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Linear electron stability for a bi-kinetic sheath model

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ABSTRACT

We establish the linear stability of an electron equilibrium for an electrostatic and collisionless plasma in interaction with a wall. The equilibrium we focus on is called in plasma physics a Debye sheath. Specifically, we consider a two species (ions and electrons) Vlasov–Poisson–Ampère system in a bounded and one dimensional geometry. The interaction between the plasma and the wall is modeled by original boundary conditions: On the one hand, ions are absorbed by the wall while electrons are partially re-emitted. On the other hand, the electric field at the wall is induced by the accumulation of charged particles at the wall. These boundary conditions ensure the compatibility with the Maxwell–Ampère equation. A global existence, uniqueness and stability result for the linearized system is proven. The main difficulty lies in the fact that (due to the absorbing boundary conditions) the equilibrium is a discontinuous function of the particle energy, which results in a linearized system that contains a degenerate transport equation at the border.

1. Introduction

1.1. A kinetic model of plasma-wall dynamics: the Vlasov–Poisson–Ampère system

We consider an electrostatic and collisionless plasma consisting of one species of ions and electrons. We use a kinetic approach to model this plasma. To this purpose, we set $\Omega = (0,1) \times \mathbb{R}$ and denote by $(x,v) \in \overline{\Omega} = [0,1] \times \mathbb{R}$ the phase space variable, where x is the particle position and v the particle velocity. This work is concerned with the linear stability of an equilibrium for the two species Vlasov–Poisson system in the presence of spatial boundaries

$$\begin{cases} \partial_t f_i + v\partial_x f_i + E\partial_v f_i = 0 & \text{in} \quad (0,+\infty) \times \Omega, \\ \partial_t f_e + v\partial_x f_e - \frac{1}{\mu} E\partial_v f_e = 0 & \text{in} \quad (0,+\infty) \times \Omega, \end{cases} \tag{1}$$

$$-\varepsilon^2 \partial_{xx}\phi = n_i - n_e \quad \text{in} \quad (0,+\infty) \times (0,1), \tag{2}$$

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where $f_i : [0, +\infty) \times \overline{\Omega} \to \mathbb{R}^+$, $f_e : [0, +\infty) \times \overline{\Omega} \to \mathbb{R}^+$ are the ions and electrons distribution functions in the phase-space and $\phi : [0, +\infty) \times [0, 1] \to \mathbb{R}$ is the electric potential. Here the physical parameters μ and ε stand respectively for the mass ratio between electrons and ions, and a normalized Debye length that will be (for simplicity) in the sequel taken equal to 1. We also denote

$$E = -\partial_x \phi, \quad n_i = \int_{\mathbb{R}} f_i dv, \quad n_e = \int_{\mathbb{R}} f_e dv,$$

the electric field, the ion density and the electron density. The boundary conditions are given for all $t \in (0, +\infty)$ by

$$\begin{cases} f_i(t, 0, v > 0) = f_i^{in}(v), \quad f_i(t, 1, v < 0) = 0, \\ f_e(t, 0, v > 0) = f_e^{in}(v), \quad f_e(t, 1, v < 0) = \alpha f_e(t, 1, -v), \end{cases} \tag{3}$$

$$\phi(t, 0) = 0, \quad E(t, 1) = E^*(t, f_i, f_e), \tag{4}$$

where f_i^{in} and f_e^{in} denote two given incoming particles velocity distributions that are time independent. The scalar parameter α belongs to the interval $[0, 1)$ and represents the rate of re-emitted electrons in the domain $(0, 1)$. The scalar $E^*(t, f_i, f_e)$ depends on the unknown (f_i, f_e, ϕ) via the formula

$$E^*(t, f_i, f_e) = \left(E_w^0 - \int_0^t \int_{\mathbb{R}} \left(f_i(\tau, 1, v) - f_e(\tau, 1, v)\right) v dv d\tau \right). \tag{5}$$

Up to our knowledge, the theory of existence and uniqueness for such a initial boundary value problem (1)–(4) has not been treated in full details. In the one dimensional case, there is the result of Bostan [5] which establishes the existence and uniqueness of the mild solution to a Vlasov–Poisson system in which the boundary conditions do not depend on the solution itself. Still in the one dimensional case, the work of Guo [10] studies the dynamic of a plane diode. Also, the result of BenAbdallah [3] shows the existence of weak solutions for the Vlasov–Poisson system in dimension greater than or equal to one, but once again the boundaries are not coupled to the solution itself. The existence and uniqueness of weak solutions in the half-space with specular reflection condition is obtained in [11]. The case of partially absorbing boundary condition is treated in [9]. The existence of a stationary solution to the system (1)–(4) was proven in [1], the stationary solution corresponds to the Debye sheath (see [17] for further physical details). This work can be considered as a continuation of the work [1], and a first step in the study of the wellposedness of the non-linear system (1)–(4).

1.2. Physical interpretation of the model

The bi-kinetic model (1)–(4) models the dynamical transition between the core of a plasma and a wall (see for instance [13]). The region of plasma we consider is modeled by the line segment $[0, 1]$ where $x = 0$ is assumed to be somewhere in the bulk plasma and thus a source of particles. The sources here are modeled by the injection of particles that are mathematically encoded in the given distributions f_i^{in} and f_e^{in}. The wall at $x = 1$ is supposed to be metallic and partially absorbing: it absorbs completely the ions and re-emits a fraction α of the electrons. The parameter α can be seen as a constitutive parameter of the wall. The accumulation of charged particles at the wall induces an electric-field that is given by (5) (the number E_w^0 denotes the initial electric field at the wall). The boundary condition of the electric-field at the wall can be formally re-written as $\partial_t E(t, 1) + j(t, 1) = 0$ where $j(t, 1) := \int_{\mathbb{R}} (f_i(t, 1, v) - f_e(t, 1, v)) v dv$ is the current density at the wall. At a formal level, one easily verifies that this boundary condition ensures the compatibility of the solutions to (1)–(4) with the Vlasov–Ampère system made of equations (1), (3) with the Maxwell–Ampère equation

$$\partial_t E = -j \text{ in } (0,+\infty)\times[0,1], \quad j := \int_{\mathbb{R}} (f_i - f_e) v dv, \tag{6}$$

provided the initial data satisfy the Poisson equation

$$\partial_x E(0,.) = \int_{\mathbb{R}} f_i(0,.,.) - f_e(0,.,.) dv.$$

Because of this equivalence, we shall rather consider the Vlasov–Ampère system (1), (3) and (6).

1.3. Statement of the main result

The mathematical and physical aspect we investigate in this work is the linear stability of the Debye sheath for the Vlasov–Ampère model (1), (3) and (6). The rigorous mathematical construction of such an equilibrium was obtained for the first time for the model (1), (3) together with (6) in [1]. The main result of this work can be roughly summarized as follows: if the initial data that is a small perturbation of the sheath equilibrium, then the solution of the system (1), (3) and (6) remains close to the sheath equilibrium for all times provided the ions are frozen. To make things more precise at this stage, we need supplementary materials. Let us denote by $(f_i^\infty, f_e^\infty, \phi^\infty)$ the sheath equilibrium to (1), (3) and (6). Let us write the solution of (1), (3) and (6) as the sum of the sheath equilibrium plus an interior perturbation that affects only the electrons and the electrostatic field, namely: $(f_i, f_e, \phi) = (f_i^\infty, f_e^\infty + \tilde{f}_e, \phi^\infty + \tilde{\phi})$. The formal linearization yields the linearized Vlasov–Ampère system (after dropping the $\tilde{}$)

$$\text{(LVA):} \begin{cases} \partial_t f_e + D f_e = E \partial_v f_e^\infty, & \text{in } (0,+\infty)\times\Omega \\ \partial_t E = \displaystyle\int_{\mathbb{R}} f_e v dv, & \text{in } (0,+\infty)\times[0,1] \\ f_e(t,0,v>0) = 0, \quad f_e(t,1,v<0) = \alpha f_e(t,1,-v) & \text{in } (0,+\infty) \end{cases}$$

where D denotes the first order linear differential operator defined formally by

$$D := v\partial_x - E^\infty \partial_v \text{ with } E^\infty = -(\phi^\infty)' \text{ being the equilibrium electric field.} \tag{7}$$

Because the equilibrium density f_e^∞ is discontinuous across the curve of equation $v = v_e(x)$ where the function v_e is defined for all $x \in [0,1]$ by

$$v_e(x) := -\sqrt{2\left(\phi^\infty(x) - \phi_w\right)}, \tag{8}$$

its velocity derivative takes the form

$$\partial_v f_e^\infty = [f_e^\infty]\delta^{v_e} - v f_e^\infty, \tag{9}$$

where δ^{v_e} is a Dirac distribution supported on the curve of equation $v = v_e(x)$, namely:

$$\langle \delta^{v_e}, \varphi \rangle = \int_0^1 \varphi(x, v_e(x)) dx \quad \forall \varphi \in \mathcal{D}(\Omega) \tag{10}$$

and where

$$[f_e^\infty] := \lim_{v \to v_e(x)^+} f_e^\infty(x,v) - \lim_{v \to v_e(x)^-} f_e^\infty(x,v) \tag{11}$$

denotes the constant jump of f_e^∞ across the characteristic curve $v = v_e(x)$. As a consequence, it is natural to look for solutions to (LVA) that decompose into a singular part plus a regular one, as

$$f_e = \eta_e(t,x)\delta^{v_e} + g_e(t,x,v) \tag{12}$$

with η_e and g_e two functions. The main result can be in rough terms stated as follows: *For any initial data (f_e^0, E^0) with f_e^0 of the form $f_e^0(x,v) = \eta^0(x)\delta^{v_e} + g_e^0(x,v)$ where η_0 and g_e^0 are two functions, there exists a couple of functions (η_e, g_e) and an electric field E such that if we define $f_e(t,x,v) = \eta_e(t,x)\delta^{v_e} + g_e(t,x,v)$, then the couple (f_e, E) is solution to the linearized Vlasov–Ampère system with initial condition $f_e(t = 0,x,v) = f_e^0(x,v)$ and $E(t = 0,x) = E^0(x)$. Moreover the non-negative energy functional defined for $t \geq 0$ by*

$$\mathcal{E}(t) = \frac{1}{2}\left(\int_0^1 \frac{\eta_e^2(t,x)|v_e(x)|dx}{[f_e^\infty]}dx + \int_\Omega \frac{g_e(t,x,v)^2}{f_e^\infty(x,v)}dxdv + \int_0^1 E(t,x)^2dx\right) \tag{13}$$

is non-increasing.

1.4. The mathematical approach and its difficulty

The main difficulty in the analysis lies in the fact that for $\alpha \in [0,1)$ the electron sheath equilibrium is a discontinuous function of the particle energy. This is due to the absorption of particles with positive velocities at the wall ($x = 1$), which creates a discontinuity that propagates into the domain along the characteristic curve $v = v_e(x)$. Thus the linearization of the Vlasov–Ampère system (1), (3), (6) around the sheath equilibrium $(f_i^\infty, f_e^\infty, E^\infty)$ with no perturbation on the ions, yields a linear system whose solution still denoted (f_e, E) is singular. Assuming a decomposition of the form

$$f_e(t,x,v) = \eta_e(t,x)\delta^{v_e} + g_e(t,x,v)$$

where η_e and g_e are two functions and δ^{v_e} is a Dirac mass supported by the characteristic curve of equation $v = v_e(x)$ yields another linear system on (η_e, g_e, E). Making a suitable change of variable leads to the system (VAL). The system (VAL) contains a degenerate transport equation because the given velocity field v_e vanishes at $x = 1$ and its derivate $\partial_x v_e$ is not essentially bounded as the Diperna–Lions theory of transport equation [8] requires. This difficulty is overcame using the fact that the velocity field is only weakly degenerated, it vanishes at the border like a square-root. This allows us to prove a Hardy–Poincaré type inequality. Ultimately by applying the Hille–Yosida theorem, we show that the linearized system (VAL) is well-posed and that the energy of the system is non-increasing.

1.5. Previous works

Stability issues for Vlasov–Poisson systems in bounded geometry are of a tremendous importance for practical applications, be it in the modeling of laboratory plasmas, or in the design of numerical methods. Unfortunately, and despite its worthy interest, it seems that it has not been studied in full details. Stability analysis for such a Vlasov–Poisson system has already been performed in the absence of spatial boundaries, that is, either in all space or in a periodical setting [16,14,12]. However, it seems that in the presence of spatial boundaries, the question of stability has not been extensively addressed. Up to our knowledge, it is only very recently, with the work of Nguyen and Strauss in [15] that the question was raised. The authors considered the Vlasov–Maxwell system in a cylindrical geometry and assumed an equilibrium that is a C^1 function of the particle energy and momentum. Additionally, they assumed as in the previous works [16,12] that the equilibrium, on its support, is monotone in the particle energy, which seem to be a key ingredient to prove stability results. Only the recent work of Ben-Artzi in [4] deals with equilibria that are not monotone in the particle energy. The author gives sufficient conditions for an instability, but on the other hand the analysis is performed in an unbounded geometry.

1.6. Organization of the paper

The plan of this paper is as follows. In section 2, we derive the system (VAL) and give a precise statement of the main result, including the functional spaces, and the notions of solutions we consider. In section 3, we state the Hardy–Poincaré type inequality and give some technical lemmas needed to prove the main result. We eventually prove the main result. In Appendix A, we briefly discuss the regularity of the solution.

2. The stability result

2.1. Description of the sheath equilibrium

The equilibrium $(f_i^\infty, f_e^\infty, \phi^\infty)$ is associated with an electron boundary condition which is a Maxwellian, namely it takes the form

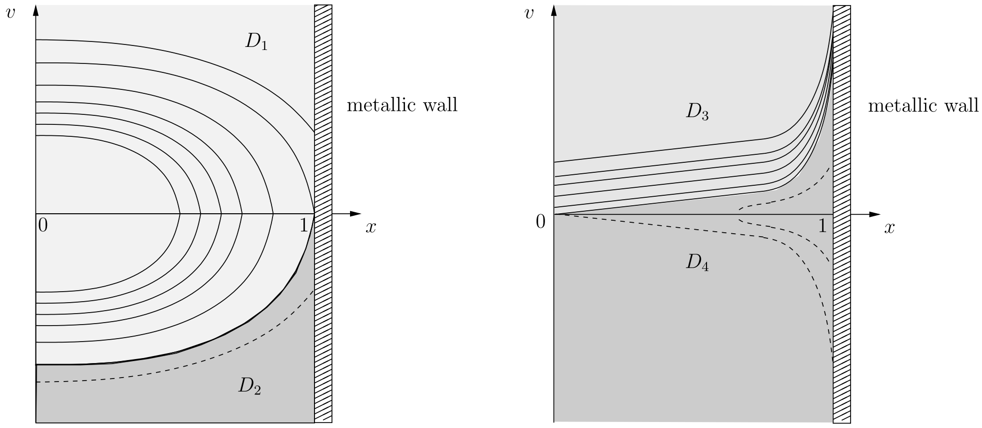
$$f_e^{in}(v) = n_0\sqrt{\frac{2}{\pi}}e^{-\frac{v^2}{2}} \text{ for } v > 0, \tag{14}$$

where $n_0 > 0$ is an electron reference density. The sheath equilibrium is a stationary and weak solution of the Vlasov–Ampère system (1), (3), (6). It belongs to the space $(L^1 \cap L^\infty(\Omega))^2 \times C^2[0,1]$ and enjoys the following properties:

1. For all $x \in [0,1]$, $(\phi^\infty)''(x) \le 0$, $E^\infty(x) := -(\phi^\infty)'(x) \ge 0$, $\phi^\infty(0) = 0$, $\phi^\infty(1) =: \phi_w$ and $E^\infty(1) =: E_w^\infty > 0$.
2. $f_i^\infty(x,v) = \begin{cases} f_i^{in}\left(\sqrt{v^2 + 2\phi^\infty(x)}\right) \text{ for } (x,v) \text{ s.t. } v > \sqrt{-2\phi^\infty(x)} \\ 0 \text{ elsewhere.} \end{cases}$
3. $f_e^\infty(x,v) = n_0\sqrt{\frac{2}{\pi}} \begin{cases} e^{-\frac{v^2}{2}}e^{\phi^\infty(x)} \text{ for } (x,v) \text{ s.t. } v \ge v_e(x) \\ \alpha e^{-\frac{v^2}{2}}e^{\phi^\infty(x)} \text{ for } (x,v) \text{ s.t. } v < v_e(x). \end{cases}$
4. $\int_{\mathbb{R}} f_i^\infty(x,v)v\,dv = \int_{\mathbb{R}} f_e^\infty(x,v)v\,dv$ for all $x \in [0,1]$.

Such an equilibrium is proven to exist in [1] under the necessary and sufficient condition that the following kinetic Bohm criterion

$$\frac{\displaystyle\int_{\mathbb{R}^+} \frac{f_i^{in}(v)}{v^2}dv}{\displaystyle\int_{\mathbb{R}^+} f_i^{in}(v)dv} < \frac{\left(\sqrt{2\pi} + (1-\alpha)\displaystyle\int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}}}{v^2}dv\right)}{\left(\sqrt{2\pi} - (1-\alpha)\displaystyle\int_{\sqrt{-2\phi_w}}^{+\infty} e^{-\frac{v^2}{2}}dv\right)}$$



(a) Schematic characteristic electron trajectories associated with the potential ϕ^∞. The electron equilibrium f_e^∞ is discontinuous across the curve $v = v_e(x)$ (bold curve).

(b) Schematic characteristic ions trajectories associated with the potential ϕ^∞. The ion equilibrium f_i^∞ vanishes outside D_3 (lighter gray).

Fig. 1. Ions and electrons phase space.

holds true for $f_i^{in} \in L^1 \cap L^\infty(\mathbb{R}^+)$. The physical meaning of this inequality is that there cannot be too many ions particles entering the domain with low velocities. It is instructive to have a representation of the ions and electrons characteristics in the phase space (see Fig. 1). We see that for the electrons, the equilibrium is a truncated Maxwellian distribution. Especially, it is discontinuous across the characteristic curve $S := \{(x, v_e(x)) \,/\, x \in [0,1]\}$ where the function v_e is defined in (8). To be more precise we have the following:

Lemma 2.1.

a) $f_e^\infty \in C^2(\Omega \setminus S)$.

b)

$$\partial_v f_e^\infty = [f_e^\infty]\delta^{v_e} - v f_e^\infty,$$

where $[f_e^\infty] = n_0\sqrt{\frac{2}{\pi}}(1-\alpha)e^{\phi_w} > 0$ *is the jump* f_e^∞ *across the characteristic curve* S *defined in* (11) *and* δ^{v_e} *is the Dirac mass supported by the function* v_e *defined in* (8).

Proof. a) It is straightforward from its definition that f_e^∞ belongs to $C^2(\Omega \setminus S)$. b) It follows from an integration by parts. Indeed, for all $\varphi \in \mathcal{D}(\Omega)$ we have

$$\begin{aligned}
\langle \partial_v f_e^\infty, \varphi \rangle &= -\langle f_e^\infty, \partial_v \varphi \rangle \\
&= -\int_0^1 \int_{\mathbb{R}} f_e^\infty(x,v)\partial_v\varphi(x,v)dvdx \\
&= -\int_0^1 \int_{v \geq v_e(x)} f_e^\infty(x,v)\partial_v\varphi(x,v)dvdx
\end{aligned}$$

$$
\begin{aligned}
&-\int_0^1 \int_{v<v_e(x)} f_e^\infty(x,v)\partial_v\varphi(x,v)dvdx \\
&= -n_0\sqrt{\frac{2}{\pi}}\int_0^1 \left[e^{-\frac{v^2}{2}}e^{\phi^\infty(x)}\varphi(x,v)\right]_{v_e(x)}^{+\infty} dx \\
&\quad - n_0\sqrt{\frac{2}{\pi}}\int_0^1 \int_{v\geq v_e(x)} ve^{-\frac{v^2}{2}}e^{\phi^\infty(x)}\varphi(x,v)dvdx \\
&\quad - n_0\sqrt{\frac{2}{\pi}}\int_0^1 \left[\alpha e^{-\frac{v^2}{2}}e^{\phi^\infty(x)}\varphi(x,v)\right]_{-\infty}^{v_e(x)} dx \\
&\quad - n_0\sqrt{\frac{2}{\pi}}\int_0^1 \int_{v<v_e(x)} \alpha ve^{-\frac{v^2}{2}}e^{\phi^\infty(x)}\varphi(x,v)dvdx.
\end{aligned}
$$

It is then easy to conclude. □

Lemma 2.2. *The function $x \in [0,1] \mapsto v_e(x) := -\sqrt{2(\phi^\infty(x)-\phi_w)}$ has the following properties:*

a) $v_e \in C^0([0,1]) \cap C^2([0,1))$.
b) *For all $x \in [0,1)$, $v_e(x) < 0$, $v_e(1) = 0$, $v_e(x) \underset{x\to 1^-}{\sim} -\nu\sqrt{1-x}$ where $\nu = \sqrt{2E_w^\infty}$.*
c) *For all $x \in (0,1)$, $v_e'(x)v_e(x) + E^\infty(x) = 0$.*
d) *$v_e \in W^{1,1}(0,1)$ and $\frac{1}{v_e} \in L^1(0,1)$.*

Proof. We skip the proof because it is essentially a consequence of the regularity of ϕ^∞. □

This lemma is important because it makes precise the regularity of v_e which must be known insofar as it will appear as the velocity field of a one dimensional transport equation of the form $(\partial_t + v_e\partial_x)u = s$. The fact that $\frac{1}{v_e} \in L^1(0,1)$ implies that the characteristics, namely the solutions of the ode

$$\dot{x}(t) = v_e(x(t))$$

have a finite incoming time into the domain $[0,1]$. We give a sketch of the characteristics in the plan (x,t).

2.2. Derivation of the linearized system (VAL)

We now derive the linear system (VAL). Let us give the definition of solution we consider for the linearized Vlasov–Ampère system.

Definition 2.3. Assume f_e^0 a measure on $\overline{\Omega}$ and $E^0 \in L^2(0,1)$. Let f_e be a measure on $[0,+\infty)\times\overline{\Omega}$ and $E \in W^{1,\infty}_{loc}\left([0,+\infty); L^2(0,1)\right)$. We say that (f_e, E) is a weak solution of (LVA) system iff:

a) For all $\varphi \in C_c^1([0,+\infty)\times\overline{\Omega})$ such that $\varphi(t,0,v\leq 0) = 0$ and $\varphi(t,1,v\geq 0) = -\alpha\varphi(t,1,-v)$

$$
\begin{aligned}
&-\langle f_e, \partial_t\varphi + D\varphi\rangle_{[0,+\infty)\times\overline{\Omega}} \\
&= \langle f_e^0, \varphi(0,.)\rangle_{\overline{\Omega}}
\end{aligned}
$$

$$+ [f_e^\infty] \int_0^{+\infty} \int_0^1 E(t,x)\varphi(t,x,v_e(x))dxdt$$
$$- \int_0^{+\infty} \int_0^1 \int_{\mathbb{R}} E(t,x) v f_e^\infty(x,v)\varphi(t,x,v)dxdvdt.$$

b) The current density $(t,x) \in (0,+\infty) \times (0,1) \mapsto j_e(t,x) := \langle f_e(t,x,.), v\rangle_{\mathbb{R}}$ belongs to $L^\infty_{loc}([0,+\infty); L^2(0,1))$, $\partial_t E(t,x) = j_e(t,x)$ for a.e. $(t,x) \in (0,+\infty) \times (0,1)$ and $E(t=0,x) = E^0(x)$ for a.e. $x \in (0,1)$.

One readily checks that f_e satisfies

$$\partial_t f_e + Df_e = [f_e^\infty]E\delta^{v_e} - Evf_e^\infty \text{ in } \mathcal{D}'((0,+\infty)\times\Omega). \tag{15}$$

Since the right hand side of (15) is the sum of Dirac mass supported by v_e plus a function, a natural idea is to look for $f_e \in \mathcal{D}'((0,+\infty)\times\Omega)$ under the following form

$$f_e = \eta_e(t,x)\delta^{v_e} + g_e(t,x,v)$$

where $\eta_e : [0,+\infty) \times [0,1] \to \mathbb{R}$ and $g_e : [0,+\infty) \times \overline{\Omega}$ are a priori two regular functions. Note that for all $\varphi \in \mathcal{D}((0+\infty)\times\Omega)$

$$\langle f_e, \varphi\rangle = \int_0^{+\infty}\int_0^1 \eta_e(t,x)\varphi(t,x,v_e(x))dxdt + \int_0^{+\infty}\int_\Omega g_e(t,x,v)\varphi(t,x,v)dxdvdt.$$

Proposition 2.4. *Let (f_e, E) be a weak solution of (LVA) with f_e of the form (12) where $\eta_e \in L^1_{loc}([0,+\infty)\times(0,1))$ and $g_e \in L^1_{loc}([0,+\infty)\times\Omega)$. Then f_e is solution of the Vlasov equation (15) if and only if η_e and g_e satisfy*

$$\partial_t\eta_e + \partial_x(v_e\eta_e) = [f_e^\infty]E \text{ in } \mathcal{D}'((0,+\infty)\times(0,1)), \tag{16}$$
$$\partial_t g_e + Dg_e = -Evf_e^\infty \text{ in } \mathcal{D}'((0,+\infty)\times\Omega). \tag{17}$$

Proof. We omit the proof because it follows from standard calculations. □

Let us now introduce the change of unknown

$$w_e(t,x) := v_e(x)\eta_e(t,x), \quad h_e(t,x,v) := \begin{cases} \frac{g_e(t,x,v)}{\sqrt{f_e^\infty(x,v)}} \text{ if } f_e^\infty(x,v) \neq 0 \\ g_e(t,x,v) \text{ if } f_e^\infty(x,v) = 0. \end{cases} \tag{18}$$

We recall that $Df_e^\infty = 0$ in $\mathcal{D}'(\Omega)$ and we note that by definition f_e^∞ never vanishes when $\alpha \in (0,1)$. One also checks by a straightforward calculation that the couple (η_e, g_e) is a solution of (16)–(17) if and only if (w_e, h_e) is a solution to

$$\partial_t w_e + v_e\partial_x w_e = [f_e^\infty]v_e E \text{ in } \mathcal{D}'((0,+\infty)\times(0,1)), \tag{19}$$
$$\partial_t h_e + Dh_e = -Ev\sqrt{f_e^\infty} \text{ in } \mathcal{D}'((0,+\infty)\times\Omega). \tag{20}$$

The transport equation (19) has a negative on $[0,1)$ and vanishing at $x=1$ velocity field v_e defined by (8). It is a priori not clear whether a boundary condition is needed for this equation. Having a closer look at

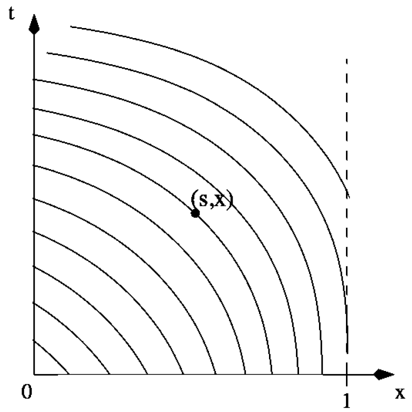


Fig. 2. Characteristics curves in the plan (x, t). The backward in time characteristics cross $x = 1$ in finite time.

the characteristics that are draw in Fig. 2, we see that a boundary condition at $x = 1$ is necessary if one wants to solve the equation on the domain $(0, +\infty) \times (0, 1)$. Physically speaking, the number $w_e(t, x)$ is the current of electrons at the position $x \in [0, 1]$ carried by the singular part of f_e notably

$$w_e(t, x) = \eta_e(t, x)\langle \delta_{v=v_e(x)}, v\rangle$$

where $\delta_{v=v_e(x)}$ denotes the classical Dirac mass supported at $v = v_e(x)$. For $\eta_e \underset{x\to 1^-}{=} o(\frac{1}{v_e})$ this yields $w_e(t, 1) = 0$. We thus impose the homogeneous Dirichlet boundary condition $w_e(t, 1) = 0$. The boundary conditions for the transport equation (20) are easily derived from the original boundary condition on f_e, they write for all $t > 0$:

$$h_e(t, 0, v > 0) = 0, \quad h_e(t, 1, v < 0) = \sqrt{\alpha} h_e(t, 1, -v). \tag{21}$$

The initial boundary value problem then writes: given $w_e^0 : [0, 1] \to \mathbb{R}$, $h_e^0 : \overline{\Omega} \to \mathbb{R}$ and $E^0 \in [0, 1] \to \mathbb{R}$, find $w_e : [0, +\infty) \times [0, 1] \to \mathbb{R}$, $h_e : [0, +\infty) \times \overline{\Omega} \to \mathbb{R}$ and $E : [0, +\infty) \times [0, 1] \to \mathbb{R}$ solution to

$$(\mathrm{VAL}) : \begin{cases} \partial_t w_e + v_e \partial_x w_e = [f_e^\infty] v_e E \text{ in } (0, +\infty) \times (0, 1), \\ \partial_t h_e + D h_e = -E v \sqrt{f_e^\infty} \text{ in } (0, +\infty) \times \Omega, \\ \partial_t E = w_e + \int_{\mathbb{R}} v \sqrt{f_e^\infty} h_e dv \text{ in } (0, +\infty) \times [0, 1], \\ h_e(t > 0, 0, v > 0) = 0,\ h_e(t > 0, 1, v < 0) = \sqrt{\alpha} h_e(t, 1, -v), \\ w_e(t > 0, 1) = 0, \end{cases}$$

satisfying $w_e(t = 0, .) = w_e^0$, $h_e(t = 0, ., .) = h_e^0$, $E(t = 0, .) = E^0$.

2.3. The main result

We are now in position to state precisely our main result. First let us begin with defining what linear stability stands for in this work.

Definition 2.5. Let G and H be two Hilbert spaces equipped respectively with the norm $\|.\|_H$, $\|.\|_G$ and with the continuous embedding $G \hookrightarrow H$. We say that the equilibrium $(f_i^\infty, f_e^\infty, \phi^\infty)$ is linearly stable by interior electron perturbation of the form (12) iff:

a) For all $(w_e^0, h_e^0, E^0) \in G$ the system (VAL) admits a unique strong solution $(w_e, h_e, E) \in C^0([0, +\infty); G) \cap C^1([0, +\infty); H)$.
b) For all $\epsilon > 0$, there is $\eta > 0$ such that $\|(w_e^0, h_e^0, E^0)\|_H < \eta \Rightarrow \|(w_e, h_e, E)\|_H < \epsilon\ \forall t \geq 0$.

The Hilbert spaces to be considered are the following

$$G := H^1_{|v_e|^{\frac{1}{2}},0}(0,1) \times W^2_{0,\alpha}(\Omega) \times L^2(0,1), \quad H := L^2_{|v_e|^{-\frac{1}{2}}}(0,1) \times L^2(\Omega) \times L^2(0,1)$$

where the spaces $L^2_{|v_e|^{-\frac{1}{2}}}(0,1)$, $H^1_{|v_e|^{\frac{1}{2}},0}(0,1)$ and $W^2_{0,\alpha}(\Omega)$ are defined in section 3.

Theorem 2.6. *The equilibrium $(f_i^\infty, f_e^\infty, \phi^\infty)$ is linearly stable in the sense of Definition 2.5. More precisely,*

a) *For all $(w_e^0, h_e^0, E^0) \in G$ there exists a unique strong solution $(w_e, h_e, E) \in C^0([0,+\infty); G) \cap C^1([0,+\infty); H)$ to (VAL).*
b) *The energy defined in (13) is well-defined and for all times $t \geq 0$, re-writes*

$$\mathcal{E}(t) = \frac{1}{2}\left(\int_0^1 \frac{w_e(t,x)^2}{[f_e^\infty]|v_e(x)|}dx + \int_\Omega h_e(t,x,v)^2 dxdv + \int_0^1 E(t,x)^2 dx\right)$$

and is non-increasing.

One of the key ingredient in the proof of Theorem 2.6 is the following energy identity.

Proposition 2.7 *(Energy dissipation). Strong solutions to (VAL) satisfy*

$$\frac{d}{dt}\mathcal{E}(t) = -\frac{1}{2}\left(\frac{1}{[f_e^\infty]}w_e^2(t,0) + (1-\alpha)\int_{\mathbb{R}^+} vh_e^2(t,1,v)dv - \int_{\mathbb{R}^-} vh_e^2(t,0,v)dv\right) \leq 0.$$

In particular, for all $0 \leq t \leq t'$, $0 \leq \mathcal{E}(t') \leq \mathcal{E}(t)$.

Proof. Let $(w_e, h_e, E) \in C^0([0,+\infty); G) \cap C^1([0,+\infty); H)$ a solution to (VAL). We set $\mathcal{E}_{w_e}(t) := \frac{1}{[f_e^\infty]}\int_0^1 \frac{w_e(t,x)^2}{|v_e(x)|}dx$, $\mathcal{E}_{h_e}(t) := \int_\Omega h_e(t,x,v)^2 dxdv$ and $\mathcal{E}_{pot}(t) := \int_0^1 E(t,x)^2 dx$. We compute each terms separately. Because of the regularity of (w_e, h_e, E) we can differentiate under the integral sign.

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_{w_e}(t) &= \frac{2}{[f_e^\infty]}\int_0^1 \frac{\partial_t w_e(t,x) w_e(t,x)}{|v_e(x)|}dx \\ &= \frac{2}{[f_e^\infty]}\int_0^1 \left([f_e^\infty]E(t,x)v_e(x) - v_e(x)\partial_x w_e(t,x)\right)\frac{w_e(t,x)}{|v_e(x)|}dx.\end{aligned}$$

Once again because $w_e(t,.) \in H^1_{|v_e|^{\frac{1}{2}},0}(0,1) \subset L^2_{|v_e|^{-\frac{1}{2}}}(0,1)$ the second integral is convergent

$$\left|\int_0^1 (v_e(x)\partial_x w_e(t,x))\frac{w_e(t,x)}{|v_e(x)|}dx\right| < +\infty,$$

and we deduce

$$\frac{d}{dt}\mathcal{E}_{w_e}(t) = -2\int_0^1 E(t,x)w_e(x)dx - \frac{1}{[f_e^\infty]}w_e^2(t,0).$$

We now compute the energy part associated with h_e. Using the boundary conditions and an integration by parts we get

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_{h_e}(t) &= 2\int_{\Omega} \partial_t h_e(t,x,v)h_e(t,x,v)dxdv \\ &= 2\int_{\Omega} \left(-E(t,x)v\sqrt{f_e^\infty(x,v)} - Df_e^\infty(x,v)\right) h_e(t,x,v)dxdv \\ &= -2\int_{\Omega} E(t,x)v\sqrt{f_e^\infty(x,v)}h_e(t,x,v)dxdv \\ &\quad - (1-\alpha)\int_{\mathbb{R}^+} vh_e^2(t,1,v)dv + \int_{\mathbb{R}^-} vh_e^2(t,0,v)dv.\end{aligned}$$

Note that since $h_e(t,.,.) \in W^2_{0,\alpha}(\Omega)$ the boundary terms make sense. We lastly turn to the electric part of the energy.

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_{pot}(t) &= 2\int_0^1 \partial_t E(t,x)E(t,x)dx \\ &= 2\int_0^1 \left(w_e(t,x) + \int_{\mathbb{R}} v\sqrt{f_e^\infty(x,v)}h_e(t,x,v)dv\right) E(t,x)dx.\end{aligned}$$

Gathering all terms together enables to get the desired identity. In particular, we deduce that $t \in [0,+\infty) \mapsto \mathcal{E}(t)$ is non-increasing. Hence $\mathcal{E}(t) \leq \mathcal{E}(t')$ for all $0 \leq t \leq t'$. □

3. Functional spaces, technical lemmas and proof of the main result

In this section, we define the functional framework that is part of the main result of Theorem 2.6. We eventually prove the main result by showing that the Hille–Yosida theorem applies.

3.1. Functional spaces

We define the following spaces

$$H^1_{|v_e|^{\frac{1}{2}}}(0,1) := \{u \in L^2(0,1) \text{ s.t. } \sqrt{|v_e|}u' \in L^2(0,1)\}$$

where v_e is the function defined by (8) and note that it is such that $\frac{1}{v_e} \in L^1(0,1)$. It is a Hilbert space endowed with the inner product

$$(u,v)_{H^1_{|v_e|^{\frac{1}{2}}}(0,1)} := \int_0^1 u(x)v(x) + |v_e(x)|u'(x)v'(x)dx \quad \forall (u,v) \in H^1_{|v_e|^{\frac{1}{2}}}(0,1)^2.$$

Moreover, we can prove the imbedding $H^1_{|v_e|^{\frac{1}{2}}}(0,1) \hookrightarrow C^0[0,1]$ so that we can define the space

$$H^1_{|v_e|^{\frac{1}{2}},0}(0,1) := \{u \in H^1_{|v_e|^{\frac{1}{2}}}(0,1) \text{ s.t. } u(1) = 0\}.$$

We also defined the weighted Lebesgue space

$$\mathcal{L}^2_{|v_e|^{-\frac{1}{2}}}(0,1) := \{u : (0,1) \to \mathbb{R} \text{ measurable s.t. } \int_0^1 \frac{u^2(x)}{|v_e(x)|}dx < +\infty\}$$

and the quotient space

$$L^2_{|v_e|^{-\frac{1}{2}}}(0,1) := \mathcal{L}^2_{|v_e|^{-\frac{1}{2}}}(0,1)/\mathcal{R}$$

where $\mathcal{R}$ denotes the usual equivalence relation of almost everywhere equality for the Lebesgue measure. The space $L^2_{|v_e|^{-\frac{1}{2}}}(0,1)$ is an Hilbert space endowed with the inner product

$$(u,v)_{L^2_{|v_e|^{-\frac{1}{2}}}(0,1)} := \int_0^1 \frac{u(x)}{\sqrt{|v_e(x)|}} \frac{v(x)}{\sqrt{|v_e(x)|}}dx \quad \forall (u,v) \in L^2_{|v_e|^{-\frac{1}{2}}}(0,1)^2.$$

An important tool in this work is the following inequality.

Lemma 3.1 *(Hardy–Poincaré type inequality). For all $\varphi \in H^1_{|v_e|^{\frac{1}{2}},0}(0,1)$,*

$$\int_0^1 \frac{\varphi(x)^2}{|v_e(x)|}dx \leq \|\frac{1}{v_e}\|^2_{L^1(0,1)} \int_0^1 |v_e(x)|\varphi'(x)^2 dx.$$

Consequently, $H^1_{|v_e|^{\frac{1}{2}},0}(0,1) \hookrightarrow L^2_{|v_e|^{-\frac{1}{2}}}(0,1)$.

Proof. We prove the inequality for $\varphi \in C_c^\infty(0,1)$. Let $\delta > 0$, one has for all $x \in [0, 1-\delta]$

$$\varphi(x) - \varphi(1-\delta) = -\int_x^{1-\delta} \frac{\varphi'(s)\sqrt{|v_e(s)|}}{\sqrt{|v_e(s)|}}ds.$$

Using the Cauchy–Schwarz inequality yields

$$\begin{aligned}|\varphi(x) - \varphi(1-\delta)| &\leq \|\sqrt{|v_e|}\varphi'\|_{L^2(x,1-\delta)}\|\frac{1}{\sqrt{|v_e|}}\|_{L^2(x,1-\delta)} \\ &\leq \left(\int_0^1 |v_e(x)|\varphi'(x)^2 dx\right)^{\frac{1}{2}} \|\frac{1}{v_e}\|^{\frac{1}{2}}_{L^1(0,1)}.\end{aligned}$$

Taking the limit as $\delta \to 0^+$ yields for all $x \in [0,1]$.

$$|\varphi(x)| \leq \left(\int_0^1 |v_e(x)|\varphi'(x)^2 dx\right)^{\frac{1}{2}} \|\frac{1}{v_e}\|^{\frac{1}{2}}_{L^1(0,1)}.$$

Therefore for all $x \in [0,1)$ we have

$$\frac{\varphi(x)^2}{|v_e(x)|} \leq \frac{1}{|v_e(x)|}\left(\int_0^1 |v_e(x)|\varphi'(x)^2 dx\right) \|\frac{1}{v_e}\|_{L^1(0,1)}.$$

One has therefore for $\delta > 0$

$$\int_0^{1-\delta} \frac{\varphi(x)^2}{|v_e(x)|}dx \leq \int_0^{1-\delta} \frac{1}{|v_e(x)|}dx \left(\int_0^1 |v_e(x)|\varphi'(x)^2 dx\right) \|\frac{1}{v_e}\|_{L^1(0,1)}$$
$$\leq \|\frac{1}{v_e}\|^2_{L^1(0,1)} \int_0^1 |v_e(x)|\varphi'(x)^2 dx.$$

Taking the limit at $\delta \to 0^+$ yields the desired inequality. The result extends to functions of the space $H^1_{|v_e|^{\frac{1}{2}},0}(0,1)$ by using the density of $C_c^\infty(0,1)$ in $H^1_{|v_e|^{\frac{1}{2}},0}(0,1)$ (see [7] Lemma 2.6 for the proof of density). $\square$

Thanks to Lemma 3.1 we can define on $H^1_{|v_e|^{\frac{1}{2}},0}(0,1)$ the following norm

$$\|u\|_{H^1_{|v_e|^{\frac{1}{2}},0}(0,1)} := \left(\int_0^1 |v_e(x)|u'(x)^2 dx\right)^{\frac{1}{2}} \qquad \forall u \in H^1_{|v_e|^{\frac{1}{2}},0}(0,1).$$

It is an equivalent norm to the $H^1_{|v_e|^{\frac{1}{2}}}(0,1)$-norm. We will also need the following density result.

Lemma 3.2. *$H^1_{|v_e|^{\frac{1}{2}},0}(0,1)$ is dense in $L^2_{|v_e|^{-\frac{1}{2}}}(0,1)$.*

Proof. Let $u \in L^2_{|v_e|^{-\frac{1}{2}}}(0,1)$. Let us build a sequence $(u_n)_{n\in\mathbb{N}} \subset H^1_{|v_e|^{\frac{1}{2}},0}(0,1)$ such that $\|u_n - u\|_{L^2_{|v_e|^{-\frac{1}{2}}}(0,1)} \underset{n\to+\infty}{\to} 0$. We firstly remark that since $\frac{u}{\sqrt{-v_e}} \in L^2(0,1)$ and because $H^1_0(0,1)$ is dense in $L^2(0,1)$ there is a sequence $(\tilde{u}_n)_{n\in\mathbb{N}} \subset H^1_0(0,1)$ such that $\int_0^1 \left|\tilde{u}_n - \frac{u}{\sqrt{-v_e}}\right|^2 (x)dx \underset{n\to+\infty}{\to} 0$. Let us then define for all $n \in \mathbb{N}$ $u_n := \tilde{u}_n\sqrt{-v_e}$. To conclude the proof, it suffices to check that for all $n \in \mathbb{N}$, $u_n \in H^1_{|v_e|^{\frac{1}{2}},0}(0,1)$. Because $v_e \in C^0[0,1]$ one readily verifies that $u_n \in C^0[0,1] \cap L^2(0,1)$. Let us now compute the derivative. One has in $\mathcal{D}'(0,1)$

$$u_n' = \tilde{u}_n'\sqrt{-v_e} - \tilde{u}_n\frac{v_e'}{2\sqrt{-v_e}} \Rightarrow \sqrt{-v_e}u_n' = -\tilde{u}_n v_e - \frac{\tilde{u}_n v_e'}{2}.$$

One has $\tilde{u}_n v_e \in L^2(0,1)$, it therefore suffices to prove that $\tilde{u}_n v_e' \in L^2(0,1)$. Using Lemma 2.2 d) we have for all $x \in (0,1)$

$$\tilde{u}_n(x)v_e'(x) = -\tilde{u}_n(x)\frac{E^\infty(x)}{v_e(x)}.$$

Since $E^\infty \in C^0[0,1]$, it suffices to show that $\frac{\tilde{u}_n}{v_e} \in L^2(0,1)$. Still using Lemma 2.2 b), we have $\frac{\tilde{u}_n^2(x)}{v_e^2(x)} \underset{x\to1^-}{\sim} \nu^2\frac{\tilde{u}_n(x)^2}{(1-x)}$. But $\tilde{u}_n \in H^1_0(0,1)$ and a classical Hardy inequality (see [6, p. 147] for instance) enables us to conclude that $\frac{\tilde{u}_n}{v_e} \in L^2(0,1)$. $\square$

We now define

$$W^2(\Omega) := \left\{h \in L^2(\Omega) \text{ s.t. } Dh \in L^2(\Omega)\right\}.$$

Following [2], for a function $h \in W^2(\Omega)$ we can define its restriction to $\Sigma_- := \{0\} \times (0, +\infty) \cup \{1\} \times (-\infty, 0)$ and $\Sigma_+ := \{0\} \times (-\infty, 0) \cup \{1\} \times (0, +\infty)$. Moreover, $h_{|\Sigma_-}$ and $h_{|\Sigma_+}$ belongs respectively to $L^2_{loc}(\Sigma_-)$ and $L^2_{loc}(\Sigma_+)$. Thus, we can also define

$$W^2_{0,\alpha}(\Omega) := \left\{h \in W^2(\Omega) \text{ s.t. } h(0, v > 0) = 0 \text{ and } h(1, v < 0) = \sqrt{\alpha} h(1, -v)\right\}.$$

Lemma 3.3. *$W^2_{0,\alpha}(\Omega)$ is dense in $L^2(\Omega)$ for the L^2-norm.*

Proof. It suffices to remark that $C^1_c(\Omega) \subset W^2_{0,\alpha}(\Omega)$. Then we deduce that $\overline{C^1_c(\Omega)} \subset \overline{W^2_{0,\alpha}(\Omega)} \subset L^2(\Omega)$ where $\overline{X}$ denotes the closure of the set X in $L^2(\Omega)$ for the L^2-norm. But, $C^1_c(\Omega)$ is dense in $L^2(\Omega)$ so $\overline{C^1_c(\Omega)} = L^2(\Omega)$ and then $\overline{W^2_{0,\alpha}(\Omega)} = L^2(\Omega)$. $\square$

To finish with this section, we state a Lemma due to Bardos [2] and justify a Green formula.

Lemma 3.4. *$C^\infty_c(\overline{\Omega}) \cap W^2_{0,\alpha}(\Omega)$ is dense in $W^2_{0,\alpha}(\Omega)$ for the norm defined by*

$$\|h\|^2_{W^2_{0,\alpha}} := \|h\|^2_{L^2(\Omega)} + \|Dh\|^2_{L^2(\Omega)} \quad \forall h \in W^2_{0,\alpha}.$$

Proof. See [2]. $\square$

Lemma 3.5 *(Traces integrability). Let $h \in W^2_{0,\alpha}(\Omega)$ then $h(1,.) \in L^2(\mathbb{R}^+, |v|dv)$ and $h(0,.) \in L^2(\mathbb{R}^-, |v|dv)$ and the following Green Formula holds:*

$$\int_{\Omega} Dh(x,v)h(x,v)dxdv = (1-\alpha)\int_{\mathbb{R}^+} \frac{v}{2} h^2(1,v)dv - \int_{\mathbb{R}^-} \frac{v}{2} h^2(0,v)dv.$$

Proof. We argue by density. Let $h \in W^2_{0,\alpha}(\Omega)$ then in virtue of Lemma 3.4 there is $(h_n)_{n\in\mathbb{N}} \subset C^\infty_c(\overline{\Omega}) \cap W^2_{0,\alpha}(\Omega)$ such that $\|h_n - h\|_{W^2_{0,\alpha}} \underset{n\to+\infty}{\to} 0$. Using the boundary conditions and a Green Formula (that is valid for regular functions) we have for all $n \in \mathbb{N}$,

$$\int_{\Omega} Dh_n(x,v)h_n(x,v)dxdv = (1-\alpha)\int_{\mathbb{R}^+} \frac{v}{2} h_n^2(1,v)dv - \int_{\mathbb{R}^-} \frac{v}{2} h_n^2(0,v)dv.$$

By standard arguments, it is easy to see that

$$\int_0^1 \int_{\mathbb{R}} Dh_n(x,v)h_n(x,v)dxdv \underset{n\to+\infty}{\to} \int_0^1 \int_{\mathbb{R}} Dh(x,v)h(x,v)dxdv.$$

Then the sequences $\left(\displaystyle\int_{\mathbb{R}^+} \frac{v}{2} h_n^2(1,v)dv\right)_{n\in\mathbb{N}}$ and $\left(\displaystyle\int_{\mathbb{R}^-} \frac{v}{2} h_n^2(0,v)dv\right)_{n\in\mathbb{N}}$ are bounded and converge (up to an extraction). Lastly, we can show that the trace operators

$$\gamma_0 : h \in C^\infty_c(\overline{\Omega}) \cap W^2_{0,\alpha}(\Omega) \mapsto h(0,.) \in L^2(\mathbb{R}^-, |v|dv),$$
$$\gamma_1 : h \in C^\infty_c(\overline{\Omega}) \cap W^2_{0,\alpha}(\Omega) \mapsto h(1,.) \in L^2(\mathbb{R}^+, |v|dv)$$

extend both continuously to $W^2_{0,\alpha}(\Omega)$. This finally enables us to pass to the limit at both side of the previous equality so that the formula holds. □

Lemma 3.6. *For all $h \in W^2_{0,\alpha}(\Omega)$ and for all $\psi \in W^2(\Omega)$ such that $\psi(0, v < 0) = 0$ and $\psi(1, v > 0) = \sqrt{\alpha}\psi(1, -v)$ a.e. we have,*

$$\int_\Omega Dh(x,v)\psi(x,v)dxdv = -\int_\Omega h(x,v)D\psi(x,v)dxdv.$$

3.2. Proof of the main result

We prove the main result by checking that the Hille–Yosida's Theorem applies (see [6, p. 105] for a precise statement). The Hilbert space $H = L^2_{|v_e|^{-\frac{1}{2}}}(0,1) \times L^2(\Omega) \times L^2(0,1)$ is equipped with the inner product

$$(U_1, U_2)_H := \frac{1}{[f_e^\infty]}(w_1, w_2)_{L^2_{|v_e|^{-\frac{1}{2}}}(0,1)} + (h_1, h_2)_{L^2(\Omega)} + (E_1, E_2)_{L^2(0,1)},$$

for all $U_1 := (w_1, h_1, E_1)$, $U_2 := (w_2, h_2, E_2)$ in H. We introduce the unbounded operator $A : D(A) \subset H \to H$ defined by:

$$\begin{cases} AU := \begin{pmatrix} v_e\partial_x w - [f_e^\infty]v_e E \\ Dh + Ev\sqrt{f_e^\infty} \\ -\left(w + \int_{\mathbb{R}} v\sqrt{f_e^\infty}hdv\right) \end{pmatrix}, \\ \forall U := \begin{pmatrix} w \\ h \\ E \end{pmatrix} \in D(A) = H^1_{|v_e|^{\frac{1}{2}},0}(0,1) \times W^2_{0,\alpha}(\Omega) \times L^2(0,1). \end{cases}$$

We are going to check the assumptions of the Hille–Yosida's Theorem. For precise definitions we refer the reader to Appendix B.

Lemma 3.7. *The unbounded operator $A : D(A) \subset H \to H$ has the following properties:*

a) *It is dissipative, in the sense that $(AU, U)_H \geq 0$.*
b) *$D(A)$ is dense in H.*

Proof. a) We prove that A is dissipative. Let $U := \begin{pmatrix} w \\ h \\ E \end{pmatrix} \in D(A)$. We compute

$$(AU, U)_H = \frac{1}{[f_e^\infty]}\underbrace{(v_e\partial_x w - [f_e^\infty]v_e E, w)_{L^2_{|v_e|^{-\frac{1}{2}}}(0,1)}}_{:=I_1}$$
$$+ \underbrace{\left(Dh + Ev\sqrt{f_e^\infty}, h\right)_{L^2(\Omega)}}_{:=I_2}$$

$$-\underbrace{\left(w+\int_{\mathbb{R}} hv\sqrt{f_e^\infty}dv, E\right)_{L^2(0,1)}}_{:=I_3}.$$

We can compute I_1 and I_2 by an integration by parts so that we obtain:

$$I_1 = \int_0^1 \left(-\partial_x w(x) + [f_e^\infty]E(x)\right) w(x)dx = \frac{w^2(0)}{2} + [f_e^\infty]\int_0^1 E(x)w(x)dx,$$

$$I_2 = \int_0^1\int_{\mathbb{R}} \left(Dh(x,v) + E(x)v\sqrt{f_e^\infty(x,v)}\right) h(x,v)dxdv$$

$$= (1-\alpha)\int_{\mathbb{R}^+} \frac{vh^2(1,v)}{2}dv - \int_{\mathbb{R}^-}\frac{vh^2(0,v)}{2}dv + \int_\Omega E(x)v\sqrt{f_e^\infty(x,v)}h(x,v)dxdv,$$

$$I_3 = \int_0^1 E(x)w(x)dx + \int_\Omega E(x)v\sqrt{f_e^\infty(x,v)}h(x,v)dxdv.$$

Collecting all terms together, we finally deduce

$$(AU,U)_H = \frac{1}{[f_e^\infty]}\frac{w^2(0)}{2} + (1-\alpha)\int_{\mathbb{R}^+}\frac{vh^2(1,v)}{2}dv - \int_{\mathbb{R}^-}\frac{vh^2(0,v)}{2}dv \geq 0.$$

b) The fact that $D(A)$ is dense in H is a consequence of Lemmas 3.2 and 3.3. $\square$

Lemma 3.8. *The unbounded operator $I + A : D(A) \subset H \to H$ is such that $R(I + A) = H$.*

Proof. We are going to apply Proposition Appendix B.2. Of course, $D(A)$ is dense in H as we have already proven. Since H is a Hilbert space, by the Riesz representation theorem, we can identify H with its dual, namely $H' \cong H$ so that the adjoint operator of $I + A$ is the unbounded operator $(I + A)^* : D(A^*) \subset H \to D(A)'$ defined by:

$$\begin{cases} (I+A)^*U := U + A^*U \text{ with } A^*U := \begin{pmatrix} -v_e\partial_x w + [f_e^\infty]v_e E \\ -Dh - Ev\sqrt{f_e^\infty} \\ \left(w + \int_{\mathbb{R}} v\sqrt{f_e^\infty}hdv\right)\end{pmatrix}, \\ \forall U := \begin{pmatrix} w \\ h \\ E\end{pmatrix} \in D(A^*) = \{u \in H^1_{v_e^{\frac{1}{2}}}(0,1) \text{ s.t. } u(0) = 0\}\times \\ \{h \in W^2(\Omega) \text{ s.t. } h(0, v<0) = 0 \text{ and } h(1, v>0) = \sqrt{\alpha}h(1,-v)\} \times L^2(0,1). \end{cases}$$

Of course, $D(A^*)$ is also dense in H. This enables to prove that $I + A$ is closed (see [6, Proposition II.16, p. 28]). Lastly, straightforward integrations by parts allow us to prove that A^* is also dissipative which in the end turns out to be sufficient to prove that

$$((I+A)^*U,U)_H = \|U\|_H^2 + (A^*U,U)_H \geq \|U\|_H^2.$$

Therefore a Cauchy–Schwarz inequality yields $\|(I+A)^*U\|_H \geq \|U\|_H$. Therefore Proposition Appendix B.2 applies, it concludes the proof. □

The proof of the main result is now straightforward. Combining Lemmas 3.7 and 3.8 we can apply the Hille–Yosida theorem. For all $(w_e^0, h_e^0, E^0) \in D(A)$ there is a unique $\begin{pmatrix} w_e \\ h_e \\ E \end{pmatrix} \in C^0([0,+\infty); D(A)) \cap C^1([0,+\infty), H)$ such that

$$\frac{d}{dt}\begin{pmatrix} w_e \\ h_e \\ E \end{pmatrix} + A\begin{pmatrix} w_e \\ h_e \\ E \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

with

$$\begin{pmatrix} w(0,.) \\ h(0,.,.) \\ E(0,.) \end{pmatrix} = \begin{pmatrix} w_e^0 \\ h_e^0 \\ E^0 \end{pmatrix}.$$

The boundary conditions are included in the space $D(A)$ so that they are satisfied by the solution. The solution also satisfies for all times $0 \leq t \leq t'$ the inequality

$$\left\| \begin{pmatrix} w_e(t',.,.) \\ h_e(t',.,.) \\ E(t',.) \end{pmatrix} \right\|_H^2 \leq \left\| \begin{pmatrix} w_e(t,.,.) \\ h_e(t,.,.) \\ E(t,.,.) \end{pmatrix} \right\|_H^2$$

which re-writes in terms of the energy $\mathcal{E}(t') \leq \mathcal{E}(t)$. It also implies the stability with respect to perturbation. Indeed, for any $\epsilon > 0$, it suffices to choose (w_e^0, h_e^0, E^0) such that $\mathcal{E}(0) < \epsilon$ to get that $\mathcal{E}(t) < \epsilon$ for all times $t \geq 0$.

Appendix A. On the regularity of η_e

We want to explain why we have worked on the flux variable w_e rather than on the density number η_e. First notice that one has the following imbedding

$$H^1_{|v_e|^{\frac{1}{2}},0}(0,1) \hookrightarrow C^0[0,1]$$

which implies that $w_e(t,.) \in C^0[0,1]$ for all $t \geq 0$. The boundary condition on w_e therefore makes sense. As far as the electron density η_e is concerned, we now observe that because $\frac{1}{v_e} \in L^1(0,1)$ one has

$$\eta_e \in C^0\left([0,+\infty); C^0[0,1) \cap L^1(0,1)\right) \cap C^1\left([0,+\infty); L^1(0,1)\right),$$

and $\eta_e \underset{x\to 1^-}{=} o(\frac{1}{v_e})$. However, it is not clear whether η_e can be extended by continuity at $x = 1$. In fact, we cannot expect any more integrability on the spatial derivative of η_e and thus the notion of boundary condition at $x = 1$ is not obvious. To illustrate this lack of regularity, a simple calculation shows that

$$v_e \partial_x \eta_e = \partial_x w_e - \frac{v_e'}{v_e} w_e \text{ in } \mathcal{D}'(0,1).$$

The first term at the right hand side of the equality belongs to $L^1(0,1)$ while the second term does not:

$$\forall t \geq 0\, \forall x \in (0,1) \quad |\frac{v_e'(x)}{v_e(x)} w_e(t,x)| \geq \min_{x\in[0,1]} |w_e(t,x)||\frac{v_e'(x)}{v_e(x)}|,$$

and $\int_0^1 |\frac{v_e'(x)}{v_e(x)}|dx = +\infty$. This lack of integrability of $\partial_x \eta_e$ is inherent in the nature of the functional space $H^1_{|v_e|^{\frac{1}{2}},0}(0,1)$. For instance, we can show that a function of the form $w : x \in [0,1] \mapsto (1-x)^s$ belongs to $H^1_{|v_e|^{\frac{1}{2}},0}(0,1)$ if and only if $s > \frac{1}{4}$. The quotient $\eta := \frac{w}{v_e}$ has the following behavior $\eta(x) \underset{x\to 1^-}{\sim} \frac{-(1-x)^{s-\frac{1}{2}}}{\nu}$ with $s - \frac{1}{2} > -\frac{1}{4}$. Then for $\frac{1}{4} < s < \frac{1}{2}$ we have $\lim_{x\to 1^-} \eta(x) = -\infty$ and $\eta \in L^1(0,1)$ but $\partial_x \eta \notin L^1(0,1)$.

Appendix B. Reminder on linear evolution equation in infinite dimension

Definition Appendix B.1. Let H be a Hilbert space and $A : D(A) \subset H \to H$ an unbounded linear operator. We say that A is dissipative if

$$(Av, v)_H \geq 0 \quad \forall v \in D(A),$$

A is maximal dissipative if moreover $R(I + A) = H$.

We recall a result that characterizes surjective operators.

Proposition Appendix B.2 *([6] Theorem II.19 page 30). Let $A : D(A) \subset H \to H$ be an unbounded linear operator, closed with $D(A)$ dense in H. Then*

$$R(A) = H \Leftrightarrow \exists C \geq 0 \text{ such that } \|v\|_{H'} \leq C\|A^* v\|_{H'} \quad \forall v \in D(A^*),$$

where A^* is the adjoint-operator of A and H' is the dual space of H.

Theorem Appendix B.3 *(Hille–Yosida). Let A be a maximal monotone operator in a Hilbert space H. Then for all $u_0 \in D(A)$ there is a unique $u \in C^1([0,+\infty); H) \cap C^0([0,+\infty); D(A))$ solution of the problem:*

$$\begin{cases} \frac{du}{dt} + Au = 0 \text{ on } [0,+\infty) \\ u(0) = u_0. \end{cases}$$

Moreover, one has

$$\|u(t)\|_H \leq \|u_0\| \text{ and } \|\frac{du}{dt}(t)\|_H \leq \|Au_0\|_H \text{ for all } t \geq 0.$$

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