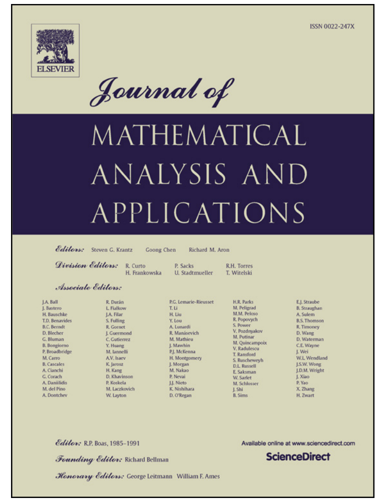


Reproducing pairs and Gabor systems at critical density

M. Speckbacher, P. Balazs

PII: S0022-247X(17)30548-6
DOI: http://dx.doi.org/10.1016/j.jmaa.2017.05.079
Reference: YJMAA 21440

To appear in: *Journal of Mathematical Analysis and Applications*

Received date: 13 January 2017

Reproducing pairs and Gabor systems at critical density

M. Speckbacher[1,*], P. Balazs[1]

Acoustics Research Institute, Austrian Academy of Sciences, Wohllebengasse 12-14, 1040 Vienna, Austria

Abstract

We use the concept of reproducing pairs to study Gabor systems at critical density. First, we present a generalization of the Balian-Low theorem to the reproducing pairs setting. Then, we prove our main result that there exists a reproducing partner for the Gabor system of integer time-frequency shifts of the Gaussian. In other words, the coefficients for this Gabor expansion of a square integrable function can be calculated using inner products with an unstructured family of vectors in $L^2(\mathbb{R})$. This solves one of the last few open questions for this system.

MSC2010: 42C15; 42C40

Keywords: Gabor systems; reproducing pairs; critical density; Zak transform; Balian-Low theorem

1. Introduction

The main problem of Gabor analysis is to understand the conditions and obstructions on the family $G(g,\Lambda) := \{T_{\lambda_1} M_{\lambda_2} g\}_{\lambda\in\Lambda} \subset L^2(\mathbb{R})$ to be a frame. There exists, however, a great abundance of windows g and lattices Λ generating Gabor families which are, on the one hand, complete and, on the other hand, violate at least one of the frame bounds. The well-known Balian-Low theorem, for example, states that the window function of a Gabor frame at the

*Corresponding author

Email addresses: speckbacher@kfs.oeaw.ac.at (M. Speckbacher), peter.balazs@oeaw.ac.at (P. Balazs)

critical density ($\Lambda = a\mathbb{Z} \times a^{-1}\mathbb{Z}$) cannot be well localized on the time-frequency plane. In fact, there are many more properties of a window function that prevent a system from being a Gabor frame, see [17] for an overview. It is therefore reasonable to change perspective and apply approaches beyond frame theory to understand Gabor families at critical density.

Several generalizations of frames, such as semi-frames or reproducing pairs have been introduced. A reproducing pair [25] consists of two vector families (not necessarily frames) instead of a single one that generates a bounded and invertible analysis/synthesis process in a Hilbert space. Observe that, for Gabor families, there is a conceptual similarity to weakly dual Gabor systems [14], where the analysis/synthesis process is considered in terms of a Gelfand triplet.

The main goal of this paper is to investigate whether the obstructions for Gabor frames at critical density still hold for reproducing pairs. That is, is there a reproducing pair where one of the two families is a Gabor system generated by a well localized window? First, we will consider the case of reproducing pairs consisting of two Gabor systems and derive a Balian-Low like result. Then we will turn our focus to the study of the Gabor family of integer time-frequency shifts of the Gaussian φ, which is probably the most studied object in Gabor analysis.

Already in 1932, von Neumann [26] claimed without proof that the system $\mathcal{G} := \{T_k M_l \varphi\}_{k,l\in\mathbb{Z}}$ is complete in $L^2(\mathbb{R})$. It was only in the 1970's that Perelomov [23], Bargmann et al. [7] and Bacry et al. [6] presented rigorous proofs of completeness in the context of coherent states.

A second problem was formulated by Gabor [15] in 1946 when he asked if there exists a linear coefficient map $A : L^2(\mathbb{R}) \to \mathbb{C}^{\mathbb{Z}\times\mathbb{Z}}$, such that the expansion

$$f = \sum_{k,l\in\mathbb{Z}} (Af)[k,l] T_k M_l \varphi, \tag{1}$$

converges for all $f \in L^2(\mathbb{R})$. The answer to this question is more subtle. Janssen [19] showed that such a coefficient map exists. The coefficients, however, grow polynomially and (1) converges only in the sense of tempered distributions.

As the Balian-Low theorem tells us that $\mathcal{G}$ cannot be a frame one might ask if there exists a dual window $\gamma \in L^2(\mathbb{R})$, such that $(Af)[k,l] := \langle f, T_k M_l \gamma \rangle$ yields (1) with weak convergence. The answer is "no". This can easily be seen using Zak transform methods which yields that γ is given by Bastiaans' dual window [8], a bounded function that is not in $L^p(\mathbb{R})$ for any $1 \leq p < \infty$. The answer also follows from the Balian-Low like result that we will prove in Section 3 or from a result by Daubechies and Janssen [11].

In this paper, we will consider a problem which is intermediate to the second and third question and that is yet unsolved: can the coefficient map A be calculated using inner products with an unstructured family in $L^2(\mathbb{R})$, that is, is there a system $\Psi = \{\psi_{k,l}\}_{k,l\in\mathbb{Z}}$, such that $(Af)(k,l) = \langle f, \psi_{k,l} \rangle$ with the series (1) converging weakly?

We will use a characterization of reproducing pairs from [5] in our proof to show our main result in Theorem 13: the existence of a reproducing partner for $\mathcal{G}$. This family of vectors however is totally unstructured and cannot have a shift-invariant structure. Our result can be reformulated in several contexts. For example, there is a dual system for the complete Bessel-sequence $\mathcal{G}$, or there is a family of vectors Ψ making $(\mathcal{G}, \Psi)$ a reproducing pair.

This paper is organized as follows. In Section 2 we present the basics of reproducing pairs and Gabor theory needed in the course of this article. Section 3 is devoted reproducing pairs of two Gabor systems and a generalization of the Balian-Low theorem. In Section 4 we investigate the existence of a reproducing partner for the system $\mathcal{G}$.

2. Preliminaries and notation

Throughout this paper we use the notation $f(\cdot)$ for functions on subsets of $\mathbb{R}^d$ and $c[\cdot]$ for sequences on $\mathbb{Z}^d$. Moreover, $\hat{f}$ or $\mathcal{F}(f)$ will denote the Fourier transform using the convention

$$\hat{f}[k] = \int_{\left[-\frac{1}{2},\frac{1}{2}\right)^d} f(\omega) e^{-2\pi i k\cdot\omega} d\omega, \quad k \in \mathbb{Z}^d,$$

for functions $f : \left[-\frac{1}{2}, \frac{1}{2}\right)^d \to \mathbb{C}$, and

$$\hat{c}(\omega) = \sum_{k \in \mathbb{Z}^d} c[k] e^{-2\pi i k \cdot \omega}, \quad \omega \in \left[-\frac{1}{2}, \frac{1}{2}\right)^d,$$

for sequences $c : \mathbb{Z}^d \to \mathbb{C}$.

2.1. Frames and semi-frames

Frames were introduced in 1952 by Duffin and Schaeffer [12] as a generalization of orthonormal bases. A countable family of vectors $\{\psi_k\}_{k\in\mathcal{I}}$ in a separable Hilbert space $\mathcal{H}$ is called a frame if there exist constants $A, B > 0$ such that

$$A \|f\|^2 \leq \sum_{k\in\mathcal{I}} |\langle f, \psi_k\rangle|^2 \leq B \|f\|^2, \ \forall\ f \in \mathcal{H}. \tag{2}$$

For a thorough introduction to frame theory, see [9]. Frame theory has proven its usefulness in many different fields such as sampling theory [1] or theoretical physics [2]. However, there exists a great reservoir of complete families that do not satisfy both frame inequalities. We will see an example of this situation in Section 2.3. Hence, several generalizations of frames such as reproducing pairs (see Section 2.2) or semi-frames [3, 4], have been introduced. The basic idea of semi-frames is to consider complete families $\{\psi_k\}_{k\in\mathcal{I}}$ that only satisfy one of the inequalities in (2). In particular, $\{\psi_k\}_{k\in\mathcal{I}}$ is called a lower semi-frame if

$$A \|f\|^2 \leq \sum_{k\in\mathcal{I}} |\langle f, \psi_k\rangle|^2, \ \forall\ f \in \mathcal{H}, \tag{3}$$

and is called an upper semi-frame if

$$0 < \sum_{k\in\mathcal{I}} |\langle f, \psi_k\rangle|^2 \leq B \|f\|^2, \ \forall\ f \in \mathcal{H}. \tag{4}$$

An upper semi-frame is often also called a complete Bessel sequence. Many results from frame theory can be extended to the setting of semi-frames.

The following Lemma can be found in [3, Lemma 2.5].

Lemma 1. *Let $\Phi = \{\phi_k\}_{k\in\mathcal{I}}$ be an upper semi-frame with bound B, and $\Psi = \{\psi_k\}_{k\in\mathcal{I}}$ be a family of vectors satisfying*

$$\langle f, g\rangle = \sum_{k\in\mathcal{I}} \langle f, \phi_k\rangle \langle \psi_k, g\rangle, \quad \forall\ f, g \in \mathcal{H},$$

then Ψ is a lower semi-frame with lower bound B^{-1}.

2.2. Reproducing pairs

The concept of reproducing pairs has been introduced recently in [25] and studied in more detail in [5]. The main idea is to omit both frame bounds and to consider two vector families (instead of a single one) that generate a bounded and boundedly invertible analysis/synthesis process.

Although the general definition in [5] is given with respect to arbitrary Borel measures, we will only present the discrete setting here as we will exclusively study discrete Gabor systems in this paper.

Definition 1. *Let $\Psi = \{\psi_k\}_{k\in\mathcal{I}}$, $\Phi = \{\phi_k\}_{k\in\mathcal{I}}$ be two families in $\mathcal{H}$. The pair (Ψ,Φ) is called a reproducing pair for $\mathcal{H}$ if the operator $S_{\Psi,\Phi} : \mathcal{H} \to \mathcal{H}$, weakly defined by*

$$\langle S_{\Psi,\Phi} f, g\rangle := \sum_{k\in\mathcal{I}} \langle f, \psi_k\rangle\langle \phi_k, g\rangle, \tag{5}$$

is bounded and boundedly invertible.
Given a family Ψ, any system Φ for which (Ψ,Φ) is a reproducing pair is called a reproducing partner for Ψ.

Remark 2. *Please note that the results on discrete reproducing pairs in this section can also be formulated if weak convergence of $S_{\Psi,\Phi}$ is replaced by norm convergence. Our main result in Section 4 however relies on Janssen's result that the Gabor series of integer time-frequency shifts of the Gaussian window can only be weakly convergent.*

Let $\mathcal{V}_\Phi(\mathcal{I})$ be the space of all sequences $\xi : \mathcal{I} \to \mathbb{C}$ such that

$$\Big|\sum_{k\in\mathcal{I}} \xi[k]\langle \phi_k, g\rangle\Big| \leq c\,\|g\|\,, \forall\, g \in \mathcal{H}.$$

In this case, the Riesz representation theorem guarantees that the synthesis operator $D_\Phi : \mathcal{V}_\Phi(\mathcal{I}) \to \mathcal{H}$, weakly given by

$$\langle D_\Phi \xi, g\rangle = \sum_{k\in\mathcal{I}} \xi[k]\langle \phi_k, g\rangle,$$

is well-defined. By definition, $\mathcal{V}_\Phi(\mathcal{I})$ is the most general domain for which the synthesis operator converges weakly. The proof of the following result can be found in [5, Theorem 4.1]. It answers the question of the existence of a reproducing partner for a given family in a Hilbert space.

Theorem 3. *Let $\Phi = \{\phi_k\}_{k\in\mathcal{I}} \subset \mathcal{H}$ be a family of vectors and $\{e_k\}_{k\in\mathcal{I}}$ be an orthonormal basis for $\mathcal{H}$. There exists another family Ψ, such that (Ψ, Φ) is a reproducing pair if and only if*

(i) $\mathsf{Range}\, D_\phi = \mathcal{H}$ *and*

(ii) there exists a family $\{\xi_k\}_{k\in\mathcal{I}} \subset \mathcal{V}_\Phi(\mathcal{I})$ such that

$$D_\Phi \xi_k = \sum_{n\in\mathcal{I}} \xi_k[n]\phi_n = e_k, \ \forall\ k \in \mathcal{I}, \tag{6}$$

and

$$\sum_{k\in\mathcal{I}} |\xi_k[n]|^2 < \infty, \ \forall\ n \in \mathcal{I}. \tag{7}$$

One instance of a reproducing partner $\Psi = \{\psi_k\}_{k\in\mathcal{I}}$ (in general there exist more than one) is then given by

$$\psi_n := \sum_{k\in\mathcal{I}} \overline{\xi_k[n]} e_k.$$

Please bear in mind that summation in equation (6) is with respect to $n \in \mathcal{I}$ for fixed $k \in \mathcal{I}$ while it is the reversed situation in equation (7). The conditions (i) and (ii) can be interpreted in several ways. First, Property (i) ensures the existence of a linear operator $A : \mathcal{H} \to \mathcal{V}_\Phi(\mathcal{I})$ satisfying $f = D_\Phi A(f)$, for every $f \in \mathcal{H}$. For an example of a complete system that does not satisfy (i), see [5, Section 6.2.3]. Property (ii) guarantees that $A(f)$ can be calculated by taking inner products of f with another family $\Psi \subset \mathcal{H}$.

Second, (i) and (ii) guarantee that $\{\xi_k\}_{k\in\mathcal{I}}$ is an orthonormal basis for its closed linear span with respect to the inner product $\langle \xi, \eta\rangle_\Phi := \langle D_\Phi \xi, D_\Phi \eta\rangle$. The second condition of (ii) then assures that this space is a reproducing kernel Hilbert space.

2.3. Gabor analysis

Let $\lambda = (x, \omega) \in \mathbb{R}^2$ be a point on the time-frequency plane. A time-frequency shift of a function g by λ is given by

$$\pi(\lambda)g(t) := T_x M_\omega g(t) = e^{2\pi i\omega(t-x)}g(t-x),$$

where translation operator is given by $T_x f(t) := f(t-x)$ and the modulation operator by $M_\omega f(t) = e^{2\pi i\omega t}g(t)$. The short-time Fourier transform V_g of a function f is defined by

$$V_g f(\lambda) := \langle f, \pi(\lambda)g\rangle.$$

A Gabor system is a discrete family of functions generated by time-frequency shifts of a single window function $g \in L^2(\mathbb{R})$, i.e.,

$$G(g,\Lambda) := \{\pi(\lambda)g\}_{\lambda\in\Lambda},$$

where Λ is a discrete subset of $\mathbb{R}^2$. Given $a, b > 0$, the Gabor system generated by a rectangular lattice is

$$G(g,a,b) := \{\pi(an, bm)g\}_{n,m\in\mathbb{Z}}.$$

The product $(ab)^{-1}$ is called the density or redundancy of the Gabor system. If $G(g,a,b)$ is a frame, then $ab \leq 1$ necessarily holds, see [16, Corollary 7.5.1]. The case $ab = 1$ is called the critical density case. Since this paper is concerned with Gabor systems at the critical density and the analysis can always be reduced to $a = b = 1$, we will write $G(g) := G(g,1,1)$ and $g_{n,m} := T_n M_m g$ in order to keep notation simple.

The Balian-Low theorem states that, at the critical density, there are no Gabor frames using a window which is well-localized both in time and frequency, see for example [16, Chapter 8.4].

Theorem 4 (Amalgam Balian-Low Theorem). *Let $ab = 1$ and assume that the Gabor system $G(g,a,b)$ is a frame. Then both $g \notin W_0(\mathbb{R})$ and $\hat{g} \notin W_0(\mathbb{R})$, where*

$$W_0(\mathbb{R}) := \{f \in C(\mathbb{R}) : \sum_{n\in\mathbb{Z}} \operatorname{ess\,sup}_{x\in[0,1]} |f(x+n)| < \infty\}.$$

This obstruction motivates our approach to use reproducing pair methods to study Gabor systems at critical density that are generated by a well localized window. First, we will investigate if it is possible to choose a second Gabor system as the reproducing partner. Such a family necessarily satisfies the lower but not the upper frame bound by Lemma 1. Since any well localized window function generates a Gabor Bessel family [16, Section 6.2], it therefore follows that the second window function cannot be well localized. Second, we will use Theorem 3 to prove the existence of unstructured reproducing partners.

The analysis of the conditions of Theorem 3 heavily depends on the particular window function. Thus, we will focus on the system of integer time-frequency shift ($a = b = 1$) of the normalized Gaussian

$$\varphi_\sigma(t) = (2/\sigma)^{1/4} e^{-\pi t^2/\sigma}.$$

However, we are convinced that the recipe for our proof also works for other window functions. As in [20, Section 2.2] we will assume that $\sigma = 1$ and use the notation $\varphi := \varphi_1$ and $\mathcal{G} := G(\varphi)$.

We end this introduction to Gabor systems by defining the modulation spaces M^p_s. Let $v_s(x, \omega) := (1 + |x| + |\omega|)^s$, $s > 0$, and g be some nonzero Schwartz function. Then M^p_s is

$$M^p_s := \{ f \in L^2(\mathbb{R}) : \ V_g f \cdot v_s \in L^p(\mathbb{R}^2) \}.$$

In particular, the spaces M^1_s are commonly seen as the appropriate class of window functions for Gabor analysis. For an overview on modulation spaces, see [13, 16].

3. A Balian-Low like theorem for reproducing pairs

The Zak transform of a function f is given by the function Zf on $\mathbb{R}^2$ defined by

$$Zf(x, \omega) := \sum_{k \in \mathbb{Z}} f(x - k) e^{2\pi i \omega k},$$

with convergence of the series depending the class of functions that contain f. For example, if $f \in L^2(\mathbb{R})$, then Zf is almost everywhere well-defined and converges in $L^2\big([0,1]^2\big)$. In the following, we list some of the most important properties of Z. An extensive description can be found for example in [16, Chapter 8]. It is easy to see that Zf is periodic in the second variable and quasiperiodic in the first variable. That is, for $k, l \in \mathbb{Z}$ one has

$$Zf(x+k, \omega+l) = e^{2\pi i k\omega} Zf(x,\omega).$$

Moreover, $Z : L^2(\mathbb{R}) \to L^2\big([0,1]^2\big)$ is a unitary operator. Regularity of a function is preserved by the Zak transform. To be more precise, if $f \in W_0(\mathbb{R})$, then $Zf \in C(\mathbb{R}^2)$ and $f \in \mathcal{S}(\mathbb{R})$ if and only if $Zf \in C^\infty(\mathbb{R}^2)$. The most important property of the Zak transform in the context of Gabor analysis is, however, that it diagonalizes the Gabor frame operator. For $a = b = 1$ and $f, g, \gamma \in L^2(\mathbb{R})$, it holds that

$$Z(S_{g,\gamma} f) = \overline{Zg} \cdot Z\gamma \cdot Zf. \tag{8}$$

Hence, $(G(g), G(\gamma))$ is a reproducing pair if and only if

$$0 < m \le |Zg \cdot Z\gamma| \le M < \infty, \text{ almost everywhere,} \tag{9}$$

and $S_{g,\gamma} = I$ if and only if $\overline{Zg} \cdot Z\gamma = 1$ almost everywhere.

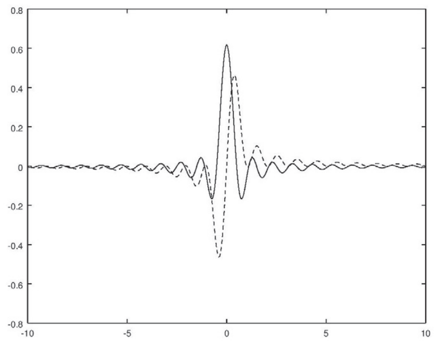
There is a close connection to Gabor Schauder bases. Heil and Powell [18, Theorem 5.10] have shown that $G(g)$ is a Gabor Schauder basis if and only if there exists $C > 0$ such that for any intervals $I, J \subset \mathbb{R}$ one has

$$\frac{1}{|I|^2|J|^2} \cdot \int_{I\times J} |Zg(x,\omega)|^2 dx d\omega \cdot \int_{I\times J} \frac{1}{|Zg(x,\omega)|^2} dx d\omega \le C.$$

In particular, any Schauder basis $G(g)$ and its dual basis $G(\gamma)$ form a reproducing pair.

In the same article [18] the authors show that the two windows of a Gabor Schauder basis and its dual cannot both be well localized. This naturally leads to the question if at least one of the two windows of a Gabor reproducing pair can be well localized.

Example 1. *Let us try to "trick" the Amalgam Balian-Low theorem by constructing a reproducing pair using window functions g, γ such that $g, \hat{g} \in W_0(\mathbb{R})$,*



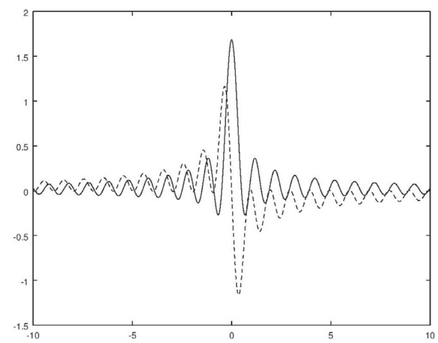


Figure 1: Plot of the functions g (left) and γ (right) in Example 1. The solid (resp. dashed) line shows the real (resp. imaginary) part of the functions g and γ.[1]

that is, g is well localized on the time-frequency plane. Observe that γ is then necessarily badly localized. Define $\vartheta(t) := t^{1/4}(1-t)^{1/4},\ t \in [0,1]$. Then define g by

$$Zg(x,\omega) := e^{2\pi i x\cdot(\omega \bmod 1)}\vartheta(\omega \bmod 1),$$

then Zg is quasiperiodic in x, periodic in ω and continuous on $\mathbb{R}^2$. Moreover, $g \in W_0(\mathbb{R})$ by [18, Theorem 6.1 (b)], $\widehat{g}(\omega) = \vartheta(\omega) \cdot \chi_{[0,1]}(\omega) \in W_0(\mathbb{R})$ and $\gamma := Z^{-1}(1/\overline{Zg}) \in L^2(\mathbb{R})$. Hence, $\big(G(g), G(\gamma)\big)$ is a reproducing pair.

However, it turns out that even rather mild decay conditions on the time-frequency distribution of the windows exclude the possibility of reproducing pairs using two Gabor systems. Daubechies and Janssen [11] obtained a first result in this direction.

Theorem 5. *Let $\big(G(g), G(\gamma)\big)$ be a reproducing pair, then neither $g \in M^2_2$ nor $\gamma \in M^2_2$.*

In this paper we will show a similar result where we replace the modulation space M^2_2 by M^1_1. Note that this is a new result as neither space embeds into the other.

[1] In the spirit of reproducible research we provide a Matlab/Octave script at `https://www.kfs.oeaw.ac.at/doc/RepPairGabor/rep_pair_gabor.m` which generates the plots of Figure 1.

The following Proposition is a simple consequence of Janssen's characterization of the modulation space M^1 via the Zak transform [22].

Proposition 6. *If $s \in \mathbb{N}_0$ and $f \in M^1_s$, then $Zf \in C^s(\mathbb{R}^2)$.*

Proof: Let $f \in M^1_s$ and $g_1, g_2 \in M^1_s$ be such that Zg_1 and Zg_2 have no common zeros. Adapting the argument of [22, Theorem 4.1] yields that $Zf \cdot \overline{Zg_n}$, $n = 1, 2$, can be expressed by an Fourier series

$$(Zf \cdot \overline{Zg_n})(x, \omega) = \sum_{k,l \in \mathbb{Z}} c_n(k, l) e^{2\pi i k x/a + 2\pi i a l \omega}, \ n = 1, 2,$$

with coefficients $\{c_n(k, l)\}_{k,l\in\mathbb{Z}} \in \ell^1_{v_s}(\mathbb{Z}^2)$. Hence, if $|\beta| = \beta_1 + \beta_2 \le s$, then $\{k^{\beta_1} l^{\beta_2} c_n(k, l)\}_{k,l\in\mathbb{Z}} \in \ell^1(\mathbb{Z}^2)$ which in turn implies that the Fourier series of the derivative

$$D^\beta (Zf \cdot \overline{Zg_n})(x, \omega) = \sum_{k,l\in\mathbb{Z}} (2\pi i)^{|\beta|} (k/a)^{\beta_1} (al)^{\beta_2} c_n(k, l) e^{2\pi i k x/a + 2\pi i a l \omega},$$

converges absolutely for all $|\beta| \le s$. Hence $Zf \cdot \overline{Zg_n} \in C^s(\mathbb{R}^2)$ for $n = 1, 2$. We may choose g_1, g_2 to be Schwartz functions, which guarantees that $Zg_n \in C^\infty(\mathbb{R}^2)$. Finally, as Zg_1 and Zg_2 have no common zeros, it follows that $Zf \in C^s(\mathbb{R}^2)$. □

Lemma 7. *Let $F \in L^2\big([0, 1]^2\big)$ be Lipschitz and assume that there exists $z^* \in [0, 1]^2$ with $F(z^*) = 0$. Then $1/F \notin L^2\big([0, 1]^2\big)$.*

Proof: Let L be the Lipschitz constant of F. Then since $F(z^*) = 0$,

$$|F(z)|^2 \le L^2 \|z - z^*\|^2, \ \forall z \in B_\delta(z^*).$$

Hence, $1/F \notin L^2\big([0, 1]^2\big)$, as $1/|F(z)|^2 \ge L^{-2} \|z - z^*\|^{-2}$ on $B_\delta(z^*)$. □

Corollary 8. *Let $\big(G(g), G(\gamma)\big)$ be a reproducing pair. Then both $g \notin M^1_1$ and $\gamma \notin M^1_1$.*

Proof: Assume without loss of generality that $g \in M_1^1$. Then $Zg \in C^1(\mathbb{R}^2)$ by Proposition 6. Also $|Z\gamma| \geq m/|Zg|$ by (9), and there exists $z^* \in [0,1]^2$ such that $Zg(z^*) = 0$ by [16, Lemma 8.4.2]. However, $Z\gamma \notin L^2\big([0,1]^2\big)$ by Lemma 7 and consequently $\gamma \notin L^2(\mathbb{R})$, a contradiction. □

4. In quest of a reproducing partner for $\mathcal{G}$

For the rest of this paper, we focus on the study of $\mathcal{G}$, the Gabor system generated by integer time-frequency shifts of the Gaussian. As already mentioned in the introduction, $\mathcal{G}$ is a complete Bessel family but not a frame. By Corollary 8, there is no dual Gabor system for $\mathcal{G}$. We will use Theorem 3 to show that the expansion coefficients can be calculated via inner products. Finally, we show that any reproducing partner for $\mathcal{G}$ cannot have a shift-invariant structure in the time or frequency domain.

4.1. The range of $D_{\mathcal{G}}$

Condition (i) in Theorem 3 is satisfied. This is a consequence of [19, Theorem 4.7], which states that for every $f \in \mathcal{S}'(\mathbb{R})$ there exists a sequence ξ such that $f = \sum_{n,m\in\mathbb{Z}} \xi[n,m] T_n M_m \varphi$ with convergence in the sense of tempered distributions. Consequently, the series converges weakly for every $f \in L^2(\mathbb{R})$ by the density of $\mathcal{S}(\mathbb{R})$ in $L^2(\mathbb{R})$. Hence, $\mathsf{Range}\, D_{\mathcal{G}} = L^2(\mathbb{R})$.

In order to verify condition (ii), that is, that there exists $\{\xi_{k,l}\}_{(k,l)\in\mathbb{Z}^2}$ such that $D_{\mathcal{G}}\xi_{k,l} = e_{k,l}$ and $\sum_{k,l\in\mathbb{Z}} |\xi_{k,l}[n,m]|^2 < \infty$ for all $(n,m) \in \mathbb{Z}^2$, we need some auxiliary results.

4.2. Solving $D_{\mathcal{G}}\xi = e_{k,l}$

Lemma 9. *Let $\gamma \in L^2(\mathbb{R})$. The sequence ξ_0 solves $D_{\mathcal{G}}\xi = \gamma$ (weakly in L^2) if and only if $\mathcal{S}_{k,l}\xi_0$ solves $D_{\mathcal{G}}\xi = \gamma_{k,l}$ (weakly in L^2), where $S_{k,l}$ denotes the index-shift operator*

$$\mathcal{S}_{k,l}c[n,m] := c[n-k, m-l].$$

Proof: Let ξ_0 be a (weak) solution to $D_{\mathcal{G}}\xi = \gamma$. Then it holds that

$$\begin{aligned}\gamma_{k,l} &= T_k M_l \gamma = T_k M_l (D_{\mathcal{G}}\xi_0) = T_k M_l \Big(\sum_{n,m\in\mathbb{Z}} \xi_0[n,m] T_n M_m \varphi \Big) \\ &= \sum_{n,m\in\mathbb{Z}} \xi_0[n,m] T_{k+n} M_{l+m}\varphi = \sum_{n,m\in\mathbb{Z}} \xi_0[n-k, m-l] T_n M_m \varphi \\ &= \sum_{n,m\in\mathbb{Z}} (\mathcal{S}_{k,l}\xi_0)[n,m] T_n M_m \varphi = D_{\mathcal{G}}(\mathcal{S}_{k,l}\xi_0).\end{aligned}$$

The reversed implication follows with the same argument. □

Hence, in order to find all solutions of $D_{\mathcal{G}}\xi = \gamma_{k,l}$, it remains to find one particular weak solution of $D_{\mathcal{G}}\xi = \gamma$ and to characterize the kernel of $D_{\mathcal{G}}$ in $\mathcal{V}_{\mathcal{G}}(\mathbb{Z}^2)$.

4.2.1. Characterizing the kernel of $D_{\mathcal{G}}$

In this section, we will see that any weak solution of $D_{\mathcal{G}}\xi = 0$ is given by a two-dimensional polynomial evaluated on $\mathbb{Z}^2$ times an oscillating sign factor. This result can already be found in [19, Section 3.5 - 3.7].

As $\mathcal{G}$ is complete in $L^2(\mathbb{R})$ it follows that $\xi \in \mathsf{Ker}\, D_{\mathcal{G}}$ if and only if $\langle D_{\mathcal{G}}\xi, \varphi_{n,m}\rangle = 0$ for every $(n,m) \in \mathbb{Z}^2$. The left-hand side of this equation can be rewritten as

$$\begin{aligned}\langle D_{\mathcal{G}}\xi, \varphi_{n,m}\rangle &= \sum_{k,l\in\mathbb{Z}} \xi[k,l] \langle T_k M_l \varphi, T_n M_m \varphi\rangle = \sum_{k,l\in\mathbb{Z}} \xi[k,l] \langle T_{k-n} M_{l-m}\varphi, \varphi\rangle \\ &= \sum_{k,l\in\mathbb{Z}} \xi[k,l] \vartheta[n-k, m-l] = (\xi * \vartheta)[n,m],\end{aligned}$$

where

$$\vartheta[n,m] := \langle T_{-n} M_{-m}\varphi, \varphi\rangle.$$

Lemma 1.5.2 in [16] shows that

$$\vartheta[n,m] = (-1)^{nm} e^{-\pi(n^2+m^2)/2}.$$

If $\xi * \vartheta = 0$, then $\hat{\xi} \cdot \hat{\vartheta} = 0$ holds at least in the sense of periodic distributions. Hence, we intend to characterize those periodic distributions Λ such that $\Lambda \cdot \Theta = 0$, where $\Theta := \hat{\vartheta}$. The function Θ can be expressed analytically as

$$\Theta(\omega) = \sum_{n,m\in\mathbb{Z}} (-1)^{nm} e^{-\pi(n^2+m^2)/2} e^{-2\pi i(n\omega_1 + m\omega_2)} \tag{10}$$

$$= \theta_3(\pi\omega_1, e^{-\pi/2}) \cdot \theta_3(2\pi\omega_2, e^{-2\pi}) + \theta_4(\pi\omega_1, e^{-\pi/2}) \cdot \theta_2(2\pi\omega_2, e^{-2\pi}),$$

where $\omega \in [0,1)^2$ and θ_k denotes the k-th Jacobi theta function, see for example [27]. This function has been studied in [19, Theorem 3.5]. We will state the results in the following Lemma.

Lemma 10. *Set* $\omega_0 := (1/2, 1/2)$. *The function* $\Theta \in C^\infty([0,1]^2)$ *has the following properties:*

(i) $\Theta(\omega) \geq 0, \ \forall\ \omega \in [0,1)^2$,

(ii) $\Theta(\omega) = 0$ *if and only if* $\omega = \omega_0$,

(iii) $D^{(2,0)}\Theta(\omega_0) = D^{(0,2)}\Theta(\omega_0) > 0$.

A particular consequence of Lemma 10 is that any periodic distribution Λ satisfying $\Lambda \cdot \Theta = 0$ is supported on $\{\omega_0\}$. Thus, by [24, Theorem 6.25] there exists $N \in \mathbb{N}_0$ and coefficients $c_\alpha \in \mathbb{C}$, $\alpha \in \mathbb{N}_0^2$, such that

$$\Lambda = \sum_{|\alpha| \leq N} c_\alpha D^\alpha \delta_{\omega_0}, \tag{11}$$

Applying the inverse Fourier transform to Λ immediately shows the following result.

Corollary 11. *Every sequence* $p \in \mathsf{Ker}\, D_{\mathcal{G}}$ *can be written as*

$$p[n,m] = (-1)^{n+m} \sum_{|\alpha| \leq N} c_\alpha \cdot n^{\alpha_1} m^{\alpha_2}.$$

4.2.2. Calculation of ξ_0

We choose the orthonormal basis $\{e_{n,m}\}_{n,m\in\mathbb{Z}}$ in Theorem 3 to be the Gabor system $G(\chi)$ with $\chi := \chi_{[-1/2,1/2]}$, the characteristic function for the interval $[-1/2, 1/2]$. By Lemma 9 it remains to find a weak solution of $D_{\mathcal{G}}\xi = \chi$.

A first attempt in finding the expansion coefficients for $\mathcal{G}$ can be found in [8]. In this paper, Bastiaans constructed a window that yields the expansion coefficients for a dense subspace of $L^2(\mathbb{R})$, see also [21, Section 4.4]. Bastiaans' dual window [8] is analytically given by

$$\psi(t) = C_\psi \; e^{\pi t^2} \sum_{n>|t|-1/2} (-1)^n e^{-\pi(n+1/2)^2}, \tag{12}$$

for some constant $C_\psi > 0$. The function ψ defined as $\psi := Z^{-1}(1/\overline{Z(\varphi)})$ has the property that ψ is bounded but not contained in $L^p(\mathbb{R})$ for any $1 \leq p < \infty$. That is, $\psi \in L^\infty(\mathbb{R}) \backslash L^p(\mathbb{R})$.

In [21, Section 4.4] Janssen showed the following result: If $f \in L^1(\mathbb{R})$ and for every fixed $l \in \mathbb{Z}$ the expression $\langle f, T_k M_l \psi\rangle$ converges to zero as $|k| \to \infty$, then the following series converges in the sense of tempered distributions

$$f = \sum_{k,l\in\mathbb{Z}} \langle f, T_k M_l \psi\rangle T_k M_l \varphi. \tag{13}$$

We will prove that this result implies that (13) holds on a dense subspace of $L^2(\mathbb{R})$ with weak convergence in $L^2(\mathbb{R})$.

Lemma 12. *For every function $f \in \mathcal{M}$, where*

$$\mathcal{M} := \Big\{ f \in L^2(\mathbb{R}) : \; f = \sum_{k,l\in\mathbb{Z}} c[k,l] T_k M_l \chi, \; \|c\|_1 < \infty \Big\},$$

equation (13) holds weakly in $L^2(\mathbb{R})$. In particular, $\xi_0[k,l] := \langle \chi, T_k M_l \psi\rangle$ is a weak solution of $D_{\mathcal{G}}\xi = \chi$.

Proof: Recall that $G(\chi)$ is an orthonormal basis. Let $f \in \mathcal{M}$, then $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ because

$$\|f\|_1 \leq \sum_{k,l\in\mathbb{Z}} |c[k,l]| \|T_k M_l \chi\|_1 = \|c\|_1 \quad \text{and} \quad \|f\|_2^2 = \sum_{k,l\in\mathbb{Z}} |c[k,l]|^2 \leq \|c\|_1^2.$$

If $k \neq 0$, then

$$
\begin{aligned}
|\langle \chi, T_k M_l \psi\rangle| &\leq \int_{-1/2}^{1/2} |\psi(t-k)|dt \leq C \int_{-1/2}^{1/2} e^{\pi(t-k)^2} dt \sum_{n \geq |k|} e^{-\pi(n+1/2)^2} \\
&= C \int_{-1/2}^{1/2} e^{\pi(t-|k|)^2 - \pi(|k|+1/2)^2} dt \sum_{n \geq |k|} e^{-\pi(n+1/2)^2 + \pi(|k|+1/2)^2} \\
&\leq C \int_{-1/2}^{1/2} e^{\pi(t^2 - 1/4 - |k|(2t+1))} dt \leq C \int_{-1/2}^{1/2} e^{-\pi|k|(2t+1)} dt \\
&= \frac{C}{2\pi|k|}(1 - e^{-2\pi|k|}) \leq \frac{C}{2\pi(1+|k|)},
\end{aligned}
$$

where we have used that $e^{\pi t^2}$ is even and that $t^2 - 1/4 < 0$ on $(-1/2, 1/2)$. Observe that the constant C is independent of k since

$$
\begin{aligned}
\sum_{n \geq |k|} e^{-\pi(n+1/2)^2 + \pi(|k|+1/2)^2} &= \sum_{n \geq 0} e^{-\pi(n+|k|+1/2)^2 + \pi(|k|+1/2)^2} \\
&= \sum_{n \geq 0} e^{-\pi(n^2 + 2n|k| + n)} \leq \sum_{n \geq 0} e^{-\pi(n^2+n)} = C < \infty.
\end{aligned}
$$

Let now $f = \sum_{n,m} c[n,m] T_n M_m \chi$ with $c \in \ell^1(\mathbb{Z}^2)$. By Young's inequality for convolutions and the previous calculations we obtain

$$
\begin{aligned}
\sum_{k \in \mathbb{Z}} |\langle f, T_k M_l \psi\rangle|^2 &= \sum_{k \in \mathbb{Z}} \Big| \sum_{n,m \in \mathbb{Z}} c[n,m] \langle \chi, T_{k-n} M_{l-m} \psi\rangle \Big|^2 \\
&\leq C \sum_{k \in \mathbb{Z}} \Big(\sum_{n,m \in \mathbb{Z}} |c[n,m]| (1 + |k-n|)^{-1} \Big)^2 \\
&= C \sum_{k \in \mathbb{Z}} \Big(\sum_{n \in \mathbb{Z}} \Big(\sum_{m \in \mathbb{Z}} c[n,m] \Big) (1 + |k-n|)^{-1} \Big)^2 \\
&\leq C \|c\|_1^2 \sum_{k \in \mathbb{Z}} (1 + |k|)^{-2} < \infty.
\end{aligned}
$$

This estimate finally implies that for each fixed $l \in \mathbb{Z}$, we have $|\langle f, T_k M_l \psi\rangle| \to 0$, as $|k| \to \infty$. □

Let us investigate the behavior of ξ_0 in more detail. We will use the notations $g(t) := C_\psi e^{\pi t^2}$, with C_ψ being the constant from (12),

$$
G_k := \sum_{n \geq 0} (-1)^n e^{-\pi(n^2 + 2|k|n + n + 1/4)},
$$

and $\mu_k(t) := (-1)^k e^{-\pi|k|} G_k e^{-2\pi k t}$. It holds that

$$\begin{aligned}
\xi_0[k,l] = \langle \gamma, T_k M_l \psi \rangle &= C_\psi \int_{-1/2}^{1/2} e^{\pi(t-k)^2} e^{-2\pi i l t} dt \sum_{n \geq |k|} (-1)^n e^{-\pi(n+1/2)^2} \\
&= C_\psi \int_{-1/2}^{1/2} e^{\pi t^2} e^{-2\pi k t} e^{-2\pi i l t} dt \sum_{n \geq |k|} (-1)^n e^{-\pi(n+1/2)^2} e^{\pi k^2} \\
&= C_\psi \int_{-1/2}^{1/2} e^{\pi t^2} e^{-2\pi k t} e^{-2\pi i l t} dt \sum_{n \geq 0} (-1)^{n+k} e^{-\pi(n^2+2|k|n+|k|+n+1/4)} \\
&= C_\psi (-1)^k e^{-\pi|k|} G_k \int_{-1/2}^{1/2} e^{\pi t^2} e^{-2\pi k t} e^{-2\pi i l t} dt \\
&= \int_{-1/2}^{1/2} g(t) \mu_k(t) e^{-2\pi i l t} dt = \widehat{g \cdot \mu_k}[l]. \qquad (14)
\end{aligned}$$

For $k = 0$, the Fourier transform of μ_0 is given by

$$\widehat{\mu_0}[l] = G_0 \delta_0[l]. \qquad (15)$$

If $k \neq 0$, then

$$\begin{aligned}
\widehat{\mu_0}[l] &= (-1)^k e^{-\pi|k|} G_k \int_{-1/2}^{1/2} e^{-2\pi(k+il)t} dt \\
&= \frac{(-1)^{k+1} e^{-\pi|k|} G_k}{2\pi(k+il)} (e^{-\pi(k+il)} - e^{\pi(k+il)}) \\
&= \frac{(-1)^{l+k+1}}{2\pi(k+il)} (e^{-\pi(|k|+k)} - e^{-\pi(|k|-k)}) G_k \\
&= \frac{(-1)^{l+k}}{2\pi(k+il)} \operatorname{sgn}(k)(1 - e^{-2\pi|k|}) G_k \\
&= \frac{(-1)^{l+k}}{2\pi(k+il)} H_k, \qquad (16)
\end{aligned}$$

where $H_k := \operatorname{sgn}(k)(1 - e^{-2\pi|k|}) G_k$. Observe that $H_k \to \pm e^{-\pi/4}$ as $k \to \pm\infty$.

4.3. Existence of reproducing partners

We are now ready to state and prove our main result.

Theorem 13. *There exists a system Ψ making $(\Psi, \mathcal{G})$ a reproducing pair. In other words, there exists a dual system for the complete Bessel sequence $\mathcal{G}$.*

Proof: Set

$$\xi_{k,l} = \mathcal{S}_{k,l} \xi_0 + p_{k,l}, \qquad (17)$$

where $p_{k,l}$ is a particular choice from $\mathsf{Ker}\, D_{\mathcal{G}}$ which we will specify later in the proof. Then $D_{\mathcal{G}}\xi_{k,l} = \gamma_{k,l}$. Let us assume that there exist a reproducing partner for $\mathcal{G}$. By Theorem 3, we can choose $\{\xi_{k,l}\}_{k,l\in\mathbb{Z}}$ in such a way that

$$\sum_{k,l\in\mathbb{Z}} \left|\xi_{k,l}[n,m]\right|^2 < \infty, \ \forall\, n,m \in \mathbb{Z}. \tag{18}$$

Recall that, by Corollary 11, there is some $N \in \mathbb{N}_0$ such that $p_{k,l}$ is given by

$$p_{k,l}[n,m] = (-1)^{n+m} \sum_{|\alpha|\leq N} c_\alpha[k,l] \cdot n^{\alpha_1} m^{\alpha_2}.$$

In the following we will choose $N = 0$ for every $(k,l) \in \mathbb{Z}^2$, that is, $p_{k,l}[n,m] = (-1)^{n+m}c[k,l]$. Since we assume that equation (18) holds, we can apply Parseval's formula to the summation with respect to $l \in \mathbb{Z}$. Using equation (14), this yields

$$\begin{aligned}
\sum_{l\in\mathbb{Z}} \left|\xi_{k,l}[n,m]\right|^2 &= \sum_{l\in\mathbb{Z}} \left|\mathcal{S}_{k,l}\xi_0[n,m] + p_{k,l}[n,m]\right|^2 \\
&= \sum_{l\in\mathbb{Z}} \left|\xi_0[n-k, m-l] + p_{k,l}[n,m]\right|^2 \\
&= \sum_{l\in\mathbb{Z}} \Big|\mathcal{F}(g\cdot\mu_{n-k})[m-l] + (-1)^{n+m}c[k,l]\Big|^2 \\
&= \int_{-1/2}^{1/2} \Big|M_m(g\cdot\mu_{n-k})(-\omega) + (-1)^{n+m}\mathcal{F}^{-1}(c[k,\cdot])(\omega)\Big|^2 d\omega \\
&\leq |g(1/2)|^2 \int_{-1/2}^{1/2} \Big|M_m\mu_{n-k}(-\omega) + (-1)^{n+m}(\mathcal{F}^{-1}(c[k,\cdot])/g)(\omega)\Big|^2 d\omega \\
&= C\sum_{l\in\mathbb{Z}} \Big|\mathcal{F}(\mu_{n-k})[m-l] + (-1)^{n+m}\mathcal{F}\big(\mathcal{F}^{-1}(c[k,\cdot])/g\big)[l]\Big|^2 =: (*).
\end{aligned} \tag{19}$$

If $k \neq n$ it follows by equation (16) that

$$\begin{aligned}
(*) &= C\sum_{l\in\mathbb{Z}} \Big|\frac{(-1)^{n+k+l+m}H_{n-k}}{2\pi(n-k+i(m-l))} + (-1)^{n+m}\mathcal{F}\big(\mathcal{F}^{-1}(c[k,\cdot])/g\big)[l]\Big|^2 \\
&= \frac{C}{4\pi^2}\sum_{l\in\mathbb{Z}} \frac{\left|H_{n-k} - \beta[k,l]\cdot\left(1-\frac{n+im}{k+il}\right)\right|^2}{(n-k)^2+(m-l)^2},
\end{aligned} \tag{20}$$

where

$$\beta[k,l] := 2\pi(-1)^{k+l}\mathcal{F}\big(\mathcal{F}^{-1}(c[k,\cdot])/g\big)[l]\cdot(k+il).$$

If $k = n$, then by (15) and (19) we get the estimate

$$\sum_{l\in\mathbb{Z}} \left|\xi_{n,l}[n,m]\right|^2 \leq C \sum_{l\in\mathbb{Z}} |G_0\delta_0[m-l] - 2\pi^{-1}(-1)^{m+l}\beta[n,l]/(n+il)|^2. \quad (21)$$

Set $c_{0,l} := 0$ for every $l \in \mathbb{Z}$ and

$$c[k,l] := (2\pi)^{-1}\mathrm{sgn}(k)e^{-\pi/4} \cdot \mathcal{F}(\mathcal{F}^{-1}(h_k)\cdot g)[l], \text{ if } k \neq 0,$$

where $h_k[l] := (-1)^{k+l}(k+il)^{-1}$. Then $\beta[k,l] = \mathrm{sgn}(k)e^{-\pi/4}$ and therefore the right hand side of equation (21) converges for every $(n,m) \in \mathbb{Z}^2$. Moreover, we have

$$\sum_{\substack{k,l\in\mathbb{Z}\\ k\neq n}} \left|\xi_{k,l}[n,m]\right|^2$$

$$\leq C \sum_{\substack{k,l\in\mathbb{Z}\\ k\neq n}} \frac{\left|H_{k-n} - \mathrm{sgn}(k)e^{-\pi/4}\right|^2}{(n-k)^2+(m-l)^2} + \frac{e^{-\pi/4}\cdot(1-\delta_0[k])\cdot(n^2+m^2)}{(k^2+l^2)\cdot\big((n-k)^2+(m-l)^2\big)}.$$

It is easy to see that summing up only the second terms yields a finite expression and the sum over the first terms can be estimated as follows

$$\sum_{\substack{k,l\in\mathbb{Z}\\ k\neq n}} \frac{\left|H_{k-n} - \mathrm{sgn}(k)e^{-\pi/4}\right|^2}{(n-k)^2+(m-l)^2} \leq \sum_{\substack{k\in\mathbb{Z}\\ k\neq n}} \left|H_{k-n} - \mathrm{sgn}(k)e^{-\pi/4}\right|^2 \sum_{l\in\mathbb{Z}} \frac{1}{1+l^2},$$

which is finite for every $(n,m) \in \mathbb{Z}^2$. All in all, we have shown that (18) holds for every $(n,m) \in \mathbb{Z}^2$. This concludes the proof. □

4.4. Non-existence of shift-invariant reproducing partners

We conclude this paper by showing that any reproducing partner for $\mathcal{G}$ is necessarily unstructured in the sense that it cannot be written as a shift-invariant system in time or in frequency. For more information on shift-invariant systems see for example [10, Chapter 8].

Proposition 14. *There exists no shift-invariant dual system for $\mathcal{G}$, that is, any reproducing partner Ψ for $\mathcal{G}$ cannot be written as $\psi_{n,m} = T_n\psi_m$ or $\psi_{n,m} = M_m\psi_n$.*

Proof: Let us write $\psi_{n,m}$ as a Gabor expansion with respect to the orthonormal basis $G(\chi)$. If we assume that $\psi_{n,m} = T_n \psi_m$, then

$$\psi_{0,m} = T_0 \psi_m = T_{-n} \psi_{n,m} = \sum_{k,l \in \mathbb{Z}} \overline{\xi_{k,l}[n,m]} T_{k-n} M_l \chi$$

$$= \sum_{k,l \in \mathbb{Z}} \overline{\xi_{k+n,l}[n,m]} T_k M_l \chi,$$

which implies that $\xi_{k,l}[0,m] = \xi_{k+n,l}[n,m]$, for all $(n,m,k,l) \in \mathbb{Z}^4$. Using (17),

$$\xi_{k+n,l}[n,m] = \mathcal{S}_{k+n,l}\xi_0[n,m] + p_{k+n,l}[n,m]$$

$$= \xi_0[-k, m-l] + p_{k+n,l}[n,m] = \mathcal{S}_{k,l}\xi_0[0,m] + p_{k+n,l}[n,m],$$

it follows that

$$p_{k,l}[0,m] = p_{k+n,l}[n,m], \ \forall \ n,m,k,l \in \mathbb{Z}. \tag{22}$$

In the following, we set $c_0 := c_{0,0}$ and let $\alpha, \beta \in \mathbb{N}_0^2$. By Corollary 11 and equation (22), one has

$$(-1)^m \sum_{\substack{|\beta| \leq N \\ \beta_1 = 0}} c_\beta[k,l] \cdot m^{\beta_2} = (-1)^{n+m} \sum_{|\alpha| \leq N} c_\alpha[k+n,l] \cdot n^{\alpha_1} m^{\alpha_2}.$$

Setting $k = s - n$, $m = 0$ then yields

$$c_0[s-n,l] = (-1)^n \sum_{\substack{|\alpha| \leq N \\ \alpha_2 = 0}} c_\alpha[s,l] \cdot n^{\alpha_1}.$$

If there exists $(s,l) \in \mathbb{Z}^2$, such that

$$\sum_{\substack{|\alpha| \leq N \\ \alpha_2 = 0}} c_\alpha[s,l] \cdot n^{\alpha_1} \neq 0 \ \text{ for some } n \in \mathbb{Z}, \tag{23}$$

then either $|c_0[s-n,l]| \to \infty$ as $|n| \to \infty$, or $c_0[s-n,l] = (-1)^n c_0[s,l] \neq 0$ for all $n \in \mathbb{Z}$, which implies $|c_0[\cdot,l]| \equiv C > 0$. As $|\xi_0[k,l]| \to 0$ for $|(k,l)| \to \infty$, we obtain for both cases that

$$\sum_{k,l \in \mathbb{Z}} |\xi_{k,l}[0,0]|^2 = \infty. \tag{24}$$

If there exist no $(s,l) \in \mathbb{Z}^2$, such that equation (23) holds, then $p_{k,l}[0,0] \equiv 0$ and consequently $\xi_{k,l}[0,0] = \xi_0[-k,-l]$. Hence, (24) holds and $\psi_{0,0}$ is not well defined.

An analogous argument can be used to show that $\psi_{n,m}$ cannot be given by $M_m\psi_n$. $\square$

5. Conclusion

It appears that the obstructions to Gabor frames as portrayed for example in [17] are preserved if one leaves the setup of frame theory and considers reproducing pairs consisting of two Gabor families instead. Already minor decay conditions on the window functions exclude the possibility to form reproducing pairs.

We have seen that the concept of reproducing pairs provides new insights on complete vector systems. In particular, the characterization given in Theorem 3 has proven to be a useful tool for different vector families, see also [5].

Here, we used Theorem 3 to show the existence of dual systems for the system of integer time-frequency shifts of the Gaussian. The crucial point in our argument is to estimate the behaviour of ξ_0 and choose appropriate elements of the kernel of $D_{\mathcal{G}}$. We believe that the same recipe will work for other windows, like the Hermite functions.

Acknowledgement

Funding: This work was funded by the Austrian Science Fund (FWF) START-project FLAME ['Frames and Linear Operators for Acoustical Modeling and Parameter Estimation'; Y 551-N13].

The authors would like to thank Tomasz Hrycak and Hartmut Führ for valuable discussions.

[1] A. Aldroubi and K. Gröchenig. Non-uniform sampling and reconstruction in shift-invariant spaces. *SIAM Review*, 43:585–620, 2001.

[2] S. T. Ali, J-P. Antoine, and J-P. Gazeau. *Coherent States, Wavelets and their Generalizations*. Graduate texts in contemporary physics. Springer, 2nd edition, 2014.

[3] J-P. Antoine and P. Balazs. Frames and semi-frames. *J. Phys. A: Math. Theor.*, 44(20):205201, 2011.

[4] J-P. Antoine and P. Balazs. Frames, semi-frames and Hilbert scales. *Numer. Func. Anal. Opt.*, 33:736–769, 2012.

[5] J-P. Antoine, M. Speckbacher, and C. Trapani. Reproducing pairs of measurable functions. *Acta Appl. Math.*, 2017.

[6] H. Bacry, A. Grossmann, and J. Zak. Proof of completeness of lattice states in the kq representation. *Phys. Rev. B*, 12(4), 1975.

[7] V. Bargmann, P. Butera, L. Girardello, and J. R. Klauder. On the completeness of the coherent states. *Rep. Math. Phys.*, 2(4):221–228, 1971.

[8] M. J. Bastiaans. Gabor's expansion of a signal into Gaussian elementary signals. *Proc. IEEE*, 68(4):538–539, 1980.

[9] O. Christensen. *An Introduction to Frames and Riesz Bases.* Applied and Numerical Harmonic Analysis. Birkhäuser, 2003.

[10] O. Christensen. *Frames and Bases - An Introductory Course*. Birkhäuser Boston, 2008.

[11] I. Daubechies and A. J. E. M. Janssen. Two theorems on lattice expansions. *IEEE Trans. Inform. Theory*, 39(1):3–6, 1993.

[12] R. J. Duffin and A. C. Schaeffer. A class of nonharmonic Fourier series. *Trans. Amer. Math. Soc.*, 72:341–366, 1952.

[13] H. G. Feichtinger. Modulation spaces: Looking back and ahead. *Samp. Theory Signal Image Process.*, 5:109–140, 2006.

[14] H. G. Feichtinger and G. Zimmermann. A Banach space of test functions for Gabor analysis. In H.G. Feichtinger and T. Strohmer, editors, *Gabor analysis and Algorithms - Theory and Applications*. Birkhäuser Boston, 1998.

[15] D. Gabor. Theory of communication. Part 1: The analysis of information. *J. Inst. Electr. Eng. 3*, 93:429–441, November 1946.

[16] K. Gröchenig. *Foundations of Time-Frequency Analysis*. Appl. Numer. Harmon. Anal. Birkhäuser Boston, 2001.

[17] K. Gröchenig. The mystery of Gabor frames. *J. Fourier Anal. Appl.*, 20(4):865–895, 2014.

[18] C. Heil and M. Powell. Gabor Schauder bases and the Balian-Low theorem. *J. Math. Phys.*, 74, 2006.

[19] A. J. E. M. Janssen. Gabor representation of generalized functions. *J. Math. Anal. Appl.*, 83:377–394, 1981.

[20] A. J. E. M. Janssen. Weighted Wigner distributions vanishing on lattices. *J. Math. Anal. Appl.*, 80:156–167, 1981.

[21] A. J. E. M. Janssen. Bargmann transform, Zak transform, and coherent states. *J. Math. Phys.*, 23(5):720–731, 1982.

[22] A. J. E. M. Janssen. Zak transform characterization of S_0. *Samp. Theory Signal Image Process.*, 5(2):141–162, 2006.

[23] A. M. Perelomov. On the completeness of a system of coherent states. *Theoret. Math. Phys.*, 6(2):156–164, 1971.

[24] W. Rudin. *Functional analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill Inc., New York, second edition, 1991.

[25] M. Speckbacher and P. Balazs. Reproducing pairs and the continuous nonstationary Gabor transform on LCA groups. *J. Phys. A: Math. Theor.*, 48(395201), 2015.

[26] J. von Neumann. *Mathematische Grundlagen der Quantenmechanik*. Springer, Berlin, 1932. English translation: "Mathematical Foundations of Quantum Mechanics," Princeton Univ. Press, 1955.

[27] E. T. Whittaker and G. N. Watson. *A Course of Modern Analysis*. Cambridge University Press, 4th edition, 1996.