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Initial behavior of solutions to the Yang–Mills heat equation

Nelia Charalambous [a,*,1], Leonard Gross [b]

[a] Department of Mathematics and Statistics, University of Cyprus, Nicosia, 1678, Cyprus
[b] Department of Mathematics, Cornell University, Ithaca, NY 14853-4201, USA

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ABSTRACT

We explore the small-time behavior of solutions to the Yang–Mills heat equation with rough initial data. We consider solutions $A(t)$ with initial value $A_0 \in H_{1/2}(M)$, where M is a bounded convex region in $\mathbb{R}^3$ or all of $\mathbb{R}^3$. The behavior, as $t \downarrow 0$, of the $L^p(M)$ norms of the time derivatives of $A(t)$ and its curvature $B(t)$ will be determined for $p = 2$ and 6, along with the $H_1(M)$ norm of these derivatives.

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* Corresponding author.
E-mail addresses: nelia@ucy.ac.cy (N. Charalambous), gross@math.cornell.edu (L. Gross).

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1. Introduction

In this article we study the initial behavior of solutions to the Yang–Mills heat equation over a region M in $\mathbb{R}^3$. Denote by K a compact connected Lie group with Lie algebra $\mathfrak{k}$. A $\mathfrak{k}$ valued 1-form over M may be written as

$$A = \sum_{j=1}^{3} A_j(x)dx^j, \tag{1.1}$$

with coefficients $A_j(x) \in \mathfrak{k}$. The curvature of A is the $\mathfrak{k}$ valued 2-form given by $B = dA + A \wedge A$. The Yang–Mills heat equation is the weakly parabolic equation for a time dependent $\mathfrak{k}$ valued 1-form $A(t)$ over M given by

$$\frac{\partial}{\partial t}A(x,t) = -d^*_{A(t)}B(x,t), \tag{1.2}$$

wherein $d^*_A = d^* +$ [the interior product by $ad\, A(t)$], and $B(x,t)$ is the curvature of $A(t)$ at x. We will always take K to be a subgroup of the orthogonal, respectively unitary, group of a finite dimensional real, respectively complex, inner product space $\mathcal{V}$.

The Yang–Mills heat equation is only weakly parabolic since the second order derivative terms on the right side of (1.2) are $-d^*dA$, which are missing 'half' of the Laplacian on 1-forms $-\Delta = d^*d + dd^*$. In [1] we proved the existence and uniqueness of solutions to this equation for initial data A_0 in $H_1(M)$. In [6] the existence and uniqueness was proven for initial data in $H_{1/2}(M)$. The Sobolev index 1/2 is the critical index for the Yang–Mills heat equation in spatial dimension three. We will be concerned with solutions to (1.2) for which the initial value A_0 is in $H_{1/2}(M)$. In this case the curvature $B(t)$ can be expected to blow up in the $L^2(M)$ sense as $t \downarrow 0$ since, informally, $B(t)$ can be expected to converge to its initial value B_0 only in the sense of the negative Sobolev space $H_{-1/2}(M)$. Higher derivatives of $A(t)$ can be expected to blow up more quickly as $t \downarrow 0$. Our study is motivated by a desire to understand the nature of the singularities of gauge covariant derivatives of a solution to the Yang–Mills heat equation as time decreases to zero. In this article we will study the $L^p(M)$ behavior of various gauge covariant derivatives of $A(t)$ as $t \downarrow 0$. The values $p = 2$ and $p = 6$ (and a fortiori all p in between) are of sole interest in this paper because only first order Sobolev inequalities can be effectively used in our energy methods. Concerning higher values of p see Remark 6.14. A priori estimates of first, second and third order spatial covariant derivatives have already been used in our previous work [1,2,6] to prove existence and uniqueness of solutions to (1.2).

A function $g : M \to K$ induces a gauge transformation of a time dependent connection form on M by the definition

$$A^g(x,t) = g(x)^{-1}A(x,t)g(x) + g(x)^{-1}dg(x). \tag{1.3}$$

If $A(\cdot,\cdot)$ is a solution to the Yang–Mills heat equation (1.2) then so is $A^g(\cdot,\cdot)$, at least if g satisfies some mild regularity conditions. It is already clear from this that the Yang–Mills heat equation does not smooth all initial data, for if $A(x,t)$ is a solution with initial value $A_0(x)$ then $A^g(x,t)$ is the solution with initial value $A^g_0(x)$, and consequently, even if $A(x,t)$ is very smooth, the solution $A^g(x,t)$ needs to be no smoother than $g^{-1}dg$. Our goal is to show that solutions to (1.2) are infinitely differentiable in a gauge covariant sense for $t > 0$, even for rough initial data, and to determine the nature of the singularities of the derivatives as $t \downarrow 0$. For the class of initial data that we are interested in, namely $A_0 \in H_{1/2}(M)$, the formula (1.3) suggests that the corresponding class of allowed gauge functions should include functions $g \in H_{3/2}(M)$. A precise definition of this class, which makes it into a complete topological group, will be given in Section 2. With

these initial data, which are in fact an invariant class under these gauge transformations, it can be seen easily from (1.3) that a solution need not be even once continuously differentiable in the ordinary sense. There are in fact solutions that are not in the Sobolev space $W_1(M)$ for any $t > 0$. We are going to address this by computing only gauge covariant derivatives. The $L^p(M)$ norm of such a derivative is fully gauge invariant and therefore descends to a function on the quotient space $\mathcal{C} \equiv \{\text{connection forms}\}/\text{Gauge group}$, which is a space of connections over M as well as a version of the configuration space for the classical Yang–Mills field theory. We will establish bounds on these gauge invariant norms by functions of the action of the solution $A(\cdot)$, which are also fully gauge invariant and which therefore also descend to functions on $\mathcal{C}$. We obtain thereby bounds on the covariant derivatives given by inequalities between functions on the quotient space $\mathcal{C}$ itself. It will be shown in [7] that the space $\mathcal{C}$ has a natural Riemannian metric on it which makes it into a complete Riemannian manifold. Our main results can be interpreted as analysis over this manifold. Remark 2.11 makes this a little more precise.

The technical problem of computing high order derivatives of non-differentiable functions will be carried out by gauge transforming a solution to a smooth function, which can be done for a short time, [6], and then gauge transforming the derivative back.

For our choice of the region $M \subset \mathbb{R}^3$ we will take either $M = R^3$ or take M to be the closure of a bounded open convex subset of $\mathbb{R}^3$ with smooth boundary. Undoubtedly our methods will apply to other regions also with minor modification as well as to other manifolds. For example, they can be applied to compact three manifolds without boundary, and compact three manifolds with convex boundary. But we are going to focus just on regions in $\mathbb{R}^3$, which we believe to be adequate for our anticipated applications to quantum field theory. In case $M \neq \mathbb{R}^3$ we must impose boundary conditions on $A(t)$ for $t > 0$. The two natural boundary conditions that we will use are the Neumann-like boundary conditions (absolute boundary conditions in the sense of Ray and Singer [12]) or the Dirichlet-like boundary conditions (relative boundary conditions). For our anticipated applications to quantum field theory we will also ultimately need to use Marini boundary conditions, introduced in [8–11], which set the normal component of the curvature to zero on the boundary. Results for Marini boundary conditions will be deduced elsewhere from our results for Neumann-like boundary conditions.

2. Statement of results

2.1. Notation

Throughout this paper M will denote either $\mathbb{R}^3$ or the closure of a bounded open set in $\mathbb{R}^3$ with smooth boundary. In the latter case we will always assume that M is convex in the sense that the second fundamental form of ∂M is everywhere non-negative.

We consider a product bundle over M, $M \times \mathcal{V} \to M$, where $\mathcal{V}$ is a finite dimensional real or complex vector space with an inner product. Let K be a compact connected subgroup of the orthogonal, respectively unitary, group in $End\ \mathcal{V}$. We denote by $\mathfrak{k}$ the Lie algebra of K, which can be identified with a real subspace of $End\ \mathcal{V}$.

Let $\langle \cdot, \cdot \rangle$ be an $Ad\ K$ invariant inner product on $\mathfrak{k}$ with associated norm $|\xi|_{\mathfrak{k}}$ for $\xi \in \mathfrak{k}$. For $\mathfrak{k}$ valued p-forms ω and ϕ the L^2 pairing is given by $(\omega, \phi) = \int_M \langle \omega(x), \phi(x) \rangle_{\Lambda^p \otimes \mathfrak{k}} d\,\mathrm{Vol}$ with induced L^2 norm $\|\omega\|_2^2 = (\omega, \omega)$. We define the W_1 norm of ω by

$$\|\omega\|^2_{W_1(M)} = \int_M |\nabla \omega|^2_{\mathbb{R}^3 \otimes \Lambda^p \otimes \mathfrak{k}} d\,\mathrm{Vol} \quad + \|\omega\|_2^2, \tag{2.1}$$

where $\nabla\omega$ is constructed from the weak derivatives. Define $W_1 = W_1(M) = \{\omega : \|\omega\|_{W_1(M)} < \infty\}$. This is the Sobolev space of order one, without boundary conditions.

If $u = \sum_{|I|=r} u_I dx^I$ and $v = \sum_{|J|=p} v_J dx^J$ are $End\ \mathcal{V}$ valued forms then their wedge product, $u \wedge v = \sum_{I,J} u_I v_J dx^I \wedge dx^J$, is another $End\ \mathcal{V}$ valued form. When the appropriate action of u on v is via $ad\ u$ then we will write $[u \wedge v] = \sum_{I,J} [u_I, v_J] dx^I \wedge dx^J$. This will be the case, for example, when u is an $End\ \mathcal{V}$ valued connection form or its time derivative. If u and v take their values in $\mathfrak{k}$ then so does $[u \wedge v]$.

The interior product, $[u \lrcorner v]$, of an element $u \in \Lambda^p \otimes \mathfrak{k}$ with an element $v \in \Lambda^{p+r} \otimes \mathfrak{k}$ is defined, for $r \geq 0$, by

$$\langle w, [u \lrcorner v] \rangle_{\Lambda^r \otimes \mathfrak{k}} = \langle [u \wedge w], v \rangle_{\Lambda^{p+r} \otimes \mathfrak{k}} \quad \text{for all} \quad w \in \Lambda^r \otimes \mathfrak{k}. \tag{2.2}$$

If u and v are both in $\Lambda^1 \otimes \mathfrak{k}$ then (2.2) gives

$$\mathfrak{k} \ni [u \lrcorner v] = -[u \cdot v] = -\sum_j [u_j, v_j]$$

in an orthonormal frame for Λ^1. In particular $[u \lrcorner u] = 0$. Moreover, if $w \in \Lambda^2 \otimes \mathfrak{k}$ then $[w \lrcorner w] = 0$.

In this paper we will be concerned with a $\mathfrak{k}$-valued 1-form A as in (1.1). For $\omega \in W_1(M; \Lambda^p \otimes \mathfrak{k})$ define $d_A \omega = d\omega + [A \wedge \omega]$. Then $d_A^* \omega = d^* \omega + [A \lrcorner \omega]$. The curvature of A can be represented as

$$B = dA + (1/2)[A \wedge A]. \tag{2.3}$$

2.2. Strong solutions and boundary conditions

We take the following definition of strong and almost strong solution from [6].

Definition 2.1. Let $0 < T \leq \infty$. A *strong solution* to the Yang–Mills heat equation over $[0, T) \times M$ is a continuous function

$$A(\cdot) : [0, T) \to L^2(M; \Lambda^1 \otimes \mathfrak{k})$$

such that

a) $A(t) \in W_1$ for all $t \in (0, T)$ and $A(\cdot) : (0, T) \to W_1$ is continuous,

b) $B(t) := dA(t) + \frac{1}{2}[A(t) \wedge A(t)] \in W_1$ for each $t \in (0, T)$,

c) the strong $L^2(M)$ derivative $A'(t) \equiv dA(t)/dt$ exists on $(0, T)$, and
$A'(\cdot) : (0, T) \to L^2(M)$ is continuous,

d) $A'(t) = -d^*_{A(t)} B(t)$ for each $t \in (0, T)$.

A solution $A(\cdot)$ that satisfies all of the above conditions except for *a)* will be called an *almost strong solution*. In this case the spatial exterior derivative $dA(t)$, which appears in the definition of the curvature, must be interpreted in the weak sense.

Definition 2.2. If $M \neq \mathbb{R}^3$ then we will impose boundary conditions on the solutions. For a strong solution to the Yang–Mills heat equation we will consider two types of boundary conditions:

Neumann boundary conditions:

$$i) \quad A(t)_{norm} = 0 \quad \text{for } t > 0 \text{ and} \tag{2.4}$$

$$ii) \quad B(t)_{norm} = 0 \quad \text{for } t > 0. \tag{2.5}$$

Dirichlet boundary conditions:

$$i)\quad A(t)_{tan} = 0 \quad \text{for } t > 0 \text{ and} \tag{2.6}$$

$$ii)\quad B(t)_{tan} = 0 \quad \text{for } t > 0. \tag{2.7}$$

In [1] we also considered Marini boundary conditions, which only require $B(t)_{norm} = 0$. Solutions satisfying these boundary conditions will be derived in a later work from solutions satisfying Neumann boundary conditions. The regularity theorems of the present paper will carry over to these. We will not consider them in this paper.

Notation 2.3. The Sobolev spaces for $\mathfrak{k}$ valued 1-forms associated to the preceding boundary conditions are most easily described in terms of the related Laplacian.

If $M = \mathbb{R}^3$ define

$$-\Delta = d^*d + dd^*, \tag{2.8}$$

where d denotes the closed version of the exterior derivative operator with $C_c^\infty(\mathbb{R}^3, \Lambda^1 \otimes \mathfrak{k})$ as a core.

If $M \neq \mathbb{R}^3$ then the Neumann and Dirichlet Laplacians are again given by $\sum_{j=1}^3 \partial_j^2$ since M is the closure of an open subset of $\mathbb{R}^3$ but subject to the following boundary conditions.

$$\omega_{norm} = 0 \ \text{ and } \ (d\omega)_{norm} = 0 \quad \text{Neumann conditions}$$

$$\omega_{tan} = 0 \quad \text{ and } \ (d^*\omega)_{\partial M} = 0 \quad \text{Dirichlet conditions.}$$

Alternatively, the Neumann, respectively Dirichlet, Laplacian can be defined by (2.8), wherein d is taken to be the maximal, respectively minimal, exterior derivative operator over M. See [1] for further discussion of these domains. In all three cases the Laplacian is a nonnegative, self-adjoint operator on the appropriate domain.

For $0 \leq a \leq 1$ we define the Sobolev spaces

$$H_a = \text{Domain of } (-\Delta)^{a/2} \text{ on } L^2(M; \Lambda^1 \otimes \mathfrak{k})$$

with norm

$$\|\omega\|_{H_a} = \|(1-\Delta)^{a/2}\omega\|_{L^2(M;\Lambda^1\otimes\mathfrak{k})}. \tag{2.9}$$

In this paper we will only be concerned with the cases $a = 1/2$ and $a = 1$. But it may be interesting to note that for each number $a \in [0,1]$ the two Sobolev spaces H_a, corresponding to Dirichlet or Neumann boundary conditions, are distinct when $1/2 \leq a \leq 1$ and are identical if $0 \leq a < 1/2$, by Fujiwara's theorem [4].

With the preceding definition of a Sobolev space, we have the following embedding property

$$\|\omega\|_{H_a} \leq c_{a,b}\|\omega\|_{H_b} \text{ whenever } 0 \leq a \leq b.$$

The constant $c_{a,b}$ is independent of M.

Definition 2.4 *(The gauge group $\mathcal{G}_{3/2}$).* A measurable function $g : M \to K \subset End\ \mathcal{V}$ is a bounded function into the linear space $End\ \mathcal{V}$ and consequently its weak derivatives are well defined. Following [6] we will write $g \in W_1(M;K)$ if $\|g - I_{\mathcal{V}}\|_2 < \infty$ and the derivatives $\partial_j g \in L^2(M; End\ \mathcal{V})$. The 1-form $g^{-1}dg := \sum_{j=1}^3 g^{-1}(\partial_j g)dx^j$ is then an a.e. defined $\mathfrak{k}$ valued 1-form. The Sobolev norm $\|g^{-1}dg\|_{H_a}$ is defined as in

(2.9). For an element $g \in W_1(M;K)$ the restriction $g|\partial M$ is well defined almost everywhere on ∂M by a Sobolev trace theorem. The three versions of $\mathcal{G}_{3/2}$ that we will need are given in the following definitions.

$$\mathcal{G}_{3/2}(\mathbb{R}^3) = \Big\{ g \in W_1(\mathbb{R}^3;K) : g^{-1}dg \in H_{1/2}(\mathbb{R}^3;\Lambda^1 \otimes \mathfrak{k}) \Big\}.$$

If $M \neq \mathbb{R}^3$ define

$$\mathcal{G}^N_{3/2}(M) = \Big\{ g \in W_1(M;K) : g^{-1}dg \in H_{1/2}(M;\Lambda^1 \otimes \mathfrak{k}) \Big\},$$
$$\mathcal{G}^D_{3/2}(M) = \Big\{ g \in W_1(M;K) : g^{-1}dg \in H_{1/2}(M;\Lambda^1 \otimes \mathfrak{k}),\ g = I_{\mathcal{V}} \text{ on } \partial M \Big\}.$$

It should be understood that the two spaces denoted $H_{1/2}(M;\Lambda^1 \otimes \mathfrak{k})$ are those determined by Neumann, respectively Dirichlet, boundary conditions. It was proved in [6, Theorem 5.3] that all three versions of $\mathcal{G}_{3/2}$ are complete topological groups in the metric $dist(g,h) = \|g^{-1}dg - h^{-1}dh\|_{H_{1/2}} + \|g-h\|_{L^2(M;End\,\mathcal{V})}$.

Definition 2.5. A solution $A(\cdot)$ to the Yang–Mills heat equation is said to have *finite action* if

$$\rho(t) := (1/2)\int_0^t s^{-1/2}\|B(s)\|_2^2\, ds < \infty \tag{2.10}$$

for some $t > 0$. If $A(\cdot)$ has finite action then, actually, $\rho(t) < \infty$ for all $t > 0$ because $\|B(s)\|_2^2$ is nonincreasing. See e.g. Lemma 5.3.

It was shown in [6] that a solution to the Yang–Mills heat equation with initial value $A_0 \in H_{1/2}$ will have finite action whenever $\|A_0\|_{H_{1/2}}$ is sufficiently small. We summarize some of the results needed from [6] in the following theorem.

Theorem 2.6. ([6, Theorem 2.11]) *Assume that $A_0 \in H_{1/2}(M;\Lambda^1 \otimes \mathfrak{k})$. Then there exists an almost strong solution $A(t)$ to the Yang–Mills heat equation over $[0,\infty)$ with initial value A_0.*

If $\|A_0\|_{H_{1/2}}$ is sufficiently small then there exists a gauge function $g_0 \in \mathcal{G}_{3/2}$ such that the connection $A(t)^{g_0}$ is a strong solution over $[0,\infty)$ with initial value $A_0^{g_0}$. It is also smooth over $(0,T)\times M$ for some $T < \infty$. The solutions $A(t)$ and $A(t)^{g_0}$ have the following properties in this case:

1. *Both $A(t)$ and $A(t)^{g_0}$ are continuous functions on $[0,\infty)$ into $H_{1/2}(M;\Lambda^1 \otimes \mathfrak{k})$.*
2. *The curvatures of $A(t)$ and $A(t)^{g_0}$ satisfy (2.5) in the Neumann case and (2.7) in the Dirichlet case for all $t > 0$. The gauge regularized solution $A(t)^{g_0}$ satisfies in addition (2.4) in the Neumann case and (2.6) in the Dirichlet case for all $t > 0$.*
3. *Both $A(t)$ and $A(t)^{g_0}$ have finite action.*

Remark 2.7. It is also proved in [6] that strong solutions with finite action are unique when $M = \mathbb{R}^3$ and, if $M \neq \mathbb{R}^3$, unique under the boundary conditions (2.5) in case of Neumann boundary conditions or (2.6) in case of Dirichlet boundary conditions. The smoothness of $A^{g_0}(t)$ on $(0,T)\times M$ may hold for the same fixed g_0 for $T = \infty$, but this is still an open question.

2.3. The main theorem

We are going to establish bounds on various gauge covariant derivatives of a solution to the Yang–Mills heat equation in terms of the action functional $\rho(t)$, defined in (2.10). The class of solutions of interest are

those for which the initial value A_0 has small $H_{1/2}$ norm. But $\|A_0\|_{H_{1/2}}$ is not a gauge invariant function of A_0. In the next theorem we will show that the gauge invariant functionals of derivatives of $A(\cdot)$ that are of interest to us are controlled by the gauge invariant functional ρ. The inequalities that implement this descend therefore to inequalities on the quotient space {initial data space}$/\mathcal{G}_{3/2}$, thereby yielding analysis on the quotient space itself. See Remark 2.11 for further discussion.

By a *standard dominating function* we will mean a function $C : [0,\infty) \to [0,\infty)$ of the form $C(t) = \hat{C}(t,\rho(t))$, where $\hat{C} : [0,\infty)^2 \to [0,\infty)$ is continuous and non-decreasing in each variable, $\hat{C}(0,0) = 0$ and $\hat{C}$ is independent of the solution $A(\cdot)$.

Our main result is the following.

Theorem 2.8. *Assume that $A_0 \in H_{1/2}(M;\Lambda^1\otimes\mathfrak{k})$. Suppose that $A(\cdot)$ is a strong solution to (1.2) over $[0,\infty)$ with initial value A_0 and having finite action. If $\|A_0\|_{H_{1/2}}$ is sufficiently small then there exist $T>0$ and standard dominating functions C_{nj} for $j=1,\dots,4$ and $n=1,2,\dots$, such that, for $0<t<T$, the following estimates hold.*

$$t^{2n-\frac{1}{2}}\|A^{(n)}(t)\|_2^2 + \int_0^t s^{2n-\frac{1}{2}}\|B^{(n)}(s)\|_2^2\,ds \le C_{n1}(t) \tag{$\mathcal{A}_n$}$$

$$t^{(2n-\frac{1}{2})}\|B^{(n-1)}(t)\|_6^2 + \int_0^t s^{2n-\frac{1}{2}}\|A^{(n)}(s)\|_6^2\,ds \le C_{n2}(t) \tag{$\mathcal{B}_n$}$$

$$t^{2n+\frac{1}{2}}\|B^{(n)}(t)\|_2^2 + \int_0^t s^{2n+\frac{1}{2}}\|A^{(n+1)}(s)\|_2^2\,ds \le C_{n3}(t) \tag{$\mathcal{C}_n$}$$

$$t^{2n+\frac{1}{2}}\|A^{(n)}(t)\|_6^2 + \int_0^t s^{2n+\frac{1}{2}}\|B^{(n)}(s)\|_6^2\,ds \le C_{n4}(t). \tag{$\mathcal{D}_n$}$$

Moreover ($\mathcal{C}_n$) holds for $n=0$.

Notation 2.9. The gauge invariant version of the Sobolev 1-norm (2.1) is defined by

$$\|A^{(n)}(t)\|_{H_1^A}^2 = \sum_{j=1}^3 \int_M |\partial_j^{A(t)} A^{(n)}(t)|^2 dx + \|A^{(n)}(t)\|_2^2, \quad n\ge 1,$$

$$\|B^{(n)}(t)\|_{H_1^A}^2 = \sum_{j=1}^3 \int_M |\partial_j^{A(t)} B^{(n)}(t)|^2 dx + \|B^{(n)}(t)\|_2^2, \quad n\ge 0,$$

where

$$\partial_j^{A(t)}\,\omega = \partial_j\,\omega + ad\,A_j(t)\,\omega$$

for a $\mathfrak{k}$ valued p-form ω.

Corollary 2.10. *Under the hypotheses of Theorem 2.8 there exist $T>0$ and standard dominating functions C_{nj} for $j=5,6$ and $n=1,2,\dots$ such that, for $0<t<T$, the following estimates hold.*

$$t^{(2n-\frac{1}{2})}\|B^{(n-1)}(t)\|_{H_1^A}^2 + \int_0^t s^{2n-\frac{1}{2}}\|A^{(n)}(s)\|_{H_1^A}^2\,ds \le C_{n5}(t) \tag{$\mathcal{E}_n$}$$

$$t^{2n+\frac{1}{2}}\|A^{(n)}(t)\|^2_{H_1^A} + \int_0^t s^{2n+\frac{1}{2}}\|B^{(n)}(s)\|^2_{H_1^A}\,ds \le C_{n6}(t). \tag{$\mathcal{F}_n$}$$

Theorem 2.8 and Corollary 2.10 will be proven in Section 6.

Remark 2.11 *(Analysis over quotient space).* Denote by $\mathcal{Y}$ the set of almost strong solutions of the Yang–Mills heat equation over M with initial value in $H_{1/2}$ and having finite action. The group $\mathcal{G}_{3/2}$ acts on $\mathcal{Y}$ through its action on $A(0)$ for each $A \in \mathcal{Y}$. For simplicity of statement let us assume that uniqueness of solutions holds in this class. All of the functionals appearing on both sides of the inequalities in Theorem 2.8 and Corollary 2.10 descend to functions of the initial values on the quotient space $\mathcal{C} \equiv \mathcal{Y}/\mathcal{G}_{3/2}$. The theorem and its corollary can and should be interpreted as regularity properties of functions on the quotient space. It will be shown in [7] that $\mathcal{C}$ is a complete metric space in a natural metric.

3. The lower order terms

Our strategy consists in computing the gauge covariant exterior derivatives and coderivatives of all the nth order time derivatives $A^{(n)}(t)$ and $B^{(n)}(t)$ and expressing them in terms of lower order time derivatives. This will be done in the next subsection. These identities, in turn, will give rise to integral identities, which will be used in Sections 5 and 6 to establish initial behavior bounds by induction on n.

3.1. Pointwise identities

In this section we assume that $A(t)$ is a time dependent $\mathfrak{k}$ valued connection form over M, which is in $C^\infty((0,T)\times M)$ and solves the Yang–Mills heat equation (1.2). $B(t)$ denotes the curvature of $A(t)$. We will derive some identities by applying d_A and d_A^* to various $\mathfrak{k}$ valued forms. In case $M \neq \mathbb{R}^3$ one needs to specify boundary conditions on a p-form ω in order for it to belong to the domain of d_A or d_A^*. These are analogous to the Dirichlet and Neumann boundary conditions for the domain of d and d^* discussed in [3]. We recall from Section 3 of [1] that for the Dirichlet boundary conditions, (D), d_A is the minimal operator. It imposes nontrivial boundary conditions on the forms in its domain. d_A^* is maximal in this case. On the other hand, for the Neumann boundary conditions, (N), d_A is maximal and the domain of d_A^* imposes nontrivial boundary conditions on its elements.

The next proposition expresses spatial derivatives of solutions in terms of time derivatives.

Proposition 3.1. *Let $A(t)$ be a smooth solution to the Yang–Mills heat equation over $(0,T)$, satisfying either (2.5) or (2.6) if $M \neq \mathbb{R}^3$. Then there exist non-negative constants c_{ni}, $\bar{c}_{ni}$, $\tilde{c}_{ni}$, $\hat{c}_{ni}$, that depend only on n and i, such that, for all $n \ge 1$ and $0 < t < T$, the following identities hold.*

$$d_{A(t)}A^{(n)}(t) = B^{(n)}(t) - P_n(t), \quad \textit{where} \tag{3.1}$$
$$P_n(t) = \sum_{i=1}^{n-1} c_{ni}[A^{(i)}(t) \wedge A^{(n-i)}(t)].$$
$$d^*_{A(t)}B^{(n-1)} = -A^{(n)}(t) - Q_n(t), \quad \textit{where} \tag{3.2}$$
$$Q_n(t) = \sum_{i=1}^{n-1} \bar{c}_{ni}[A^{(i)}(t) \lrcorner B^{(n-1-i)}(t)].$$
$$d^*_{A(t)}A^{(n)}(t) = -R_n(t), \quad \textit{where} \tag{3.3}$$
$$R_n(t) = \sum_{i=1}^{n-2} \tilde{c}_{ni}[A^{(i)}(t) \lrcorner A^{(n-i)}(t)].$$

Moreover, for all $n \geq 0$ there holds

$$d_{A(t)}B^{(n)}(t) = S_n(t), \quad \text{where } S_0(t) = 0,\ S_1(t) = [B(t) \wedge A'(t)] \text{ and} \tag{3.4}$$

$$S_n(t) = [B(t) \wedge A^{(n)}(t)] + \sum_{i=1}^{n-1} \hat{c}_{ni}\,[(B^{(i)}(t) - P_i(t)) \wedge A^{(n-i)}(t)]$$

for $n \geq 2$.

The functions $P_n(t)$, $Q_n(t)$, $R_n(t)$ are polynomials in the time derivatives of A and B of order at most $n-1$ in A and at most $n-2$ in B. Empty sums are to be interpreted as zero. In particular,

$$P_1(t) = Q_1(t) = R_1(t) = R_2(t) = 0.$$

In the above identities d_A is the exterior derivative with domain matching the boundary conditions and d_A^* is its adjoint.

The next lemma carries out the inductive computation, ignoring domain issues for the operators d_A and d_A^*. These issues, which are relevant if $M \neq \mathbb{R}^3$, will be addressed in the succeeding lemmas.

Lemma 3.2. *The identities (3.1)–(3.4) hold, ignoring boundary conditions.*

Proof. We will prove the identities (3.1)–(3.3) by induction on n. Recall the identity

$$d_A A' = B' \tag{3.5}$$

proved in [1, Section 5], which is (3.1) for $n = 1$, since $P_1(t) = 0$. Let $k \geq 1$. Assume that the identity (3.1) holds for $n = k$ and differentiate both sides with respect to t to find $(d_A A^{(k)})' = B^{(k+1)} - P_k'$. Therefore

$$\begin{aligned} d_A A^{(k+1)} &= B^{(k+1)} - [A' \wedge A^{(k)}] - \sum_{i=1}^{k-1} c_{ki}([A^{(i)} \wedge A^{(k-i)}])' \\ &= B^{(k+1)} - [A' \wedge A^{(k)}] - \sum_{i=1}^{k-1} c_{ki}([A^{(i)} \wedge A^{(k+1-i)}] + [A^{(i+1)} \wedge A^{(k-i)}]). \end{aligned}$$

Thus (3.1) holds with $c_{(k+1)1} = 1 + c_{k1}$ and $c_{(k+1)i} = c_{k(i-1)} + c_{ki}$ for $2 \leq i \leq k$. Notice that $[A^{(i)} \wedge A^{(j)}] = [A^{(j)} \wedge A^{(i)}]$ for any i, j. The coefficients c_{ni} are the ones obtained from the inductive process above. This proves (3.1).

To prove (3.2) observe that for $n = 1$ this is the Yang–Mills heat equation since $Q_1(t) = 0$. For $n = 2$, the identity $d_A^* B' = -A'' - [A' \lrcorner B]$, proved in [1, Section 5] gives (3.2) with $\bar{c}_{21} = 1$. Assume that (3.2) holds for $n = k \geq 2$ and differentiate both sides with respect to t to obtain $d_A^* B^{(k)} + [A' \lrcorner B^{(k-1)}] = -A^{(k+1)} - Q_k'$. Therefore

$$\begin{aligned} d_A^* B^{(k)} &= -A^{(k+1)} - [A' \lrcorner B^{(k-1)}] - \sum_{i=1}^{k-1} \bar{c}_{ki}([A^{(i)} \lrcorner B^{(k-1-i)}])' \\ &= -A^{(k+1)} - [A' \lrcorner B^{(k-1)}] - \sum_{i=1}^{k-1} \bar{c}_{ki}([A^{(i)} \lrcorner B^{(k-i)}] + [A^{(i+1)} \lrcorner B^{(k-1-i)}]). \end{aligned}$$

This is (3.2) with $n = k+1$ and coefficients given by $\bar{c}_{(k+1)1} = 1 + \bar{c}_{k1}$ and $\bar{c}_{(k+1)i} = \bar{c}_{ki} + \bar{c}_{k(i-1)}$ for $2 \leq i \leq k$.

For the proof of (3.3) we observe that

$$d_A^* A' = -d_A^* d_A^* B = 0$$

by the Bianchi identity. Differentiating both sides with respect to t we get

$$0 = (d_A^* A')' = d_A^* A'' + [A' \lrcorner A'] = d_A^* A''$$

since $[\omega \lrcorner \omega] = 0$ for any 1-form ω. Differentiating once again with respect to t we obtain

$$d_A^* A''' + [A' \lrcorner A''] = 0.$$

This proves (3.3) for $n = 1$ and $n = 2$ because $R_1 = R_2 = 0$. Let $k \geq 2$ and assume that (3.3) holds for $n = k$. Differentiate both sides with respect to t we get

$$d_A^* A^{(k+1)} + [A' \lrcorner A^{(k)}] + \sum_{i=1}^{k-2} \tilde{c}_{ki} ([A^{(i)} \lrcorner A^{(k+1-i)}] + [A^{(i+1)} \lrcorner A^{(k-i)}]) = 0.$$

This is (3.3) with $n = k + 1$.

Finally we will derive (3.4) by applying d_A to both sides of (3.1) rather than proceeding by induction. For $n = 0$ the identity (3.4) is just the Bianchi identity. For $n \geq 1$ we find

$$d_A B^{(n)} = d_A d_A A^{(n)} + \sum_{i=1}^{n-1} c_{ni} d_A \big([A^{(i)} \wedge A^{(n-i)}] \big). \tag{3.6}$$

By the Bianchi identity we have $d_A d_A A^{(n)} = [B \wedge A^{(n)}]$. Moreover, $d_A[\omega \wedge \eta] = [d_A \omega \wedge \eta] - [\omega \wedge d_A \eta]$ for 1-forms ω, η and $[u \wedge v] = -[v \wedge u]$ whenever u is a $\mathfrak{k}$ valued 1-form and v is a $\mathfrak{k}$ valued 2-form. Therefore (3.6) gives

$$\begin{aligned} d_A B^{(n)} &= [B \wedge A^{(n)}] + \sum_{i=1}^{n-1} c_{ni} \big\{ [d_A A^{(i)} \wedge A^{(n-i)}] + [d_A A^{(n-i)} \wedge A^{(i)}] \big\} \\ &= [B \wedge A^{(n)}] + \sum_{i=1}^{n-1} (c_{ni} + c_{n(n-i)}) [d_A A^{(i)} \wedge A^{(n-i)}]. \end{aligned}$$

Using (3.1) to substitute for the term $d_A A^{(i)}$ we arrive at (3.4) with $\hat{c}_{ni} = c_{ni} + c_{n(n-i)}$. □

Although we applied the exterior derivative operator and its adjoint to smooth forms in the preceding lemma, we need to verify that the boundary conditions satisfied by these forms match with the domains of these operators when $M \neq \mathbb{R}^3$. To this end we recall here some properties of these domains, established in Section 3 of [1].

Lemma 3.3. ([1, Lemma 3.4]) *Suppose that $\omega \in W_1(M; \Lambda^p \otimes \mathfrak{k})$ and $A \in L^\infty(M)$. Then*

$$(D) \qquad \omega \in Dom(d_A) \ \text{ if and only if } \ \omega_{tan} = 0$$

$$(N) \qquad \omega \in Dom(d_A^*) \ \text{ if and only if } \ \omega_{norm} = 0.$$

Moreover we proved the following:

Lemma 3.4. ([1, Proposition 3.5]) *Assume that ω is a $\mathfrak{k}$ valued form and that $A \in W_1 \cap L^\infty$. Denote the curvature of A by B, as in (2.3).*

If $[B \wedge \omega] \in L^2$ then

$$(N) \quad \omega \in Dom(d_A) \text{ implies } \omega \in Dom((d_A)^2) \text{ and } d_A^2\omega = [B \wedge \omega]$$
$$\text{and } (D) \quad \omega \in Dom(d_A) \text{ implies } \omega \in Dom((d_A)^2) \text{ and } d_A^2\omega = [B \wedge \omega].$$

If $[B \lrcorner \omega] \in L^2$ then

$$(D) \quad \omega \in Dom(d_A^*) \text{ implies } \omega \in Dom((d_A^*)^2) \text{ and } (d_A^*)^2\omega = [B \lrcorner \omega]$$
$$\text{and } (N) \quad \omega \in Dom(d_A^*) \text{ implies } \omega \in Dom((D_A^*)^2) \text{ and } \quad (d_A^*)^2\omega = [B \lrcorner \omega].$$

For the remainder of this section we will assume that $A(t) \in C^\infty(\,(0,T) \times M : \Lambda^1 \otimes \mathfrak{k})$ is a smooth solution to the Yang–Mills heat equation which satisfies (1.2) and one of the boundary conditions (2.5) or (2.6) if $M \neq \mathbb{R}^3$.

Lemma 3.5. *Let $A(t)$ be a smooth solution to the Yang–Mills heat equation over $(0,T)$, satisfying either (2.5) or (2.6). Denote by $A^{(n)}(t)$, $B^{(n)}(t)$ the nth order time derivatives of A and B respectively. If $A(\cdot)$ satisfies (2.6) then for all $n \geq 0$ and $0 < t < T$*

$$\begin{aligned} A^{(n)}(t)_{tan} &= 0 \quad \text{and} \quad A^{(n)}(t) \in Dom(d_A). \\ B^{(n)}(t)_{tan} &= 0 \quad \text{and} \quad B^{(n)}(t) \in Dom(d_A). \end{aligned} \tag{3.7}$$

If $A(t)$ satisfies (2.5), then for all $n \geq 1$ and $0 < t < T$

$$\begin{aligned} B^{(n)}(t)_{norm} &= 0 \quad \text{and} \quad B^{(n)}(t) \in Dom(d_A^*). \\ A^{(n)}(t)_{norm} &= 0 \quad \text{and} \quad A^{(n)}(t) \in Dom(d_A^*). \end{aligned} \tag{3.8}$$

Proof. We begin with the Dirichlet case. By (2.6) we have $A(t)_{tan} = 0$ for all $t \in (0,T)$. We may differentiate $A(t)_{tan}$ with respect to t on the boundary to get $A^{(n)}(t)_{tan} = 0$ for all $n \geq 0$. Therefore, $A^{(n)}(t)$ belongs to the domain of the minimal operator d_A in this case by Lemma 3.3. By Corollary 3.7 in [1], $A(t)_{tan} = 0$ also implies that $B(t)_{tan} = 0$. As a result, $B^{(n)}(t)_{tan} = 0$ for all $n \geq 0$ and $B^{(n)}(t)$ therefore also belongs to the domain of d_A.

Similarly, in the Neumann case, since $B(t)_{norm} = 0$ for all $t \in (0,T)$, it follows that $B^{(n)}(t)_{norm} = 0$ for all $t \in (0,T)$ and therefore $B^{(n)}(t)$ belongs to the domain of the minimal operator d_A^* by Lemma 3.3. By Lemma 3.4, $B(t)_{norm} = 0$ implies that $d_A^* B(t)$ also belongs to the domain of d_A^*. Since $d_A^* B(t) = A'(t)$, we can apply Lemma 3.4 to find $A'(t)_{norm} = 0$ for all $t \in (0,T)$ as well. As a result, $A^{(n)}(t)_{norm} = 0$ for all $n \geq 1$ and therefore $A^{(n)}(t)$ also belongs to the domain of d_A^*. □

Lemma 3.6. *In case $M \neq \mathbb{R}^3$ the operators d_A and d_A^* act only on elements in their domains in the identities (3.1)–(3.4).*

Proof. The proof is similar to the proof of Lemma 5.1 in [1]. For the Dirichlet case, (3.7) implies that for all $n \geq 1$ and $t \in (0,T)$, $A^{(n)}(t)$ belongs to the domain of the minimal operator d_A. This justifies the use of d_A in (3.1). Similarly (3.7) shows that $B^{(n)}(t)$ is in the domain of d_A, which justifies its use in (3.4). Since d_A^* is the maximal operator, $B^{(n)}$ and $A^{(n)}$ both belong to its domain. This justifies its use in (3.2) and (3.3).

For the Neumann case (3.8) of Lemma 3.5 shows that $B^{(n)}(t)$ and $A^{(n)}(t)$ belong to the domain of the minimal operator d_A^* for all $n \geq 1$ and $t \in (0,T)$. Therefore the application of d_A^* in (3.2) and (3.3) is justified. The application of d_A in (3.1) and (3.4) is also justified, since it is the maximal operator in this case. □

Proof of Proposition 3.1. For the case $M \neq \mathbb{R}^3$ the identities (3.1)–(3.3) are justified by the proof of Lemma 3.2 and Lemma 3.6. For (3.4) it suffices to justify the application of d_A to both sides of (3.1) under both sets of boundary conditions. In the case of Dirichlet boundary conditions observe that, for all $n \geq 1$, $B^{(n)}(t) \in Dom(d_A(t))$, by Lemma 3.5, as is $d_{A(t)}A^{(n)}(t)$ by Lemmas 3.5 and 3.4. Moreover, since all $A^{(i)}(t)_{tan} = A^{(n-i)}(t)_{tan} = 0$ and $[A^{(i)}(t) \wedge A^{(n-i)}(t)]_{tan} = 0$, the application of d_A to each term on the right side of (3.1) is justified. The Neumann case is trivial because d_A is the maximal operator.

For $M = \mathbb{R}^3$ the identities are justified since we are considering smooth solutions to the Yang–Mills heat equation. Boundary conditions are not an issue. □

3.2. Integral identities

We will use the pointwise identities of the previous subsection to prove integral identities for smooth solutions to the Yang–Mills heat equation.

Lemma 3.7. *Let $A(t)$ be a smooth solution to the Yang–Mills heat equation over $(0,T)$, satisfying either (2.5) or (2.6) if $M \neq \mathbb{R}^3$. Then, for any integer $n \geq 0$,*

$$\frac{d}{dt}\|B^{(n)}(t)\|_2^2 + \|A^{(n+1)}(t)\|_2^2 = -\|d_A^* B^{(n)}(t)\|_2^2 + \|Q_{(n+1)}(t)\|_2^2 + 2(P_{n+1}(t), B^{(n)}(t)) \tag{3.9}$$

and, for any integer $n \geq 1$,

$$\frac{d}{dt}\|A^{(n)}(t)\|_2^2 + \|B^{(n)}(t)\|_2^2 = -\|d_A A^{(n)}(t)\|_2^2 + \|P_n(t)\|_2^2 - 2(Q_{n+1}(t), A^{(n)}(t)). \tag{3.10}$$

d_A represents the exterior derivative with domain matching the boundary conditions and d_A^* is its adjoint.

Proof. By identity (3.1)

$$\begin{aligned}(d/dt)\|B^{(n)}\|_2^2 &= 2(B^{(n+1)}, B^{(n)}) \\ &= 2(d_A A^{(n+1)} + P_{n+1},\, B^{(n)}) \\ &= 2(A^{(n+1)}, d_A^* B^{(n)}) + 2(P_{n+1},\, B^{(n)}),\end{aligned}$$

where we observe that the integration by parts is justified for both boundary conditions. The first term on the right side may be written in two different ways using (3.2)

$$\begin{aligned}(A^{(n+1)}, d_A^* B^{(n)}) &= -(d_A^* B^{(n)}, d_A^* B^{(n)}) - (Q_{n+1}, d_A^* B^{(n)}) \\ \text{and also } &= -(A^{(n+1)}, A^{(n+1)}) - (A^{(n+1)}, Q_{n+1}).\end{aligned}$$

Adding the two we obtain

$$\begin{aligned}2(A^{(n+1)}, d_A^* B^{(n)}) &= -\|A^{(n+1)}\|_2^2 - \|d_A^* B^{(n)}\|_2^2 - (Q_{n+1}, A^{(n+1)} + d_A^* B^{(n)}) \\ &= -\|A^{(n+1)}\|_2^2 - \|d_A^* B^{(n)}\|_2^2 + \|Q_{n+1}\|_2^2,\end{aligned}$$

where for the last equality we have applied (3.2) once more. (3.9) follows.

The second identity is proved in a similar manner. Using (3.2)

$$\begin{aligned}(d/dt)\|A^{(n)}\|_2^2 &= 2(A^{(n+1)}, A^{(n)}) \\ &= -2(d_A^* B^{(n)},\, A^{(n)}) - 2(Q_{n+1},\, A^{(n)}) \\ &= -2(B^{(n)}, d_A A^{(n)}) - 2(Q_{n+1},\, A^{(n)}),\end{aligned}$$

noting that the integration by parts is again justified for both boundary conditions. We rewrite the first term in two different ways using (3.1)

$$\begin{aligned}(B^{(n)}, d_A A^{(n)}) &= (d_A A^{(n)}, d_A A^{(n)}) + (P_n\,, d_A A^{(n)}) \\ &= (B^{(n)}, B^{(n)}) - (P_n\ , B^{(n)}).\end{aligned}$$

Adding the two we obtain

$$\begin{aligned}2(B^{(n)}, d_A A^{(n)}) &= \|B^{(n)}\|_2^2 + \|d_A A^{(n)}\|_2^2 + (P_n\ , d_A A^{(n)} - B^{(n)}) \\ &= \|B^{(n)}\|_2^2 + \|d_A A^{(n)}\|_2^2 - \|P_n\|_2^2\end{aligned}$$

by applying once again (3.1) for the last equality. (3.10) follows. □

4. Differential inequalities

4.1. Gaffney–Friedrichs–Sobolev inequalities in three dimensions

In our estimates, the embedding of W_1 into L^6 will be critical. Define the gauge invariant version of the W_1 norm on M by

$$\|\omega\|^2_{W_1^A(M)} = \|\nabla^A\omega\|^2_{L^2(M)} + \|\omega\|^2_{L^2(M)}$$

for any $\mathfrak{k}$ valued p-form ω on M.

On a compact three-dimensional manifold M with smooth boundary, as well as on $\mathbb{R}^3$, the Sobolev embedding theorem implies that for any $\omega \in W_1(M)$

$$\|\omega\|_6^2 \le (\kappa^2/2)\Big(\int_M |grad|\omega|\,|^2 + \|\omega\|_2^2\Big)$$

for some constant κ that depends on the geometry of M, but not on A (see, for example, [5, Theorem 7.26]). It holds also for $M = \mathbb{R}^3$. In view of Kato's inequality,

$$\int_M |grad|\omega|\,|^2 \le \|\nabla^A\omega\|_2^2,$$

it follows that

$$\|\omega\|_6^2 \le (\kappa^2/2)\|\omega\|^2_{W_1^A(M)} \quad \text{for } \omega \text{ and } A \in W_1(M). \tag{4.1}$$

We recall the following gauge invariant Gaffney–Friedrichs inequality:

Theorem 4.1. ([1, Theorem 2.17]) *Suppose that M is a compact three-dimensional Riemannian manifold with smooth boundary or that $M = \mathbb{R}^3$ and that A is a $\mathfrak{k}$ valued 1-form in $W_1(M)$ with curvature B such that $\|B\|_2 < \infty$. Then there exist constants λ_M and γ that depend only on the geometry of M and not on A, such that, for*

$$\lambda(B) := \lambda_M + \gamma \|B\|_2^4, \tag{4.2}$$

there holds

$$(1/2)\|\omega\|^2_{W_1^A(M)} \leq \|d_A\omega\|_2^2 + \|d_A^*\omega\|_2^2 + \lambda(B)\|\omega\|_2^2 \tag{4.3}$$

for any $\mathfrak{k}$ valued p-form ω in $W_1(M)$ satisfying either

$$\omega_{tan} = 0 \quad or \quad \omega_{norm} = 0$$

if $M \neq \mathbb{R}^3$. Here d_A is the covariant exterior derivative with domain matching the boundary condition on ω and d_A^* is its adjoint.

We recall from [1] that $\gamma = (1/4)(3\kappa^2)^3 c^4$ where $c \equiv \sup\{\|ad\ x\|_{\mathfrak{k}\to\mathfrak{k}} : |x|_{\mathfrak{k}} \leq 1\}$ is a constant that measures the non-commutativity of K and which is zero if K is commutative. The constant κ is the Sobolev constant from (4.1). The constant λ_M is given by

$$\lambda_M = 1 + \|W\|_\infty + \theta,$$

where W is the Weitzenböck tensor on p-forms and θ is a constant determined by the lower bound of the second fundamental form on ∂M. If M is convex then we can take $\theta = 0$ and if $M = \mathbb{R}^3$ or is a convex subset of $\mathbb{R}^3$ then we can take $\lambda_M = 1$. Thus in this paper we take $\lambda_M = 1$.

Corollary 4.2 *(Gaffney–Friedrichs–Sobolev inequality). Suppose that $M = \mathbb{R}^3$ or M is the closure of a bounded convex open subset of $\mathbb{R}^3$ with smooth boundary. Let $A \in W_1(M)$ and suppose that $\|B\|_2 < \infty$. If ω is a p-form in $W_1(M) \cap Dom(d_A) \cap Dom(d_A^*)$ then*

$$\|\omega\|_6^2 \leq \kappa^2(\|d_A\omega\|_2^2 + \|d_A^*\omega\|_2^2 + \lambda\|\omega\|_2^2) \tag{4.4}$$

with $\lambda = \lambda(B) = 1 + \gamma\|B\|_2^4$.

Note: If $M \neq \mathbb{R}^3$ and $\omega \in W_1(M)$ then the domain restrictions are equivalent to $\omega_{tan} = 0$ or $\omega_{norm} = 0$.

Proof. Combine (4.3) and (4.1). □

In the following lemma lower order time derivatives are singled out in what is otherwise the standard Gaffney–Friedrichs–Sobolev inequality. We will use the notation H_1^A instead of W_1^A because the argument of these norms always satisfies the relevant boundary conditions when $M \neq \mathbb{R}^3$. Moreover agreement of time between the argument and A will also be understood. Thus $\|A^{(n)}(t)\|^2_{H_1^A} = \|\nabla^{A(t)}A^{(n)}(t)\|_2^2 + \|A^{(n)}(t)\|_2^2$ as in Notation 2.9. These Sobolev norms are gauge invariant.

Lemma 4.3 *(GFS). Let $A(t)$ be a smooth solution to the Yang–Mills heat equation as in Proposition 3.1. Taking γ as the constant defined after Theorem 4.1, define*

$$\lambda(t) = 1 + \gamma\|B(t)\|_2^4. \tag{4.5}$$

Then for any $n \geq 1$ we have

$$\kappa^{-2}\|A^{(n)}(t)\|_6^2 \leq (1/2)\|A^{(n)}(t)\|_{H_1^A}^2$$
$$\leq \|R_n(t)\|_2^2 + \|d_A A^{(n)}(t)\|_2^2 + \lambda(t)\|A^{(n)}(t)\|_2^2 \tag{4.6}$$
$$\leq \|R_n(t)\|_2^2 + 2\|P_n(t)\|_2^2 + 2\|B^{(n)}(t)\|_2^2 + \lambda(t)\|A^{(n)}(t)\|_2^2. \tag{4.7}$$

For any $n \geq 0$ we have

$$\kappa^{-2}\|B^{(n)}(t)\|_6^2 \leq (1/2)\|B^{(n)}(t)\|_{H_1^A}^2$$
$$\leq \|S_n(t)\|_2^2 + \|d_A^* B^{(n)}(t)\|_2^2 + \lambda(t)\|B^{(n)}(t)\|_2^2 \tag{4.8}$$
$$\leq \|S_n(t)\|_2^2 + 2\|Q_{n+1}(t)\|_2^2 + 2\|A^{(n+1)}(t)\|_2^2 + \lambda(t)\|B^{(n)}(t)\|_2^2. \tag{4.9}$$

Proof. Lemma 3.5 shows that for either boundary value problem, $A^{(n)}(t)$ satisfies the correct boundary condition that allows us to apply the Gaffney–Friedrichs inequality (4.3). Using also the Sobolev inequality (4.1) we find

$$\kappa^{-2}\|A^{(n)}(t)\|_6^2 \leq (1/2)\|A^{(n)}(t)\|_{H_1^A}^2$$
$$\leq \|d_A A^{(n)}(t)\|_2^2 + \|d_A^* A^{(n)}(t)\|_2^2 + \lambda(t)\|A^{(n)}(t)\|_2^2. \tag{4.10}$$

(4.6) now follows from the identity (3.3). The identity (3.1) shows that $\|d_A A^{(n)}\|_2^2 = \|B^{(n)} - P_n\|_2^2 \leq 2\|P_n\|_2^2 + 2\|B^{(n)}\|_2^2$, which proves (4.7).

Similarly, the Sobolev inequality (4.1) and Gaffney–Friedrichs inequality (4.3) show that

$$\kappa^{-2}\|B^{(n)}\|_6^2 \leq (1/2)\|B^{(n)}(t)\|_{H_1^A}^2$$
$$\leq \|d_A B^{(n)}\|_2^2 + \|d_A^* B^{(n)}\|_2^2 + \lambda(t)\|B^{(n)}\|_2^2, \tag{4.11}$$

which yields (4.8) in view of the identity (3.4). Moreover, in accordance with (3.2) we have $\|d_A^* B^{(n)}\|_2^2 = \|A^{(n+1)} + Q_{n+1}\|_2^2 \leq 2\|Q_{n+1}\|_2^2 + 2\|A^{(n+1)}\|_2^2$, from which (4.9) follows. □

Remark 4.4. The Gaffney–Friedrichs–Sobolev inequalities take a very simple form in case $n = 0$ or 1. Thus we have

$$\kappa^{-2}\|B(t)\|_6^2 \leq \|A'(t)\|_2^2 + \lambda(t)\|B(t)\|_2^2, \tag{4.12}$$
$$\kappa^{-2}\|A'(t)\|_6^2 \leq \|B'(t)\|_2^2 + \lambda(t)\|A'(t)\|_2^2 \qquad \text{and} \tag{4.13}$$
$$\kappa^{-2}\|B'(t)\|_6^2 \leq \|\,[B(t) \wedge A'(t)]\,\|_2^2 + \|d_A^* B'(t)\|_2^2 + \lambda(t)\|B'(t)\|_2^2. \tag{4.14}$$

The first of these follows from (4.11) with $n = 0$ because $d_A B = 0$ by the Bianchi identity and $d_A^* B = -A'$ by the Yang–Mills heat equation. The second follows directly from (4.10) because $d_A A' = B'$ and $d_A^* A' = 0$. The third follows from (4.8) because $S_1(t) = [B(t) \wedge A'(t)]$.

4.2. Differential inequalities

For the remainder of this section we let $A(t)$ be a smooth solution to the Yang–Mills heat equation as in Lemma 3.7 and define $\lambda(t)$ as in (4.5). We will be using the Gaffney–Friedrichs–Sobolev inequalities to estimate the right side of the integral identities of Lemma 3.7.

Lemma 4.5 *(Estimate of (3.10) for $n \geq 1$). For each integer $n \geq 1$ there are a constant c_n depending only on n, the manifold M and the vector bundle $\mathcal{V}$ such that*

$$\begin{aligned}\frac{d}{dt}\|A^{(n)}(t)\|_2^2 + \|B^{(n)}(t)\|_2^2 \leq \Big(\lambda(t) + c_n\|B(t)\|_2^4\Big)\|A^{(n)}(t)\|_2^2 + \|P_n(t)\|_2^2 \\ + 2\kappa^2\|\hat{Q}_{n+1}(t)\|_{6/5}^2 + \|R_n(t)\|_2^2\end{aligned} \tag{4.15}$$

where $\hat{Q}_{n+1}(t)$ is defined in (4.16).

Note: All time derivatives of A or B that occur in $P_n(t)$, $\hat{Q}_{n+1}(t)$ and $R_n(t)$ are of order less than n.

Proof. We need to bound the right hand side of the integral equality (3.10). We will derive a bound for the last term in (3.10) which will include a term that cancels with the term $-\|d_A A^{(n)}(t)\|_2^2$. Define

$$\begin{aligned}\hat{Q}_{n+1}(t) &= \sum_{i=1}^{n-1} \overline{c}_{(n+1)i}[A^{(i)}(t) \lrcorner B^{(n-i)}(t)], \quad \text{for} \quad n \geq 2 \quad \text{and} \\ \hat{Q}_2(t) &= 0.\end{aligned} \tag{4.16}$$

Then (3.2), with n replaced by $n+1$, shows that $Q_{n+1} = \hat{Q}_{n+1} + \bar{c}_{(n+1)n}[A^{(n)} \lrcorner B]$. The only time derivatives $A^{(i)}$ in $\hat{Q}_{n+1}$ are of order less than n.

For non-negative functions f and g Hölder's inequality gives $\|f^2 g\|_1 = \|f^{3/2} f^{1/2} g\|_1 \leq \|f^{3/2}\|_4 \|f^{1/2}\|_4 \times \|g\|_2 = \|f\|_6^{3/2}\|f\|_2^{1/2}\|g\|_2$. Therefore, for any $\epsilon > 0$ we have

$$\begin{aligned}\|f^2 g\|_1 &\leq \|f\|_6^{3/2}\|f\|_2^{1/2}\|g\|_2 \\ &\leq (3/4)(\epsilon^{-1}\|f\|_6^{3/2})^{4/3} + (1/4)(\epsilon\|f\|_2^{1/2}\|g\|_2)^4 \\ &= (3/4)\epsilon^{-4/3}\|f\|_6^2 + (1/4)\epsilon^4\|f\|_2^2\|g\|_2^4.\end{aligned} \tag{4.17}$$

Let $c_n' = 2c\bar{c}_{(n+1)n}$. Then

$$\begin{aligned}\Big|2(Q_{n+1}, A^{(n)})\Big| &= 2\Big|(\hat{Q}_{n+1}, A^{(n)}) + \bar{c}_{(n+1)n}([A^{(n)} \lrcorner B], A^{(n)})\Big| \\ &\leq \Big\{2\|\hat{Q}_{n+1}\|_{6/5}\|A^{(n)}\|_6\Big\} + \Big\{c_n'\| \, |A^{(n)}|^2|B| \, \|_1\Big\} \\ &\leq \Big\{2\kappa^2\|\hat{Q}_{n+1}\|_{6/5}^2 + (1/2)\kappa^{-2}\|A^{(n)}\|_6^2\Big\} \\ &\quad + \Big\{\frac{c_n'}{4}\,\epsilon^4\|B\|_2^4\,\|A^{(n)}\|_2^2 + (\frac{3c_n'}{4}\,\epsilon^{-4/3}\kappa^2)\,\kappa^{-2}\|A^{(n)}\|_6^2\Big\},\end{aligned}$$

wherein we used (4.17) with $f = |A^{(n)}(t)|$ and $g = |B(t)|$.

Choose ϵ such that $(\frac{3c_n'}{4}\,\epsilon^{-4/3}\kappa^2) = 1/2$. The two $\|A^{(n))}\|_6^2$ terms add to $\kappa^{-2}\|A^{(n))}\|_6^2$. Using the Gaffney–Friedrichs–Sobolev inequality (4.6), we find

$$\begin{aligned}\Big|2(Q_{n+1}, A^{(n)})\Big| &\leq 2\kappa^2\|\hat{Q}_{n+1}\|_{6/5}^2 + c_n\|B\|_2^4\|A^{(n)}\|_2^2 + \Big(\kappa^{-2}\|A^{(n)}\|_6^2\Big) \\ &\leq 2\kappa^2\|\hat{Q}_{n+1}\|_{6/5}^2 + c_n\|B\|_2^4\|A^{(n)}\|_2^2 + \Big(\|R_n\|_2^2 + \|d_A A^{(n)}\|_2^2 + \lambda(t)\|A^{(n)}\|_2^2\Big),\end{aligned}$$

where $c_n = (c_n'/4)\epsilon^4 = (1/4)(3/2)^3\kappa^6(c_n')^4$. Insert this bound into (3.10), canceling the terms $\|d_A A^{(n)}\|_2^2$, to find

$$\frac{d}{dt}\|A^{(n)}(t)\|_2^2 + \|B^{(n)}(t)\|_2^2$$
$$\leq \|P_n(t)\|_2^2 + 2\kappa^2\|\hat{Q}_{n+1}\|_{6/5}^2 + c_n\|B\|_2^4\|A^{(n)}\|_2^2 + \left(\|R_n\|_2^2 + \lambda(t)\|A^{(n)}\|_2^2\right)$$

which is (4.15). □

Lemma 4.6 *(Estimate of (3.9) for $n \geq 0$). For each integer $n \geq 0$ there holds*

$$\frac{d}{dt}\|B^{(n)}(t)\|_2^2 + \|A^{(n+1)}(t)\|_2^2 \leq \lambda(t)\|B^{(n)}(t)\|_2^2 + \|Q_{n+1}(t)\|_2^2 + \kappa^2\|P_{n+1}(t)\|_{6/5}^2 + \|S_n(t)\|_2^2. \tag{4.18}$$

Note: All time derivatives of B that occur in $Q_n(t)$ are of order less than n. All time derivatives of A in the right side are of order less than $n+1$.

Proof. We need to bound the right hand side of (3.9). We have

$$\begin{aligned} 2|(P_{n+1}, B^{(n)})| &\leq 2\|P_{n+1}\|_{6/5}\|B^{(n)}\|_6 \\ &\leq \kappa^2\|P_{n+1}\|_{6/5}^2 + \kappa^{-2}\|B^{(n)}\|_6^2 \\ &\leq \kappa^2\|P_{n+1}\|_{6/5}^2 + \|S_n\|_2^2 + \|d_A^* B^{(n)}\|_2^2 + \lambda(t)\|B^{(n)}\|_2^2 \end{aligned}$$

by virtue of (4.8). Therefore

$$-\|d_A^* B^{(n)}\|_2^2 + 2|(P_{n+1}, B^{(n)})| \leq \kappa^2\|P_{n+1}\|_{6/5}^2 + \|S_n\|_2^2 + \lambda(t)\|B^{(n)}\|_2^2$$

This proves (4.18). □

Remark 4.7. In case $n = 0$ the inequality (4.18) gives

$$\frac{d}{dt}\|B(t)\|_2^2 + \|A'(t)\|_2^2 \leq \lambda(t)\|B(t)\|_2^2 \tag{4.19}$$

since $Q_1 = P_1 = S_0 = 0$ by Proposition 3.1. But the identity (3.9) shows that $\frac{d}{dt}\|B(t)\|_2^2 + 2\|A'(t)\|_2^2 = 0$. There is a loss of information, therefore, in (4.18), which we allow in order to get a simple inequality for all $n \geq 0$.

Under the assumption of finite action we will be able to use the preceding differential equalities to obtain integral estimates in our main result, Theorem 2.8. Proposition 4.9 below will be critical in this transition.

Notation 4.8. For a smooth solution $A(\cdot)$ on $(0,T)$ to the Yang–Mills heat equation (1.2) that has finite action let c_n be the constant appearing in (4.15) and define

$$\psi(t) = \int_0^t \lambda(\sigma)\,d\sigma = t + \gamma\int_0^t \|B(\sigma)\|_2^4 d\sigma \quad \text{and} \tag{4.20}$$

$$\psi_n(t) = t + (\gamma + c_n)\int_0^t \|B(\sigma)\|_2^4 d\sigma. \tag{4.21}$$

Lemma 5.6 will show that $\int_0^t \|B(\sigma)\|_2^4 d\sigma < \infty$ when $A(\cdot)$ has finite action. It follows from this that $\psi(t)$ and $\psi_n(t)$ are bounded, differentiable and nondecreasing functions on the interval $(0,T)$. Then, for $0 \leq s \leq t$ the functions

$$\psi^{t,s} := \psi(t) - \psi(s) \quad \text{and} \quad \psi_n^{t,s} := \psi_n(t) - \psi_n(s)$$

are non-negative.

Proposition 4.9.

$$\frac{d}{ds}\left(e^{-\psi_n(s)}\|A^{(n)}(s)\|_2^2\right) + e^{-\psi_n(s)}\|B^{(n)}(s)\|_2^2 \le e^{-\psi_n(s)}X_n(s), \quad n \ge 1 \tag{4.22}$$

and

$$\frac{d}{ds}\left(e^{-\psi(s)}\|B^{(n)}(s)\|_2^2\right) + e^{-\psi(s)}\|A^{(n+1)}(s)\|_2^2 \le e^{-\psi(s)}Y_n(s), \quad n \ge 0, \tag{4.23}$$

where

$$X_n(t) = \|P_n(t)\|_2^2 + 2\kappa^2\|\hat{Q}_{n+1}(t)\|_{6/5}^2 + \|R_n(t)\|_2^2\ ,\ n \ge 1 \quad \textit{and} \tag{4.24}$$

$$Y_n(t) = \|Q_{n+1}(t)\|_2^2 + \kappa^2\|P_{n+1}(t)\|_{6/5}^2 + \|S_n(t)\|_2^2,\ n \ge 0. \tag{4.25}$$

Note that $Y_0(t) = X_1(t) = 0$ *by virtue of Proposition 3.1 and the definition* (4.16) *of* $\hat{Q}_2$.

Proof. Since $\psi'(s) = \lambda(s)$ and $\psi_n'(s) = \lambda(s) + c_n\|B(s)\|_2^4$, the inequality (4.15) can be written as

$$\frac{d}{ds}\|A^{(n)}(s)\|_2^2 - \psi_n'(s)\|A^{(n)}(s)\|_2^2 + \|B^{(n)}(s)\|_2^2 \le X_n(s).$$

This is equivalent to (4.22), as one can see by differentiating the product and then multiplying by $e^{\psi_n(s)}$. The inequality (4.23) follows from (4.18) similarly. □

5. Initial behavior

5.1. Initial behavior from differential inequalities

From the differential inequalities (4.22) and (4.23) we are going to derive initial behavior bounds in the form of integral estimates with the help of the following elementary lemma.

Lemma 5.1 *(Initial behavior from differential inequalities). Suppose that* f, g, h *are nonnegative continuous functions on* $(0, t]$ *and that* f *is differentiable. Suppose also that*

$$(d/ds)f(s) + g(s) \le h(s), \quad 0 < s \le t. \tag{5.1}$$

Let $-1 < b < \infty$ *and assume that*

$$\int_0^t s^b f(s)ds < \infty. \tag{5.2}$$

Then

$$t^{1+b}f(t) + \int_0^t s^{(1+b)}g(s)ds \le \int_0^t s^{(1+b)}h(s)ds + (1+b)\int_0^t s^b f(s)ds. \tag{5.3}$$

If equality holds in (5.1) *then equality holds in* (5.3).

Proof. Assumption (5.1) implies that

$$(d/ds)\left(s^{(1+b)}f(s)\right) + s^{(1+b)}g(s) \leq s^{(1+b)}h(s) + (1+b)s^{(1+b)}f(s)$$

for all $0 < s \leq t$. The result follows after integrating both sides over the interval $(0,t]$ if one knows that $\lim_{t\downarrow 0} t^{(1+b)}f(t) = 0$. See [6, Lemma 4.8] for a proof without this assumption. □

Corollary 5.2. *Define $X_n(t)$ and $Y_n(t)$ by (4.24) and (4.25) respectively. The inequalities*

$$t^{2n-\frac{1}{2}}\|A^{(n)}\|_2^2 + \int_0^t s^{2n-\frac{1}{2}}\|B^{(n)}(s)\|_2^2 ds \tag{5.4}$$
$$\leq \left\{(2n-\frac{1}{2})\int_0^t s^{2n-\frac{3}{2}}\|A^{(n)}(s)\|_2^2 ds + \int_0^t s^{2n-\frac{1}{2}}X_n(s)ds\right\}e^{\psi_n(t)}, \quad n \geq 1$$

$$t^{2n+\frac{1}{2}}\|B^{(n)}(t)\|_2^2 + \int_0^t s^{2n+\frac{1}{2}}\|A^{(n+1)}(s)\|_2^2\, ds \tag{5.5}$$
$$\leq \left\{(2n+\frac{1}{2})\int_0^t s^{2n-\frac{1}{2}}\|B^{(n)}(s)\|_2^2 ds + \int_0^t s^{2n+\frac{1}{2}}Y_n(s)ds\right\}e^{\psi(t)}, \quad n \geq 0$$

hold whenever their right sides are finite, for $\psi(t)$, $\psi_n(t)$ as in (4.20) and (4.21) respectively.

Proof. Starting with the differential inequality (4.22), we can apply Lemma 5.1 with $f(s) = e^{-\psi_n(s)}\|A^{(n)}(s)\|_2^2$, $g(s) = e^{-\psi_n(s)}\|B^{(n)}(s)\|_2^2$, $h(s) = e^{-\psi_n(s)}X_n(s)$ and $b = 2n - \frac{3}{2}$. Upon multiplying the resulting inequality (5.3) by $e^{\psi_n(t)}$ we find

$$t^{2n-\frac{1}{2}}\|A^{(n)}\|_2^2 + \int_0^t e^{\psi_n^{t,s}}s^{2n-\frac{1}{2}}\|B^{(n)}(s)\|_2^2 ds \tag{5.6}$$
$$\leq (2n-\frac{1}{2})\int_0^t e^{\psi_n^{t,s}}s^{2n-\frac{3}{2}}\|A^{(n)}(s)\|_2^2 ds + \int_0^t e^{\psi_n^{t,s}}s^{2n-\frac{1}{2}}X_n(s)ds, \quad n \geq 1.$$

Since $1 \leq e^{\psi_n^{t,s}} \leq e^{\psi_n(t)}$, the inequality (5.6) continues to hold if we drop the factor $e^{\psi_n^{t,s}}$ from the integrand on the left and replace it in the integrands on the right by $e^{\psi_n(t)}$. This yields (5.4).

The same method shows that (5.5) follows from (4.23) if, in Lemma 5.1, one chooses $f(s) = e^{-\psi(s)}\|B^{(n)}(s)\|_2^2$, $g(s) = e^{-\psi(s)}\|A^{(n+1)}(s)\|_2^2$, $h(s) = e^{-\psi(s)}Y_n(s)$ and $b = 2n - \frac{1}{2}$. □

The remainder of the paper will be devoted to proving that the right hand sides of the inequalities (5.4) and (5.5) are finite. This will be done by induction on n. But the induction hypothesis will include two other inequalities besides these two.

5.2. Initial behavior of the curvature and A'

We review a few well known a priori bounds for solutions of the Yang–Mills heat equation in the presence of finite action.

Lemma 5.3. *Let $A(t)$ be a smooth solution to the Yang–Mills heat equation over $(0,T)\times M$, satisfying either (2.5) or (2.6) if $M \neq \mathbb{R}^3$. Then $\|B(t)\|_2$ is nonincreasing on $(0,T)$. Moreover, if $\|B_0\|_2 < \infty$ then*

$$\|B(t)\|_2 \leq \|B_0\|_2 \tag{5.7}$$

for $0 \leq t < T$.

Proof. Identity (3.9) for $n = 0$ gives $(d/dt)\|B(t)\|_2^2 = -2\|A'(t)\|_2^2 \leq 0$ since $P_1 = Q_1 = 0$. Therefore $\|B(t)\|_2^2$ is non-increasing. (5.7) follows from the continuity of $\|B(t)\|_2$ at $t = 0$ in this finite energy case. □

Remark 5.4. If $A(\cdot)$ has finite action then $\rho(t)$, defined in (2.10), is finite for small t and therefore for all t, since the integrand is decreasing by Lemma 5.3. Further, if $A(\cdot)$ is a solution with finite energy, i.e. $\|B_0\|_2 < \infty$, then (5.7) shows that A has finite action.

Proposition 5.5. *Let $A(t)$ be a smooth solution to the Yang–Mills heat equation over $(0,T)\times M$, satisfying either (2.5) or (2.6) if $M \neq \mathbb{R}^3$. If $A(\cdot)$ has finite action then*

$$t^{1/2}\|B(t)\|_2^2 + 2\int_0^t s^{1/2}\|A'(s)\|_2^2\, ds = \rho(t), \tag{5.8}$$

for any $t \in [0,T)$. In particular $(\mathcal{C}_n)$ holds for $n = 0$.

Proof. For $n = 0$ identity (3.9) becomes $(d/ds)\|B(s)\|_2^2 + 2\|A'(s)\|_2^2 = 0$. We can apply Lemma 5.1, taking $f(s) = \|B(s)\|_2^2$, $g(s) = 2\|A'(s)\|_2^2$, $h(s) = 0$ and $b = -1/2$. Then (5.3) asserts that

$$t^{1/2}\|B(t)\|_2^2 + 2\int_0^t s^{1/2}\|A'(s)\|_2^2\, ds = (1/2)\int_0^t s^{-1/2}\|B(s)\|_2^2 ds,$$

which is (5.8), in view of the definition (2.10) of $\rho(t)$. The hypothesis (5.2) is satisfied by the assumption of finite action. This proves that $(\mathcal{C}_n)$ holds for $n = 0$. We can take $C_{03}(t) = \rho(t)$. □

Lemma 5.6. *Let $A(t)$ be a smooth solution to the Yang–Mills heat equation over $(0,T)\times M$, satisfying either (2.5) or (2.6) if $M \neq \mathbb{R}^3$. If $A(\cdot)$ has finite action then*

$$t\|B(t)\|_2^4 \leq \rho(t)^2 \qquad \textit{and} \tag{5.9}$$

$$\int_0^t \|B(s)\|_2^4\, ds \leq 2\,\rho(t)^2. \tag{5.10}$$

Moreover, for $0 < t \leq T$ there holds

$$t\,\lambda(t) \leq t + \gamma\,\rho(t)^2, \tag{5.11}$$

$$\psi(t) \leq t + 2\gamma\,\rho(t)^2 \qquad \textit{and} \tag{5.12}$$

$$\psi_n(t) \leq t + 2(\gamma + c_n)\,\rho(t)^2 \tag{5.13}$$

where $\lambda(t)$ is defined in (4.5) and $\psi(t)$ and $\psi_n(t)$ are defined in (4.20) and (4.21) respectively. In particular these three functions are non-decreasing and are bounded by standard dominating functions.

Proof. Identity (5.8) implies that $s^{1/2}\|B(s)\|_2^2 \leq \rho(s) \leq \rho(t)$ for all $s \leq t$ since $\rho(t)$ is nondecreasing. In particular (5.9) holds. Further,

$$\int_0^t \|B(s)\|_2^4\, ds = \int_0^t \left(s^{1/2}\|B(s)\|_2^2\right)\, \left(s^{-1/2}\|B(s)\|_2^2\right)\, ds$$

$$\leq \rho(t) \int_0^t s^{-1/2}\|B(s)\|_2^2\, ds = 2\,\rho(t)^2$$

proving (5.10). The inequalities (5.11)–(5.13) now follow from their definitions and from (5.9) and (5.10). □

6. Proof of the main theorem

Remark 6.1 *(Strategy).* We will first prove the theorem under the technical assumption that $A(t)$ is a smooth solution to the Yang–Mills equation over $(0,T)\times M$. The proof will proceed by induction on n. We have already shown that $(\mathcal{C}_n)$ holds for $n=0$ in Proposition 5.5. We will show that all four inequalities $(\mathcal{A}_n)$, $(\mathcal{B}_n)$, $(\mathcal{C}_n)$, $(\mathcal{D}_n)$ in Theorem 2.8 hold for $n=1$. We will then show that if $k \geq 2$ and all four inequalities hold for $1 \leq n < k$ then all four inequalities hold for $n=k$. We will then remove the hypothesis of smoothness.

6.1. Proof for $n = 1$

Proposition 6.2 *(Proof of $\mathcal{A}_1$). Let $A(t)$ be a smooth solution to the Yang–Mills heat equation over $(0,T)\times M$, satisfying either (2.5) or (2.6). If $A(\cdot)$ has finite action for $0 \leq t < T$ then*

$$t^{3/2}\|A'(t)\|_2^2 + \int_0^t s^{3/2}\|B'(s)\|_2^2 ds \leq C_{11}(t) \tag{6.1}$$

for some standard dominating function C_{11}. In particular $\mathcal{A}_1$ holds.

Proof. Since $X_1(t) = 0$ the inequality (5.4) with $n=1$ shows that

$$t^{3/2}\|A'(t)\|_2^2 + \int_0^t s^{3/2}\|B'(s)\|_2^2 ds \leq \frac{3}{2}\, e^{\psi_1(t)} \int_0^t s^{1/2}\|A'(s)\|_2^2\, ds \leq \frac{3}{4}\, e^{\psi_1(t)}\, \rho(t)$$

wherein we have used (5.8) in the last inequality. The bound (5.13) shows that the right hand side is bounded by a standard dominating function. □

We see here that the integrability of $t^{1/2}\|A'(t)\|_2^2$ in time implies the boundedness of $t^{3/2}\|A'(t)\|_2^2$ when $A(\cdot)$ is a solution to the Yang–Mills heat equation. This reflects a frequently occurring theme.

For the remainder of this section we will assume that $A(t)$ satisfies the assumptions of Proposition 6.2.

Corollary 6.3 *(Proof of $\mathcal{B}_1$). There exists a standard dominating function C_{12} such that*

$$t^{3/2}\|B(t)\|_6^2 + \int_0^t s^{3/2}\|A'(s)\|_6^2 ds \leq C_{12}(t) \tag{6.2}$$

for all $0 \leq t < T$.

Proof. From the two GFS inequalities (4.12) and (4.13) we find

$$\begin{aligned}
&\kappa^{-2}\Big(t^{3/2}\|B(t)\|_6^2+\int_0^t s^{3/2}\|A'(s)\|_6^2 ds\Big)\\
&\quad\le \Big\{t^{3/2}\,\|A'(t)\|_2^2+(t\lambda(t))t^{1/2}\|B(t)\|_2^2\Big\}+\int_0^t\Big\{s^{3/2}\|B'(s)\|_2^2+(s\lambda(s))s^{1/2}\|A'(s)\|_2^2\Big\}ds\\
&\quad\le \Big(t^{3/2}\,\|A'(t)\|_2^2+\int_0^t s^{3/2}\|B'(s)\|_2^2\Big)+t\lambda(t)\,\Big(t^{1/2}\|B(t)\|_2^2+\int_0^t s^{1/2}\|A'(s)\|_2^2 ds\Big)\\
&\quad\le C_{11}(t)+\,t\lambda(t)\,\rho(t)
\end{aligned}$$

wherein we have used (6.1), (5.8) and the nondecreasing property of $t\lambda(t)$. The bound (5.11) shows that $t\lambda(t)\rho(t)$ is bounded by a standard dominating function. □

To prove $\mathcal{C}_1$ we need the following integral estimate.

Lemma 6.4. *Define $Y_1(t)$ as in (4.25) with $n=1$. There is a standard dominating function $\tilde{C}_{12}$ such that*

$$\int_0^t s^{5/2}Y_1(s)ds\le \tilde{C}_{12}(t) \tag{6.3}$$

for all $0\le t<T$.

Proof. The definitions in Proposition 3.1 give $Q_2(t)=\bar{c}_{21}[A'(t)\lrcorner B(t)]$ and $P_2(t)=c_{21}[A'(t)\wedge A'(t)]$. Hence, by the definition (4.25) we have

$$\begin{aligned}
Y_1(t)&=\|Q_2(t)\|_2^2+\kappa^2\|P_2(t)\|_{6/5}^2+\|S_1(t)\|_2^2\\
&=\|\bar{c}_{21}[A'(t)\lrcorner B(t)]\,\|_2^2+\kappa^2\|c_{21}[A'(t)\wedge A'(t)]\,\|_{6/5}^2+\|\,[B(t)\wedge A'(t)]\,\|_2^2\\
&\le \tilde{c}\|\;|A'(t)|\,|B(t)|\,\|_2^2+\tilde{c}\|\,|A'(t)|^2\|_{6/5}^2
\end{aligned} \tag{6.4}$$

for some constant $\tilde{c}$ that only depends on the manifold and the bundle. By Hölder's inequality

$$\|\,|A'(s)|\,|B(s)|\,\|_2^2\le\|A'(s)\|_6^2\|B(s)\|_3^2\le\|A'(s)\|_6^2\|B(s)\|_2\|B(s)\|_6.$$

Hence

$$\begin{aligned}
s^{5/2}\|\,|A'(s)|\,|B(s)|\,\|_2^2&\le(s^{3/2}\|A'(s)\|_6^2)\,(s^{1/4}\|B(s)\|_2)\,(s^{3/4}\|B(s)\|_6)\\
&\le s^{3/2}\|A'(s)\|_6^2\,\sqrt{\rho(t)\,C_{12}(t)}
\end{aligned}$$

by (5.8) and (6.2). Therefore

$$\begin{aligned}
\int_0^t s^{5/2}\|\,|A'(s)|\,|B(s)|\,\|_2^2\,ds&\le\sqrt{\rho(t)\,C_{12}(t)}\int_0^t s^{3/2}\|A'(s)\|_6^2\,ds\\
&\le\sqrt{\rho(t)\,(C_{12}(t)\,)^3}=:\tilde{\tilde{C}}_{12}(t)
\end{aligned} \tag{6.5}$$

by (6.2), giving an upper bound by a standard dominating function.

For the second term in (6.4) we also apply Hölder's inequality twice to obtain

$$\| \, |A'(s)|^2 \|_{6/5}^2 \leq \|A'(s)\|_3^2 \|A'(s)\|_2^2 \leq \| \, A'(s)\|_6 \|A'(s)\|_2 \|A'(s)\|_2^2.$$

Hence,

$$\begin{aligned} s^{5/2} \| \, |A'(s)|^2 \|_{6/5}^2 &\leq (s^{3/4} \| \, A'(s)\|_6) \, (s^{1/4} \|A'(s)\|_2) \, (s^{3/2} \|A'(s)\|_2^2) \\ &\leq (s^{3/4} \| \, A'(s)\|_6) \, (s^{1/4} \|A'(s)\|_2) \, C_{11}(t) \end{aligned}$$

by (6.1). Therefore

$$\begin{aligned} \int_0^t s^{5/2} \| \, |A'(s)|^2 \|_{6/5}^2 ds &\leq C_{11}(t) \int_0^t (s^{3/4} \| \, A'(s)\|_6) \, (s^{1/4} \|A'(s)\|_2) \, ds \\ &\leq C_{11}(t) \left[\int_0^t s^{3/2} \| \, A'(s)\|_6^2 \, ds \right]^{1/2} \left[\int_0^t s^{1/2} \|A'(s)\|_2^2 \, ds \right]^{1/2} \\ &\leq C_{11}(t) \, \sqrt{C_{12}(t) \, \rho(t)} \end{aligned}$$

by (5.8), (6.2) and Hölder's inequality for the time integral. □

We are now ready to prove $\mathcal{C}_1$.

Corollary 6.5 *(Proof of $\mathcal{C}_1$). There is a standard dominating function C_{13} such that*

$$t^{5/2} \|B'(t)\|_2^2 + \int_0^t s^{5/2} \|A''(s)\|_2^2 ds \leq C_{13}(t) \tag{6.6}$$

for all $0 \leq t < T$.

Proof. From (5.5) with $n = 1$ we get

$$\begin{aligned} t^{5/2} \|B'(t)\|_2^2 &+ \int_0^t s^{5/2} \|A''(s)\|_2^2 ds \\ &\leq \left\{ \frac{5}{2} \int_0^t s^{3/2} \|B'(s)\|_2^2 \, ds + \int_0^t s^{5/2} Y_1(s) \, ds \right\} e^{\psi(t)} \\ &\leq \left\{ \frac{5}{2} \, C_{11}(t) + \tilde{C}_{12}(t) \right\} e^{\psi(t)} \end{aligned}$$

by (6.1) and (6.3). This is bounded by a standard dominating function in view of (5.12). □

Corollary 6.6 *(Proof of $\mathcal{D}_1$). There is a standard dominating function C_{14} such that*

$$t^{5/2} \|A'(t)\|_6^2 + \int_0^t s^{5/2} \|B'(s)\|_6^2 ds \leq C_{14}(t) \tag{6.7}$$

for all $0 \leq t < T$.

Proof. Multiply the GFS inequality (4.13) by $t^{5/2}$ to find

$$\kappa^{-2}t^{5/2}\,\|A'(t)\|_6^2 \le t^{5/2}\,\|B'(t)\|_2^2 + t\lambda(t)\,(t^{3/2}\|A'(t)\|_2^2)$$
$$\le C_{13}(t) + t\lambda(t)\,C_{11}(t)$$

by (6.6) and (6.1). This is bounded by a standard dominating function in view of (5.11).

For the second term in (6.7) observe that the identity (3.2) reduces, for $n = 2$, to the identity $d_A^* B' = -A'' - \bar{c}_{21}[A' \lrcorner B]$. Replace $d_A^* B'$ by this in the GFS inequality (4.14) to find

$$\kappa^{-2}\|B'(t)\|_6^2 \le \|\,[B \wedge A']\,\|_2^2 + 2\|A''(t)\|_2^2 + 2\bar{c}_{21}^2\|\,[A' \lrcorner B]\,\|_2^2 + \lambda(t)\|B'(t)\|_2^2.$$

It follows that for some constant $\bar{c}$ that only depends on the manifold and the bundle,

$$\int_0^t s^{5/2}\|B'(s)\|_6^2 ds \le \bar{c}\int_0^t s^{5/2}\Big\{\|A''(s)\|_2^2 + \|\,|A'(s)|\,|B(s)|\,\|_2^2 + \lambda(s)\|B'(s)\|_2^2\Big\}\,ds$$
$$\le \bar{c}\Big\{\,C_{13}(t) + \tilde{\tilde{C}}_{12}(t) + t\lambda(t)\int_0^t s^{3/2}\|B'(s)\|_2^2 ds\Big\}$$
$$\le \bar{c}\Big\{\,C_{13}(t) + \tilde{\tilde{C}}_{12}(t) + t\lambda(t)C_{11}(t)\Big\}$$

by (6.6), (6.5), and (6.1). This is bounded by a standard dominating function in view of (5.11). □

This completes the proof of Theorem 2.8 for $n = 1$ when $A(t)$ is smooth.

6.2. Bounds on lower order terms

The induction mechanism in the next section will give us information about the initial behavior of the time derivatives of A and B. We will use this information with the help of the following bounds.

Lemma 6.7 *(Bounds on lower order terms). For all $n \ge 1$ there exist constants $d_{n,r}$ independent of M and A such that*

$$\|P_n(t)\|_2^2 \le d_{n,1}c^2\sum_{i=1}^{n-1}\|A^{(i)}(t)\|_2\|A^{(i)}(t)\|_6\|A^{(n-i)}(t)]\|_6^2 \tag{6.8}$$

$$\|P_n(t)\|_{6/5}^2 \le d_{n,2}c^2\sum_{i=1}^{n-1}\|A^{(i)}(t)\|_6\|A^{(i)}(t)\|_2\|A^{(n-i)}(t)\|_2^2 \tag{6.9}$$

$$\|Q_n(t)\|_2^2 \le d_{n,3}c^2\sum_{i=1}^{n-1}\|A^{(i)}(t)\|_6^2\|B^{(n-1-i)}(t)\|_2\|B^{(n-1-i)}(t)\|_6 \tag{6.10}$$

$$\|\hat{Q}_n(t)\|_{6/5}^2 \le d_{n,4}c^2\sum_{i=1}^{n-2}\|A^{(i)}(t)\|_2\|A^{(i)}(t)\|_6\|B^{(n-1-i)}(t)\|_2^2 \tag{6.11}$$

$$\|R_n(t)\|_2^2 \le d_{n,5}c^2\sum_{i=1}^{n-2}\|A^{(i)}(t)\|_2\|A^{(i)}(t)\|_6\|A^{(n-i)}(t)\|_6^2 \tag{6.12}$$

$$\|S_n(t)\|_2^2 \le d_{n,6}\Big(\sum_{i=1}^{n} \|\,[A^{(i)}(t) \wedge B^{(n-i)}(t)]\,\|_2^2 + \sum_{i=1}^{n-1} \|\,[A^{(i)}(t) \wedge P_{n-i}(t)]\,\|_2^2\Big). \tag{6.13}$$

Note: It will be clear from the proof that $d_{n,2} = d_{n,1}$ and $d_{n,4} \le d_{n,3}$.

Proof. The Lemma is a simple application of Hölder's inequality. From (3.1) we have

$$\begin{aligned}\|P_n(t)\|_2^2 &= \|\sum_{i=1}^{n-1} c_{ni}[A^{(i)}(t) \wedge A^{(n-i)}(t)]\,\|_2^2 \le d_{n,1}\sum_{i=1}^{n-1} \|\,[A^{(i)}(t) \wedge A^{(n-i)}(t)]\|_2^2 \\ &\le d_{n,1}c^2 \sum_{i=1}^{n-1} \|A^{(i)}(t)\|_3^2 \|A^{(n-i)}(t)]\|_6^2 \\ &\le d_{n,1}c^2 \sum_{i=1}^{n-1} \|A^{(i)}(t)\|_2 \|A^{(i)}(t)\|_6 \|A^{(n-i)}(t)]\|_6^2.\end{aligned}$$

This proves (6.8). For the second estimate we have

$$\begin{aligned}\|P_n(t)\|_{6/5}^2 &= \|\sum_{i=1}^{n-1} c_{ni}[A^{(i)}(t) \wedge A^{(n-i)}(t)]\|_{6/5}^2 \\ &\le d_{n,2}\sum_{i=1}^{n-1} \|\,[A^{(i)}(t) \wedge A^{(n-i)}(t)]\,\|_{6/5}^2 \\ &\le d_{n,2}c^2 \sum_{i=1}^{n-1} \|A^{(i)}(t)\|_3^2 \|A^{(n-i)}(t)\|_2^2 \\ &\le d_{n,2}c^2 \sum_{i=1}^{n-1} \|A^{(i)}(t)\|_6 \|A^{(i)}(t)\|_2 \|A^{(n-i)}(t)\|_2^2.\end{aligned}$$

Similarly, from (3.2)

$$\begin{aligned}\|Q_n(t)\|_2^2 &= \|\sum_{i=1}^{n-1} \bar{c}_{ni}[A^{(i)}(t) \lrcorner B^{(n-1-i)}(t)]\,\|_2^2 \le d_{n,3}\sum_{i=1}^{n-1} \|\,[A^{(i)}(t) \lrcorner B^{(n-1-i)}(t)]\,\|_2^2 \\ &\le d_{n,3}c^2 \sum_{i=1}^{n-1} \|A^{(i)}(t)\|_6^2 \|B^{(n-1-i)}(t)\|_3^2 \\ &\le d_{n,3}c^2 \sum_{i=1}^{n-1} \|A^{(i)}(t)\|_6^2 \|B^{(n-1-i)}(t)\|_2 \|B^{(n-1-i)}(t)\|_6.\end{aligned}$$

This proves (6.10). From the definition (4.16) we find, for $n \ge 3$

$$\begin{aligned}\|\hat{Q}_n(t)\|_{6/5}^2 &= \|\sum_{i=1}^{n-2} \overline{c}_{ni}[A^{(i)}(t) \lrcorner B^{(n-1-i)}(t)]\,\|_{6/5}^2 \\ &\le d_{n,4}\sum_{i=1}^{n-2} \|\,[A^{(i)}(t) \lrcorner B^{(n-1-i)}(t)]\,\|_{6/5}^2 \\ &\le d_{n,4}c^2 \sum_{i=1}^{n-2} \|A^{(i)}(t)\|_3^2 \|B^{(n-1-i)}(t)\|_2^2\end{aligned}$$

$$\leq d_{n,4}c^2 \sum_{i=1}^{n-2} \|A^{(i)}(t)\|_2 \|A^{(i)}(t)\|_6 \|B^{(n-1-i)}(t)\|_2^2$$

proving (6.11).

From (3.3) we have

$$\|R_n(t)\|_2^2 = \| \sum_{i=1}^{n-2} \tilde{c}_{ni} [A^{(i)}(t) \lrcorner A^{(n-i)}(t)] \|_2^2 \leq d_{n,5} \sum_{i=1}^{n-2} \| [A^{(i)}(t) \lrcorner A^{(n-i)}(t)] \|_2^2$$

$$\leq d_{n,5}c^2 \sum_{i=1}^{n-2} \|A^{(i)}(t)\|_2 \|A^{(i)}\|_6 \|A^{(n-i)}(t)\|_6^2,$$

proving (6.12).

Finally, from (3.4) we have

$$\|S_n(t)\|_2^2 = \| [B(t) \wedge A^{(n)}(t)] + \sum_{i=1}^{n-1} \hat{c}_{ni} [(B^{(i)}(t) - P_i(t)) \wedge A^{(n-i)}(t)] \|_2^2$$

$$\leq d'_{n,6} \Big\{ \| [B(t) \wedge A^{(n)}(t)] \|_2^2 + \sum_{i=1}^{n-1} \| [A^{(i)}(t) \wedge (B^{(n-i)}(t) - P_{n-i}(t))] \|_2^2 \Big\}$$

$$\leq d_{n,6} \Big(\sum_{i=1}^{n} \| [A^{(i)} \wedge B^{(n-i)}] \|_2^2 + \sum_{i=1}^{n-1} \| [A^{(i)} \wedge P_{n-i}] \|_2^2 \Big),$$

proving (6.13). □

6.3. Proof of the induction step

In Section 6.1 we proved the four inequalities of Theorem 2.8 for $n = 1$. In this subsection we will assume that $k \geq 2$ and that the four inequalities $(\mathcal{A}_n)$, $(\mathcal{B}_n)$, $(\mathcal{C}_n)$, $(\mathcal{D}_n)$ of Theorem 2.8 hold for $1 \leq n < k$. We will prove that they then also hold for $n = k$. For this purpose, we will need to show that the integrals involving X_n and Y_n in the inequalities (5.4) and (5.5) are finite under this induction hypothesis. As in Section 6.1, we will initially assume that $A(t)$ is smooth over $(0, T) \times M$.

Lemma 6.8. *If in Theorem 2.8 the inequalities $(\mathcal{A}_n)$, $(\mathcal{B}_n)$, $(\mathcal{C}_n)$, $(\mathcal{D}_n)$ hold for $1 \leq n < k$ then*

$$\int_0^t s^{2k-\frac{1}{2}} X_k(s) ds \leq \bar{C}_{k1}(t) \quad \text{and} \tag{6.14}$$

$$\sup_{0<t<T} t^{2k+\frac{1}{2}} X_k(t) \leq \bar{C}_{k1}(t) \tag{6.15}$$

for some standard dominating function $\bar{C}_{k1}$.

Proof. For the proof of (6.14) it suffices to show that

$$\int_0^t s^{2k-\frac{1}{2}} \Big(\|P_k(s)\|_2^2 + 2\kappa^2 \|\hat{Q}_{k+1}(s)\|_{6/5}^2 + \|R_k(s)\|_2^2 \Big) ds \leq \tilde{C}_{k1}(t) \tag{6.16}$$

by virtue of the definition (4.24) for X_k. In view of the inequalities (6.8), (6.12) with $n = k$ and (6.11) with $n = k + 1$, we need only show that

$$\int_0^t s^{2k-\frac{1}{2}} \|A^{(i)}(s)\|_2 \|A^{(i)}(s)\|_6 \Big(\|A^{(k-i)}(s)\|_6^2 + \|B^{(k-i)}(s)\|_2^2 \Big) ds \leq \tilde{C}_{k1}(t)$$

for $1 \leq i \leq k-1$ and for some standard dominating functions $\tilde{C}_{k1}$. But for $1 \leq i \leq k-1$, the inductive hypotheses $\mathcal{A}_i$ and $\mathcal{D}_i$ of Theorem 2.8 hold. Hence

$$\begin{aligned} & s^{2k-\frac{1}{2}} \|A^{(i)}(s)\|_2 \, \|A^{(i)}(s)\|_6 \, \Big\{ \|A^{(k-i)}(s)\|_6^2 + \|B^{(k-i)}(s)\|_2^2 \Big\} \\ & \quad = (s^{i-\frac{1}{4}} \|A^{(i)}(s)\|_2) \, (s^{i+\frac{1}{4}} \|A^{(i)}(s)\|_6) \\ & \qquad \cdot \Big\{ s^{2(k-i)-\frac{1}{2}} \, \|A^{(k-i)}(s)\|_6^2 + s^{2(k-i)-\frac{1}{2}} \, \|B^{(k-i)}(s)\|_2^2 \Big\} \\ & \quad \leq \sqrt{C_{i1}(t)} \sqrt{C_{i4}(t)} \, \Big\{ s^{2(k-i)-\frac{1}{2}} \, \|A^{(k-i)}(s)\|_6^2 + s^{2(k-i)-\frac{1}{2}} \, \|B^{(k-i)}(s)\|_2^2 \Big\} . \end{aligned} \tag{6.17}$$

The factor in braces is integrable over $(0, t)$ by the inductive hypotheses $\mathcal{B}_{k-i}$ and $\mathcal{A}_{k-i}$ of Theorem 2.8, since $k - i < k$. This proves (6.14).

For the proof of (6.15) multiply the last line of (6.17) by s and set $s = t$ to find

$$\begin{aligned} & t^{2k+\frac{1}{2}} \|A^{(i)}(t)\|_2 \, \|A^{(i)}(t)\|_6 \, \Big\{ \|A^{(k-i)}(t)\|_6^2 + \|B^{(k-i)}(t)\|_2^2 \Big\} \\ & \quad \leq \sqrt{C_{i1}(t)} \sqrt{C_{i4}(t)} \, \Big\{ t^{2(k-i)+\frac{1}{2}} \, \|A^{(k-i)}(t)\|_6^2 + t^{2(k-i)+\frac{1}{2}} \, \|B^{(k-i)}(t)\|_2^2 \Big\} \\ & \quad \leq \sqrt{C_{i1}(t)} \sqrt{C_{i4}(t)} \, \big\{ C_{(k-i)4}(t) + C_{(k-i)3}(t) \big\} \end{aligned}$$

where we have used the inductive hypothesis $\mathcal{D}_{k-i}$ of Theorem 2.8 for the first summand in braces, and the inductive hypothesis $\mathcal{C}_{k-i}$ of Theorem 2.8, for the second term. These hold because $k - i < k$. Using the inequalities (6.8), (6.11) and (6.12) as before, we conclude that

$$t^{2k+\frac{1}{2}} \Big(\|P_k(t)\|_2^2 + 2\kappa^2 \|\hat{Q}_{k+1}(t)\|_{6/5}^2 + \|R_k(t)\|_2^2 \Big) \leq \tilde{C}_{k1}(t). \tag{6.18}$$

This completes the proof of Lemma 6.8. □

Proposition 6.9 *($\mathcal{A}_k$ holds). Let $A(t)$ be a smooth solution to the Yang–Mills heat equation over $(0, T) \times M$ with finite action and satisfying either (2.5) or (2.6) when $M \neq \mathbb{R}^3$. Assume that ($\mathcal{A}_n$), ($\mathcal{B}_n$), ($\mathcal{C}_n$), ($\mathcal{D}_n$) hold for $1 \leq n < k$. Then there exists a standard dominating function C_{k1} such that $\mathcal{A}_k$ holds.*

Proof. Take $n = k$ in (5.4) to find

$$\begin{aligned} & t^{2k-\frac{1}{2}} \|A^{(k)}\|_2^2 + \int_0^t s^{2k-\frac{1}{2}} \|B^{(k)}(s)\|_2^2 ds \\ & \quad \leq \Big\{ (2k - \frac{1}{2}) \int_0^t s^{2k-\frac{3}{2}} \|A^{(k)}(s)\|_2^2 ds + \int_0^t s^{2k-\frac{1}{2}} X_k(s) ds \Big\} e^{\psi_k(t)} \\ & \quad \leq \Big\{ (2k - \frac{1}{2}) \, C_{(k-1)3}(t) + \bar{C}_{k1}(t) \Big\} e^{\psi_k(t)} \end{aligned}$$

where we have used the inductive hypothesis $\mathcal{C}_{k-1}$ to bound the first term on the right, and Lemma 6.8 to bound the second term. Using (5.13) it follows that there is a standard dominating function C_{k1} for which $\mathcal{A}_k$ holds. □

Proposition 6.10 *($\mathcal{B}_k$ holds). Let $A(t)$ be as in Proposition 6.9. If in Theorem 2.8, $(\mathcal{A}_n)$, $(\mathcal{B}_n)$, $(\mathcal{C}_n)$, $(\mathcal{D}_n)$ hold for $n < k$ then $\mathcal{B}_k$ holds for some standard dominating function C_{k2}.*

Proof. From (4.9) with n replaced by $k-1$ we find

$$\kappa^{-2}\|B^{(k-1)}(t)\|_6^2 \le \|S_{k-1}\|_2^2 + 2\|Q_k\|_2^2 + 2\|A^{(k)}\|_2^2 + \lambda(t)\|B^{(k-1)}\|_2^2. \tag{6.19}$$

To prove boundedness of the first term in $\mathcal{B}_k$ it suffices therefore to show that

$$t^{2k-\frac{1}{2}}\Big(\|S_{k-1}\|_2^2 + 2\|Q_k\|_2^2 + 2\|A^{(k)}\|_2^2 + \lambda(t)\|B^{(k-1)}\|_2^2\Big) \le \tilde{C}_{k2}(t). \tag{6.20}$$

Concerning the second term in (6.20), observe that, for $1 \le i < k$, there holds

$$\begin{aligned} &t^{2k-\frac{1}{2}}\|A^{(i)}(t)\|_6^2\|B^{(k-1-i)}(t)\|_2\|B^{(k-1-i)}(t)\|_6 \\ &\quad = \Big(t^{2i+\frac{1}{2}}\|A^{(i)}(t)\|_6^2\Big)\Big(t^{k-i-\frac{3}{4}}\|B^{(k-1-i)}(t)\|_2\Big)\Big(t^{k-i-\frac{1}{4}}\|B^{(k-1-i)}(t)\|_6\Big) \\ &\quad \le C_{i4}(t)\sqrt{C_{(k-1-i)3}(t)\,C_{(k-i)2}(t)}, \end{aligned}$$

where we have used the inductive assumption D_i with $i < k$ in the first factor, the inductive assumption $\mathcal{C}_{k-1-i}$ with $k-1-i < k$ in the second factor, and the inductive assumption $\mathcal{B}_{k-i}$ with $k-i < k$ in the third factor. It follows from (6.10) that $t^{2k-\frac{1}{2}}\|Q_k(t)\|_2^2$ is bounded on $(0,T)$.

By (6.13) the first sum in $\|S_{k-1}(t)\|_2^2$ has similar bounds as the terms in $\|Q_k(t)\|_2^2$ since $\|\,[A^{(i)} \wedge B^{(k-1-i)}]\,\|_2^2 \le c^2\|A^{(i)}\|_6^2\|B^{(k-1-i)}\|_2\|B^{(k-1-i)}\|_6$, just as in the proof of (6.10). Therefore we need only address the terms of the form $\|\,[A^{(i)}(t) \wedge P_{k-1-i}(t)]\,\|_2^2$ in (6.13) for $1 \le i \le k-2$. Replace n by $k-1-i$ in the definition (3.1) to find

$$P_{k-1-i}(s) = \sum_{j=1}^{k-i-2} c_{(k-1-i)j}[A^{(j)}(s) \wedge A^{(k-1-i-j)}(s)]$$

In view of (6.13) it suffices to show that

$$t^{2k-\frac{1}{2}}\|\,[A^{(i)}(t) \wedge [A^{(j)}(t) \wedge A^{(k-1-i-j)}(t)]\,]\,\|_2^2$$

is bounded on $(0,T)$ for $1 \le i \le k-2$ and $1 \le j \le k-i-2$. But

$$\begin{aligned} &t^{2k-\frac{1}{2}}\|\,|A^{(i)}(t)|\,|A^{(j)}(t)|\,|A^{(k-1-i-j)}(t)|\,\|_2^2 \\ &\quad \le \Big(t^{2i+\frac{1}{2}}\|A^{(i)}(t)\|_6^2\Big)\Big(t^{2j+\frac{1}{2}}\|A^{(j)}(t)\|_6^2\Big)\Big(t^{2k-2-2i-2j+\frac{1}{2}}\|A^{(k-1-i-j)}(t)\|_6^2\Big) \\ &\quad \le C_{i4}(t)\,C_{j4}(t)\,C_{(k-1-i-j)4}(t) \end{aligned}$$

by the induction hypothesis $(\mathcal{D}_n)$ for various values of $n < k$, since $i, j \le k-2$ and $k-1-i-j \le k-3$. This gives us the boundedness of the first term in (6.20).

For the third term in (6.20), we use the inequality $\mathcal{A}_k$ of Theorem 2.8, which has already been proven in Proposition 6.9, to find

$$t^{2k-\frac{1}{2}}\|A^{(k)}(t)\|_2^2 \le C_{k1}(t).$$

Finally,

$$t^{2k-\frac{1}{2}}\lambda(t)\|B^{(k-1)}(t)\|_2^2 = \Big(t\lambda(t)\Big)\Big(t^{2(k-1)+\frac{1}{2}}\|B^{(k-1)}(t)\|_2^2\Big),$$

which is a product of a bounded function, in accordance with (5.11) and another bounded function, in accordance with the induction hypothesis $\mathcal{C}_{k-1}$. Their product is bounded by a standard dominating function by the usual argument.

We now turn our attention to the integral term of $\mathcal{B}_k$. We need to prove that

$$\int_0^t s^{2k-\frac{1}{2}}\|A^{(k)}(s)\|_6^2 ds \leq \tilde{C}_{k2}(t) \tag{6.21}$$

for some standard dominating function $\tilde{C}_{k2}$. By the inequality (4.7) it suffices to find $\tilde{C}_{k2}$ such that

$$\int_0^t s^{2k-\frac{1}{2}}\Big(\lambda(s)\|A^{(k)}(s)\|_2^2 + 2\|B^{(k)}(s)\|_2^2 + \|R_k(s)\|_2^2 + 2\|P_k(s)\|_2^2\Big)ds \leq \tilde{C}_{k2}(t).$$

Now

$$\begin{aligned}\int_0^t s^{2k-\frac{1}{2}}\lambda(s)\|A^{(k)}(s)\|_2^2 ds &= \int_0^t \Big(s\lambda(s)\Big)s^{2k-\frac{3}{2}}\|A^{(k)}(s)\|_2^2 ds \\ &\leq t\lambda(t)\, C_{(k-1)3}(t)\end{aligned}$$

by the inductive hypothesis $\mathcal{C}_{k-1}$. Moreover $\int_0^t s^{2k-\frac{1}{2}}\|B^{(k)}(s)\|_2^2 ds \leq C_{k1}(t)$ by $\mathcal{A}_k$, whose validity has been proven in Proposition 6.9. The remaining integrals are finite by (6.16). This proves $\mathcal{B}_k$ holds. □

Lemma 6.11. *If in Theorem 2.8 the inequalities $(\mathcal{A}_n)$, $(\mathcal{B}_n)$, $(\mathcal{C}_n)$, $(\mathcal{D}_n)$ hold for $1 \leq n < k$ then*

$$\int_0^t s^{2k+\frac{1}{2}} Y_k(s) ds \leq \bar{C}_{k3}(t)$$

for some standard dominating function $\bar{C}_{k3}$ with Y_k defined by (4.25).

Proof. In view of the definition (4.25) of Y_k we need to show that there is a standard dominating function $\bar{C}_{k3}$ such that

$$\int_0^t s^{2k+\frac{1}{2}}\Big(\|Q_{k+1}(s)\|_2^2 + \kappa^2\|P_{k+1}(s)\|_{6/5}^2 + \|S_k(s)\|_2^2\Big)ds \leq \bar{C}_{k3}(t). \tag{6.22}$$

By the bounds (6.10), (6.9), (6.13) it suffices to show that each of the following integrals is bounded by a standard dominating function.

$$\int_0^t s^{2k+\frac{1}{2}}\|A^{(i)}(s)\|_6^2\|B^{(k-i)}(s)\|_2\|B^{(k-i)}(s)\|_6 ds, \quad 1 \leq i \leq k \tag{6.23}$$

$$\int_0^t s^{2k+\frac{1}{2}}\|A^{(i)}(s)\|_6\|A^{(i)}(s)\|_2\|A^{(k+1-i)}(s)\|_2^2 ds, \quad 1 \leq i \leq k \tag{6.24}$$

$$\int_0^t s^{2k+\frac{1}{2}} \| [A^{(i)}(s) \wedge B^{(k-i)}(s)\|_2^2 ds, \quad 1 \le i \le k \tag{6.25}$$

$$\int_0^t s^{2k+\frac{1}{2}} \| [A^{(i)}(s) \wedge P_{k-i}(s)] \|_2^2 ds, \quad 1 \le i < k. \tag{6.26}$$

For (6.23) observe that

$$\begin{aligned} & s^{2k+\frac{1}{2}} \|A^{(i)}(s)\|_6^2 \|B^{(k-i)}(s)\|_2 \|B^{(k-i)}(s)\|_6 \\ & \quad = (s^{2i-\frac{1}{2}} \|A^{(i)}(s)\|_6^2)\, (s^{k-i+\frac{1}{4}} \|B^{(k-i)}(s)\|_2)\, (s^{k-i+\frac{3}{4}} \|B^{(k-i)}(s)\|_6) \\ & \quad \le (s^{2i-\frac{1}{2}} \|A^{(i)}(s)\|_6^2) \sqrt{C_{(k-i)3}(t)\ C_{(k-i+1)2}(t)} \end{aligned}$$

by the inductive hypothesis $\mathcal{C}_{k-i}$ of Theorem 2.8, since $k-i<k$, and by $\mathcal{B}_{k-i+1}$, because $k-i+1 \le k$ for $i = 1, \dots, k$. The integrability of the first factor is also assured by $\mathcal{B}_i$, which holds for $i \le k$ by Proposition 6.10. Therefore the integral in (6.23) is finite for $1 \le i \le k$.

The integral in (6.24) can be estimated as follows.

$$\begin{aligned} & s^{2k+\frac{1}{2}} \|A^{(i)}(s)\|_6 \|A^{(i)}(s)\|_2 \|A^{(k+1-i)}(s)\|_2^2 \\ & \quad = (s^{i-\frac{1}{4}} \|A^{(i)}(s)\|_6)\, (s^{i-\frac{3}{4}} \|A^{(i)}(s)\|_2)\, (s^{2k-2i+\frac{3}{2}} \|A^{(k+1-i)}(s)\|_2^2) \\ & \quad \le (s^{i-\frac{1}{4}} \|A^{(i)}(s)\|_6)\, (s^{i-\frac{3}{4}} \|A^{(i)}(s)\|_2)\, C_{(k-i+1)1}(t) \end{aligned}$$

by $\mathcal{A}_{k-i+1}$, which holds for $i = 1, \dots, k$ by the hypotheses of this lemma and Proposition 6.9. Therefore, by Hölder's inequality for the time integral, we have

$$\begin{aligned} & \int_0^t s^{2k+\frac{1}{2}} \|A^{(i)}(s)\|_6 \|A^{(i)}(s)\|_2 \|A^{(k+1-i)}(s)\|_2^2 ds \\ & \quad \le \left(\int_0^t s^{2i-\frac{1}{2}} \|A^{(i)}(s)\|_6^2\, ds \right)^{\frac{1}{2}} \left(\int_0^t s^{2i-\frac{3}{2}} \|A^{(i)}(s)\|_2^2\, ds \right)^{\frac{1}{2}} C_{(k-i+1)1}(t) \\ & \quad \le \sqrt{C_{i2}(t)\, C_{(i-1)3}(t)}\, C_{(k-i+1)1}(t) \end{aligned}$$

wherein the first integral is dominated by $\mathcal{B}_i$ of Theorem 2.8, which is valid for all $i \le k$ by the hypotheses of this lemma and Proposition 6.10, and the second integral is dominated in accordance with $\mathcal{C}_{i-1}$, which is valid for $i \le k$ because $i-1 < k$. Hence the integral in (6.24) is bounded by a standard dominating function.

The integral in (6.25) can be treated exactly as the integral in (6.23), since our use of Hölder's inequality in deriving (6.10) applies equally well here.

To estimate the integral in (6.26) replace n by $k-i$ in the definition (3.1) to find

$$P_{k-i}(s) = \sum_{j=1}^{k-i-1} c_{(k-i)j} [A^{(j)}(s) \wedge A^{(k-i-j)}(s)]$$

From this we see that it suffices to show that

$$\int_0^t s^{2k+\frac{1}{2}} \| [A^{(i)}(s) \wedge [A^{(j)}(s) \wedge A^{(k-i-j)}(s)]\,] \|_2^2 ds \le \tilde{C}_{k3}(t)$$

for some standard dominating function $\tilde{C}_{k3}$ for $1 \leq i < k$ and $1 \leq j \leq k-i-1$. But, by Hölder's inequality,

$$\begin{aligned} & s^{2k+\frac{1}{2}} \| \, |A^{(i)}(s)| \, |A^{(j)}(s)| \, |A^{(k-i-j)}(s)| \, \|_2^2 \\ & \quad \leq \left(s^{2i-\frac{1}{2}} \|A^{(i)}(s)\|_6^2 \right) \left(s^{2j+\frac{1}{2}} \|A^{(j)}(s)\|_6^2 \right) \left(s^{2k-2i-2j+\frac{1}{2}} \|A^{(k-i-j)}(s)\|_6^2 \right) \\ & \quad \leq \left(s^{2i-\frac{1}{2}} \|A^{(i)}(s)\|_6^2 \right) C_{j4}(t) C_{(k-i-j)4}(t) \end{aligned}$$

by $\mathcal{D}_j$ and $\mathcal{D}_{k-i-j}$, both of which hold in accordance with the hypotheses of this lemma, since both subscripts are strictly less than k. The integrability of the first factor follows from $\mathcal{B}_i$, which holds because $i < k$.

This completes the proof of Lemma 6.11. □

Proposition 6.12 *($\mathcal{C}_k$ holds). Let $A(t)$ be as in Proposition 6.9. If in Theorem 2.8, $(\mathcal{A}_n)$, $(\mathcal{B}_n)$, $(\mathcal{C}_n)$, $(\mathcal{D}_n)$ hold for $1 \leq n < k$ then $\mathcal{C}_k$ holds.*

Proof. Setting $n = k$ in (5.5), we see that it suffices to show that

$$\left\{ (2k + \frac{1}{2}) \int_0^t s^{2k-\frac{1}{2}} \|B^{(k)}(s)\|_2^2 ds + \int_0^t s^{2k+\frac{1}{2}} Y_k(s) ds \right\} e^{\psi(t)} \leq C_{k3}(t)$$

for some standard dominating function C_{k3}. By Lemma 6.11 the second integral is bounded by $\bar{C}_{k3}(t)$. The first integral is also bounded by a standard bounding function since $\mathcal{A}_k$ holds, as was proven in Proposition 6.9. This proves that $\mathcal{C}_k$ holds in view of (5.12). □

Proposition 6.13 *($\mathcal{D}_k$ holds). Let $A(t)$ be as in Proposition 6.9. If $(\mathcal{A}_n)$, $(\mathcal{B}_n)$, $(\mathcal{C}_n)$, $(\mathcal{D}_n)$ hold for $1 \leq n < k$ then $\mathcal{D}_k$ holds.*

Proof. From (4.7) and (4.9) we find

$$\kappa^{-2} t^{2k+\frac{1}{2}} \|A^{(k)}(t)\|_6^2 + \kappa^{-2} \int_0^t s^{2k+1} \|B^{(k)}(s)\|_6^2 ds$$

$$\leq t^{2k+\frac{1}{2}} \left(\lambda(t) \|A^{(k)}(t)\|_2^2 + 2\|B^{(k)}(t)\|_2^2 + \|R_k(t)\|_2^2 + 2\|P_k(t)\|_2^2 \right) \tag{6.27}$$

$$+ \int_0^t s^{2k+\frac{1}{2}} \left(\lambda(s) \|B^{(k)}(s)\|_2^2 + 2\|A^{(k+1)}(s)\|_2^2 + \|S_k(s)\|_2^2 + 2\|Q_{k+1}(s)\|_2^2 \right) ds \tag{6.28}$$

In order to prove $\mathcal{D}_k$ we need to show that this sum is bounded by a standard bounding function. Concerning the line (6.27), the identity

$$t^{2k+\frac{1}{2}} \lambda(t) \|A^{(k)}(t)\|_2^2 = \left(t\lambda(t) \right) \left(t^{2k-\frac{1}{2}} \|A^{(k)}(t)\|_2^2 \right),$$

together with (5.11) and the already established bound $\mathcal{A}_k$ show this term is bounded by a standard dominating function. Moreover,

$$t^{2k+\frac{1}{2}} \|B^{(k)}(t)\|_2^2 \leq C_{k3}(t)$$

by $\mathcal{C}_k$, which has already been proven in Proposition 6.12. Further, $t^{2k+\frac{1}{2}}(\|R_k(t)\|_2^2 + 2\|P_k(t)\|_2^2)$ is suitably dominated, as has been shown in (6.18). Thus the line (6.27) is bounded by a standard dominating function.

With respect to the line (6.28) observe that

$$\int_0^t s^{2k+\frac{1}{2}}\lambda(s)\|B^{(k)}(s)\|_2^2 ds = \int_0^t \Big(s\lambda(s)\Big)\Big(s^{2k-\frac{1}{2}}\|B^{(k)}(s)\|_2^2\Big)ds \leq t\lambda(t)\, C_{k1}(t)$$

by $\mathcal{A}_k$, which has been proved in Proposition 6.9. Furthermore

$$\int_0^t s^{2k+\frac{1}{2}}\|A^{(k+1)}(s)\|_2^2 ds \leq C_{k3}(t)$$

by $\mathcal{C}_k$, which has been proved in Proposition 6.12. Thus in view of (6.28) we need only show that

$$\int_0^t s^{2k+\frac{1}{2}}\Big(\|S_k(s)\|_2^2 + 2\|Q_{k+1}(s)\|_2^2\Big)ds \leq \bar{C}_{k4}(t)$$

for some standard dominating function $\bar{C}_{k4}$. But this has already been shown in (6.22). This concludes the proof of Proposition 6.13. □

Proof of Theorem 2.8. All of the inequalities $(\mathcal{A}_n)$–$(\mathcal{D}_n)$ have been established by induction under the assumption that the solution $A(\cdot)$ has finite action and under the technical assumption that the solution is smooth. The first assumption is necessary because the bounds are given in terms of the action $\rho(t)$. The second assumption is needed to justify the computations. Here the additional hypothesis that $\|A_0\|_{H_{1/2}}$ is small enters because it ensures, as in Theorem 2.6, that there is a gauge function $g_0 \in \mathcal{G}_{3/2}$ which transforms the solution to a smooth solution. Having such a gauge function enables the following argument. Let $A(\cdot)$ denote the finite action solution specified in Theorem 2.8 and let $\hat{A}(t) = A(t)^{g_0} \equiv g_0^{-1}A(t)g_0 + g_0^{-1}dg_0$ be the smooth solution obtained, as in Theorem 2.6 and satisfying either (2.5) or (2.6) when $M \neq \mathbb{R}^3$. Since g_0 is time independent we have $(d/dt)^n\hat{A}(t) = g_0^{-1}A^{(n)}(t)g_0$ for $n \geq 1$ (but not for $n = 0$). Similarly, $(d/dt)^n\hat{B}(t) = g_0^{-1}B^{(n)}g_0$ for $n \geq 0$. Hence $\|(d/dt)^n\hat{A}(t)\|_2 = \|A^{(n)}(t)\|_2$ and $\|(d/dt)^n\hat{B}(t)\|_2 = \|B^{(n)}(t)\|_2$. Moreover $\partial_j^{\hat{A}(t)}(g^{-1}\omega g) = g^{-1}(\partial_j^{A(t)}\omega)g$ for any $\mathfrak{k}$ valued p-form ω on M. Taking e.g. $\omega = A^{(n)}(t)$, this shows that $\|\partial_j^{\hat{A}(t)}(d/dt)^n\hat{A}(t)\|_2 = \|\partial_j^{A(t)}A^{(n)}(t)\|_2$ and in particular $\|(d/dt)^n\hat{A}(t)\|_{H_1^{\hat{A}(t)}} = \|A^{(n)}(t)\|_{H_1^{A(t)}}$. (In Notation 2.9 we have suppressed t in the subscripts.) In this way all of the quantities estimated in Theorem 2.8 and Corollary 2.10 can be estimated instead for the same gauge invariant functionals of the smooth solution $\hat{A}(\cdot)$. Since all of the dominating functions C_{nj} are also gauge invariant, the inequalities of Theorem 2.8 and Corollary 2.10, having been established for $\hat{A}$ apply equally to A. This completes the proof of Theorem 2.8. □

Proof of Corollary 2.10. In the proofs of the inequalities $(\mathcal{B}_n)$ and $(\mathcal{D}_n)$ of Theorem 2.8 we used the bounds (4.6), (4.7), (4.8) and (4.9) to bound the L^6 norms of $A^{(n)}(t)$ and $B^{(n)}(t)$. But the same right hand sides also bound the H_1^A norms of these quantities. Thus if in (6.19) one replaces $\kappa^{-2}\|B^{(k-1)}(t)\|_6^2$ by $(1/2)\|B^{(k-1)}(t)\|_{H_1^A}^2$ and one replaces in (6.21) $\|A^{(k)}(s)\|_6^2$ by $(\kappa^2/2)\|A^{(k)}(s)\|_{H_1^A}^2$ then the proof of Proposition 6.10 proves that the inequality $(\mathcal{E}_n)$ of Corollary 2.10 holds for $n = k$. Similarly, one needs only replace the L^6 norms on the left hand side of (6.27) plus (6.28) by H_1^A norms to find correct inequalities which yield the inequality $(\mathcal{F}_n)$ of Corollary 2.10 with $n = k$, via the proof of Proposition 6.13. No further induction is needed because the L^2 and L^6 bounds needed in these two proofs have already been proven in Theorem 2.8. □

Remark 6.14 *(Pointwise bounds).* In [2] we derived pointwise bounds on $A'(t,x)$ and $B(t,x)$ by a Neumann domination technique in the case $A(0)$ was in $H_1(M)$. In that instance we took M to be a compact three manifold with convex boundary. Pointwise bounds for $B(t,x)$ were derived in [6] in the case $A(0)$ is in $H_{1/2}(M)$ and M is either all of $\mathbb{R}^3$ or is a bounded convex set in $\mathbb{R}^3$ with smooth boundary. It seems likely that these techniques could yield pointwise bounds on all of the derivatives $A^{(n)}(t,x)$ and $B^{(n)}(t,x)$ with the help of the results in this paper if $M = \mathbb{R}^3$. We have not pursued this. But if $M \neq \mathbb{R}^3$ then some steps in the Neumann domination technique break down because of boundary value problems for derivatives of B. For example if one wishes to obtain pointwise bounds on $B'(t,x)$ when the solution $A(\cdot)$ satisfies Dirichlet boundary conditions then the technique requires that $(d_A^* B')_{tan} = 0$. But this boundary condition need not hold when the solution $A(\cdot)$ merely satisfies Dirichlet boundary conditions. Moreover failure to obtain the behavior of $\|B'(t)\|_{L^\infty(M)}$ as $t \downarrow 0$ leads, in turn, to failure to obtain pointwise bounds on A'', even though the required boundary conditions hold for A''.

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