{L^2} + \|\mathcal{F}_q(u, \partial_x u)\|_{L^2}. \end{aligned} \quad (5.2)$$

Then, integrating on the interval $(0, t)$, we get that

$$\begin{aligned} \|\Delta_q u(t)\|_{L^2} &\leq \|\Delta_q u_0\|_{L^2} + k \int_0^t \|u^{k-1}\|_{L^\infty} \|\partial_x u\|_{L^\infty} \|\Delta_q u\|_{L^2} ds \\ &\quad + \int_0^t \|u^k \Delta_q u \partial_x u - \Delta_q [u^k \partial_x u]\|_{L^2} ds + \int_0^t \|\mathcal{F}_q(u, \partial_x u)\|_{L^2} ds. \end{aligned} \quad (5.3)$$

Using the commutator estimate (Lemma 2.4), we have that

$$\begin{aligned} &\|2^{\frac{3}{2}q} \|u^k \partial_x \Delta_q u - \Delta_q [u^k \partial_x u]\|_{L^2}\|_{l^1} \\ &\leq C \left(\|u^{k-1} \partial_x u\|_{L^\infty} \|u\|_{B_{2,1}^{\frac{3}{2}}} + \|\partial_x u\|_{L^\infty} \|u^{k-1} \partial_x u\|_{B_{2,1}^{\frac{1}{2}}} \right) \\ &\leq C \left(\|u\|_{L^\infty}^{k-1} \|\partial_x u\|_{L^\infty} \|u\|_{B_{2,1}^{\frac{3}{2}}} + \|\partial_x u\|_{L^\infty} \|u^{k-1}\|_{L^\infty} \|\partial_x u\|_{B_{2,1}^{\frac{1}{2}}} \right) \\ &\leq C \|u\|_{L^\infty}^{k-1} \|\partial_x u\|_{L^\infty} \|u\|_{B_{2,1}^{\frac{3}{2}}}. \end{aligned} \quad (5.4)$$

Since $\partial_x P(D)$ is the S^{-1} -multiplier and the Besov space $B_{2,1}^{\frac{1}{2}}$ is a Banach space, using the inequality (4.7) again, we have that

$$\begin{aligned} \|\mathcal{F}_q(u, \partial_x u)\|_{B_{2,1}^{\frac{3}{2}}} &\leq C \left(\|\Delta_q P(D) u^{k-2} (\partial_x u)^3\|_{B_{2,1}^{\frac{3}{2}}} + \|\Delta_q \partial_x P(D) u^{k+1}\|_{B_{2,1}^{\frac{3}{2}}} \right. \\ &\quad \left. + \|\Delta_q \partial_x P(D) u^{k-1} (\partial_x u)^2\|_{B_{2,1}^{\frac{3}{2}}} \right) \\ &\leq C \left(\|u^{k-2} (\partial_x u)^3\|_{B_{2,1}^{\frac{1}{2}}} + \|u^{k+1}\|_{B_{2,1}^{\frac{1}{2}}} + \|u^{k-1} (\partial_x u)^2\|_{B_{2,1}^{\frac{1}{2}}} \right) \\ &\leq C \left(\|u\|_{L^\infty}^{k-2} \|\partial_x u\|_{L^\infty}^2 + \|u\|_{L^\infty}^k + \|u\|_{L^\infty}^{k-1} \|\partial_x u\|_{L^\infty} \right) \|u\|_{B_{2,1}^{\frac{3}{2}}}. \end{aligned} \quad (5.5)$$

Hence, multiplying both sides of (5.3) by $2^{\frac{3}{2}q}$ and taking the l^1 -norm, we get that

$$\begin{aligned} \|u(t)\|_{B_{2,1}^{\frac{3}{2}}} &\leq \|u_0\|_{B_{2,1}^{\frac{3}{2}}} + C \int_0^t \|u\|_{L^\infty}^{k-1} \|\partial_x u\|_{L^\infty} \|u\|_{B_{2,1}^{\frac{3}{2}}} ds \\ &\quad + \int_0^t \|2^{\frac{3}{2}q} \|u^k \Delta_q u \partial_x u - \Delta_q [u^k \partial_x u]\|_{L^2} \|l^1\| ds + \int_0^t \|2^{\frac{3}{2}q} \|F_q\|_{L^2} \|l^1\| ds \\ &\leq \|u_0\|_{B_{2,1}^{\frac{3}{2}}} + C \int_0^t \|u\|_{L^\infty}^{k-1} \|\partial_x u\|_{L^\infty} \|u\|_{B_{2,1}^{\frac{3}{2}}} ds \\ &\quad + C \int_0^t \|u\|_{L^\infty}^{k-1} \|\partial_x u\|_{L^\infty} \|u\|_{B_{2,1}^{\frac{3}{2}}} ds + C \int_0^t \left(\|u\|_{L^\infty}^{k-2} \|\partial_x u\|_{L^\infty}^2 \right. \\ &\quad \left. + \|u\|_{L^\infty}^k + \|u\|_{L^\infty}^{k-1} \|\partial_x u\|_{L^\infty} \right) \|u\|_{B_{2,1}^{\frac{3}{2}}} ds \\ &\leq \|u_0\|_{B_{2,1}^{\frac{3}{2}}} + C \int_0^t \|u\|_{L^\infty}^{k-2} (\|u\|_{L^\infty}^2 + \|\partial_x u\|_{L^\infty} \|u\|_{L^\infty} + \|\partial_x u\|_{L^\infty}^2) \|u\|_{B_{2,1}^{\frac{3}{2}}} ds \\ &\leq \|u_0\|_{B_{2,1}^{\frac{3}{2}}} + C \int_0^t \|u\|_{L^\infty}^{k-2} (\|\partial_x u\|_{L^\infty}^2 + \|u\|_{L^\infty}^2) \|u\|_{B_{2,1}^{\frac{3}{2}}} ds. \end{aligned} \quad (5.6)$$

By applying the Gronwall inequality, we get that

$$\|u(t)\|_{B_{2,1}^{\frac{3}{2}}} \leq C \|u_0\|_{B_{2,1}^{\frac{3}{2}}} \exp \left\{ \int_0^t \|u\|_{L^\infty}^{k-2} \|\partial_x u\|_{L^\infty}^2 + \|u\|_{L^\infty}^k ds \right\}. \quad (5.7)$$

Assume that the maximum existence of time T^* is finite, but

$$\int_0^{T^*} \|u\|_{L^\infty}^{k-2} \|\partial_x u\|_{L^\infty}^2 ds < \infty \quad \text{and} \quad \int_0^{T^*} \|u\|_{L^\infty}^k ds < \infty. \quad (5.8)$$

Then it follows from (5.7) that $\sup_{t \uparrow T^*} \|u(t)\|_{B_{2,1}^{\frac{3}{2}}} < \infty$, which contradicts the assumption that T^* is the maximum existence time of the solution. This completes the proof of Theorem 5.1. \square

Remark 5.2. By the Sobolev embedding $H^s = B_{2,2}^s \hookrightarrow B_{2,1}^{\frac{3}{2}}$ with $s > \frac{3}{2}$, the [Theorem 5.1](#) also holds true when the initial data take values in H^s ($s > \frac{3}{2}$), where the existence of the strong solution is ensured by the [Theorem 3.2](#).

Especially, by means of the conservative law when $b = \frac{k}{2}$, we have the following result.

Corollary 5.3. Assume that $b = \frac{k}{2}$ and $u_0 \in H^s$ ($s > \frac{3}{2}$). If T^* is the maximum existence time of the strong solution to the Cauchy problem (1.7)–(1.8), then we have that

$$T^* < \infty \implies \int_0^{T^*} \|\partial_x u(s)\|_{L^\infty}^2 ds = \infty. \quad (5.9)$$

Proof. By using a density argument, one just only considers the case $s \geq 3$. Multiplying the Equ. (1.1) by $u - \partial_x^2 u$ and integrating by parts on \mathbb{R} , one can get the following conservation law

$$\int_{\mathbb{R}} u^2(t) + (\partial_x u)^2(t) + (\partial_x^2 u)^2(t) dx = \int_{\mathbb{R}} u_0^2 + (\partial_x u_0)^2 + (\partial_x^2 u_0)^2(t) dx, \quad (5.10)$$

for each t in the lifespan. Therefore, it follows from the Sobolev embedding that

$$\|u(t)\|_{L^\infty} \leq C\|u(t)\|_{H^1} \leq C \int_{\mathbb{R}} u_0^2 + (\partial_x u_0)^2 + (\partial_x^2 u_0)^2(t) dx < \infty.$$

On the other hand, following the same argument as in [Theorem 5.1](#), one can obtain that

$$\begin{aligned} \|u(t)\|_{B_{2,1}^{\frac{3}{2}}} &\leq \|u_0\|_{B_{2,1}^{\frac{3}{2}}} + C \int_0^t \|u\|_{L^\infty}^{k-2} (\|\partial_x u\|_{L^\infty}^2 + \|u\|_{L^\infty}^2) \|u\|_{B_{2,1}^{\frac{3}{2}}} ds \\ &\leq \|u_0\|_{B_{2,1}^{\frac{3}{2}}} + C \int_0^t (\|\partial_x u\|_{L^\infty}^2 + 1) \|u\|_{B_{2,1}^{\frac{3}{2}}} ds, \end{aligned}$$

which combined with the Gronwall inequality yields

$$\|u(t)\|_{B_{2,1}^{\frac{3}{2}}} \leq C\|u_0\|_{B_{2,1}^{\frac{3}{2}}} e^{C \int_0^t (\|\partial_x u\|_{L^\infty}^2 + 1) ds}.$$

The desired blow-up criterion is a direct consequence of the above inequality. \square

Remark 5.4. The maximum existence time T^* of the solution can be chosen independent of the regularity index s . Indeed, let $u_0 \in H^s$ ($s > \frac{3}{2}$) and consider some $s' \in (\frac{3}{2}, s)$. The [Theorem 3.2](#) ensures the existence of unique solution u_s (resp., $u_{s'}$) in H^s (resp., $H^{s'}$) with the maximum existence time T_s^* (resp., $T_{s'}^*$). Since $H^s \hookrightarrow H^{s'}$, it follows from the uniqueness that $T_s^* \leq T_{s'}^*$, and also $u_s \equiv u_{s'}$ on $[0, T_s^*)$. On the other hand, if $T_s^* < T_{s'}^*$, then $u_{s'} \in C([0, T_s^*]; H^{s'}) \hookrightarrow L^2(0, T_s^*; L^\infty)$, which contradicts to the [Corollary 5.3](#). Hence, we have $T_s^* = T_{s'}^*$.

By [Corollary 5.3](#) and the Sobolev embedding theorem, we can readily obtain the following blow-up criterion.

Corollary 5.5. Assume that $b = \frac{k}{2}$ and $u_0 \in H^s (s > \frac{3}{2})$. If T^* is the maximum existence time of the solution, then the solution will blow up in finite time if and only if

$$\limsup_{t \uparrow T^*} \|\partial_x u(x, t)\|_{L^\infty} = +\infty.$$

The following result shows that if the slope of the equation is unbounded from below or above under some conditions, the wave breaking phenomena of the solution will occur in finite time.

Theorem 5.6. (1) Let $u_0 \in H^s$, $s > \frac{3}{2}$, and let T^* be the maximum existence time of the strong solution u to the Cauchy problem (1.7)–(1.8). Then the solution u blows up in finite time if and only if one of the following cases happens:

- If $b > \frac{k}{2}$, the slope of u^k (i.e., $ku^{k-1}\partial_x u$) is unbounded from below, that is

$$\lim_{t \uparrow T^*} \liminf_{x \in \mathbb{R}} (u^{k-1}\partial_x u)(x, t) = -\infty. \quad (5.11)$$

- If $b < \frac{k}{2}$, the slope of u^k is unbounded from above, that is

$$\lim_{t \uparrow T^*} \limsup_{x \in \mathbb{R}} (u^{k-1}\partial_x u)(x, t) = +\infty. \quad (5.12)$$

- (2) If $b = \frac{k}{2}$, the Cauchy problem (1.7)–(1.8) admits a global strong solution.

Proof. (1) By a simple density argument, it suffices to consider the case $s \geq 3$. In this case, as a direct consequence of Theorem 3.2, the Cauchy problem (1.7)–(1.8) admits a unique solution in $C([0, T^*]; H^s) \cap C^1([0, T^*]; H^{s-1})$. An essential observation is that one can rewrite the Equ. (1.1) in the form of

$$\partial_t m + u^k \partial_x m + bu^{k-1} u_x m = 0, \quad m = u - \partial_x^2 u. \quad (5.13)$$

Multiplying both sides of Equ. (5.12) by $2m$, and integrating by parts with respect to x on \mathbb{R} , we get

$$\frac{d}{dt} \int_{\mathbb{R}} m^2 dx = k \int_{\mathbb{R}} u^{k-1} u_x m^2 dx - 2 \int_{\mathbb{R}} bu^{k-1} u_x m^2 dx = (k - 2b) \int_{\mathbb{R}} u^{k-1} u_x m^2 dx. \quad (5.14)$$

In the case of $b > \frac{k}{2}$, if the slope $u^{k-1}\partial_x u$ is bounded from below, then there exists a $C > 0$ such that $u^{k-1}\partial_x u \geq -C$. On the contrary, if $b < \frac{k}{2}$ and the slope $u^{k-1}\partial_x u$ is bounded from above, then there exists a $C > 0$ such that $u^{k-1}\partial_x u \leq C$.

In either case, we can deduce from (5.14) that

$$\int_{\mathbb{R}} m^2(x, t) dx \leq \int_{\mathbb{R}} m_0^2(x) dx + C \int_0^t \int_{\mathbb{R}} m^2(x, t) dx dt.$$

Applying the Gronwall inequality, we get

$$\|m(t)\|_{L^2} \leq \|m_0\|_{L^2} e^{Ct} \quad \text{for } t \in [0, T), \quad (5.15)$$

which implies that the H^2 -norm of the solution u is bounded if the slope of u^k is bounded from below when $b > \frac{k}{2}$, or bounded from above when $b < \frac{k}{2}$.

On the other hand, by using the fact of $u = G * m$ and $\partial_x u = (\partial_x G) * m$, we deduce from the Young inequality that

$$\|u^{k-1} \partial_x u\|_{L^\infty} \leq \|u\|_{L^\infty}^{k-1} \|\partial_x u\|_{L^\infty} \leq \|G\|_{L^2}^{k-1} \|\partial_x G\|_{L^2} \|m\|_{L^2}^k. \quad (5.16)$$

Integrating by parts, we have

$$\|m\|_{L^2}^2 = \int_{\mathbb{R}} (u - \partial_x^2 u)^2 dx = \int_{\mathbb{R}} u^2 + 2(\partial_x u)^2 + (\partial_x^2 u)^2 dx.$$

That is to say,

$$\|u\|_{H^2}^k \leq \|m\|_{L^2}^k \leq 2^k \|u\|_{H^2}^k. \quad (5.17)$$

Combining (5.16) and (5.17), we see that the L^∞ -norm of the slop $u^{k-1} \partial_x u$ is bounded if the H^2 -norm of the solution u is bounded.

(2) If $b = \frac{k}{2}$, it follows from (5.14) that $\int_{\mathbb{R}} m^2 dx = \int_{\mathbb{R}} m_0^2 dx$. So we have that

$$\|\partial_x u\|_{L^\infty} \leq C \|\partial_x u\|_{H^1} \leq C \int_{\mathbb{R}} m^2 dx = C \int_{\mathbb{R}} m_0^2 dx < \infty,$$

which combined with the Corollary 5.5 yields the global existence of the strong solution. Thus the proof of Theorem 5.6 is completed. \square

Let $\Phi(x, t)$ be the particle line evolved by the differential equation

$$\begin{cases} \frac{d}{dt} \Phi(x, t) = u^k(\Phi(x, t), t), \\ \Phi(x, 0) = x. \end{cases} \quad (5.18)$$

Applying classical results in the theory of ordinary differential equations, one can obtain the following conservative property on the strong solution.

Theorem 5.7. *Let $u_0 \in H^s$, $s \geq 3$, and let T^* be the maximal existence time of the corresponding solution $u(x, t)$ to (1.7). Then (5.17) admits a unique solution $\Phi \in C^1([0, T^*) \times \mathbb{R}, \mathbb{R})$. Moreover, the map $\Phi(\cdot, t)$ is an increasing diffeomorphism of \mathbb{R} with*

$$\partial_x \Phi(x, t) = \exp \left\{ k \int_0^t (u^{k-1} \partial_x u)(\Phi(x, s), s) ds \right\} > 0,$$

for $(x, t) \in [0, T^*) \times \mathbb{R}$, and we have the following conservative law property of m :

$$m(\Phi(x, t), t) (\partial_x \Phi)^{\frac{b}{k}}(x, t) = m_0(x), \quad (5.19)$$

where $(x, t) \in [0, T^*) \times \mathbb{R}$ and $m = u - \partial_x^2 u$. Moreover, if $u_0(x)$ is compactly supported in an interval $[c, d] \subseteq \mathbb{R}$, then the function $m(x, t)$ has also compact support in $[\Phi(c, t), \Phi(d, t)] \subseteq \mathbb{R}$ for every $t \in [0, T^*)$.

Proof. By utilizing the Theorem 3.2, we have $u \in C([0, T^*]; H^s) \cap C^1([0, T^*]; H^{s-1})$. Then the Sobolev embedding theorem ensures that $u(x, t)$ is bounded, Lipschitz in the space variable x and of class C^1

in time. Therefore, by applying the classical results in the theory of ordinary differential equations, the equation (5.17) admits a unique solution $\Phi \in C([0, T^*) \times \mathbb{R})$.

Differentiating (5.17) with respect to x , we get that

$$\begin{cases} \frac{d}{dt} \partial_x \Phi(x, t) = (ku^{k-1} \partial_x u)(\Phi(x, t), t) \partial_x \Phi(x, t), \\ \partial_x \Phi(x, 0) = 1, \end{cases} \quad (5.20)$$

which leads to

$$\partial_x \Phi(x, t) = \exp \left\{ k \int_0^t (u^{k-1} \partial_x u)(\Phi(x, s), s) ds \right\},$$

for all $t \in [0, T^*)$, $x \in \mathbb{R}$. For $\forall \bar{T} < T^*$, it follows from the Sobolev embedding that

$$\sup_{(x,t) \in [0, \bar{T}] \times \mathbb{R}} |(u^{k-1} \partial_x u)(x, t)| < \infty,$$

which infers that there exists $K > 0$ such that $\partial_x \Phi(x, t) \geq e^{-kKt} > 0$, for $(x, t) \in [0, T^*) \times \mathbb{R}$.

Applying the particle method to $\Phi(x, t)$, we have that

$$\begin{aligned} \frac{d}{dt} \left(m(\Phi(x, t), t) \Phi_x^{\frac{b}{k}}(x, t) \right) &= (\partial_t m + \partial_x m \partial_t \Phi)(\Phi(x, t), t) (\partial_x \Phi)^{\frac{b}{k}}(x, t) \\ &\quad + \frac{b}{k} m(\Phi(x, t), t) (\partial_x \Phi)^{\frac{b}{k}-1}(x, t) \frac{\partial}{\partial t} (\partial_x \Phi)(x, t) \\ &= (\partial_t m + u^k \partial_x m + bu^{k-1} m \partial_x u)(\Phi(x, t), t) (\partial_x \Phi)^{\frac{b}{k}}(x, t) = 0. \end{aligned}$$

Since $\Phi_x(x, 0) = 1$, integrating the above equality on $[0, t]$, one obtains (5.19). The final result follows from the fact that the function $\Phi(x, t)$ is increasing with respect to the space variable for every t in the lifespan of the solution. This completes the proof of Theorem 5.7. \square

Remark 5.8. Since the map $\Phi(\cdot, t)$ is an increasing diffeomorphism from \mathbb{R} to \mathbb{R} for every $t \in [0, T^*)$, then for $\forall w \in L^\infty$, the L^∞ -norm of w is preserved under the diffeomorphisms $\Phi(\cdot, t)$, i.e., $\|w(\cdot, t)\|_{L^\infty} = \|w(\Phi(\cdot, t), t)\|_{L^\infty}$.

Remark 5.9. The equality (5.19) in Theorem 5.7 implies that the sign of the function $m(x, t)$ along the particle line $\Phi(x, t)$ is the same as that of $m_0(x)$ for every t in the lifespan T^* .

6. Gevrey regularity and analyticity of solutions

In this section, we investigate the Gevrey regularity and analyticity of the solutions to the Cauchy problem (1.7)–(1.8), and show that the solution mapping is continuous in the Sobolev–Gevrey spaces. To this end, let us first introduce the Sobolev–Gevrey spaces.

Definition 6.1. Let $\sigma, \delta > 0$ and let s be a real number. The Sobolev–Gevrey space $G_{\sigma, s}^\delta$ on \mathbb{R} is defined by

$$G_{\sigma, s}^\delta = \left\{ u \in C^\infty; \|u\|_{G_{\sigma, s}^\delta} = \left(\int_{\mathbb{R}} (1 + |\xi|^2)^s e^{2\delta|\xi|^{\frac{1}{\sigma}}} |\widehat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}} < \infty \right\}.$$

Remark 6.2. If $\sigma > 1$, the function in $G_{\sigma,s}^\delta$ is the Gevrey class function. If $\sigma = 1$, it becomes the usual analytic function and δ is the radius of analyticity. If $0 < \sigma < 1$, it is called the ultra-analytic function. Moreover, by using the Fourier multiplier $e^{\delta(-\Delta)^{\frac{1}{2\sigma}}}$, it is easy to see that $\|f\|_{G_{\sigma,s}^\delta} = \|e^{\delta(-\Delta)^{\frac{1}{2\sigma}}} f\|_{H^s}$.

Lemma 6.3. [35] (1) Let $0 < \delta' < \delta$, $0 < \sigma' < \sigma$ and $s' < s$, then we have

$$G_{\sigma,s}^\delta \hookrightarrow G_{\sigma,s}^{\delta'}, \quad G_{\sigma',s}^\delta \hookrightarrow G_{\sigma,s}^\delta, \quad G_{\sigma,s}^\delta \hookrightarrow G_{\sigma,s'}^\delta.$$

Moreover, for every $u \in G_{\sigma,s}^\delta$, we have the following basic estimates:

$$\begin{aligned} \|\partial_x u\|_{G_{\sigma,s}^{\delta'}} &\leq \frac{e^{-\sigma} \sigma^\sigma}{(\delta - \delta')^\sigma} \|u\|_{G_{\sigma,s}^\delta}, \\ \|(I - \partial_x^2)^{-1} u\|_{G_{\sigma,s}^\delta} &= \|u\|_{G_{\sigma,s-2}^\delta} \leq \|u\|_{G_{\sigma,s}^\delta}, \\ \|\partial_x (I - \partial_x^2)^{-1} u\|_{G_{\sigma,s}^\delta} &\leq \|u\|_{G_{\sigma,s-1}^\delta}, \\ \|\partial_x (I - \partial_x^2)^{-1} u\|_{G_{\sigma,s}^\delta} &\leq \frac{1}{2} \|u\|_{G_{\sigma,s}^\delta}. \end{aligned}$$

(2) Let $s > \frac{1}{2}$, $\sigma \geq 1$ and $\delta > 0$. The space $G_{\sigma,s}^\delta$ is a Banach algebra, and there exist constants $C_s, C'_s > 0$ such that, for $\forall u, v \in G_{\sigma,s}^\delta$,

$$\begin{aligned} \|uv\|_{G_{\sigma,s}^\delta} &\leq C_s \|u\|_{G_{\sigma,s}^\delta} \|v\|_{G_{\sigma,s}^\delta}, \\ \|uv\|_{G_{\sigma,s-1}^\delta} &\leq C'_s \|u\|_{G_{\sigma,s-1}^\delta} \|v\|_{G_{\sigma,s}^\delta}. \end{aligned}$$

Lemma 6.4 (Generalized Ovsyannikov theorem). [35] Let $\{\mathcal{X}_\delta\}_{0 < \delta < 1}$ be a scale of decreasing Banach spaces, i.e., for any $\delta' < \delta$ we have $\mathcal{X}_\delta \hookrightarrow \mathcal{X}_{\delta'}$. Consider the Cauchy problem

$$\begin{cases} \frac{du}{dt} = H(u(t), t), \\ u(0) = u_0. \end{cases} \quad (6.1)$$

Let $\sigma \geq 1$ and let T, R be positive constants. Assume that F satisfies the following conditions:

(1) If $0 < \delta' < \delta < 1$, the function $u(t)$ is holomorphic in $|t| < T$ and continuous on $|t| \leq T$ with values in \mathcal{X}_δ and

$$\sup_{|t| < T} \|u(t)\|_\delta < R,$$

then the mapping $t \mapsto H(u(t), t)$ is holomorphic function $|t| < T$ with values in $\mathcal{X}_{\delta'}$.

(2) There exists $M > 0$ only depending on u_0 and R such that, for any $0 < \delta < 1$,

$$\sup_{|t| < T} \|H(u_0, t)\|_\delta \leq \frac{M}{(1 - \delta)^\sigma}.$$

(3) For $0 < \delta' < \delta < 1$ and any $u, v \in \mathcal{X}_\delta$ with $\|u - u_0\|_\delta < R$, $\|v - v_0\|_\delta < R$, there exists $K > 0$ only depending on u_0 and R such that

$$\sup_{|t| < T} \|H(u, t) - H(v, t)\|_{\delta'} \leq \frac{K}{(\delta - \delta')^\sigma} \|u - v\|_\delta.$$

Then there exists $\bar{T} \in (0, T)$ such that the Cauchy problem (6.1) has a unique function $u(t)$ in \mathcal{X}_δ . Moreover, for every $\delta \in (0, 1)$, it is holomorphic in $|t| < \frac{\bar{T}(1-\delta)^\sigma}{2^\sigma - 1}$, where

$$\bar{T} = \min \left\{ \frac{1}{2^{2\sigma+4}K}, \frac{(2^\sigma - 1)R}{(2^\sigma - 1)2^{2\sigma+3}KR + M} \right\}.$$

Remark 6.5. The abstract Cauchy–Kovalevsky theorem was first proposed by Ovsyannikov in [43–45]. Later, Nirenberg [40], Nishida [41], etc., developed a lot of different versions of this theorem. Very recently, W. Luo and Z. Yin [35] proved the generalized Ovsyannikov theorem (Lemma 6.4), which is weaker than the classical Cauchy–Kovalevsky theorem due to the appearance of the index σ in the Gevrey class.

Theorem 6.6. Let $\sigma \geq 1$, $s > \frac{3}{2}$ and the initial data $u_0 \in G_{\sigma,s}^1$. Then for $\forall \sigma \in (0, 1)$, there exists a $\bar{T} > 0$ such that the Cauchy problem (1.7)–(1.8) admits a unique solution u which is holomorphic in $|t| < \frac{\bar{T}(1-\delta)^\sigma}{2^\sigma - 1}$ with values in $G_{\sigma,s}^\delta$. Especially, the time \bar{T} can be formulated by

$$\bar{T} = \frac{1}{2^{2\sigma+2k+4}(e^{-\sigma}\sigma^\sigma + 1)(1 + c_3 + 2c_2 + 3c_1)C_s^k \|\bar{u}_0\|_{G_{\sigma,s}^1}^k} \approx \frac{1}{\|\bar{u}_0\|_{G_{\sigma,s}^1}^k}, \quad (6.2)$$

where $c_1 = \frac{(k-1)|b-k|}{2}$, $c_2 = \frac{|3k-b|}{2}$, $c_3 = \frac{|b|}{k+1}$ and $C_s > 0$ is the constant given in Lemma 6.3.

Remark 6.7. If $\sigma > 1$, by virtue of the Remark 6.2, we see that Theorem 6.6 implies the Gevrey regularity of the corresponding solutions. If $\sigma = 1$, the generalized Ovsyannikov theorem coincides with the Cauchy–Kovalevsky theorem (see, e.g., [44]). In other words, in the case of $\sigma = 1$, the Theorem 6.6 ensures the existence and uniqueness of the analytic solution, which is analytic in two variables, globally in space and locally in time.

Proof of Theorem 6.6. Let us begin the proof of Theorem 6.6 for the Equ. (1.1) by writing it in the following nonlocal form:

$$\frac{du}{dt} = H(u), \quad u(0) = u_0, \quad (6.3)$$

where the nonlinear autonomous term is given by

$$\begin{aligned} H(u) := & \underbrace{-u^k \partial_x u}_{H_1(u)} - \underbrace{\frac{(k-1)(b-k)}{2}(I - \partial_x^2)^{-1}[u^{k-2}(\partial_x u)^3]}_{H_2(u)} \\ & - \underbrace{\partial_x(I - \partial_x^2)^{-1}\left[\frac{3k-b}{2}u^{k-1}(\partial_x u)^2 + \frac{b}{k+1}u^{k+1}\right]}_{H_3(u)}. \end{aligned}$$

For fixed $\sigma \geq 1$ and $s > \frac{3}{2}$, we consider $\mathcal{X}_\delta = G_{\sigma,s}^\delta$. Thanks to the property (1) of Lemma 6.3, we see that the \mathcal{X}_δ is the scale of decreasing Banach spaces. To prove the Theorem 6.6, we need to verify the conditions (1)–(3) in the Lemma 6.4.

Let us begin with the condition (1). Firstly, we note that if the mapping $t \mapsto u(t)$ is holomorphic, so is the mapping $t \mapsto H(u(t))$. It remains to show that the functional $H(u) \in G_{\sigma,s}^{\delta'}$ provided $0 < \delta' < \delta$. Let $C_s > 0$ be the same constant as that given in Lemma 6.3, we can estimate that

$$\|H_1(u)\|_{G_{\sigma,s}^{\delta'}} \leq C_s \|u^k\|_{G_{\sigma,s}^{\delta'}} \|\partial_x u\|_{G_{\sigma,s}^{\delta'}} \leq \frac{e^{-\sigma} \sigma^\sigma C_s^k}{(\delta - \delta')^\sigma} \|u\|_{G_{\sigma,s}^{\delta}}^{k+1}, \quad (6.4)$$

$$\|H_2(u)\|_{G_{\sigma,s}^{\delta'}} \leq c_1 C_s^k \|u\|_{G_{\sigma,s-2}^{\delta'}}^{k-2} \|\partial_x u\|_{G_{\sigma,s-2}^{\delta'}}^3 \leq \frac{e^{-\sigma} \sigma^\sigma c_1 C_s^k}{(\delta - \delta')^\sigma} \|u\|_{G_{\sigma,s}^{\delta}}^k, \quad (6.5)$$

$$\begin{aligned} \|H_3(u)\|_{G_{\sigma,s}^{\delta'}} &\leq \left\| \frac{3k-b}{2} u^{k-1} (\partial_x u)^2 + \frac{b}{k+1} u^{k+1} \right\|_{G_{\sigma,s-1}^{\delta'}} \\ &\leq c_2 C_s^k \|u\|_{G_{\sigma,s-1}^{\delta'}}^{k-1} \|\partial_x u\|_{G_{\sigma,s-1}^{\delta'}}^2 + c_3 C_s^k \|u\|_{G_{\sigma,s-1}^{\delta'}}^{k+1} \\ &\leq \frac{e^{-\sigma} \sigma^\sigma c_2 C_s^k}{(\delta - \delta')^\sigma} \|u\|_{G_{\sigma,s}^{\delta'}}^{k+1} + \frac{e^{-\sigma} \sigma^\sigma c_3 C_s^k}{(\delta - \delta')^\sigma} \|u\|_{G_{\sigma,s}^{\delta}}^{k+1} \\ &\leq \frac{e^{-\sigma} \sigma^\sigma C_s^k (c_2 + c_3)}{(\delta - \delta')^\sigma} \|u\|_{G_{\sigma,s}^{\delta}}^{k+1}. \end{aligned} \quad (6.6)$$

Thus, it follows from (6.4)–(6.6) that

$$\begin{aligned} \|H(u)\|_{G_{\sigma,s}^{\delta'}} &\leq \|H_1(u)\|_{G_{\sigma,s}^{\delta'}} + \|H_3(u)\|_{G_{\sigma,s}^{\delta'}} + \|H_2(u)\|_{G_{\sigma,s}^{\delta'}} \\ &\leq \frac{e^{-\sigma} \sigma^\sigma C_s^k}{(\delta - \delta')^\sigma} (1 + c_1 + c_2 + c_3) \|u\|_{G_{\sigma,s}^{\delta}}^{k+1}, \end{aligned} \quad (6.7)$$

which implies that $H(u)$ satisfies the condition (1) of Lemma 6.4. By the same argument, for $\bar{u}_0 \in G_{\sigma,s}^1$, we can obtain that

$$\|H(\bar{u}_0)\|_{G_{\sigma,s}^{\delta}} \leq \frac{e^{-\sigma} \sigma^\sigma C_s^k}{(1 - \delta)^\sigma} (1 + c_1 + c_2 + c_3) \|\bar{u}_0\|_{G_{\sigma,s}^1}^{k+1}. \quad (6.8)$$

It is clear that $H(u)$ satisfies the condition (2) of Lemma 6.4 if we choose

$$M := e^{-\sigma} \sigma^\sigma C_s^k (1 + c_1 + c_2 + c_3) \|\bar{u}_0\|_{G_{\sigma,s}^1}^{k+1}.$$

Next, we shall verify the condition (3). To this end, let $u, v \in G_{\sigma,s}^{\delta}$ satisfying $\|u - \bar{u}_0\|_{G_{\sigma,s}^{\delta}} \leq R$ and $\|v - \bar{u}_0\|_{G_{\sigma,s}^{\delta}} \leq R$. For $0 < \delta' < \delta$, by using the properties of the spaces $G_{\sigma,s}^{\delta}$ stated in the Lemma 6.3, we deduce that

$$\begin{aligned} \|H(u) - H(v)\|_{G_{\sigma,s}^{\delta'}} &\leq \underbrace{\|u^k \partial_x u - v^k \partial_x v\|_{G_{\sigma,s}^{\delta'}}}_{E_1(u)} + \underbrace{c_3 \|u^{k+1} - v^{k+1}\|_{G_{\sigma,s-1}^{\delta'}}}_{E_2(u)} \\ &\quad + \underbrace{c_1 \|u^{k-2} (\partial_x u)^3 - v^{k-2} (\partial_x v)^3\|_{G_{\sigma,s-2}^{\delta'}}}_{E_3(u)} \\ &\quad + \underbrace{c_2 \|u^{k-1} (\partial_x u)^2 - v^{k-1} (\partial_x v)^2\|_{G_{\sigma,s-1}^{\delta'}}}_{E_4(u)}. \end{aligned} \quad (6.9)$$

Let us estimate the terms $E_i(u)$ in (6.9) one by one. For the terms $E_1(u)$ and $E_2(u)$, by using the following identities

$$\begin{aligned} u^k \partial_x u - v^k \partial_x v &= \sum_{i=0}^{k-1} u^i v^{k-1-i} (u - v) \partial_x v + u^k \partial_x (u - v), \\ u^{k+1} - v^{k+1} &= (u - v) \sum_{i=0}^k u^i v^{k-i} \end{aligned}$$

and the algebra property of space $G_{\sigma,s}^\delta$, we obtain that

$$\begin{aligned}
 E_1(u) &\leq C_s^k \sum_{i=0}^{k-1} \|u\|_{G_{\sigma,s}^{\delta'}}^i \|v\|_{G_{\sigma,s}^{\delta'}}^{k-1-i} \|u-v\|_{G_{\sigma,s}^{\delta'}} \|\partial_x v\|_{G_{\sigma,s}^{\delta'}} + C_s^k \|u\|_{G_{\sigma,s}^{\delta'}}^k \|\partial_x(u-v)\|_{G_{\sigma,s}^{\delta'}} \\
 &\leq \left[\frac{e^{-\sigma} \sigma^\sigma C_s^k}{(\delta-\delta')^\sigma} \sum_{i=0}^k \|u\|_{G_{\sigma,s}^\delta}^i \|v\|_{G_{\sigma,s}^\delta}^{k-i} + \frac{C_s^k}{(\delta-\delta')^\sigma} \|u\|_{G_{\sigma,s}^\delta}^k \right] \|u-v\|_{G_{\sigma,s}^\delta} \\
 &\leq \frac{(e^{-\sigma} \sigma^\sigma + 1) C_s^k}{(\delta-\delta')^\sigma} (\|u\|_{G_{\sigma,s}^\delta} + \|v\|_{G_{\sigma,s}^\delta})^k \|u-v\|_{G_{\sigma,s}^\delta} \\
 &\leq \frac{(e^{-\sigma} \sigma^\sigma + 1) C_s^k}{(\delta-\delta')^\sigma} (2\|\bar{u}_0\|_{G_{\sigma,s}^1} + 2R)^k \|u-v\|_{G_{\sigma,s}^\delta}, \tag{6.10}
 \end{aligned}$$

and

$$\begin{aligned}
 E_2(u) &\leq C_s^k c_3 \sum_{i=0}^k \|u\|_{G_{\sigma,s}^\delta}^i \|v\|_{G_{\sigma,s}^\delta}^{k-i} \|u-v\|_{G_{\sigma,s}^\delta} \\
 &\leq \frac{C_s^k c_3}{(\delta-\delta')^\sigma} (\|u\|_{G_{\sigma,s}^\delta} + \|v\|_{G_{\sigma,s}^\delta})^k \|u-v\|_{G_{\sigma,s}^\delta} \\
 &\leq \frac{C_s^k c_3}{(\delta-\delta')^\sigma} (2\|\bar{u}_0\|_{G_{\sigma,s}^1} + 2R)^k \|u-v\|_{G_{\sigma,s}^\delta}. \tag{6.11}
 \end{aligned}$$

For the term $E_3(u)$, we have

$$\begin{aligned}
 E_3(u) &\leq \frac{3c_1}{2} \|u\|_{G_{\sigma,s}^{\delta'}}^{k-2} \|\partial_x(u-v)\|_{G_{\sigma,s}^{\delta'}} (\|\partial_x u\|_{G_{\sigma,s-2}^{\delta'}}^2 + \|\partial_x v\|_{G_{\sigma,s-2}^{\delta'}}^2) \\
 &\quad + c_1 C_s^k \|\partial_x v\|_{G_{\sigma,s-2}^{\delta'}}^3 \sum_{i=0}^{k-3} \|u\|_{G_{\sigma,s}^{\delta'}}^i \|v\|_{G_{\sigma,s}^{\delta'}}^{k-3-i} \|u-v\|_{G_{\sigma,s}^{\delta'}} \\
 &\leq \frac{3c_1 e^{-\sigma} \sigma^\sigma C_s^k}{2(\delta-\delta')^\sigma} \|u\|_{G_{\sigma,s}^\delta}^{k-2} (\|u\|_{G_{\sigma,s}^\delta}^2 + \|v\|_{G_{\sigma,s}^\delta}^2) \|u-v\|_{G_{\sigma,s}^\delta} \\
 &\quad + \frac{c_1 e^{-\sigma} \sigma^\sigma C_s^k}{(\delta-\delta')^\sigma} \sum_{i=0}^{k-3} \|u\|_{G_{\sigma,s}^\delta}^i \|v\|_{G_{\sigma,s}^\delta}^{k-3-i} \|u-v\|_{G_{\sigma,s}^\delta} \\
 &\leq \frac{3c_1 e^{-\sigma} \sigma^\sigma C_s^k}{(\delta-\delta')^\sigma} (\|u\|_{G_{\sigma,s}^\delta} + \|v\|_{G_{\sigma,s}^\delta})^k \|u-v\|_{G_{\sigma,s}^\delta} \\
 &\leq \frac{3c_1 e^{-\sigma} \sigma^\sigma C_s^k}{(\delta-\delta')^\sigma} (2\|\bar{u}_0\|_{G_{\sigma,s}^1} + 2R)^k \|u-v\|_{G_{\sigma,s}^\delta} \tag{6.12}
 \end{aligned}$$

Similar to the estimation of $E_3(u)$, for the term $E_4(u)$, we have that

$$\begin{aligned}
 E_4(u) &\leq c_2 C_s^k \|u-v\|_{G_{\sigma,s-1}^\delta} \sum_{i=0}^{k-2} \|u\|_{G_{\sigma,s-1}^{\delta'}}^i \|v\|_{G_{\sigma,s-1}^{\delta'}}^{k-2-i} \|\partial_x u\|_{G_{\sigma,s-1}^\delta}^2 \\
 &\quad + c_2 C_s^k \|v\|_{G_{\sigma,s-1}^{\delta'}}^{k-1} \|\partial_x(u-v)\|_{G_{\sigma,s-1}^{\delta'}} \|\partial_x(u+v)\|_{G_{\sigma,s-1}^{\delta'}} \\
 &\leq \frac{c_2 e^{-\sigma} \sigma^\sigma C_s^k}{(\delta-\delta')^\sigma} \|u-v\|_{G_{\sigma,s}^\delta} \sum_{i=0}^{k-2} \|u\|_{G_{\sigma,s}^{\delta'}}^i \|v\|_{G_{\sigma,s}^{\delta'}}^{k-2-i} \|u\|_{G_{\sigma,s}^\delta}^2
 \end{aligned}$$

$$\begin{aligned}
& + \frac{c_2 e^{-\sigma} \sigma^\sigma C_s^k}{(\delta - \delta')^\sigma} \|v\|_{G_{\sigma,s}^{\delta'}}^{k-1} \|u - v\|_{G_{\sigma,s}^{\delta'}} (\|u\|_{G_{\sigma,s}^{\delta'}} + \|v\|_{G_{\sigma,s}^{\delta'}}) \\
& \leq \frac{2c_2 e^{-\sigma} \sigma^\sigma C_s^k}{(\delta - \delta')^\sigma} (\|u\|_{G_{\sigma,s}^\delta} + \|v\|_{G_{\sigma,s}^\delta})^k \|u - v\|_{G_{\sigma,s}^\delta} \\
& \leq \frac{2c_2 e^{-\sigma} \sigma^\sigma C_s^k}{(\delta - \delta')^\sigma} (2\|\bar{u}_0\|_{G_{\sigma,s}^1} + 2R)^k \|u - v\|_{G_{\sigma,s}^\delta}
\end{aligned} \tag{6.13}$$

Combing the estimates (6.9)–(6.13), we get that

$$\begin{aligned}
& \|H(u) - H(v)\|_{G_{\sigma,s}^{\delta'}} \\
& \leq \frac{(e^{-\sigma} \sigma^\sigma + 1)(1 + c_3 + 2c_2 + 3c_1)C_s^k}{(\delta - \delta')^\sigma} (2\|\bar{u}_0\|_{G_{\sigma,s}^1} + 2R)^k \|u - v\|_{G_{\sigma,s}^\delta},
\end{aligned} \tag{6.14}$$

which implies that $H(u)$ satisfies the condition (2) of Lemma 6.4 by taking $K := (e^{-\sigma} \sigma^\sigma + 1)(1 + c_3 + 2c_2 + 3c_1)C_s^k(2\|\bar{u}_0\|_{G_{\sigma,s}^1} + 2R)^k$. Moreover, we have that $\bar{T} = \min\{\frac{1}{2^{2\sigma+4}K}, \frac{(2^\sigma-1)R}{(2^\sigma-1)2^{2\sigma+3}KR+M}\}$. If we take $R = \|\bar{u}_0\|_{G_{\sigma,s}^1}$, then $K = (e^{-\sigma} \sigma^\sigma + 1)(1 + c_3 + 2c_2 + 3c_1)C_s^k 2^{2k} \|\bar{u}_0\|_{G_{\sigma,s}^1}^k$ and $M \leq 2^{2\sigma+3}KR$, which implies that $\bar{T} = \frac{1}{2^{2\sigma+2k+4}(e^{-\sigma} \sigma^\sigma + 1)(1 + c_3 + 2c_2 + 3c_1)C_s^k \|\bar{u}_0\|_{G_{\sigma,s}^1}^k}$. Thus the proof of Theorem 6.6 is completed. \square

To study the continuity of the data-to-solution mapping, we introduce

Definition 6.8. [35] Let $\sigma \geq 1$, $s > \frac{1}{2}$. The solution mapping for the Eq. (1.1) is said to be continuous if for a $u_0^\infty \in G_{\sigma,s}^1$ there exists $T := T(\|u_0\|_{G_{\sigma,s}^1}) > 0$ such that for any sequence u_0^n in $G_{\sigma,s}^1$ and $\|u_0^n - u_0^\infty\|_{G_{\sigma,s}^1} \rightarrow 0$ as $n \rightarrow \infty$, the solution of Eq. (1.1) satisfies $\|u^n - u^\infty\|_{E_T} \rightarrow 0$ as $n \rightarrow \infty$, where

$$\|u\|_{E_T} := \sup_{|t| < \frac{T(1-\delta)^\sigma}{2^\sigma}, 0 < \delta < 1} \left\{ \|u\|_{G_{\sigma,s}^\delta} (1-\delta)^\sigma \sqrt{1 - \frac{|t|}{T(1-\delta)^\sigma}} \right\}. \tag{6.15}$$

Remark 6.9. Let $\sigma \geq 1$ and let E_T consist of all functions which are holomorphic and continuous with respect to t in $G_{\sigma,s}^\delta$ for every $0 < \delta < 1$ and $|t| < \frac{T(1-\delta)^\sigma}{2^\sigma-1}$, then E_T is a Banach space equipped with the norm (6.15), see [35, Proposition 3.5].

Theorem 6.10. Under the conditions of Theorem 6.6, the data-to-solution mapping $G_{\sigma,s}^1 \ni u_0 \mapsto u \in E_T$ of the Cauchy problem (1.7)–(1.8) is continuous.

Proof. The existence and uniqueness of the solution has been proved in Theorem 6.6, and it follows from (6.2) that the lifespan for the solutions with respect to the data u_0^n and u_0^∞ are given by $\bar{T}^n = \frac{1}{2^{2\sigma+2k+4}(e^{-\sigma} \sigma^\sigma + 1)(1 + c_3 + 2c_2 + 3c_1)C_s^k \|u_0^n\|_{G_{\sigma,s}^1}^k}$ and $\bar{T}^\infty = \frac{1}{2^{2\sigma+2k+4}(e^{-\sigma} \sigma^\sigma + 1)(1 + c_3 + 2c_2 + 3c_1)C_s^k \|u_0^\infty\|_{G_{\sigma,s}^1}^k}$ respectively, C_s is given in Lemma 6.3 and $c_i, i = 1, 2, 3$, are given in Theorem 6.6.

Since $u^n \rightarrow u^\infty$ in $G_{\sigma,s}^1$, there exists a $N > 0$ such that

$$\|u_0^n\|_{G_{\sigma,s}^1} \leq \|u_0^\infty\|_{G_{\sigma,s}^1} + 1, \quad \forall n > N. \tag{6.16}$$

Let us choose $\tilde{T} < \min\{\bar{T}^\infty, \bar{T}^N\}$ for $\forall n > N$, as a direct consequence of the Theorem 6.6, we get that for $\forall n > N$ and $\forall |t| < \frac{\tilde{T}(1-\delta)^\sigma}{2^\sigma-1}$

$$u^n(t) = u_0^n + \int_0^t H(u^n(s))ds \quad \text{and} \quad u^\infty(t) = u_0^\infty + \int_0^t H(u^\infty(s))ds, \tag{6.17}$$

where $H(u)$ is the same as (6.3). Indeed, the \tilde{T} can be chosen as

$$\tilde{T} = \frac{1}{2^{2\sigma+2k+4}(e^{-\sigma}\sigma^\sigma + 1)(1 + c_3 + 2c_2 + 3c_1)C_s^k(\|u_0^\infty\|_{G_{\sigma,s}^1} + 1)^k}.$$

It follows from (6.17) that, for $\forall |t| < \frac{\tilde{T}(1-\delta)^\sigma}{2^\sigma - 1}$,

$$\|u^n(t) - u^\infty(t)\|_{G_{\sigma,s}^\delta} \leq \|u_0^n - u_0^\infty\|_{G_{\sigma,s}^\delta} + \int_0^t \|H(u^n(s)) - H(u^\infty(s))\|_{G_{\sigma,s}^\delta} ds. \quad (6.18)$$

By the same taken in the estimate (6.14), we have

$$\|H(u^n) - H(u^\infty)\|_{G_{\sigma,s}^{\delta'}} \leq \frac{K}{(\delta - \delta')^\sigma} \|u^n - u^\infty\|_{G_{\sigma,s}^\delta}, \quad \delta' < \delta, \quad (6.19)$$

with $K = (e^{-\sigma}\sigma^\sigma + 1)(1 + c_3 + 2c_2 + 3c_1)C_s^k 2^{2k} \|u_0^\infty\|_{G_{\sigma,s}^1}^k$. In order to estimate the second term on the right hand side of (6.18), by applying the Lemma 3.7 in [35] with $\delta(\nu) = \frac{1}{2}(1 + \delta) + (\frac{1}{2})^{2+\frac{1}{\sigma}} \{[(1 - \delta)^\sigma - \frac{t}{a}]^{\frac{1}{\sigma}} - [(1 - \delta)^\sigma + (2^{\sigma+1} - 1)\frac{t}{a}]^{\frac{1}{\sigma}}\}$, we get that

$$\int_0^t \frac{\|u^n(\nu) - u^\infty(\nu)\|_{G_{\sigma,s}^{\delta(\nu)}}}{(\delta(\nu) - \delta)^\sigma} d\nu \leq \frac{\tilde{T} 2^{2\sigma+3} \|u^n(t) - u^\infty(t)\|_{E_{\tilde{T}}}}{(1 - \delta)^\sigma} \sqrt{\frac{\tilde{T}(1 - \delta)^\sigma}{\tilde{T}(1 - \delta)^\sigma - t}}. \quad (6.20)$$

By the estimates (6.18)–(6.20), we get that

$$\begin{aligned} \|u^n(t) - u^\infty(t)\|_{G_{\sigma,s}^\delta} &\leq \|u_0^n - u_0^\infty\|_{G_{\sigma,s}^\delta} \\ &\quad + \frac{K \tilde{T} 2^{2\sigma+3} \|u^n(t) - u^\infty(t)\|_{E_{\tilde{T}}}}{(1 - \delta)^\sigma} \sqrt{\frac{\tilde{T}(1 - \delta)^\sigma}{\tilde{T}(1 - \delta)^\sigma - t}}. \end{aligned} \quad (6.21)$$

Noting that $K \tilde{T} 2^{2\sigma+3} < \frac{1}{2}$, then (6.21) implies that

$$\begin{aligned} (1 - \delta)^\sigma \sqrt{\frac{\tilde{T}(1 - \delta)^\sigma - t}{\tilde{T}(1 - \delta)^\sigma}} \|u^n(t) - u^\infty(t)\|_{G_{\sigma,s}^\delta} \\ \leq \frac{1}{2} \|u^n(t) - u^\infty(t)\|_{E_{\tilde{T}}} + (1 - \delta)^\sigma \sqrt{\frac{\tilde{T}(1 - \delta)^\sigma - t}{\tilde{T}(1 - \delta)^\sigma}} \|u_0^n - u_0^\infty\|_{G_{\sigma,s}^\delta} \\ \leq \frac{1}{2} \|u^n(t) - u^\infty(t)\|_{E_{\tilde{T}}} + \|u_0^n - u_0^\infty\|_{G_{\sigma,s}^1}. \end{aligned} \quad (6.22)$$

By taking the supremum over $0 < \delta < 1$, $0 < t < \frac{\tilde{T}(1-\delta)^\sigma}{2^\sigma - 1}$ on both sides of (6.22), it follows from the definition of the space E_T that

$$\|u^n(t) - u^\infty(t)\|_{E_T} \leq \frac{1}{2} \|u^n(t) - u^\infty(t)\|_{E_{\tilde{T}}} + \|u_0^n - u_0^\infty\|_{G_{\sigma,s}^1}, \quad \forall n > N. \quad (6.23)$$

Hence, we have

$$\|u^n(t) - u^\infty(t)\|_{E_T} \leq 2 \|u_0^n - u_0^\infty\|_{G_{\sigma,s}^1} \longrightarrow 0, \quad \text{as } n \rightarrow \infty, \quad (6.24)$$

which implies the desired result, and this completes the proof of Theorem 6.10. \square

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