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Higher-order Cahn–Hilliard equations with dynamic boundary conditions

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Dedicated to Jerry Goldstein and Rainer Nagel on the occasion of their 75th birthday with great friendship and admiration

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ABSTRACT

Our aim in this paper is to study the well-posedness and the dissipativity of higher-order Cahn–Hilliard equations with dynamic boundary conditions. More precisely, we prove the existence and uniqueness of solutions and the existence of the global attractor.

1. Introduction

The Cahn–Hilliard system,

$$\frac{\partial u}{\partial t} = \Delta w, \ w = -\Delta u + f(u), \tag{1.1}$$

plays an essential role in materials science as it describes important qualitative features of two-phase systems related with phase separation processes. This can be observed, e.g., when a binary alloy is cooled down sufficiently. One then observes a partial nucleation (i.e., the apparition of nucleides in the material) or a total nucleation, the so-called spinodal decomposition: the material quickly becomes inhomogeneous,

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forming a fine-grained structure in which each of the two components appears more or less alternatively. In a second stage, which is called coarsening, occurs at a slower time scale and is less understood, these microstructures coarsen. We refer the reader to, e.g., [4,5,28,31–33,43] and [44] for more details. Here, u is the order parameter (it corresponds to a (rescaled) density of atoms) and w is the chemical potential. Furthermore, f is the derivative of a double-well potential.

This system, endowed with Neumann boundary conditions for both u and w (meaning that the interface is orthogonal to the boundary and that there is no mass flux at the boundary) or with periodic boundary conditions, has been extensively studied and one now has a rather complete picture as far as the existence, uniqueness and regularity of solutions and the asymptotic behavior of the solutions are concerned. We refer the reader to the review paper [8] and the references therein.

Recently, dynamic boundary conditions, which take into account the interactions with the walls in confined systems, were proposed in [14–17,21] and [27]; these boundary conditions also yield a dynamic contact angle with the walls. The Cahn–Hilliard equation, together with such boundary conditions, was studied in, e.g., [11,17,19–21,38,40,49] and [57]; see also [9,10,41] and [42] for the numerical analysis and simulations.

We consider in this paper the higher-order Cahn–Hilliard system

$$\frac{\partial u}{\partial t} = \Delta w, \ \ w = P(-\Delta)u + f(u), \tag{1.2}$$

where $P(s) = \sum_{i=1}^{k} a_i s^i$, $a_k > 0$, $k \geq 2$.

Such higher-order equations follow from higher-order (anisotropic) phase-field models recently proposed by G. Caginalp and E. Esenturk in [3] in the context of phase-field systems. Assuming isotropy and a constant temperature, one finds (1.2). Furthermore, (1.2) was studied in [7], with Dirichlet–Navier boundary conditions.

These models also contain sixth-order Cahn–Hilliard models. We can note that there is currently a strong interest in the study of sixth-order Cahn–Hilliard equations. These equations arise in situations such as strong anisotropy effects being taken into account in phase separation processes (see [53]), atomistic models of crystal growth (see [1,2,13] and [18]), the description of growing crystalline surfaces with small slopes which undergo faceting (see [50]), oil–water-surfactant mixtures (see [22] and [23]) and mixtures of polymer molecules (see [12]). We refer the reader to [6,24–26,29,30,34–37,45–48,54,55] and [56] for the mathematical and numerical analysis of such models. In particular, dynamic boundary conditions for several sixth-order Cahn–Hilliard equations were proposed in [37].

Our aim in this paper is to propose dynamic boundary conditions for the more general higher-order model (1.2). To do so, we follow the approach proposed in [21], i.e., we start with the total (in the bulk and on the boundary) mass conservation

$$\frac{d}{dt}\Big(\int_\Omega u\,dx + \int_\Gamma u\,d\Sigma\Big) = 0, \tag{1.3}$$

instead of the sole bulk mass conservation

$$\frac{d}{dt}\int_\Omega u\,dx = 0,$$

as in the previous approaches. Here, Ω is the domain occupied by the system (we assume that it is a bounded and regular domain of $\mathbb{R}^N$, $N = 2$ or 3) and $\Gamma = \partial\Omega$. We further assume that f is regular enough.

Following [21], a first dynamic boundary condition, which is compatible with the mass conservation (1.3), reads

$$\frac{\partial u}{\partial t} = \eta \Delta_\Gamma w - \frac{\partial w}{\partial \nu} \text{ on } \Gamma,\ \eta \geq 0, \tag{1.4}$$

where Δ_Γ denotes the Laplace–Beltrami operator ($\eta = 0$ corresponds to the case where there is no diffusion on the boundary). Indeed, integrating (formally) the first of (1.2) over Ω, we obtain

$$\frac{d}{dt} \int_\Omega u \, dx = \int_\Gamma \frac{\partial w}{\partial \nu} \, d\Sigma,$$

hence the result, owing to (1.4).

Next, we rewrite (1.2) in the form

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta w_k, \\
w_k &= -\Delta w_{k-1} + f(u), \\
w_{k-1} &= -\Delta w_{k-2} + a_1 u, \\
w_{k-2} &= -\Delta w_{k-3} + a_2 u, \\
&\cdots \\
w_2 &= -\Delta w_1 + a_{k-2} u, \\
w_1 &= -a_k \Delta u + a_{k-1} u.
\end{aligned}$$

Following again [21], we can then consider the following dynamic boundary conditions:

$$\begin{aligned}
w_k &= -\sigma \Delta_\Gamma w_{k-1} + \frac{\partial w_{k-1}}{\partial \nu} + g(u) \text{ on } \Gamma, \\
w_{k-1} &= -\sigma \Delta_\Gamma w_{k-2} + \frac{\partial w_{k-2}}{\partial \nu} + a_1 u \text{ on } \Gamma, \\
w_{k-2} &= -\sigma \Delta_\Gamma w_{k-3} + \frac{\partial w_{k-3}}{\partial \nu} + a_2 u \text{ on } \Gamma, \\
&\cdots \\
w_2 &= -\sigma \Delta_\Gamma w_1 + \frac{\partial w_1}{\partial \nu} + a_{k-2} u \text{ on } \Gamma, \\
w_1 &= -a_k \sigma \Delta_\Gamma u + \frac{\partial u}{\partial \nu} + a_{k-1} u \text{ on } \Gamma,
\end{aligned}$$

where $\sigma \geq 0$ (again, when $\sigma = 0$, there is no diffusion on the boundary) and g is regular enough.

We now set

$$U = \begin{pmatrix} u \\ u|_\Gamma \end{pmatrix},\ W_i = \begin{pmatrix} w_i \\ w_i|_\Gamma \end{pmatrix},\ i = 1,\ \cdots,\ k,$$

and

$$A_\kappa \begin{pmatrix} \varphi \\ \varphi|_\Gamma \end{pmatrix} = \begin{pmatrix} -\Delta \varphi \\ (-\kappa \Delta_\Gamma \varphi + \frac{\partial \varphi}{\partial \nu})|_\Gamma \end{pmatrix},\ \kappa \geq 0.$$

We thus have, in view of (1.4) and the above,

$$\frac{\partial U}{\partial t} = -A_\eta W_k,$$
$$W_k = A_\sigma W_{k-1} + \begin{pmatrix} f(u) \\ g(u)|_\Gamma \end{pmatrix},$$
$$W_{k-1} = A_\sigma W_{k-2} + a_1 U,$$
$$W_{k-2} = A_\sigma W_{k-3} + a_2 U,$$
$$\dots$$
$$W_2 = A_\sigma W_1 + a_{k-2} U,$$
$$W_1 = A_\sigma U + a_{k-1} U,$$

so that

$$W_k = P(A_\sigma)U + \begin{pmatrix} f(u) \\ g(u)|_\Gamma \end{pmatrix}.$$

Setting $W = \begin{pmatrix} w \\ w|_\Gamma \end{pmatrix}$, we are thus lead to the study of the boundary value problem

$$\frac{\partial U}{\partial t} = -A_\eta W, \tag{1.5}$$
$$W = P(A_\sigma)U + \begin{pmatrix} f(u) \\ g(u)|_\Gamma \end{pmatrix}. \tag{1.6}$$

Remark 1.1. The Cahn–Hilliard system (1.2) follows from the bulk (Ginzburg–Landau type) free energy

$$\Psi_\Omega = \int_\Omega \Big(\sum_{i=1}^{k} a_i |(-\Delta)^{\frac{i}{2}} u|^2 + F(u)\Big)\, dx,$$

where we keep the operator $(-\Delta)^{\frac{i}{2}}$ formal when i is odd and $F' = f$. We then introduce the surface free energy

$$\Psi_\Gamma = \int_\Gamma \Big(\sum_{i=1}^{k} b_i |(-\Delta_\Gamma)^{\frac{i}{2}} u|^2 + G(u)\Big)\, d\Sigma,$$

where $G' = g$. We define the total free energy as the sum

$$\Psi = \Psi_\Omega + \Psi_\Gamma.$$

Equations (1.5)–(1.6) are then related to Ψ in the sense that

$$W = \frac{\delta \Psi}{\delta u},$$

where $\frac{\delta}{\delta u}$ denotes a variational derivative with respect to u (see [21]), in the particular case

$$b_i = \sigma a_i, \ i = 1, \ \cdots, \ k.$$

Of course, it is also important to consider general b_i's. In that case, however, the corresponding higher-order Cahn–Hilliard system can no longer be rewritten in the compact form (1.5)–(1.6) and is more difficult to study; this will be considered elsewhere.

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Our aim in this paper is to study the higher-order model (1.5)–(1.6). In particular, we obtain the existence and uniqueness of solutions, as well as the existence of the global attractor.

We will focus here on the case η, $\sigma > 0$ only (actually, for simplicity, we will take $\eta = \sigma = 1$), i.e., we assume diffusion on the walls. The case $\eta = 0$ and/or $\sigma = 0$ will be studied elsewhere.

Notation. Throughout the paper, the same letters c, c', c'' and c''' denote (generally positive) constants which may vary from line to line. Similarly, the same letter Q denotes (positive) monotone increasing and continuous functions which may vary from line to line.

2. Setting of the problem

We first introduce the following spaces.

1) $H = L^2(\Omega)$, $H_\Gamma = L^2(\Gamma)$, $\mathcal{H} = H \times H_\Gamma$. Here, H and H_Γ are endowed with their usual scalar products and associated norms, denoted by $((\cdot,\cdot))$, $\|\cdot\|$, $((\cdot,\cdot))_\Gamma$ and $\|\cdot\|_\Gamma$, and $\mathcal{H}$ is endowed with its usual scalar product and associated norm, denoted by $((\cdot,\cdot))_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}}$. More generally, $\|\cdot\|_X$ denotes the norm on the Banach space X.

2) $V = H^1(\Omega)$, $V_\Gamma = H^1(\Gamma)$, $\mathcal{V} = \{\begin{pmatrix}\varphi\\ \psi\end{pmatrix} \in V \times V_\Gamma,\ \varphi|_\Gamma = \psi\}$ which we again endow with their usual scalar products and associated norms.

3) We set, for $\phi = \begin{pmatrix}\varphi\\ \psi\end{pmatrix} \in H \times H_\Gamma$,

$$\langle\phi\rangle = \frac{1}{\mathrm{Vol}(\Omega) + |\Gamma|}\Big(\int_\Omega \varphi\,dx + \int_\Gamma \psi\,d\Sigma\Big).$$

We then set

$$\dot{\mathcal{H}} = \{\phi = \begin{pmatrix}\varphi\\ \psi\end{pmatrix} \in \mathcal{H},\ \langle\phi\rangle = 0\}$$

and

$$\dot{\mathcal{V}} = \mathcal{V} \cap \dot{\mathcal{H}}.$$

In particular, the inclusions $\dot{\mathcal{V}} \subset \dot{\mathcal{H}} \subset \dot{\mathcal{V}}'$ are dense, continuous and compact.

We have the

Lemma 2.1. *The norm* $\|\cdot\|^2_{\dot{\mathcal{V}}} = \|\nabla\cdot\|^2 + \|\nabla_\Gamma\cdot\|^2$ *is equivalent to the usual* $H^1(\Omega) \times H^1(\Gamma)$*-norm on* $\dot{\mathcal{V}}$. *Here,* ∇_Γ *denotes the surface gradient.*

Proof. It suffices to prove that there exists a positive constant c such that

$$\|\varphi\|^2_{H^1(\Omega)} + \|\varphi|_\Gamma\|^2_{H^1(\Gamma)} \le c\|\phi\|^2_{\dot{\mathcal{V}}},\ \forall\,\phi = \begin{pmatrix}\varphi\\ \varphi|_\Gamma\end{pmatrix} \in \dot{\mathcal{V}}.$$

For the sake of simplicity, we omit the symbol $|_\Gamma$ in what follows.

Suppose not. Then, for any $n \in \mathbb{N}$, there exists $\phi_n = \begin{pmatrix}\varphi_n\\ \varphi_n\end{pmatrix} \in \dot{\mathcal{V}}\backslash\{0\}$ such that

$$\|\phi_n\|^2_{\dot{\mathcal{V}}} < \frac{1}{n}(\|\varphi_n\|^2_{H^1(\Omega)} + \|\varphi_n\|^2_{H^1(\Gamma)}).$$

We set

$$\Theta_n = \begin{pmatrix} \theta_n \\ \theta_n \end{pmatrix} = \frac{\phi_n}{(\|\varphi_n\|^2_{H^1(\Omega)} + \|\varphi_n\|^2_{H^1(\Gamma)})^{\frac{1}{2}}},$$

so that

$$\|\theta_n\|^2_{H^1(\Omega)} + \|\theta_n\|^2_{H^1(\Gamma)} = 1, \tag{2.1}$$

$$\|\Theta_n\|^2_{\dot{\mathcal{V}}} \le \frac{1}{n}. \tag{2.2}$$

It then follows from (2.2) that

$$\nabla\theta_n \to 0 \text{ in } L^2(\Omega) \text{ and } \nabla_\Gamma\theta_n \to 0 \text{ in } L^2(\Gamma). \tag{2.3}$$

Furthermore, it follows from (2.1) and the compact embeddings $H^1(\Omega) \subset L^2(\Omega)$ and $H^1(\Omega) \subset L^2(\Gamma)$ that, up to a subsequence which we do not relabel,

$$\theta_n \to \theta \text{ in } L^2(\Omega) \text{ and } L^2(\Gamma) \tag{2.4}$$

and

$$0 = \int_\Omega \theta_n\, dx + \int_\Gamma \theta_n\, d\Sigma \to \int_\Omega \theta\, dx + \int_\Gamma \theta\, d\Sigma, \tag{2.5}$$

for some $\Theta = \begin{pmatrix} \theta \\ \theta \end{pmatrix}$. Next, it follows from (2.4) that

$$\theta_n \to \theta \text{ in } \mathcal{D}'(\Omega),$$

so that

$$\nabla\theta_n \to \nabla\theta \text{ in } \mathcal{D}'(\Omega),$$

whence, in view of (2.3),

$$\theta = c \text{ (constant)}.$$

We finally deduce from (2.5) that $c = 0$, so that $\Theta = 0$, hence a contradiction, since, passing to the limit in (2.1), there holds

$$\|\theta\|^2_{H^1(\Omega)} + \|\theta\|^2_{H^1(\Gamma)} = 1.$$

This finishes the proof. □

Next, we introduce the bilinear form

$$a : \dot{\mathcal{V}} \times \dot{\mathcal{V}} \to \mathbb{R},\ (\phi, \Theta) \mapsto ((\nabla\varphi, \nabla\theta)) + ((\nabla_\Gamma\varphi, \nabla_\Gamma\theta))_\Gamma,$$

$\phi = \begin{pmatrix} \varphi \\ \varphi \end{pmatrix}$, $\Theta = \begin{pmatrix} \theta \\ \theta \end{pmatrix}$.

It follows from Lemma 2.1 that a is symmetric, continuous and coercive in $\dot{\mathcal{V}}$. This allows us to define the linear operator

$$A : \dot{\mathcal{V}} \to \dot{\mathcal{V}}'$$

by

$$\langle A\phi, \Theta\rangle_{\dot{\mathcal{V}}',\dot{\mathcal{V}}} = a(\phi,\Theta),\ \phi,\ \Theta \in \dot{\mathcal{V}}.$$

The operator A is a strictly positive, selfadjoint and unbounded linear operator. Furthermore, we can define the domain of A in $\dot{\mathcal{H}}$, $D(A) = \{\phi \in \dot{\mathcal{V}},\ \exists\,\Xi \in \dot{\mathcal{H}},\ ((\Xi,\Theta))_{\mathcal{H}} = a(\phi,\Theta),\ \forall\,\Theta \in \dot{\mathcal{V}}\}$. Note that A is an isomorphism from $\dot{\mathcal{V}}$ onto $\dot{\mathcal{V}}'$ and from $D(A)$ onto $\dot{\mathcal{H}}$. Finally, noting that A^{-1} can be considered as a selfadjoint and compact operator in $\dot{\mathcal{H}}$, we can define the powers A^s, together with their domains $D(A^s)$, $s \in \mathbb{R}$ (being understood that $D(A^0) = \dot{\mathcal{H}}$), and A^s, $s > 0$, can be extended as an isomorphism from $\dot{\mathcal{H}}$ onto $D(A^{-s})$ and, more generally, from $D(A^{s_1})$ onto $D(A^{s_1-s})$, $s_1 \in \mathbb{R}$. We refer the interested reader to, e.g., [51] for more details. In particular, there holds

Proposition 2.2. *For $k \in \mathbb{N}$, $D(A^k) = \dot{\mathcal{V}} \cap (H^{2k}(\Omega) \times H^{2k}(\Gamma))$ and the norm $\|A^k \cdot\|_{\mathcal{H}}$ is equivalent to the usual $H^{2k}(\Omega) \times H^{2k}(\Gamma)$-one on $D(A^k)$.*

Proof. We proceed by induction.

First case: $k = 1$. Let $\phi = \begin{pmatrix}\varphi\\ \varphi\end{pmatrix} \in \dot{\mathcal{V}}$ be the solution to

$$a(\phi,\Theta) = ((\mathcal{F},\Theta))_{\mathcal{H}},\ \forall\,\Theta \in \dot{\mathcal{V}}, \tag{2.6}$$

where $\mathcal{F} = \begin{pmatrix}f_1\\ f_2\end{pmatrix} \in \dot{\mathcal{H}}$. Then, it is easy to see that

$$a(\phi,\Theta) = ((\mathcal{F},\Theta))_{\mathcal{H}},\ \forall\,\Theta \in \mathcal{V}. \tag{2.7}$$

Let us first assume that $\phi \in H^2(\Omega) \times H^2(\Gamma)$. Then, integrating by parts, we have

$$-\int\limits_{\Omega} \Delta\varphi\theta\,dx + \int\limits_{\Gamma} (-\Delta_\Gamma\varphi + \frac{\partial\varphi}{\partial\nu})\theta\,d\Sigma = ((f_1,\theta)) + ((f_2,\theta))_\Gamma,\ \forall \begin{pmatrix}\theta\\ \theta\end{pmatrix} \in H^1(\Omega) \times H^1(\Gamma).$$

Taking $\theta \in \mathcal{D}(\Omega)$, this yields that

$$-\Delta\varphi = f_1 \text{ in } \mathcal{D}'(\Omega),\ L^2(\Omega) \text{ and a.e.} \tag{2.8}$$

There thus remains

$$\int\limits_{\Gamma} (-\Delta_\Gamma\varphi + \frac{\partial\varphi}{\partial\nu})\theta\,d\Sigma = ((f_2,\theta))_\Gamma,\ \forall\,\theta \in H^1(\Gamma),$$

which yields that

$$-\Delta_\Gamma\varphi + \frac{\partial\varphi}{\partial\nu} = f_2 \ \text{ in } L^2(\Gamma) \text{ and a.e.} \tag{2.9}$$

We thus deduce that $\dot{\mathcal{V}} \cap (H^2(\Omega) \times H^2(\Gamma)) \subset D(A)$ and, if $\phi = \begin{pmatrix} \varphi \\ \varphi \end{pmatrix}$ belongs to this space, then

$$A\phi = \begin{pmatrix} -\Delta\varphi \\ -\Delta_\Gamma\varphi + \frac{\partial\varphi}{\partial\nu} \end{pmatrix}.$$

Let now $\phi \in \dot{\mathcal{V}}$ and $\mathcal{F} \in \dot{\mathcal{H}}$ be such that (2.6) and, thus, (2.7) are satisfied. Then, taking $\Theta = \begin{pmatrix} \theta \\ 0 \end{pmatrix}$, $\theta \in \mathcal{D}(\Omega)$, in (2.7), we can see that (2.8) still holds. Furthermore, since $\varphi \in H^1(\Omega)$ and $\Delta\varphi \in L^2(\Omega)$, the trace $\frac{\partial\varphi}{\partial\nu}$ can be defined in $H^{-\frac{1}{2}}(\Gamma)$ and a generalized form of Green's formula is valid for every $\theta \in H^1(\Omega)$ (see [52]; see also [51], Chapter II, Example 2.5), yielding

$$-((\Delta\varphi, \theta)) = -\langle \frac{\partial\varphi}{\partial\nu}, \theta \rangle_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)} + ((\nabla\varphi, \nabla\theta)),\ \forall\theta \in H^1(\Omega),$$

whence, in view of (2.7) and (2.8),

$$\langle \frac{\partial\varphi}{\partial\nu} - f_2, \theta \rangle_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)} + ((\nabla_\Gamma\varphi, \nabla_\Gamma\theta))_\Gamma = 0,\ \forall\theta \in H^1(\Omega),\ \theta \in H^1(\Gamma). \tag{2.10}$$

Actually, (2.10) also holds for any $\theta \in H^1(\Gamma)$ (take $\theta \in H^{\frac{3}{2}}(\Omega)$ and note that Ω is regular enough) and we see that

$$\langle -\Delta_\Gamma\varphi + \frac{\partial\varphi}{\partial\nu} - f_2, \theta \rangle_{H^{-1}(\Gamma), H^1(\Gamma)} = 0,\ \forall\theta \in H^1(\Gamma),$$

so that (2.9) is again valid, in a weak form.

Next, it follows from Lemma 2.1 and the beginning of the proof of [38], Lemma A.1, that, if $\phi = \begin{pmatrix} \varphi \\ \varphi \end{pmatrix} \in D(A)$,

$$\|\varphi\|^2_{H^1(\Omega)} + \|\varphi\|^2_{H^1(\Gamma)} \le c\|A\phi\|^2_{\mathcal{H}}.$$

Rewriting then our elliptic problem in the form

$$-\Delta\varphi = f_1,\ -\Delta_\Gamma\varphi + \frac{\partial\varphi}{\partial\nu} + \varphi = f_2 + \varphi, \tag{2.11}$$

it follows from [38], Lemma A.1, that

$$\|\varphi\|^2_{H^2(\Omega)} + \|\varphi\|^2_{H^2(\Gamma)} \le c(\|A\phi\|^2_{\mathcal{H}} + \|\phi\|^2_{\mathcal{V}}) \le c'\|A\phi\|^2_{\mathcal{H}},$$

which completes the proof in the case $k = 1$.

Second case: $k \ge 2$. We assume that

$$\|\phi\|_{H^{2(k-1)}(\Omega) \times H^{2(k-1)}(\Gamma)} \le c\|A^{k-1}\phi\|_{\mathcal{H}},\ \forall\phi \in D(A^{k-1}). \tag{2.12}$$

Let ϕ belong to $D(A^k)$. Noting that $A^k\phi = \mathcal{F}$, $\mathcal{F} \in \dot{\mathcal{H}}$, is equivalent to

$$A^{k-1}A\phi = \mathcal{F},\ A\phi \in D(A^{k-1}),$$

we deduce from (2.12) that

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$$\|A\phi\|_{H^{2(k-1)}(\Omega)\times H^{2(k-1)}(\Gamma)} \le c\|A^k\phi\|_{\mathcal{H}}. \tag{2.13}$$

On the other hand, it follows from the elliptic regularity result given in [38], Corollary A.1 (once more applied to the slightly modified elliptic system (2.11)), and (2.12)–(2.13) that

$$\begin{aligned}\|\phi\|^2_{H^{2k}(\Omega)\times H^{2k}(\Gamma)} &\le c(\|A\phi\|^2_{H^{2(k-1)}(\Omega)\times H^{2(k-1)}(\Gamma)} + \|\phi\|^2_{H^{2(k-1)}(\Omega)\times H^{2(k-1)}(\Gamma)})\\ &\le c(\|A^k\phi\|^2_{\mathcal{H}} + \|A^{k-1}\phi\|^2_{\mathcal{H}}),\end{aligned}$$

whence

$$\|\phi\|_{H^{2k}(\Omega)\times H^{2k}(\Gamma)} \le c\|A^k\phi\|_{\mathcal{H}},$$

noting that $D(A^k)$ is continuously embedded into $D(A^{k-1})$. This finishes the proof. □

Noting that, by definition, $D(A^{\frac{1}{2}}) = \dot{\mathcal{V}}$ and proceeding in a similar way (writing, in particular, that $A^{k+\frac{1}{2}} = A^{k-\frac{1}{2}}A$, $k \ge 1$), we can also prove the

Proposition 2.3. *For $k \in \mathbb{N}\cup\{0\}$, $D(A^{k+\frac{1}{2}}) = \dot{\mathcal{V}} \cap (H^{2k+1}(\Omega) \times H^{2k+1}(\Gamma))$ and the norm $\|A^{k+\frac{1}{2}} \cdot \|_{\mathcal{H}}$ is equivalent to the usual $H^{2k+1}(\Omega) \times H^{2k+1}(\Gamma)$-one on $D(A^{k+\frac{1}{2}})$.*

We finally note that $D(A^{-\frac{1}{2}}) = \dot{\mathcal{V}}'$ and, since the norm $\|A^{\frac{1}{2}} \cdot \|_{\mathcal{H}}$ is equivalent to the $\dot{\mathcal{V}}$-one, it follows that the norm $\|\cdot\|_{-1} = \|A^{-\frac{1}{2}} \cdot \|_{\mathcal{H}}$ is equivalent to the usual $\dot{\mathcal{V}}'$-one.

Remark 2.4. We can note that the bilinear form a can also be defined on $\mathcal{V} \times \mathcal{V}$; in that case however, it is still continuous, but not coercive. This allows us to also consider the operator A as an operator from $\mathcal{V}$ onto $\mathcal{V}'$.

We now consider the following initial and boundary value problem:

$$\frac{\partial U}{\partial t} = -AW \text{ in } \mathcal{V}', \tag{2.14}$$

$$W = P(A)U + \mathcal{F}(U) \text{ in } \mathcal{V}', \tag{2.15}$$

$$U|_{t=0} = U_0, \tag{2.16}$$

where $U = \begin{pmatrix} u \\ u \end{pmatrix}$, $W = \begin{pmatrix} w \\ w \end{pmatrix}$ and $\mathcal{F}(U) = \begin{pmatrix} f(u) \\ g(u) \end{pmatrix}$. Furthermore,

$$P(s) = \sum_{i=1}^{k} a_i s^i,\ a_k > 0,\ k \ge 2.$$

As far as the functions f and g are concerned, we assume that

$$f,\ g \in \mathcal{C}^2(\mathbb{R}), \tag{2.17}$$

$$f' \ge -c_0,\ g' \ge -c_1,\ c_0,\ c_1 \ge 0, \tag{2.18}$$

$$f(s)(s-m) \ge c_2F(s) - c_3(m) \ge -c_4(m),\ c_2 > 0,\ c_3,\ c_4 \ge 0,\ s,\ m \in \mathbb{R}, \tag{2.19}$$

$$g(s)(s-m) \ge c_5G(s) - c_6(m) \ge -c_7(m),\ c_5 > 0,\ c_6,\ c_7 \ge 0,\ s,\ m \in \mathbb{R}, \tag{2.20}$$

where $F(s) = \int_0^s f(\xi)\, d\xi$, $G(s) = \int_0^s g(\xi)\, d\xi$ and the constants c_3, c_4, c_6 and c_7 depend continuously on m;

$$F(s) \geq c_8 s^4 - c_9,\ G(s) \geq c_{10} s^4 - c_{11},\ c_8,\ c_{10} > 0,\ c_9,\ c_{11} \geq 0,\ s \in \mathbb{R}. \tag{2.21}$$

Remark 2.5. In particular, the above assumptions are satisfied by the usual cubic bulk nonlinear term $f(s) = s^3 - s$. However, it was proposed in [14,15] and [16] that g be affine, $g(s) = as + b$, $a > 0$. Unfortunately, such surface "nonlinear" terms do not satisfy (2.21), which is essential to have coercivity in the proof of existence given below (see (3.16)–(3.18)), and cannot be considered in general. However, when $a_i \geq 0$, $i = 1, \cdots, k-1$ (recall that $a_k > 0$), the coercivity is straightforward and (2.21) is no longer needed, so that the affine surface terms can be considered.

Setting, whenever it makes sense,

$$\overline{\phi} = \phi - \begin{pmatrix} \langle \phi \rangle \\ \langle \phi \rangle \end{pmatrix},$$

so that $\langle \overline{\phi} \rangle = 0$, we can rewrite (2.14) in the following equivalent form

$$A^{-1} \frac{\partial U}{\partial t} = -\overline{W} \text{ in } \dot{\mathcal{V}}', \tag{2.22}$$

noting that $\langle \frac{\partial U}{\partial t} \rangle = 0$. Furthermore, it follows from (2.15) that

$$\langle W \rangle = \langle \mathcal{F}(U) \rangle. \tag{2.23}$$

We finally note that $A\phi = A\overline{\phi}$ (see Remark 2.4).

3. A priori estimates

The estimates derived in this section are formal. They can however easily be justified within, e.g., a Galerkin scheme.

We multiply (2.14) by W, scalarly in $\mathcal{H}$, and have

$$((\frac{\partial U}{\partial t}, W))_{\mathcal{H}} + \|W\|^2_{\mathcal{V}} = 0. \tag{3.1}$$

We then multiply (2.15) by $\frac{\partial U}{\partial t}$ to obtain

$$((\frac{\partial U}{\partial t}, W))_{\mathcal{H}} = ((P(A)U, \frac{\partial U}{\partial t}))_{\mathcal{H}} + ((\mathcal{F}(U), \frac{\partial U}{\partial t}))_{\mathcal{H}}. \tag{3.2}$$

We note that

$$((P(A)U, \frac{\partial U}{\partial t}))_{\mathcal{H}} = \frac{1}{2} \frac{d}{dt} \sum_{i=1}^{k} a_i \|A^{\frac{i}{2}} \overline{U}\|^2_{\mathcal{H}} \tag{3.3}$$

and

$$((\mathcal{F}(U), \frac{\partial U}{\partial t}))_{\mathcal{H}} = \frac{d}{dt}(\int_\Omega F(u)\, dx + \int_\Gamma G(u)\, d\Sigma). \tag{3.4}$$

It then follows from (3.2)–(3.4) that

$$((\frac{\partial U}{\partial t}, W))_{\mathcal{H}} = \frac{1}{2}\frac{d}{dt}(\sum_{i=1}^{k} a_i \|A^{\frac{i}{2}}\overline{U}\|^2_{\mathcal{H}} + 2\int_{\Omega} F(u)\, dx + 2\int_{\Gamma} G(u)\, d\Sigma). \tag{3.5}$$

We finally deduce from (3.1) and (3.5) that

$$\frac{d}{dt}(\sum_{i=1}^{k} a_i \|A^{\frac{i}{2}}\overline{U}\|^2_{\mathcal{H}} + 2\int_{\Omega} F(u)\, dx + 2\int_{\Gamma} G(u)\, d\Sigma) + 2\|W\|^2_{\mathcal{V}} = 0. \tag{3.6}$$

We further note that it follows from the interpolation inequality

$$\|\phi\|_{H^i(\Omega)\times H^i(\Gamma)} \le c(i)\|\phi\|^{\frac{i}{m}}_{H^m(\Omega)\times H^m(\Gamma)}\|\phi\|^{1-\frac{i}{m}}_{\mathcal{H}}, \tag{3.7}$$
$$\phi \in H^m(\Omega)\times H^m(\Gamma),\ i\in\{1,\ \cdots\ ,\ m-1\},\ m\in\mathbb{N},\ m\ge 2,$$

and Propositions 2.2 and 2.3 that

$$\frac{a_k}{2}\|\overline{U}\|^2_{H^k(\Omega)\times H^k(\Gamma)} - c\|\overline{U}\|^2_{\mathcal{H}} \le \sum_{i=1}^{k} a_i \|A^{\frac{i}{2}}\overline{U}\|^2_{\mathcal{H}} \le c'\|\overline{U}\|^2_{H^k(\Omega)\times H^k(\Gamma)}. \tag{3.8}$$

Next, we multiply (2.22) by $\overline{U}$ to find

$$\frac{1}{2}\frac{d}{dt}\|\overline{U}\|^2_{-1} = -((W, \overline{U}))_{\mathcal{H}}. \tag{3.9}$$

We then multiply (2.15) by $\overline{U}$ and have

$$((W, \overline{U}))_{\mathcal{H}} = \sum_{i=1}^{k} a_i \|A^{\frac{i}{2}}\overline{U}\|^2_{\mathcal{H}} + ((\mathcal{F}(U), \overline{U}))_{\mathcal{H}}. \tag{3.10}$$

It follows from (3.9)–(3.10) that

$$\frac{d}{dt}\|\overline{U}\|^2_{-1} + 2\sum_{i=1}^{k} a_i \|A^{\frac{i}{2}}\overline{U}\|^2_{\mathcal{H}} + 2((\mathcal{F}(U), \overline{U}))_{\mathcal{H}} = 0. \tag{3.11}$$

We assume from now on that

$$|\langle U_0\rangle| \le M,\ M \ge 0 \text{ given}. \tag{3.12}$$

Therefore,

$$|\langle U(t)\rangle| \le M,\ \forall\, t \ge 0. \tag{3.13}$$

We thus deduce from (2.19)–(2.20) and (3.13) that

$$((\mathcal{F}(U), \overline{U}))_{\mathcal{H}} \ge c(\int_{\Omega} F(u)\, dx + \int_{\Gamma} G(u)\, d\Sigma) - c'_M,\ c > 0,\ c'_M \ge 0. \tag{3.14}$$

For simplicity, we omit the dependence of the constants on M in what follows.

Summing (3.6) and (3.11), we deduce from (3.14) a differential inequality of the form

$$\frac{dE_1}{dt} + c(E_1 + \|W\|^2_{\dot{\mathcal{V}}}) \le c', \; c > 0, \tag{3.15}$$

where

$$E_1 = \|\overline{U}\|^2_{-1} + \sum_{i=1}^{k} a_i \|A^{\frac{i}{2}} \overline{U}\|^2_{\mathcal{H}} + 2 \int_\Omega F(u)\, dx + 2 \int_\Gamma G(u)\, d\Sigma. \tag{3.16}$$

Furthermore, it follows from (2.21) and (3.8) that

$$E_1 \ge c(\|\overline{U}\|^2_{H^k(\Omega)\times H^k(\Gamma)} + \int_\Omega F(u)\, dx + \int_\Gamma G(u)\, d\Sigma) + c'\|U\|^4_{L^4(\Omega)\times L^4(\Gamma)} - c''\|U\|^2_{\mathcal{H}} - c''',$$

so that

$$E_1 \ge c(\|\overline{U}\|^2_{H^k(\Omega)\times H^k(\Gamma)} + \int_\Omega F(u)\, dx + \int_\Gamma G(u)\, d\Sigma) - c', \tag{3.17}$$

noting that

$$\|U\|^2_{\mathcal{H}} \le \epsilon \|U\|^4_{L^4(\Omega)\times L^4(\Gamma)} + c_\epsilon, \; \forall\, \epsilon > 0. \tag{3.18}$$

Moreover,

$$E_1 \le c(\|\overline{U}\|^2_{H^k(\Omega)\times H^k(\Gamma)} + \int_\Omega F(u)\, dx + \int_\Gamma G(u)\, d\Sigma) + c'. \tag{3.19}$$

In particular, it follows from (3.13), (3.15), (3.17), (3.19) and Gronwall's lemma that

$$\begin{aligned} &\|U(t)\|^2_{H^k(\Omega)\times H^k(\Gamma)} \\ &\le ce^{-c't}(\|U_0\|^2_{H^k(\Omega)\times H^k(\Gamma)} + \int_\Omega F(u_0)\, dx + \int_\Gamma G(u_0)\, d\Sigma) + c'', \; c' > 0, \; t \ge 0, \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} &\int_t^{t+r} (\|\frac{\partial U}{\partial t}\|^2_{-1} + \|W\|^2_{\dot{\mathcal{V}}})\, ds \\ &\le ce^{-c't}(\|U_0\|^2_{H^k(\Omega)\times H^k(\Gamma)} + \int_\Omega F(u_0)\, dx + \int_\Gamma G(u_0)\, d\Sigma) + c'', \; c' > 0, \; t \ge 0, \end{aligned} \tag{3.21}$$

$r > 0$ given, where $U_0 = \begin{pmatrix} u_0 \\ u_0 \end{pmatrix}$. Note indeed that it follows from (2.14) that

$$\|\frac{\partial U}{\partial t}\|_{-1} = \|W\|_{\dot{\mathcal{V}}}. \tag{3.22}$$

We now rewrite (2.14)–(2.15) in the equivalent form

$$A^{-1}\frac{\partial U}{\partial t} + P(A)\overline{U} + \mathcal{F}(U) - \langle \mathcal{F}(U)\rangle = 0 \text{ in } \dot{\mathcal{V}}'. \tag{3.23}$$

We multiply (3.23) by $A^k\overline{U}$, scalarly in $\mathcal{H}$, and have, owing to the interpolation inequality (3.7),

$$\frac{d}{dt}\|A^{\frac{k-1}{2}}\overline{U}\|^2 + c\|\overline{U}\|^2_{H^{2k}(\Omega)\times H^{2k}(\Gamma)} \leq c(\|U\|^2_{\mathcal{H}} + \|f(u)\|^2 + \|g(u)\|^2_\Gamma). \tag{3.24}$$

Recalling that $k \geq 2$, it follows from the continuous embeddings $H^k(\Omega) \subset \mathcal{C}(\overline{\Omega})$ and $H^k(\Gamma) \subset \mathcal{C}(\Gamma)$ and the continuity of f and g that

$$\|U\|^2_{\mathcal{H}} + \|f(u)\|^2 + \|g(u)\|^2_\Gamma \leq Q(\|U\|_{H^k(\Omega)\times H^k(\Gamma)}), \tag{3.25}$$

whence, owing to (3.20),

$$\|U\|^2_{\mathcal{H}} + \|f(u)\|^2 + \|g(u)\|^2_\Gamma \leq e^{-ct}Q(\|U_0\|_{H^k(\Omega)\times H^k(\Gamma)}) + c',\ c>0,\ t\geq 0, \tag{3.26}$$

and (3.24) and (3.26) yield

$$\frac{d}{dt}\|A^{\frac{k-1}{2}}\overline{U}\|^2 + c\|\overline{U}\|^2_{H^{2k}(\Omega)\times H^{2k}(\Gamma)} \leq e^{-ct}Q(\|U_0\|_{H^k(\Omega)\times H^k(\Gamma)}) + c',\ c>0,\ t\geq 0. \tag{3.27}$$

Summing (3.15) and (3.27), we obtain a differential inequality of the form

$$\frac{dE_2}{dt} + c(E_2 + \|\frac{\partial U}{\partial t}\|^2_{-1} + \|W\|^2_{\dot{\mathcal{V}}}) \leq e^{-c't}Q(\|U_0\|_{H^k(\Omega)\times H^k(\Gamma)}) + c'',\ c'>0,\ t\geq 0, \tag{3.28}$$

where

$$E_2 = E_1 + \|A^{\frac{k-1}{2}}\overline{U}\|^2 \tag{3.29}$$

satisfies

$$c(\|\overline{U}\|^2_{H^k(\Omega)\times H^k(\Gamma)} + \int_\Omega F(u)\,dx + \int_\Gamma G(u)\,d\Sigma) - c' \leq E_2 \tag{3.30}$$
$$\leq c''(\|\overline{U}\|^2_{H^k(\Omega)\times H^k(\Gamma)} + \int_\Omega F(u)\,dx + \int_\Gamma G(u)\,d\Sigma) + c''',\ c>0.$$

In a next step, we differentiate (3.23) with respect to time to find

$$A^{-1}\frac{\partial}{\partial t}\frac{\partial U}{\partial t} + P(A)\frac{\partial U}{\partial t} + \mathcal{F}'(U)\cdot\frac{\partial U}{\partial t} - \langle\mathcal{F}'(U)\cdot\frac{\partial U}{\partial t}\rangle = 0 \text{ in } \dot{\mathcal{V}}', \tag{3.31}$$

where

$$\mathcal{F}'(U)\cdot\frac{\partial U}{\partial t} = \begin{pmatrix} f'(u)\frac{\partial u}{\partial t} \\ g'(u)\frac{\partial u}{\partial t}\end{pmatrix}.$$

We multiply (3.31) by $\frac{\partial U}{\partial t}$, scalarly in $\mathcal{H}$, and have, owing to (2.18) and the interpolation inequality (3.7) (also recall that $\langle\frac{\partial U}{\partial t}\rangle = 0$),

$$\frac{d}{dt}\|\frac{\partial U}{\partial t}\|^2_{-1} + c\|\frac{\partial U}{\partial t}\|^2_{H^k(\Omega)\times H^k(\Gamma)} \leq c'\|\frac{\partial U}{\partial t}\|^2_{\mathcal{H}},\ c>0. \tag{3.32}$$

Noting that

$$\|\frac{\partial U}{\partial t}\|_{\mathcal{H}}^2 = ((A^{-\frac{1}{2}}\frac{\partial U}{\partial t}, A^{\frac{1}{2}}\frac{\partial U}{\partial t}))_{\mathcal{H}},$$

we see that

$$\|\frac{\partial U}{\partial t}\|_{\mathcal{H}}^2 \leq c\|\frac{\partial U}{\partial t}\|_{-1}\|\frac{\partial U}{\partial t}\|_{H^1(\Omega)\times H^1(\Gamma)} \tag{3.33}$$

and (3.32)–(3.33) yield

$$\frac{d}{dt}\|\frac{\partial U}{\partial t}\|_{-1}^2 + c\|\frac{\partial U}{\partial t}\|_{H^k(\Omega)\times H^k(\Gamma)}^2 \leq c'\|\frac{\partial U}{\partial t}\|_{-1}^2,\ c > 0. \tag{3.34}$$

We thus deduce from (3.21), (3.34) and the uniform Gronwall's lemma (see, e.g., [51], Chapter III, Lemma 1.1) that

$$\|\frac{\partial U}{\partial t}(t)\|_{-1}^2 \leq e^{-ct}Q(\|U_0\|_{H^k(\Omega)\times H^k(\Gamma)}) + c',\ c > 0,\ t \geq r, \tag{3.35}$$

$r > 0$ given.

We finally rewrite (3.23) as an elliptic system, for $t > 0$ fixed,

$$P(A)\overline{U} = h(t) \text{ in } \dot{\mathcal{V}}', \tag{3.36}$$

where $h(t) = -A^{-1}\frac{\partial U}{\partial t}(t) - \overline{\mathcal{F}(U(t))}$ satisfies, owing to (3.26) and (3.35),

$$\|h(t)\|_{\mathcal{H}} \leq e^{-ct}Q(\|U_0\|_{H^k(\Omega)\times H^k(\Gamma)}) + c',\ c > 0,\ t \geq r, \tag{3.37}$$

$r > 0$ given. Multiplying (3.36) by $A^k\overline{U}$, scalarly in $\mathcal{H}$, we obtain, owing to (3.13), (3.20), (3.37) and the interpolation inequality (3.7),

$$\|U(t)\|_{H^{2k}(\Omega)\times H^{2k}(\Gamma)} \leq e^{-ct}Q(\|U_0\|_{H^k(\Omega)\times H^k(\Gamma)}) + c',\ c > 0,\ t \geq r, \tag{3.38}$$

$r > 0$ given.

Remark 3.1. If we further assume that $U_0 \in H^{2k+1}(\Omega) \times H^{2k+1}(\Gamma)$, then $\frac{\partial U}{\partial t}(0) \in \mathcal{H}$ and

$$\|\frac{\partial U}{\partial t}(0)\|_{-1} \leq Q(\|U_0\|_{H^{2k+1}(\Omega)\times H^{2k+1}(\Gamma)}). \tag{3.39}$$

In that case, it follows from (3.34) and Gronwall's lemma that

$$\|\frac{\partial U}{\partial t}(t)\|_{-1}^2 \leq Q(\|U_0\|_{H^{2k+1}(\Omega)\times H^{2k+1}(\Gamma)}),\ t \in [0, 1], \tag{3.40}$$

which, combined with (3.38) (for $r = 1$), yields

$$\|U(t)\|_{H^{2k}(\Omega)\times H^{2k}(\Gamma)} \leq e^{-ct}Q(\|U_0\|_{H^{2k+1}(\Omega)\times H^{2k+1}(\Gamma)}) + c',\ c > 0,\ t \geq 0. \tag{3.41}$$

4. The dissipative semigroup

We first give the definition of a weak solution to (2.14)–(2.16).

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Definition 4.1. We assume that $U_0 \in \mathcal{V}'$. A weak solution to (2.14)–(2.16) is a pair (U, W) such that, for any given $T > 0$,

$$U \in \mathcal{C}([0,T]; \langle U_0 \rangle + \dot{\mathcal{V}}') \cap L^2(0,T; H^k(\Omega) \times H^k(\Gamma)),$$
$$W \in L^2(0,T; \mathcal{V}),$$
$$U(0) = U_0$$

and

$$\frac{d}{dt}((A^{-1}\overline{U}, \phi))_{\mathcal{H}} = -((\overline{W}, \phi))_{\mathcal{H}}, \ \forall\, \phi \in \dot{\mathcal{V}},$$
$$((\overline{W}, \Theta))_{\mathcal{H}} = \sum_{i=1}^{k} a_i((A^{\frac{i}{2}}\overline{U}, A^{\frac{i}{2}}\Theta))_{\mathcal{H}} + ((\overline{\mathcal{F}(U)}, \Theta))_{\mathcal{H}}, \ \forall\, \Theta \in \dot{\mathcal{V}} \cap (H^k(\Omega) \times H^k(\Gamma)),$$

in the sense of distributions, with

$$U = \overline{U} + \langle U_0 \rangle,$$
$$\langle W \rangle = \langle \mathcal{F}(U) \rangle.$$

Here, it is understood that $\langle U_0 \rangle = \frac{1}{\mathrm{Vol}(\Omega)+|\Gamma|}\langle U_0, 1 \rangle_{\mathcal{V}',\mathcal{V}}$.

We have the

Theorem 4.2. *(i) We assume that $U_0 \in H^k(\Omega) \times H^k(\Gamma)$ and $|\langle U_0 \rangle| \le M$, $M \ge 0$ given. Then, (2.14)–(2.16) possesses a unique weak solution (U, W) such that, for any $T > 0$,*

$$U \in L^\infty(\mathbb{R}^+; H^k(\Omega) \times H^k(\Gamma)) \cap L^2(0,T; H^{2k}(\Omega) \times H^{2k}(\Gamma))$$

and

$$\frac{\partial U}{\partial t} \in L^2(0,T; \dot{\mathcal{V}}').$$

(ii) If we further assume that $U_0 \in H^{k+1}(\Omega) \times H^{k+1}(\Gamma)$, then, for any $T > 0$,

$$U \in L^\infty(\mathbb{R}^+; H^{k+1}(\Omega) \times H^{k+1}(\Gamma))$$

and

$$\frac{\partial U}{\partial t} \in L^\infty(0,T; \dot{\mathcal{V}}') \cap L^2(\mathbb{R}^+; H^k(\Omega) \times H^k(\Gamma)).$$

Proof. The proofs of existence and regularity in (i) and (ii) follow from the a priori estimates derived in the previous section and, e.g., a standard Galerkin scheme. In particular, we can consider a Galerkin basis based on the spectrum of the operator A (see, e.g., [51]).

Let then (U_1, W_1) and (U_2, W_2) be two weak solutions to (2.14)–(2.15) such that

$$\langle U_1(0) \rangle = \langle U_2(0) \rangle.$$

We set $U = U_1 - U_2$ and $W = W_1 - W_2$ and have, noting that $\langle U(0) \rangle = 0$,

$$\frac{d}{dt}((A^{-1}U,\phi))_{\mathcal{H}} = -((\overline{W},\phi))_{\mathcal{H}},\ \forall\,\phi \in \dot{\mathcal{V}}, \tag{4.1}$$

$$((\overline{W},\Theta))_{\mathcal{H}} = \sum_{i=1}^{k} a_i((A^{\frac{i}{2}}U, A^{\frac{i}{2}}\Theta))_{\mathcal{H}} + ((\overline{\mathcal{F}(U)},\Theta))_{\mathcal{H}},\ \forall\,\Theta \in \dot{\mathcal{V}} \cap (H^k(\Omega)\times H^k(\Gamma)), \tag{4.2}$$

$$\langle W\rangle = \langle \mathcal{F}(U_1) - \mathcal{F}(U_2)\rangle. \tag{4.3}$$

Taking $\phi = U$ in (4.1), we obtain

$$\frac{1}{2}\frac{d}{dt}\|U\|_{-1}^2 = -((\overline{W},U))_{\mathcal{H}}. \tag{4.4}$$

Taking then $\Theta = U$ in (4.2), we find

$$((\overline{W},U))_{\mathcal{H}} = \sum_{i=1}^{k} a_i\|A^{\frac{i}{2}}U\|_{\mathcal{H}}^2 + ((\mathcal{F}(U_1)-\mathcal{F}(U_2),U))_{\mathcal{H}}. \tag{4.5}$$

Noting that

$$((\mathcal{F}(U_1)-\mathcal{F}(U_2),U))_{\mathcal{H}} = ((f(u_1)-f(u_2),u)) + ((g(u_1)-g(u_2),u))_{\Gamma},$$

it follows from (2.18), (3.7) and (4.5) that

$$\begin{aligned}((\overline{W},U))_{\mathcal{H}} &\geq \frac{a_k}{2}\|A^{\frac{k}{2}}U\|_{\mathcal{H}}^2 - c\|U\|_{\mathcal{H}}^2\\ &\geq \frac{a_k}{2}\|A^{\frac{k}{2}}U\|_{\mathcal{H}}^2 - c\|U\|_{-1}\|A^{\frac{k}{2}}U\|_{\mathcal{H}}\\ &\geq c\|U\|_{H^k(\Omega)\times H^k(\Gamma)}^2 - c'\|U\|_{-1}^2,\ c>0,\end{aligned}$$

which, combined with (4.4), yields

$$\frac{d}{dt}\|U\|_{-1}^2 \leq c\|U\|_{-1}^2. \tag{4.6}$$

Gronwall's lemma finally gives

$$\|u_1(t)-u_2(t)\|_{-1} \leq e^{ct}\|u_1(0)-u_2(0)\|_{-1},\ t\geq 0, \tag{4.7}$$

whence the uniqueness, as well as the continuous dependence with respect to the initial data in the $\dot{\mathcal{V}}'$-norm. □

It follows from Theorem 4.2 that we can define the family of solving operators

$$S(t) : \Phi_M \to \Phi_M,\ U_0 \mapsto U(t),\ t\geq 0,$$

where

$$\Phi_M = \{\Theta \in H^k(\Omega)\times H^k(\Gamma),\ \langle\Theta\rangle = M\},$$

$M\in\mathbb{R}$ given. This family of solving operators forms a semigroup, i.e., $S(0)=I$ and $S(t+\tau)=S(t)\circ S(\tau)$, $\forall\, t,\ \tau\geq 0$, which is continuous with respect to the $\dot{\mathcal{V}}'$-topology (more precisely, one writes $S(t) = M + \overline{S}(t)$, where $\overline{S}(t) : \overline{U}_0 \mapsto \overline{U}(t),\ t\geq 0$, is continuous with respect to the $\dot{\mathcal{V}}'$-topology).

Remark 4.3. It follows from (4.7) that we can extend $S(t)$ (by continuity and in a unique way) to $M + \dot{\mathcal{V}}'$.

It follows from (3.20) that we have the

Theorem 4.4. *The semigroup $S(t)$ is dissipative in Φ_M, in the sense that it possesses a bounded absorbing set $\mathcal{B}_0 \subset \Phi_M$ (i.e., $\forall B \subset \Phi_M$ bounded, $\exists t_0 = t_0(B)$ such that $t \geq t_0$ implies $S(t)B \subset \mathcal{B}_0$).*

Remark 4.5. (i) The dissipativity is a first step in view of the study of the (temporal) asymptotic behavior of the associated dynamical system. In particular, an important issue is to prove the existence of finite-dimensional attractors: such objects describe all possible dynamics of the system; furthermore, the finite-dimensionality means, very roughly speaking, that, even though the initial phase space Φ_M has infinite dimension, the reduced dynamics can be described by a finite number of parameters (we refer the interested reader to, e.g., [39] and [51] for discussions on this subject).
(ii) Actually, it follows from (3.38) that we have a bounded absorbing set $\mathcal{B}_1$ which is compact in Φ and bounded in $H^{2k}(\Omega) \times H^{2k}(\Gamma)$. This yields the existence of the global attractor $\mathcal{A}_M$ which is compact in Φ, bounded in $H^{2k}(\Omega) \times H^{2k}(\Omega)$ and attracts the bounded sets of Φ_M in the topology of $\dot{\mathcal{V}}'$ (see [39] and [51] for more details).
(iii) We recall that the global attractor $\mathcal{A}_M$ is the smallest (for the inclusion) compact set of the phase space which is invariant by the flow (i.e., $S(t)\mathcal{A}_M = \mathcal{A}_M$, $\forall t \geq 0$) and attracts all bounded sets of initial data as time goes to infinity; it thus appears as a suitable object in view of the study of the asymptotic behavior of the system. We refer the reader to, e.g., [39] and [51] for more details and discussions on this.
(iv) We can also prove, based on standard arguments (see, e.g., [39] and [51]) that $\mathcal{A}_M$ has finite dimension, in the sense of covering dimensions such as the Hausdorff and the fractal dimensions.

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