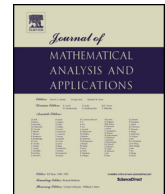




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Spectrum of the Iwatsuka Hamiltonian at thresholds

Pablo Miranda^a, Nicolas Popoff^{b,*}

^a Departamento de Matemática y Ciencia de la Computación, Universidad de Santiago de Chile, Las Sophoras 173, Santiago, Chile

^b Université de Bordeaux, IMB, UMR 5251, 33405 Talence cedex, France

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ABSTRACT

We consider the bi-dimensional Schrödinger operator with unidirectionally constant magnetic field, H_0 , sometimes known as the “Iwatsuka Hamiltonian”. This operator is analytically fibered, with band functions converging to finite limits at infinity. We first obtain the asymptotic behavior of the band functions and its derivatives. Using this results we give estimates on the current and on the localization of states whose energy value is close to a given *threshold* in the spectrum of H_0 . In addition, for non-negative electric perturbations V we study the spectral properties of $H_0 \pm V$, by considering the Spectral Shift Function associated to the operator pair $(H_0 \pm V, H_0)$. We compute the asymptotic behavior of the Spectral Shift Function at the thresholds, which are the only points where it can grow to infinity.

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1. Introduction

1.1. Context and motivation

Let b be a scalar magnetic field in $\mathbb{R}^2$ that is translationally invariant in the sense that it does not depend on one of the spatial variables: $b(x, y) = b(x)$. Let $a : \mathbb{R} \rightarrow \mathbb{R}$ be the function $a(x) = \int_0^x b(t)dt$. Then the potential $A(x) := (0, a(x))$ satisfies $\text{curl } A = b$. In this article we study the magnetic Schrödinger operator

$$H_0 := (-i\nabla - A)^2 = -\partial_x^2 + (-i\partial_y - a(x))^2$$

acting in $L^2(\mathbb{R}^2)$.

In [20] Akira Iwatsuka considered this class of Hamiltonians in order to show new examples of magnetic Schrödinger operators with purely absolutely continuous spectrum. This model has many interesting properties, and some of them have been well studied in the past. For instance, in the physics literature it

* Corresponding author.

E-mail addresses: pablo.miranda.r@usach.cl (P. Miranda), Nicolas.Popoff@math.u-bordeaux1.fr (N. Popoff).

can be seen as an open quantum waveguide in the sense that the presence of such magnetic field generates transport for a spinless particle in a plane (see [25] for references). Rigorous propagation properties of this model are described for example in [23,22,10,19]. Additionally, the spectral properties of H_0 have also been studied. In this sense, one important conjecture on the spectrum of H_0 is that it should be purely absolutely continuous as long as b is non-constant. However, although it have been found different conditions on b that ensure this property, the general result remains unproved (see [20,23,14,32]).

In order to describe the problems that we propose to study in this article it is necessary to write a well known decomposition of the operator H_0 .

Fiber decomposition of H_0 . Let $\mathcal{F}$ be the partial Fourier transform with respect to the translationally invariant variable $y \in \mathbb{R}$:

$$(\mathcal{F}u)(x, k) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} e^{-iky} u(x, y) dy, \quad \text{for } u \in C_0^\infty(\mathbb{R}^2).$$

Then

$$\mathcal{F}H_0\mathcal{F}^* = \int_{\mathbb{R}}^{\oplus} h(k) dk, \quad (1)$$

where $h(k)$ is a self-adjoint operator acting in $L^2(\mathbb{R})$, defined by

$$h(k) = -\frac{d^2}{dx^2} + (a(x) - k)^2, \quad k \in \mathbb{R}. \quad (2)$$

For any $k \in \mathbb{R}$, under hypotheses (3) below, $h(k)$ has compact resolvent, therefore its spectrum is discrete, and moreover, it is simple. We denote the increasing sequence of eigenvalues by $\{E_n(k)\}_{n=1}^\infty$. For any $n \in \mathbb{N}$, the *band function* $E_n(\cdot)$ is analytic as a function of $k \in \mathbb{R}$ [20].

In a broad sense, the results of this article are valid for magnetic fields that satisfy:

- a) $b \in C^\infty(\mathbb{R})$.
 - b) b is increasing.
 - c) $\lim_{x \rightarrow \pm\infty} b(x) = b_\pm$, for $b_+ > b_- > 0$.
- (3)

Even more, each one of these hypotheses can be relaxed, but the results are much easier to read for these kind of magnetic fields.

Set $\underline{\mathcal{E}}_n := b_-(2n-1)$ and $\overline{\mathcal{E}}_n := b_+(2n-1)$, then, condition (3) implies that $\underline{\mathcal{E}}_n \leq E_n(k) \leq \overline{\mathcal{E}}_n$ for all $k \in \mathbb{R}$. Furthermore, $E_n(\cdot)$ is strictly increasing for any $n \in \mathbb{N}$, and $\lim_{k \rightarrow +\infty} E_n(k) = \overline{\mathcal{E}}_n$; $\lim_{k \rightarrow -\infty} E_n(k) = \underline{\mathcal{E}}_n$ for all $n \in \mathbb{N}$, see [20,23]. In particular, the spectrum of H_0 , $\sigma(H_0)$, is purely absolutely continuous and

$$\sigma(H_0) = \bigcup_{n \geq 1} \overline{E_n(\mathbb{R})} = \bigcup_{n \geq 1} [\underline{\mathcal{E}}_n, \overline{\mathcal{E}}_n]. \quad (4)$$

Moreover, using decomposition (1) and the monotonicity of E_n it is possible to see that the multiplicity of the spectrum of H_0 changes at any point in the set of points $\{\underline{\mathcal{E}}_n, \overline{\mathcal{E}}_n\}_{n=1}^\infty$. This set of points will be referred as *thresholds* in $\sigma(H_0)$. We denote it by $\mathcal{T}_{H_0}$.

1.2. Description of the main results of the article

The first step in the physical description of our magnetic system at energies near thresholds, requires to precise the behavior of the band functions when $k \rightarrow \pm\infty$. Therefore, our first task in this article is to

describe qualitatively this behavior. We will do this for magnetic fields converging to their limits as power decaying functions, with similar hypotheses on its derivatives (see (5)). Under this generic hypothesis, the asymptotics of the band functions enters in the framework of the semi-classical harmonic approximation, see Remark 2.5. Note that magnetic fields converging exponentially to their limit, or constant at infinity, do not enter in our analysis and would require different tools for the asymptotics of the band functions at infinity. Nevertheless, step magnetic fields could be considered for the purposes of this article (see Remark 4.6).

In Theorem 2.2 we obtain the first three asymptotic terms of $k \mapsto E_n(k)$, which are roughly given by $(2n-1)b(\frac{k}{b_+})$. We also give the asymptotic behavior of the first and second derivative, which can be formally obtained as the derivatives of the asymptotic term of E_n . A byproduct of our analysis in section 2 is an estimate of the eigenfunctions by a linear combination of Hermite's functions.

In section 3, we exploit these results to obtain properties of functions localized in energy near a given threshold. This problematic arises from physical consideration: it consists in taking a quantum state of a given energy, and to study its evolution under the magnetic system considered. We are particularly interested in the propagation properties in the translationally invariant direction (that is the y direction). As explained in [23,19], a state localized in energy away from thresholds bear a current which can be bounded from below. Moreover, it is localized in the x variable, i.e. it decays in the direction in which the magnetic field varies. On the contrary, when the energy interval contains a thresholds, no lower bound on the current is available. Inspired by the approach of [18], we study the *bulk component* of a quantum states localized in energy near a given threshold. Using the asymptotics of the band functions and its derivatives, we are able to provide a quantitative control on the current carried by the given state, as the distance from its energy to the threshold goes to 0 (see Proposition 3.1). Further, the asymptotics of the eigenfunctions of the fiber operator allow us to prove that a state localized in a energy near a threshold is concentrated in a zone where the magnetic field is close to its limit. See Theorem 3.3.

In sections 4–6 we will perturb the operator H_0 , and consider the spectral properties of the resulting operator. More specifically, we will take a non-negative decaying electric potential V and will define $H := H_0 \pm V$. Under these hypotheses the essential spectrum of H coincide with the essential spectrum of H_0 . However, discrete eigenvalues may appear. Counting the number of such eigenvalues has been considered for b constant and non-constant (for the Landau case, $b = \text{constant}$, see for instance [28,29,27,30], and for Iwatsuka [31,11,24]). In this article we study an extension of this problem to the continuous spectrum of H , namely, we describe the properties of the spectral shift function (SSF) for the operator pair (H, H_0) . (The connection between the eigenvalue counting function and the SSF is given by (39).)

In Theorem 4.1 we prove that the SSF is continuous everywhere except at the thresholds in $\mathcal{T}_{H_0}$ and the eigenvalues of H . Moreover, we show that it is a bounded function in compact subsets of $\mathbb{R} \setminus \mathcal{T}_{H_0}$. Subsequently, Theorem 4.3 states that the SSF is bounded at the thresholds as well, as long as the rate of convergence of V to zero is large compared to the rate of convergence of $b(x)$ to $b_{\pm}$. On the contrary, in Theorem 4.4 we describe the behavior of the SSF at the thresholds, for certain potentials V that decay moderately at infinity. In this case, the SSF is unbounded as a fixed threshold is approached. This result is given by the semiclassical formula (46) which relates the behavior of the SSF with a volume in phase space determined by the potential V , and partly extend some of the results obtained in the above articles on the Landau and Iwatsuka Hamiltonians.

The approach we use here to study the SSF comes somehow from [28], where the discrete spectrum of the Landau Hamiltonian was described. This approach was also used to analyze the discrete spectrum of the Iwatsuka Hamiltonian in [24]. However, in the aforementioned articles it was not necessary to make a detailed analysis of the corresponding band functions; they are constants in the Landau case and some variational estimates are enough for the Iwatsuka case. There are other models where the asymptotics of band functions was necessary to make the spectral analysis, the closest one being a half-plane submitted to a constant magnetic field ([5,18,4]). One of the differences with our situation is that we consider a whole

class of varying magnetic field, and in these references the magnetic field is constant, allowing, for instance, the use of special functions to study the fiber operator.

In this article, we will state almost all of our results for the thresholds $\{\bar{\mathcal{E}}_n\}_{n=1}^\infty$. Since these are thresholds corresponding to the fact that $b(x) \rightarrow b_+$ as $x \rightarrow +\infty$, we only write the assumptions on b for $x \rightarrow +\infty$. It is clear that the same assumptions as $x \rightarrow -\infty$ would lead to symmetric results for the thresholds $\{\underline{\mathcal{E}}_n\}_{n=1}^\infty$.

2. Asymptotics of the band functions

We will assume:

- a) $\exists x_0 \in \mathbb{R}, \forall p \in \mathbb{N}, \forall x \geq x_0, \quad b^{(p)}(x) \neq 0.$
- b) $b'(x) = o(b_+ - b(x)),$ and $\forall p \in \mathbb{N}, \quad b^{(p+1)}(x) = o(b^{(p)}(x)); \quad x \rightarrow +\infty.$
- c) $(b')^2(x) = o(b''(x)); \quad x \rightarrow +\infty.$

These hypothesis will be naturally satisfied for magnetic fields that converge to b_+ as a negative power of x . Therefore, in the following sections we will exploit our results for this kind of magnetic fields (see assumption (10) below).

Notation 2.1. Denote by $x_k := a^{-1}(k)$, the point where the potential $(a(x) - k)^2$ attains its minimum; $b_k := b(x_k)$ and for $p \geq 1, b_k^{(p)} := b^{(p)}(x_k)$.

For $\sigma > 0$ and $\tau > 0$, define the function

$$\epsilon_{\sigma,\tau}(k) := (b'_k)^2 + \sup_{x \in (\sigma k, +\infty)} \{|b'b''(x)| + |b^{(3)}(x)|\} + e^{-\tau k^2}. \quad (6)$$

For readability, we will write $\epsilon(k) := \epsilon_{\sigma,\tau}(k)$.

Theorem 2.2. Assume that b satisfies (3) and (5), then for any $n \in \mathbb{N}$, there exist $\sigma > 0$ and $\tau > 0$ such that

$$E_n(k) = b_k \Lambda_n + \gamma_n b_k^{-1} b''_k + O(\epsilon(k)), \quad k \rightarrow \infty, \quad (7)$$

with $\Lambda_n := 2n - 1$ and $\gamma_n := \frac{1}{4}(2n^2 - 2n + 1)$.

Moreover, the derivatives have an asymptotic expansion that is the formal derivative of E_n , up to the remainder:

$$E'_n(k) = b_k^{-1} \Lambda_n b'_k + O(\epsilon(k)), \quad k \rightarrow \infty, \quad (8)$$

and

$$E''_n(k) = b_k^{-2} \Lambda_n b''_k + O(\epsilon(k)), \quad k \rightarrow \infty. \quad (9)$$

In the above asymptotics, the remainders depends on n .

Remark 2.3. It is possible to precise the range of values of the constants involved in Definition (6): The theorem holds true at least for all constants $(\sigma, \tau) \in (0, \frac{1}{b_+} - \frac{\sigma_0}{(b_+)^{1/2}}) \times (0, \frac{\sigma_0^2}{2})$, where $\sigma_0 \in (0, \frac{1}{(b_+)^{1/2}})$.

We will exploit Theorem 2.2 in the case that $b_+ - b(x)$ is power-like decaying i.e., we are interested in magnetic fields satisfying $b_+ - b(x) \sim x^{-M}$. However, instead of making assumptions on the derivatives of b ,

as in (5), and in order to write the results of the following sections concisely, we will consider the model case:

$$\exists x_0 > 0, \forall x \geq x_0, \quad b_+ - b(x) = \frac{1}{x^M} \quad \text{with } M > 0. \quad (10)$$

Corollary 2.4. Assume (3) and (10). Set $\eta = \min(2M + 2, M + 3)$, then, as $k \rightarrow +\infty$:

$$E_n(k) - \bar{\mathcal{E}}_n = -\frac{\Lambda_n b_+^M}{k^M} + O\left(\frac{1}{k^{M+2}}\right).$$

$$E'_n(k) = M \frac{\Lambda_n b_+^M}{k^{M+1}} + O\left(\frac{1}{k^\eta}\right) \quad \text{and} \quad E''_n(k) = -M(M+1) \frac{\Lambda_n b_+^M}{k^{M+2}} + O\left(\frac{1}{k^\eta}\right).$$

Although this corollary is a direct consequence of Theorem 2.2, it can also be seen as the first terms of the asymptotics provided by the harmonic approximation for a semi-classical Schrödinger operator (see Remark 2.5).

2.1. Construction of quasi-modes

In order to prove Theorem 2.2 we will apply the standard procedure of constructing quasi-modes, for $h(k)$ in (2) which is a Sturm–Liouville operator. The potential $(a(x) - k)^2$ vanishes only at the point $x = x_k = a^{-1}(k)$. Near this point, its formal Taylor expansion is $b_k^2(x - x_k)^2 + O(x^3)$, where the remainder depends on k . Therefore, guided by the semiclassical approximation, we perform the change of variables

$$t = b_k^{1/2}(x - x_k), \quad (11)$$

and the operator $h(k)$ becomes

$$b_k(-\partial_t^2 + w(t, k)), \quad t \in \mathbb{R}, \quad (12)$$

where

$$w(t, k) := b_k^{-1}(a(b_k^{-1/2}t + x_k) - k)^2.$$

We compute the derivatives of w , with the notation $x = b_k^{-1/2}t + x_k$:

$$\begin{aligned} w'(t, k) &= 2b_k^{-3/2}(a(x) - k)b(x) \\ w^{(2)}(t, k) &= 2b_k^{-2}(b(x)^2 + (a(x) - k)b'(x)) \\ w^{(3)}(t, k) &= 2b_k^{-5/2}(3b(x)b'(x) + (a(x) - k)b''(x)) \\ w^{(4)}(t, k) &= 2b_k^{-3}(3b'(x)^2 + 4b(x)b''(x) + (a(x) - k)b^{(3)}(x)). \end{aligned}$$

In consequence, the Taylor expansion of the normalized potential w near $t = 0$ writes

$$w(t, k) = t^2 + \alpha_1(k)t^3 + \alpha_2(k)t^4 + R_4(t, k), \quad (13)$$

with

$$\alpha_1(k) = b_k^{-3/2}b'_k \quad \text{and} \quad \alpha_2(k) = \frac{1}{4}b_k^{-3}(b'_k)^2 + \frac{1}{3}b_k^{-2}b''_k. \quad (14)$$

Remark 2.5. The tools used here are closed to those of the semi-classical harmonic approximation ([9]). Notice that by trying the scaling $X = x_k x$ and setting $\hbar = x_k^{-2}$, we are led (up to factor) to an operator of the form $-\hbar^2 \partial_X^2 + W(X, \hbar)$, where the potential W has a unique non degenerate minimum, and a Taylor expansion near the minimum in powers of X , depending on $\hbar$ but also on $b_k^{(p)}$. The hypotheses (5) express then conditions of the potential W so that this operator enters indeed in the framework of harmonic approximation. Under condition (10), the potential W has an expansion with powers of X , and with various powers of $\hbar$ (and therefore in k^{-1}), but due to the different scales, it is not clear that a full asymptotic expansion in power of k^{-1} exists.

If Hypothesis (5) is not fulfilled, such as in the case of a magnetic field constant at infinity, the asymptotics of the band function does not depend anymore only on the behavior of W at its minimum, but on more global quantities, and would require different tools in its analysis. Up to our knowledge, there is no work on the low-lying spectrum of $-\hbar^2 \partial_X^2 + W$ when W is flat at its minimum.

From (12) and (13) we are led to compute the spectrum of the formal operator $h_0 + w_1 + w_2 + R_4$ with

$$h_0 = -\partial_t^2 + t^2, \quad w_1 = \alpha_1 t^3, \quad w_2 = \alpha_2 t^4, \quad R_4(t, k) = w(t, k) - (t^2 + \alpha_1 t^3 + \alpha_2 t^4). \quad (15)$$

We will consider these operators formally, and construct a quasi eigenpair of the form

$$(\mu_0 + \mu_1 + \mu_2, \varphi_0 + \varphi_1 + \varphi_2), \quad (16)$$

where this ansatz is adapted to perturbation theory, as follows:

2.1.1. The engine

Let Ψ_n be the n -th normalized Hermite's function, starting from $n = 1$ (see for example [1]).

The order 0 term leads to $(h_0 - \mu_0)\varphi_0 = 0$, which is solved by taking

$$\mu_0 = \Lambda_n \quad \text{and} \quad \varphi_0 = \Psi_n.$$

2.1.2. First order

We look for a pair (μ_1, φ_1) such that $(h_0 - \mu_0)\varphi_1 = (\mu_1 - w_1)\varphi_0$. The Fredholm alternative writes $\langle (\mu_1 - w_1)\Psi_n, \Psi_n \rangle = 0$, therefore, using the symmetry of Ψ_n and the oddness of w_1 , we get

$$\mu_1 = 0 \quad \text{and} \quad \varphi_1 = -(h_0 - \mu_0)^{-1}(w_1 \Psi_n).$$

Taking into account that

$$t\Psi_n(t) = \sqrt{\frac{n-1}{2}}\Psi_{n-1} + \sqrt{\frac{n}{2}}\Psi_{n+1}, \quad (17)$$

we obtain

$$\begin{aligned} t^3\Psi_n &= 2^{-3/2}(\sqrt{(n-1)(n-2)(n-3)}\Psi_{n-3} + 3(n-1)\sqrt{n-1}\Psi_{n-1} \\ &\quad + 3n\sqrt{n}\Psi_{n+1} + \sqrt{n(n+1)(n+2)}\Psi_{n+3}). \end{aligned} \quad (18)$$

Hence

$$\begin{aligned} \varphi_1 &= -\alpha_1(k)2^{-5/2}\left(-\frac{1}{3}\sqrt{(n-1)(n-2)(n-3)}\Psi_{n-3} - 3(n-1)\sqrt{n-1}\Psi_{n-1} \right. \\ &\quad \left. + 3n\sqrt{n}\Psi_{n+1} + \frac{1}{3}\sqrt{n(n+1)(n+2)}\Psi_{n+3}\right). \end{aligned} \quad (19)$$

2.1.3. Second order

We have that $w_1\varphi_1 = O(\alpha_1^2)$, therefore we shall give priority to the term $w_2\varphi_0$ since $\alpha_1^2 = o(\alpha_2)$, see (5). These considerations bring us to solve the equation $(h_0 - \mu_0)\varphi_2 = (\mu_2 - w_2)\varphi_0$, and as above we get $\mu_2 = \langle w_2\varphi_0, \varphi_0 \rangle$ and $\varphi_2 = (h_0 - E_0)^{-1}(\mu_2 - w_2)\varphi_0$. Computations using (17) and (18) provides

$$\langle t^4\Psi_n, \Psi_n \rangle = \langle t^3\Psi_n, t\Psi_n \rangle = \frac{3}{4}(2n^2 - 2n + 1) := \gamma_n,$$

implying that

$$\mu_2 = \frac{3}{4}(2n^2 - 2n + 1)\alpha_2.$$

Moreover, φ_2 has the form

$$\varphi_2 = \alpha_2(k) \sum_{p=-2}^2 c_p \Psi_{n+2p},$$

where, by construction $c_0 = 0$ and

$$\forall n \geq 1, \begin{cases} c_{-2} = -\frac{1}{32}\sqrt{(n-1)(n-2)(n-3)(n-4)} \\ c_{-1} = -\frac{1}{16}\sqrt{(n-1)(n-2)(4n-6)} \\ c_1 = \frac{1}{16}\sqrt{n(n+1)(4n+2)} \\ c_2 = \frac{1}{32}\sqrt{n(n+1)(n+2)(n+3)}. \end{cases} \quad (20)$$

2.2. Application of the spectral theorem and proof of the asymptotics of the band function

Denote by $(\mu, v_n^{\text{qm}}(\cdot, k))$ the quasimodes constructed above according to the ansatz (16). Note that v_n^{qm} is obviously in the domain of $-\partial_t^2 + w$. Then, by construction

$$((-\partial_t^2 + w) - \mu)v_n^{\text{qm}} = w_1\varphi_1 + w_1\varphi_2 + w_2\varphi_1 - \mu_2\varphi_1 - \mu_2\varphi_2 + \omega_2\varphi_2 + R_4v_n^{\text{qm}}.$$

Let $\eta_n(k)$ be the norm of the above quantity. Accordingly

$$\eta_n(k) = \|R_4v_n^{\text{qm}}\| + O(\alpha_1^2) + O(\alpha_1\alpha_2) = \|R_4v_n^{\text{qm}}\| + O(\alpha_1^2), \quad (21)$$

where we have used condition (5). Recalling that R_4 is the fourth order Taylor remainder of w , we get a constant $C > 0$ such that

$$\forall k \in \mathbb{R}, \quad \|R_4v_n^{\text{qm}}(\cdot, k)\|^2 \leq C \int_{\mathbb{R}} \left(\sup_{|s| < t} |w^{(5)}(s, k)| |t^5| \right)^2 v_n^{\text{qm}}(t, k)^2 dt,$$

with

$$w^{(5)}(t, k) = 2b_k^{-7/2}(10b'(x)b''(x) + 5b(x)b^{(3)}(x) + (a(x) - k)b^{(4)}(x)).$$

Let $\sigma_0 \in (0, \frac{1}{(b_+)^{1/2}})$, we consider the above integral in two pieces, $|t| < \sigma_0 k$ and $|t| > \sigma_0 k$. First we treat the part $|t| < \sigma_0 k$, where the potential $w^{(5)}$ is small: We have $|a(x) - k| \leq c|t|$ for all $(t, k) \in \mathbb{R} \times \mathbb{R}$ with $c > 0$ a constant, see (11). Therefore, there exists a polynomial P_n (whose coefficients do not depend on k) such that

$$\begin{aligned}
I_1 &:= \int_{|t| < \sigma_0 k} \left(\sup_{|s| < t} |w^{(5)}(s, k)| |t|^5 \right)^2 v_n^{\text{qm}}(t, k)^2 dt \\
&\leq C \left(\sup_{|t| < \sigma_0 k} (|b'b''| + |b^{(3)}| + |b^{(4)}|)(x) \right)^2 \int_{|t| < \sigma_0 k} P_n(t) e^{-t^2} dt.
\end{aligned}$$

Let $\sigma \in (0, \frac{1}{b_+} - \frac{\sigma_0}{(b_+)^{1/2}})$. Recalling (11), we see that $|t| < \sigma_0 k$ is equivalent to $|x - x_k| \leq \frac{\sigma_0}{b_k^{1/2}} k$. Using that $x_k \geq \frac{k}{b_+}$, we obtain $x > (\frac{1}{b_+} - \frac{\sigma_0}{b_k^{1/2}})k \geq \sigma k$ for k large enough, since $b_k \rightarrow b_+$ as $k \rightarrow \infty$. Then, by assumption on b , we have $b^{(4)} = o(b^{(3)})$ and we find another constant $C > 0$ such that for k large enough

$$I_1 \leq C \sup_{(\sigma k, +\infty)} (|b'b''| + |b^{(3)}|)^2.$$

For the part $|t| > \sigma_0 k$, we will use that the quasi-mode has exponential decay. By similar estimates as above, we obtain

$$\begin{aligned}
I_2 &:= \int_{|t| > \sigma_0 k} \left(\sup_{|s| < t} w^{(5)}(s, k) t^5 \right)^2 v_n^{\text{qm}}(t, k)^2 dt \\
&\leq C \int_{|t| > \sigma_0 k} P_n(t) e^{-t^2} dt \leq C(\tau) e^{-\tau k^2},
\end{aligned}$$

for all $\tau \in (0, \sigma_0^2)$. Therefore, using that $I_1 + I_2 = \|R_4 v_n^{\text{qm}}\|^2$, we get

$$\|R_4 v_n^{\text{qm}}\| = O\left(\sup_{(\sigma k, +\infty)} (|b'b''| + |b^{(3)}|) + e^{-\tau \frac{k^2}{2}}\right). \quad (22)$$

Combining (22) with (21), and recalling the definition of ϵ in (6), we can see that

$$\eta_n(k) = O(\epsilon(k)). \quad (23)$$

Finally, noticing that $\|v_n^{\text{qm}}\| = 1 + o(1)$ and applying the spectral theorem

$$|E_n(k) - b_k \mu| = O(\epsilon(k)),$$

which together with (5) c) implies the asymptotics (7).

Denote by q_k the quadratic form associated with $h(k)$. Each $E_n(k)$ is a simple eigenvalue of $h(k)$, and since each $E_n(k)$ converge to $\overline{\mathcal{E}_n}$ from below, there exists $c > 0$ and k_n such that

$$\forall k \geq k_n, \quad \text{dist}(E_n(k), \sigma(h(k)) \setminus \{E_n(k)\}) \geq c.$$

This spectral gap allows us to precise that the constructed quasi-modes are closed to the eigenfunctions of $h(k)$ in the following sense (see [16, Proposition 2.5]):

$$\|u_n - u_n^{\text{qm}}\| + \sqrt{q_k(u_n - u_n^{\text{qm}})} = O(\epsilon(k)), \quad (24)$$

where u_n^{qm} is the quasi-mode in the original variable, i.e.,

$$u_n^{\text{qm}}(x, k) := b_k^{1/4} v_n^{\text{qm}}(b_k^{1/2}(x - x_k), k), \quad (25)$$

and u_n is a normalized eigenfunction of $h(k)$ associated with $E_n(k)$.

2.3. Asymptotics of the derivatives

The standard Feynman–Hellman (FH) formula writes

$$\begin{aligned} E'_n(k) &= -2 \int_{\mathbb{R}} (a(x) - k) u_n(x, k)^2 dx \\ &= -2 \int_{\mathbb{R}} (a(x) - k) u_n^{\text{qm}}(x, k)^2 dx - 2 \int_{\mathbb{R}} (a(x) - k) (u_n(x, k)^2 - u_n^{\text{qm}}(x, k)^2) dx. \end{aligned}$$

The last term is easily controlled:

$$\begin{aligned} &\left| \int_{\mathbb{R}} (a(x) - k) (u_n(x, k)^2 - u_n^{\text{qm}}(x, k)^2) dx \right| \\ &\leq \left(\int_{\mathbb{R}} (a(x) - k)^2 (u_n(x, k) - u_n^{\text{qm}}(x, k))^2 dx \int_{\mathbb{R}} (u_n(x, k) + u_n^{\text{qm}}(x, k))^2 dx \right)^{1/2} \\ &\leq C q_k (u_n - u_n^{\text{qm}})^{1/2}, \end{aligned}$$

and is therefore $O(\epsilon(k))$, see (24). Now we have to compute the main term. We do the same change of variable (11) as above and we expand the potential:

$$a(tb_k^{-1/2} + x_k) - k = b_k^{1/2} (t + \beta_1 t^2 + \beta_2 t^3 + \tilde{R}_3),$$

with $\beta_1 = \frac{\alpha_1}{2}$ and $\beta_2 = \frac{\alpha_2}{2} - \frac{\alpha_1^2}{8}$, see (14). Recall that the quasi mode in these variables writes $v_n^{\text{qm}} = \varphi_0 + \varphi_1 + \varphi_2$, where φ_p are defined in Subsection 2.1. Then we get

$$\begin{aligned} &-2 \int_{\mathbb{R}} (a(x) - k) u_n^{\text{qm}}(x, k)^2 dx \\ &= -2b_k^{1/2} \int_{\mathbb{R}} (t + \beta_1 t^2 + \beta_2 t^3 + \tilde{R}_3(t)) (\varphi_0(t) + \varphi_1(t) + \varphi_2(t))^2 dt. \end{aligned}$$

Expanding the above integrand, we use that some terms are odd functions and their integral cancels, obtaining

$$E'_n(k) = -2b_k^{1/2} (T_1 + T_2 + T_3) + O(\epsilon_n(k)), \quad (26)$$

where

$$\begin{aligned} T_1 &= \int_{\mathbb{R}} (t + \beta_1 t^2) (\varphi_0(t) + \varphi_1(t))^2 dt, \\ T_2 &= \int_{\mathbb{R}} 2t\varphi_1\varphi_2 + 2\beta_1 t^2 \varphi_0\varphi_2 + \beta_1 t^2 \varphi_2^2 + 2\beta_2 t^3 \varphi_1(\varphi_0 + \varphi_2) dt, \quad \text{and} \quad T_3 = \int_{\mathbb{R}} \tilde{R}_3(t) v_n^{\text{qm}}(t, k)^2 dt. \end{aligned}$$

The term T_1 satisfies

$$T_1 = 2\langle t\varphi_0, \varphi_1 \rangle + \beta_1 \|t\varphi_0\|^2 + O(\alpha_1^2),$$

and it is known that $\|t\varphi_0\|^2 = \frac{\Lambda_n}{2}$. More tedious computations provide $\langle t\varphi_0, \varphi_1 \rangle = -\frac{3}{8}\alpha_1\Lambda_n$, so that

$$T_1 = -\frac{1}{2}\alpha_1\Lambda_n + O(\alpha_1^2).$$

Moreover, using our assumptions on the magnetic field, $T_2 = O(\alpha_1\alpha_2) = O(b'_k b''_k) = o((b'_k)^2)$. Finally, we estimate the term T_3 using the same arguments leading to (22), which give us $T_3 = O(\sup_{(\tilde{\sigma}k, +\infty)} b^{(3)}(k) + e^{-\tau k^2})$. Equation (8) is obtained by combining these last estimates with (26).

Concerning the second derivative, we have the formula:

$$E''_n(k) = -2 \left(-1 + 2 \int_{\mathbb{R}} (a(x) - k) \partial_k u_n(x, k) u_n(x, k) dx \right).$$

Note that $\partial_k u_n$ satisfies the following ODE:

$$(h(k) - E_n(k)) \partial_k u_n(x, k) = (E'_n(k) + 2(a(x) - k)) u_n(x, k).$$

Due to the FH formula, the r.h.s. is orthogonal to u_n and therefore we get

$$\partial_k u_n(\cdot, k) = (h(k) - E_n(k))^{-1} ((E'_n(k) + 2(a(\cdot) - k)) u_n(\cdot, k)).$$

We are left with the task of estimating

$$\int_{\mathbb{R}} (a(x) - k) u_n(x, k) (h(k) - E_n(k))^{-1} ((E'_n(k) + 2(a(\cdot) - k)) u_n(\cdot, k)) dx. \quad (27)$$

We proceed in the same change of variables and Taylor expansion as above and we use resolvent formulas in order to apply $(h(k) - E_n(k))^{-1}$. We don't give all details here since these computations are rather tedious and redundant with those given above. First, arguing as before, we can replace u_n by u_n^{qm} in formula (27). The factor $(a(x) - k) u_n^{\text{qm}}(x, k)$ is easily expressed with Hermite's function as in the previous section. Further, we get (the terms are ordered and grouped by decreasing order as $k \rightarrow +\infty$):

$$\begin{aligned} & (2(a(x) - k) + E'_n(k)) u_n^{\text{qm}}(x, k) \\ &= 2b_k^{1/2} t \varphi_0 + \left(E'_n \varphi_0 + 2b_k^{1/2} t \varphi_1 + b_k^{1/2} \alpha_1 t^2 \varphi_0 \right) + \left(2b_k^{1/2} t \varphi_2 + 2b_k^{1/2} \beta_2 t^3 \varphi_0 \right) + R_3(t, k) \\ &:= \tau_0 + \tau_1 + \tau_2 + R_3, \end{aligned} \quad (28)$$

with obvious notations, and where R_3 is a remainder. We want to apply the resolvent $(h(k) - E_n(k))^{-1}$ to the above expression. Using the resolvent formula we obtain formally

$$\begin{aligned} & (h(k) - E_n(k))^{-1} \\ &= b_k^{-1} (h_0 - \Lambda_n)^{-1} (\mathbb{I} - (b_k^{-1} E_n(k) - \Lambda_n - (\alpha_1 t^3 + \alpha_2 t^4 + R_4)) (h_0 - \Lambda_n)^{-1})^{-1} \\ &= b_k^{-1} (h_0 - \Lambda_n)^{-1} (\mathbb{I} + (b_k^{-1} E_n(k) - \Lambda_n - (\alpha_1 t^3 + \alpha_2 t^4 + R_4)) (h_0 - \Lambda_n)^{-1} + O(\alpha_1^2)) \end{aligned}$$

This formula is useful for applying the resolvent on τ_0 given in (28). For the terms τ_1 and τ_2 , which are respectively $O(\alpha_1)$ and $O(\alpha_2)$, we can apply directly $b_k^{-1}(h_0 - \Lambda_n)^{-1}$ instead of $(h(k) - E_n(k))^{-1}$ since the remainder will be $O(\epsilon)$.

As above, we express everything in the basis of Hermite's functions, neglecting all the term that are controlled by α_1^2 . The above constructions insures that there is no φ_0 in each of these terms, so that it is possible to apply $(h_0 - \Lambda_n)^{-1}$. The same estimates as above lead to the asymptotics of $E_n''(k)$ announced in Theorem 2.2.

3. States localized near the thresholds

To begin we introduce some notations. Let $\mathcal{U}_n : L^2(\mathbb{R}) \mapsto L^2(\mathbb{R} \times \mathbb{R})$ be the isometry defined by $(\mathcal{U}_n w)(x, k) := w(k)u_n(x, k)$. Its adjoint is $(\mathcal{U}_n^* v)(k) = \langle u_n(\cdot, k), v(\cdot, k) \rangle$, and we consider the projectors $\Pi_n := \mathcal{U}_n \mathcal{U}_n^*$ and $\pi_n := \mathcal{F}^* \Pi_n \mathcal{F}$. We know that $\sum_{n \geq 1} \pi_n = \mathbb{I}$.

Let I be a Borel subset of $\sigma(H_0)$. A function $\varphi \in \text{Dom}(H_0)$ is localized in I if it satisfies $\mathbb{P}_I(H_0)\varphi = \varphi$, where $\mathbb{P}_I(H_0)$ is the spectral projector for H_0 associated with I . It is easy to see that φ is localized in I if and only if for all $n \in \mathbb{N}$, $\text{supp}(\mathcal{U}_n^* \mathcal{F}\varphi) \subset E_n^{-1}(I)$. Assume that I is bounded and that I contains exactly one threshold $\bar{E}_n$ (or that the distance from I to $\bar{E}_n$ is small). Then all $\pi_m \varphi$ with $m \neq n$ have standard *edge state* properties (see [19]), whereas $\pi_n \varphi$ would be called the *bulk* component of φ in the terminology of [8, 18] (notice that the *bulk-edge* terminology is a priori irrelevant here since the system has no boundary). The edge states are known to be functions localized in a bounded region (in the x direction) of the plane, whose current (representing their transport properties in the y direction) is bounded from below by a constant that depends only of the energy interval I , see (29) below. On the contrary, current of bulk states cannot be bounded from below, and they are not localized in general.

The appearance of bulk states is linked to one of the main original properties of this model: the band functions are not proper (therefore this model does not enter the theory developed in [15]). Indeed, if I contains a threshold $\bar{E}_n$, the set $E_n^{-1}(I)$ is not bounded (or gets large if I gets close to $\bar{E}_n$). To treat the properties of quantum states localized in energy near a given threshold $\bar{E}_n$, it is necessary to have good understanding of the corresponding band function at infinity. In this section we will describe some properties of the bulk component of $\pi_n \varphi$, namely we will show that its current is small and that it is localized in the zone of configuration space where H_0 is closed to the Landau Hamiltonian with constant magnetic field b_+ .

3.1. Estimates on the current

The current operator is $J_y := -i[H_0, y]$, defined as a self adjoint operator on $\text{Dom}(H_0)$. Easy computations show that $J_y = -i\partial_y - a$. We introduce some notations in order to link J_y with the derivative of the band functions.

The Feynman–Hellmann formula shows that the velocity operator satisfies

$$\langle J_y \pi_n \varphi, \pi_n \varphi \rangle = \langle E'_n \mathcal{U}_n^* \mathcal{F} \varphi, \mathcal{U}_n^* \mathcal{F} \varphi \rangle,$$

see [23]. Notice that this form is reminiscent of [34, Section 5.2], stating that in a Fourier basis adapted to the diagonalization of H_0 , the velocity operator is unitarily equivalent to the multiplication by the derivative of the band functions.

Let $I \subset \sigma(H_0)$ and let φ be such that $\mathbb{P}_I(H_0)\varphi = \varphi$. Then the above identities yield

$$\inf_{E_n^{-1}(I)} E'_n \leq \frac{\langle J_y \pi_n \varphi, \pi_n \varphi \rangle}{\|\pi_n \varphi\|^2} \leq \sup_{E_n^{-1}(I)} E'_n. \quad (29)$$

Therefore, as explained in [23,19], a state $\pi_n \varphi$ localized in energy away from the thresholds $\bar{\mathcal{E}}_n$ and $\underline{\mathcal{E}}_n$ has a positive current, whereas when I contains such a threshold, no lower bound on the current is available.

Define $\varrho_n : (0, \bar{\mathcal{E}}_n - \underline{\mathcal{E}}_n) \rightarrow \mathbb{R}$, as the inverse function of $\bar{\mathcal{E}}_n - E_n$. Under the assumption (10), we deduce from Corollary 2.4 the following asymptotics, as $\delta \rightarrow 0$:

$$\varrho_n(\delta) = \alpha \delta^{-\frac{1}{M}} + O(\delta^{\frac{1}{M}}) \quad \text{and} \quad \varrho'_n(\delta) = -\beta \delta^{-\frac{1}{M}-1} + O(\delta^{\frac{\eta}{M}}), \quad (30)$$

with η defined in Corollary 2.4, and

$$\alpha = b_+ \Lambda_n^{\frac{1}{M}} \quad \text{and} \quad \beta = \frac{b_+ \Lambda_n^{\frac{1}{M}}}{M}. \quad (31)$$

Following the approach introduced in [18], we consider an energy interval $I_b := (\bar{\mathcal{E}}_n - \delta_1, \bar{\mathcal{E}}_n - \delta_2)$ with $0 < \delta_1 < \delta_2$, and provide quantitative upper and lower bound for the current of states localized in energy in I_b , as $\delta_i \rightarrow 0$.

Proposition 3.1. Assume (3) and (10). Then there exists $\eta > 0$ such that for all $\varphi \in D(H_0)$ satisfying $\mathbb{P}_{I_b}(H_0)\varphi = \varphi$, there holds, as $\delta_i \rightarrow 0$:

$$\frac{\delta_1^{1+\frac{1}{M}}}{\beta} + O(\delta_1^{\frac{\eta}{M}}) \leq \frac{\langle J_y \pi_n \varphi, \pi_n \varphi \rangle}{\|\pi_n \varphi\|^2} \leq \frac{\delta_2^{1+\frac{1}{M}}}{\beta} + O(\delta_2^{\frac{\eta}{M}}),$$

where β is defined in (31). Here the remainders are uniform with respect to δ_i and φ .

Proof. Note that $(E'_n \circ E_n^{-1})(y) = -\frac{1}{\varrho'_n(\bar{\mathcal{E}}_n - y)}$. Let $I \subset \sigma(H_0)$, assume that $\mathcal{T}_{H_0} \cap I = \emptyset$, and define the set $\tau(I) := -I + \bar{\mathcal{E}}_n$. Let φ be such that $\mathbb{P}_I(\varphi) = \varphi$. Then, (29) provides directly

$$\inf_{y \in \tau(I)} \frac{-1}{\varrho'_n(y)} \leq \frac{\langle J_y \pi_n \varphi, \pi_n \varphi \rangle}{\|\pi_n \varphi\|^2} \leq \sup_{y \in \tau(I)} \frac{-1}{\varrho'_n(y)}. \quad (32)$$

We now apply this estimate with $I = I_b$ and we conclude using (30). $\square$

Remark 3.2. Formula (32) is very general and gives estimates on the current depending on intrinsic quantities. Using Theorem 2.2, we could show under more general conditions that ϱ'_n behaves roughly as $\delta \mapsto b_+ / (\Lambda_n b'(\frac{\varrho_n(\delta)}{b_+}))$, as $\delta \downarrow 0$, and we could give the corresponding version of Proposition 3.1. However, to keep the statements simple, we preferred to consider more explicit magnetic fields.

3.2. (De)localization of states at energy near thresholds

It is known that states whose energy is far from thresholds are localized in x -direction, in the sense that they have exponential decay at infinity, and this decay can be estimated using the distance from their energy to the threshold, see [19]. Such states are usually called *edge states*, because they typically arise in quantum systems with a boundary, although it is known that open magnetic system such as the one considered here can exhibit this kind of behavior ([13]). On the contrary, states whose energy is close to the thresholds are called *bulk states* since they are usually located far from the boundary, if this exists. Here we give quantitative estimates on states, as their energy tends to the limit of a band function, showing that they are located in a region where the variation of the magnetic field is small.

Theorem 3.3. Set $I_b(\delta) := (\bar{\mathcal{E}}_n - \delta, \bar{\mathcal{E}}_n)$, and assume (3) and (10). Then, there exist positive constants c, C and δ_0 , such that $\forall \delta \in (0, \delta_0)$:

$$\forall \varphi \in \text{Ran}(\mathbb{P}_{I_b(\delta)}(H_0)), \quad \int_{-\infty}^{c\delta^{-\frac{1}{M}}} \int_{\mathbb{R}} |\pi_n \varphi(x, y)|^2 dy dx \leq C \delta^{\frac{\eta}{M}} \|\pi_n \varphi\|^2, \quad (33)$$

where η has been defined in [Corollary 2.4](#).

Proof. Consider the function ϵ defined in [\(6\)](#), and set $r_n(\delta) := \sup_{k \in \varrho_n((0, \delta))} \epsilon(k)$ and $k_-(\delta) := \inf \varrho_n((0, \delta)) = \varrho_n(\delta)$. For $\nu > 1$, let

$$x(\delta) := x_{k_-(\delta)} - \frac{\nu}{b^+} \sqrt{|\log r_n(\delta)|}.$$

We will prove the more precise result:

$$\forall \varphi \in \text{Ran}(\mathbb{P}_{I_b(\delta)}), \quad \int_{-\infty}^{x(\delta)} \|\pi_n \varphi(x, \cdot)\|_{L^2(\mathbb{R})}^2 dx \leq C r_n(\delta) \|\pi_n \varphi\|^2. \quad (34)$$

From [Corollary 2.4](#) one can easily prove that there exist $A, B > 0$ such that

$$\begin{aligned} x_{k_-(\delta)} &= A \delta^{-1/M} (1 + o(1)) \\ r_n(\delta) &= B \delta^{\frac{\eta}{M}} (1 + o(1)), \quad \delta \downarrow 0, \end{aligned} \quad (35)$$

and therefore $x(\delta) \geq c \delta^{-\frac{1}{M}}$ for δ small enough with $c > 0$ a constant. Thus, [\(33\)](#) is implied using [\(34\)](#) and [\(35\)](#).

Now, let us prove [\(34\)](#). We use that $\|\pi_n \varphi(x, \cdot)\|_{L^2(\mathbb{R})} = \|\varphi_n u_n(x, \cdot)\|_{L^2(\mathbb{R})}$, where $\varphi_n := \mathcal{U}_n^* \mathcal{F} \varphi$. Moreover, $\mathbb{P}_{I_b(\delta)}(H_0) \varphi = \varphi$ implies that $\text{supp}(\varphi_n) \subset \varrho_n((0, \delta))$, and therefore

$$\begin{aligned} \int_{-\infty}^{x(\delta)} \|\pi_n \varphi(x, \cdot)\|_{L^2(\mathbb{R})}^2 dx &= \int_{\varrho_n((0, \delta))} \varphi_n(k)^2 \int_{-\infty}^{x(\delta)} u_n(x, k)^2 dx dk \\ &= \int_{\varrho_n((0, \delta))} \varphi_n(k)^2 \int_{-\infty}^{x(\delta)} u_n^{\text{qm}}(x, k)^2 dx dk + O(\epsilon(k)), \end{aligned} \quad (36)$$

where u_n^{qm} is the quasimode constructed in [Section 2.1](#), see [\(25\)](#). Thus, we are left with the task of estimating $\int_{-\infty}^{x(\delta)} u_n^{\text{qm}}(x, k)^2 dx$ for $k \in \varrho_n((0, \delta))$. To this end, we use the same change of variables [\(11\)](#), and noticing that $x(\delta) \rightarrow +\infty$ as $\delta \rightarrow 0$ we found that

$$\int_{-\infty}^{x(\delta)} u_n^{\text{qm}}(x, k)^2 dx = \int_{-\infty}^{b_k^{1/2}(x(\delta) - x_k)} v_n^{\text{qm}}(t, k)^2 b_k^{-1/2} dt = (1 + O(\alpha_1(k)) b_k^{-1/2} \int_{-\infty}^{b_k^{1/2}(x(\delta) - x_k)} \Psi_n(t)^2 dt.$$

Since a^{-1} is increasing, by definition of k^- , $x_{k^-} - x_k \leq 0$ for all $k \in \varrho_n((0, \delta))$. Then,

$$b_k^{1/2}(x(\delta) - x_k) \leq b_k^{1/2}(x(\delta) - x_{k^-}) = -\nu \left(\frac{b_k}{b^+} \right)^{1/2} \sqrt{|\log r_n(\delta)|} < 0.$$

Moreover, as $L \rightarrow -\infty$, there holds

$$\int_{t \leq L} \Psi_n(t)^2 dt = O(L^{2n+1} e^{-L^2}).$$

Accordingly, for a constant $C > 0$ and δ small:

$$\forall k \in \varrho_n((0, \delta)), \quad \int_{-\infty}^{x(\delta)} u_n^{\text{qm}}(x, k)^2 dx \leq C \sqrt{|\log r_n(\delta)|}^{2n+1} r_n(\delta)^{\nu^2 \left(\frac{b_k}{b_+}\right)}.$$

Finally, for δ small enough, $\nu^2 \left(\frac{b_k}{b_+}\right) > 1$ for all $k \in \varrho_n((0, \delta))$, which implies

$$\forall k \in \varrho_n((0, \delta)), \quad \int_{-\infty}^{x(\delta)} u_n^{\text{qm}}(x, k)^2 dx \leq C r_n(\delta).$$

We get (34) by combining the last inequality with (36) and $\int_k \varphi_n(k)^2 dk = \|\pi_n \varphi\|^2$. $\square$

Remark 3.4. Using the more precise (and more general) estimate (34), it is possible to get the best possible constant c in (33), up to a correction term in $\sqrt{\log \delta}$. We have preferred to state (33) for readability.

4. The spectral shift function

Consider an electric potential $V : \mathbb{R}^2 \rightarrow [0, \infty)$, that is a Lebesgue measurable function satisfying

$$V(x, y) \leq C \langle x, y \rangle^{-m}, \quad (x, y) \in \mathbb{R}^2, \quad (37)$$

for some positive constant C , $m > 2$, and where $\langle x, y \rangle := (1 + x^2 + y^2)^{1/2}$. On the domain of H_0 introduce the operator

$$H := H_0 + V,$$

self-adjoint in $L^2(\mathbb{R}^2)$. Estimate (37) combined with the diamagnetic inequality imply that for any real $E_0 < \inf \sigma(H)$ the operator $V^{1/2}(H_0 - E_0)^{-1}$ is Hilbert–Schmidt, and hence the resolvent difference $(H - E_0)^{-1} - (H_0 - E_0)^{-1}$ is a trace-class operator. This last property implies that there exists a unique function $\xi = \xi(\cdot; H, H_0) \in L^1(\mathbb{R}; (1 + E^2)^{-1} dE)$, called the Spectral Shift Function (SSF) for the operator pair (H, H_0) , that satisfies the Lifshits–Kreĭn trace formula:

$$\text{Tr}(f(H) - f(H_0)) = \int_{\mathbb{R}} \xi(E; H, H_0) f'(E) dE,$$

for each $f \in C_0^\infty(\mathbb{R})$, and vanishes identically in $(-\infty, \inf \sigma(H))$ [33].

The SSF can be seen as the scattering phase of the operator pair (H, H_0) , namely we have the Birman–Kreĭn formula:

$$\det(S(E)) = e^{-2\pi i \xi(E)}, \quad E \in \sigma_{ac}(H_0) \text{ a.e.}, \quad (38)$$

where $S(E)$ is the scattering matrix of the operator pair (H, H_0) .

In addition, we have the following relation between the SSF and the eigenvalue counting function. Condition (37) implies that the essential spectrum of H is given by $\bigcup_{n \geq 1} [\underline{\mathcal{E}}_n, \overline{\mathcal{E}}_n]$. Then, if we suppose that there is a finite gap in this set, let say $\overline{\mathcal{E}}_n < \underline{\mathcal{E}}_{n+1}$, we may define:

$$\mathcal{N}_n(\lambda) = \text{Rank } \mathbb{P}_{(\mathcal{E}_n + \lambda, \mathcal{E}_{n+1})}(H),$$

which is the function that counts the number of discrete eigenvalues of H in the interval $(\bar{\mathcal{E}}_n + \lambda, \underline{\mathcal{E}}_{n+1})$. From (66) below and the Birman–Schwinger principle we can see that

$$\mathcal{N}_n(\lambda) = \xi(\bar{\mathcal{E}}_n + \lambda; H_0 + V, H_0) + O(1), \quad \lambda \in (0, \underline{\mathcal{E}}_{n+1} - \bar{\mathcal{E}}_n) \quad (39)$$

i.e., outside the essential spectrum both functions agree up to a bounded term, and therefore the SSF could be seen as an extension of $\mathcal{N}_n$ to the whole real line². As we already mentioned in the introduction, there exist considerable works that study the distribution of the discrete eigenvalues of H by giving the asymptotic behavior of $\mathcal{N}_n(\lambda)$, particularly for the constant magnetic field case. For b non-constant there exist some works as well, but to the best knowledge of the authors, the spectral shift function inside the continuous spectrum has not been considered yet.

In this article, for the class defined by the SSF in $L^1(\mathbb{R}; (1 + E^2)^{-1} dE)$, we are going to take Pushnitski's representative given by (66) below, and our main theorems will refer to it.

Our goal is to study the behavior of the SSF when a thresholds is approached by below or by above, for negative or positive perturbations. In what follows, we have stated our results for thresholds $\bar{\mathcal{E}}_n$, corresponding to the limit in $+\infty$ of the band functions, in link with the fact that $b(x) \rightarrow b^+$ as $x \rightarrow +\infty$. If there exists $p \in \mathbb{N}$ such that $\bar{\mathcal{E}}_n = \underline{\mathcal{E}}_p$, the results are more difficult to read, because the possible singularity of the SSF at $\bar{\mathcal{E}}_n$ will not come only from the n -th band function, but also from the fact that $\lim_{k \rightarrow -\infty} E_p(k) = \bar{\mathcal{E}}_n$. For that reason, from now on we will assume

$$\forall p \in \mathbb{N}, \quad \bar{\mathcal{E}}_n \neq \underline{\mathcal{E}}_p, \quad (40)$$

which is equivalent to say that $\frac{b}{b_+}$ is not a ratio of odd integers.

The following theorem shows that for some non-constant magnetic fields and suitable conditions on V , the SSF “extends” the properties of continuity and boundedness of the eigenvalue counting function into the continuous spectrum of H .

Theorem 4.1. *Suppose that $V \geq 0$ satisfies (37) and that b satisfies (3) and (10). Set $H_{\pm} := H_0 \pm V$, then:*

- (1) *On any compact set $\mathcal{C} \subset \mathbb{R} \setminus \mathcal{T}_{H_0}$, $\sup_{E \in \mathcal{C}} \xi(E; H_{\pm}, H_0) < \infty$, i.e. the SSF is bounded away from the thresholds. Moreover, as $\lambda \downarrow 0$:*

$$\xi(\bar{\mathcal{E}}_n - \lambda; H_{\pm}, H_0) = O(\lambda^{-1} + \lambda^{-2/M}), \quad \xi(\bar{\mathcal{E}}_n + \lambda; H_{+}, H_0) = O(\lambda^{-1}), \quad (41)$$

$$\xi(\bar{\mathcal{E}}_n + \lambda; H_{-}, H_0) = O(1). \quad (42)$$

- (2) *The SSF $\xi(\cdot; H_{\pm}, H_0)$ is continuous on $\mathbb{R} \setminus (\sigma_p(H_{\pm}) \cup \mathcal{T}_{H_0})$, where $\sigma_p(H_{\pm})$ denote the set of eigenvalues of $H_{\pm}$.*

Remark 4.2. It is obvious that under conditions of Theorem 4.1, the function $\mathcal{N}_n(\lambda)$ satisfies the same boundedness and continuity properties, wherever it is defined. Similar results for different magnetic Hamiltonians have been obtained before, see for example [6,3,4].

From Theorem 4.1 we know that the only possible points for $\xi(E; H_{\pm}, H_0)$ to be unbounded, are the thresholds in $\mathcal{T}_{H_0}$. In the following two Theorems we show that this also depends on the relation between the decaying rate of V and the convergence rate of $b(x)$ to b_+ . Furthermore, in the unbounded case we will obtain the explicit asymptotic behavior of ξ at the thresholds.

² Although we have to be aware that the SSF is only defined as an element of $L^1(\mathbb{R}; (1 + E^2)^{-1} dE)$.

Theorem 4.3. Assume that $V \geq 0$ satisfies (37). Suppose also that b satisfies (3) and (10) with $m > M + 2$. Then, as $\lambda \downarrow 0$,

$$\xi(\bar{\mathcal{E}}_n + \lambda; H_{\pm}, H_0) = O(1) \quad \text{and} \quad \xi(\bar{\mathcal{E}}_n - \lambda; H_{\pm}, H_0) = O(1).$$

This theorem covers four different cases. One of these was already stated, under weaker assumption on V , in (42), and is conceptually very different. For the proof of this theorem is not necessary that condition (37) holds isotropically. It is enough if for some $x_0 \in \mathbb{R}$ it holds for $x > x_0$ with $m > M + 2$, and for $x < x_0$ with $m > 2$. Notice that the conditions of the Theorem include the case where V is compactly supported.

To obtain the asymptotic behavior of ξ in the unbounded case, we need to impose more restrictive conditions on V . First, we will assume that for any pair $(\alpha, \beta) \in \mathbb{Z}_+^2$ ($\mathbb{Z}_+ := \{0, 1, 2, \dots\}$), there exists a positive constant $C_{\alpha, \beta}$ satisfying

$$|\partial_x^\beta \partial_y^\alpha V(x, y)| \leq C_{\alpha, \beta} \langle x, y \rangle^{-m-\alpha-\beta} \quad \text{for all } (x, y) \in \mathbb{R}^2, \quad (43)$$

where $m > 2$.

Next, let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a measurable function and for $\lambda > 0$ set

$$N_0(\lambda, f) := \frac{1}{2\pi} \text{vol}\{(x, y) \in \mathbb{R}^2; f(x, y) > \lambda, x > 0\},$$

where vol denotes the Lebesgue measure in $\mathbb{R}^2$.

Then, we will suppose that for some positive constants C and λ_0

$$N_0(\lambda, V) \geq C\lambda^{-2/m}, \quad 0 < \lambda < \lambda_0, \quad (44)$$

and that $N_0(\lambda, V)$ satisfies a homogeneity condition of the form

$$\lim_{\epsilon \downarrow 0} \limsup_{\lambda \downarrow 0} \lambda^{2/m} (N_0(\lambda(1 - \epsilon), V) - N_0(\lambda(1 + \epsilon), V)) = 0. \quad (45)$$

Conditions (43), (44) are necessary to have some control of the function V at infinity, from above and below (Note that if V satisfies (43), then $N_0(\lambda, V) = O(\lambda^{-\frac{2}{m}})$, as $\lambda \rightarrow 0$). Meanwhile, (45) is a regularity condition. They are usually assumed in the study of this kind of asymptotics (see for example [7,28,21,31]). The existence of a limit of the form

$$\lim_{(x,y) \rightarrow \infty} \left(\langle x, y \rangle^m V(x, y) - \omega \left(\frac{(x, y)}{|(x, y)|} \right) \right) = 0,$$

where $\omega : S^1 \rightarrow [\epsilon, \infty)$ is smooth and $\epsilon > 0$, guarantees that V satisfy (43), (44) and (45).

Theorem 4.4. Assume that $V \geq 0$ satisfies (43) with $m > 2$ and $N_0(\lambda, V)$ satisfies (44), (45). Suppose also that b satisfies (3) and (10) with $M > m$. Then, the following asymptotic formula at the thresholds holds true:

$$\xi(\bar{\mathcal{E}}_n \pm \lambda; H_{\pm}, H_0) = \pm b_{\pm} N_0(\lambda, V)(1 + o(1)), \quad \lambda \downarrow 0. \quad (46)$$

Remark 4.5. Equation (46) is similar to the results obtained in [4] for the SSF of some magnetic Schrödinger operators defined in a Half-plane, and in [28], [24] for the eigenvalue counting function of the perturbed Landau Hamiltonian and Iwatsuka Hamiltonian, respectively.

Moreover, (46) could be written in the form

$$\xi(\bar{\mathcal{E}}_n \pm \lambda; H_{\pm}, H_0) = \pm \int_{\{(x,y) \in \mathbb{R}^2; V(x,y) > \lambda, x > 0\}} b(x) dx dy (1 + o(1)), \quad \lambda \downarrow 0, \quad (47)$$

which is in accordance with the formula obtained for the function that counts number of discrete eigenvalues of the perturbed Pauli operator near zero (see [21]). This formula shows that the asymptotic behavior of ξ depends on the whole magnetic field, but if $b(x)$ converges to b_+ fast enough it is only necessary to consider the limit b_+ . On the contrary, if the convergence of $b(x)$ to b_+ is slow in comparison with the decaying rate of V , the formula (47) is no longer true, as is shown by Theorem 4.3, because the term in the right hand side of (47) is unbounded as $\lambda \downarrow 0$, for example when $V(x, y) = \langle x, y \rangle^{-m}$ and $m > M + 2$. Then Theorem 4.3 is somehow a non-semiclassical result.

Remark 4.6. Let $b_+ > b_- > 0$, and consider the magnetic field satisfying $b(x) = b_+$ for $x > 0$, and $b(x) = b_-$ for $x < 0$. Then, for this model it should be possible to obtain a formula similar to (46), using the results on the band functions of [17] combined with the ideas used to analyze the SSF that appear in [4]. Note that this magnetic field does not satisfy (10) nor (5). Further, this may suggests that a result like (46) is true as long as $b(x)$ converges fast enough to b_+ . However, such a result would need the analysis of band function in the general case, which is not in the scope of this article.

The proofs of Theorems 4.1, 4.3 and 4.4 appear in the following two sections. They are inspired by [4] and [3].

5. A preliminary limiting absorption principle and other results

Let $(\mathfrak{S}_p, \|\cdot\|_p)$, $p \geq 1$, be the Schatten-von Neumann class of compact operators. Define the operator valued function $G_n : \mathbb{R} \rightarrow \mathfrak{S}_2(L^2(\mathbb{R}^2); \mathbb{C})$, by

$$G_n(k)f := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-iky} V^{1/2}(x, y) u_n(x, k) f(x, y) dx dy, \quad f \in L^2(\mathbb{R}^2),$$

the function u_n being the normalized eigenfunction of $h(k)$ as in section 2.

Lemma 5.1. *For any $n \in \mathbb{N}$:*

- (1) *The function $\|G_n^* G_n\|$ is in $L^1(\mathbb{R})$.*
- (2) *The operator function $G_n^* G_n$ satisfies a Lipschitz condition at infinity: There exist $C > 0$ and $k_0 \in \mathbb{R}$ such that for all k and k' in (k_0, ∞)*

$$\|G_n^* G_n(k) - G_n^* G_n(k')\| \leq C|k - k'|.$$

Proof. Since $G_n^* G_n(k)$ is of finite rank, we can use the different norms indistinctly. The first statement follows using that for some $\tau > 0$

$$\sup_{(x,k) \in \mathbb{R}^2} |u_n(x, k) e^{\tau(x-x_k)^2}| < \infty, \quad (48)$$

[31, Lemma 4.7], and (37), which imply

$$\begin{aligned}
\|G_n(k)^*G_n(k)\|_2^2 &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} V(x, y) V(x', y') |u_n(x, k) u_n(x', k)|^2 dx dx' dy dy' \\
&\leq \frac{1}{2\pi} \left(\int_{\mathbb{R}} \langle y \rangle^{-m/2} dy \right)^2 \left(\int_{\mathbb{R}} \langle x \rangle^{-m/2} u_n(x, k)^2 dx \right)^2 \\
&\leq C \left(\int_{\mathbb{R}} \langle x \rangle^{-m/2} e^{-2\tau(x-x_k)^2} dx \right)^2.
\end{aligned}$$

The last integral is in $L^1(\mathbb{R}_k)$ by Young's inequality and $m > 2$.

For the second statement, recalling the definition of Π_n at the beginning of section 3, we can see that

$$\begin{aligned}
&\|G_n(k)^*G_n(k) - G_n(k')^*G_n(k')\|_2 \\
&\leq \frac{1}{2\pi} \left(\sup_{x \in \mathbb{R}} \int_{\mathbb{R}} V(x, y) dy \right) \|\Pi_n(k) - \Pi_n(k')\|_2 \\
&\leq C \oint_{\gamma} \|(h(k) - \omega)^{-1} - (h(k') - \omega)^{-1}\| d|\omega| \\
&\leq C|k - k'| \oint_{\gamma} \|(h(k) - \omega)^{-1}(a(x) - k')(h(k') - \omega)^{-1}\| d|\omega|,
\end{aligned}$$

where the contour γ is such that the only eigenvalues of $h(k)$ and $h(k')$ inside it are $E_n(k)$ and $E_n(k')$. In order to find a uniform bound of $\oint_{\gamma} \|(h(k) - \omega)^{-1}(a(x) - k')(h(k') - \omega)^{-1}\| d|\omega|$ for k, k' big, it is enough to use the inequalities

$$\begin{aligned}
&\|(a(x) - k)(h(k) - \omega)^{-1}g\|_{L^2(\mathbb{R})} \\
&\leq \|(h(k) - \omega)^{-1}g\|_{L^2(\mathbb{R})} + \|(a(x) - k)^2(h(k) - \omega)^{-1}g\|_{L^2(\mathbb{R})} \\
&\leq \|(h(k) - \omega)^{-1}g\|_{L^2(\mathbb{R})} + C\|(h(k) + 2b_+)(h(k) - \omega)^{-1}g\|_{L^2(\mathbb{R})},
\end{aligned}$$

valid for any $g \in L^2(\mathbb{R})$. In the last inequality we have used [12, Theorem 1]. $\square$

By Lemma 5.1, for any $z \in \mathbb{C}_+$ and $l \in \mathbb{N}$, it is possible to define the integral $\int_{\mathbb{R}} \frac{G_l(k)^*G_l(k)}{E_l(k) - z} dk$, which is a trace class operator. On the other side, we can define

$$T(z) := V^{1/2}(H_0 - z)^{-1}V^{1/2},$$

and, from (1) it is easy to see that

$$T(z) = \sum_{l \in \mathbb{N}} \int_{\mathbb{R}} \frac{G_l(k)^*G_l(k)}{E_l(k) - z} dk. \quad (49)$$

We will show that $T(z)$ admits a limit as $z \downarrow E$, for $E \in \sigma(H_0) \setminus \mathcal{T}_{H_0}$. This limit is linked to the SSF through the Pushnitski representation (66) given below.

Recall that $\varrho_l : (0, \bar{\mathcal{E}}_l - \underline{\mathcal{E}}_l] \rightarrow \mathbb{R}$ is defined as the inverse function of $\bar{\mathcal{E}}_l - E_l$. Therefore, for any $l \in \mathbb{N}$:

$$\forall z \in \mathbb{C} \setminus \sigma(H_0), \quad \int_{\mathbb{R}} \frac{G_l(k)^*G_l(k)}{E_l(k) - z} dk = \int_0^{\bar{\mathcal{E}}_l - \underline{\mathcal{E}}_l} \frac{G_l(\varrho_l(s))^*G_l(\varrho_l(s))}{s + z - \bar{\mathcal{E}}_l} \varrho'_l(s) ds.$$

Proposition 5.2. Assume that V satisfies (37). Then, for $E \in \mathbb{R} \setminus \mathcal{T}_{H_0}$ the limit

$$\lim_{\delta \downarrow 0} T(E + i\delta) = T(E + i0)$$

exists in the norm sense. Moreover:

- I. The operator function $\operatorname{Im} T(\cdot + i0)$ is continuous in the trace class norm in $\mathbb{R} \setminus \mathcal{T}_{H_0}$. Further, the rank of $\operatorname{Im} T(E + i0)$ is finite and constant between two consecutive thresholds.
- II. The real part $\operatorname{Re} T(\cdot + i0)$ is continuous in the operator norm in $\mathbb{R} \setminus \mathcal{T}_{H_0}$.

Proof. Define γ_n^- as the nearest threshold below $\bar{\mathcal{E}}_n$ i.e., $\gamma_n^- := \max\{\gamma \in \mathcal{T}_{H_0}, \gamma < \bar{\mathcal{E}}_n\}$, and consider the set

$$L_n := \{l \in \mathbb{N}; (\gamma_n^-, \bar{\mathcal{E}}_n) \cap E_l(\mathbb{R}) \neq \emptyset\}. \quad (50)$$

Note that, for instance, $L_n = \{n\}$, when $\gamma_n^- = \mathcal{E}_n$ and $L_n = \{n, n+1\}$, when $\gamma_n^- = \mathcal{E}_{n+1}$.

Setting $\gamma_1^- := -\infty$, we have $\mathbb{R} \setminus \mathcal{T}_H = \bigcup_{n \in \mathbb{N}} (\gamma_n^-, \bar{\mathcal{E}}_n) \cup (\bar{\mathcal{E}}_n, \gamma_{n+1}^-)$. We will show the proof of the proposition on the set $(\gamma_n^-, \bar{\mathcal{E}}_n)$. The other intervals can be treated similarly.

Define the projection $P_n := \sum_{l \notin L_n} \pi_l$, where π_l was defined at the beginning of section 3. Then,

$$\sum_{l \notin L_n} \int_{\mathbb{R}} \frac{G_l(k)^* G_l(k)}{E_l(k) - z} dk = V^{1/2} P_n (H_0 - z)^{-1} P_n V^{1/2}$$

depends analytically on $z \in \mathbb{C}^+ \cup (\gamma_n^-, \bar{\mathcal{E}}_n)$, in consequence, the limit

$$\lim_{\delta \downarrow 0} V^{1/2} P_n (H_0 - E - i\delta)^{-1} P_n V^{1/2} = V^{1/2} ((H_0 - E) P_n)^{-1} V^{1/2} \quad (51)$$

exists for $E \in (\gamma_n^-, \bar{\mathcal{E}}_n)$. The operator in the rhs of the last equality is a self-adjoint Hilbert–Schmidt operator since $T(z)$ is Hilbert–Schmidt. Additionally, it is continuous with respect to E .

Further, thanks to Lemma 5.1 and the definition of L_n , for $E \in (\gamma_n^-, \bar{\mathcal{E}}_n)$ we have the following limit:

$$\begin{aligned} \lim_{\delta \downarrow 0} \sum_{l \in L_n} \int_{\mathbb{R}} \frac{G_l(k)^* G_l(k)}{E_l(k) - E - i\delta} dk &= \sum_{l \in L_n} \text{p.v.} \int_0^{\bar{\mathcal{E}}_l - \mathcal{E}_l} \frac{G_l(\varrho_l(s))^* G_l(\varrho_l(s))}{s - \bar{\mathcal{E}}_l + E} \varrho'_l(s) ds \\ &+ i\pi \sum_{l \in L_n} G_l(\varrho_l(\bar{\mathcal{E}}_l - E))^* G_l(\varrho_l(\bar{\mathcal{E}}_l - E)) \varrho'_l(\bar{\mathcal{E}}_l - E), \end{aligned} \quad (52)$$

which combined with (51) and (49) give us the existence of $\lim_{\delta \downarrow 0} T(E + i\delta)$. Therefore, from the previous decomposition we get that

$$\forall E \in (\gamma_n^-, \bar{\mathcal{E}}_n), \quad \operatorname{Im} T(E + i0) = \pi \sum_{l \in L_n} G_l(\varrho_l(\bar{\mathcal{E}}_l - E))^* G_l(\varrho_l(\bar{\mathcal{E}}_l - E)) \varrho'_l(\bar{\mathcal{E}}_l - E),$$

thus

$$\forall E \in (\gamma_n^-, \bar{\mathcal{E}}_n), \quad \operatorname{Rank}(\operatorname{Im} T(E + i0)) = \#L_n. \quad (53)$$

Furthermore, the continuity of $G_l^* G_l$ and of ϱ'_l for any $l \in \mathbb{N}$, imply the trace norm continuity of $\operatorname{Im}(T(\cdot + i0))$. We have the results concerning $\operatorname{Im}(T(\cdot + i0))$ altogether.

Standard properties of the principal value show that the real part of the r.h.s. of (52) is continuous (in the operator norm) with respect to $E \in (\gamma_n^-, \bar{\mathcal{E}}_n)$. In consequence, $\operatorname{Re} T(\cdot + i0)$ is continuous in $(\gamma_n^-, \bar{\mathcal{E}}_n)$. $\square$

To continue we need the following lemma:

Lemma 5.3. For $\lambda \in (0, \overline{\mathcal{E}_n} - \underline{\mathcal{E}_n})$ and $\epsilon_\lambda \in (0, \lambda)$ we define:

$$P(\lambda) := \text{p.v.} \int_{\lambda - \epsilon_\lambda}^{\lambda + \epsilon_\lambda} \frac{G_n(\varrho_n(s))^* G_n(\varrho_n(s))}{s - \lambda} \varrho'_n(s) ds.$$

Then, there exists $C_0 > 0$ independent of λ such that

$$\forall \lambda \in (0, \overline{\mathcal{E}_n} - \underline{\mathcal{E}_n}), \quad \|P(\lambda)\|_1 \leq C_0 \frac{\epsilon_\lambda}{\lambda^{2/M+1}}. \quad (54)$$

Proof. Write

$$P(\lambda) = \text{p.v.} \int_{\lambda - \epsilon_\lambda}^{\lambda + \epsilon_\lambda} \frac{G_n(\varrho_n(s))^* G_n(\varrho_n(s))}{s - \lambda} \varrho'_n(s) ds = \text{p.v.} (\mathcal{M}_1(\lambda) + \mathcal{M}_2(\lambda)),$$

with

$$\begin{cases} \mathcal{M}_1(\lambda) = \int_0^{\epsilon_\lambda} (G_n(\varrho_n(\lambda + s))^* G_n(\varrho_n(\lambda + s)) - G_n(\varrho_n(\lambda - s))^* G_n(\varrho_n(\lambda - s))) \varrho'_j(\lambda + s) \frac{ds}{s} \\ \mathcal{M}_2(\lambda) = \int_0^{\epsilon_\lambda} G_n(\varrho_n(\lambda - s))^* G_n(\varrho_n(\lambda - s)) (\varrho'_n(\lambda + s) - \varrho'_n(\lambda - s)) \frac{ds}{s}. \end{cases}$$

Since both $\mathcal{M}_1(\lambda)$ and $\mathcal{M}_2(\lambda)$ are convergent integrals, there holds

$$\text{p.v.} \int_{\lambda - \epsilon_\lambda}^{\lambda + \epsilon_\lambda} \frac{G_n(\varrho_n(s))^* G_n(\varrho_n(s))}{s - \lambda} \varrho'_n(s) ds = \mathcal{M}_1(\lambda) + \mathcal{M}_2(\lambda).$$

From Lemma 5.1, we have

$$\|\mathcal{M}_1(\lambda)\|_1 \leq C \int_0^{\epsilon_\lambda} \left| \frac{\varrho_n(\lambda + s) - \varrho_n(\lambda - s)}{s} \right| |\varrho'_n(\lambda + s)| ds$$

For each $\lambda > 0$ and $\epsilon_\lambda \in (0, \lambda)$ fixed, we have that for $s \in (0, \epsilon_\lambda)$, $\varrho_n(\lambda + s) - \varrho_n(\lambda - s) = \varrho'(c_{\lambda,s})s$ where $c_{\lambda,s} \in (\lambda - s, \lambda + s) \subset (\lambda - \epsilon_\lambda, \lambda + \epsilon_\lambda) \subset (0, 2\lambda)$. From (30), we get $|\varrho'_n(\lambda + s)| \leq C(\lambda + s)^{-1-1/M} < C(\lambda - s)^{-1-1/M}$ and $|\varrho'_n(c_{\lambda,s})| \leq C(c_{\lambda,s})^{-1-1/M} < C(\lambda - s)^{-1-1/M}$. Then, we deduce

$$\begin{aligned} \|\mathcal{M}_1(\lambda)\|_1 &\leq C \int_0^{\epsilon_\lambda} (\lambda - s)^{-2-2/M} ds \\ &\leq \tilde{C} \frac{\epsilon_\lambda}{\lambda^{2/M+1}}. \end{aligned}$$

For $\mathcal{M}_2$ we compute the asymptotics of $\varrho_n''(s)$ in 0 as in (30) and we get

$$\begin{aligned}\|\mathcal{M}_2(\lambda)\|_1 &\leq C \int_0^{\epsilon_\lambda} \left| \frac{\varrho_n'(\lambda+s) - \varrho_n'(\lambda-s)}{s} \right| ds \\ &\leq \tilde{C} \frac{\epsilon_\lambda}{\lambda^{2/M+1}}. \quad \square\end{aligned}$$

For $A \in (0, \overline{\mathcal{E}}_n - \underline{\mathcal{E}}_n]$ and $\lambda \in (-\infty, A) \setminus \{0\}$, set

$$T_{n,A}(\lambda) := \text{p.v.} \int_0^A \frac{G_n(\varrho_n(s))^* G_n(\varrho_n(s))}{s-\lambda} \varrho_n'(s) ds. \quad (55)$$

Note that for $\lambda < 0$, it is not necessary to define this operator as a principal value integral.

In the same manner as γ_n^- is the nearest thresholds below $\overline{\mathcal{E}}_n$, define γ_n^+ as nearest thresholds above $\overline{\mathcal{E}}_n$.

Proposition 5.4. Assume the hypotheses of Theorem 4.1. Let $n \in \mathbb{N}$, and $A_- \in (0, \overline{\mathcal{E}}_n - \gamma_n^-)$, $A_+ \in (0, \gamma_n^+ - \overline{\mathcal{E}}_n)$. For $\delta \in (0, 1)$, take $I_\delta^\pm \subset (0, \delta A_\pm)$. Then, for $\lambda \in I_\delta^\pm$, we can write

$$\text{Re } T(\overline{\mathcal{E}}_n \pm \lambda + i0) = T_{n,A_\pm}(\mp \lambda) + \tilde{T}_n^\pm(\lambda), \quad (56)$$

where the operators $T_{n,A_\pm}$, $\tilde{T}_n^\pm$ satisfy that there exists $C_0 > 0$ such that

$$\begin{aligned}\|T_{n,A_+}(-\lambda)\|_1 &\leq C_0 |\lambda|^{-1}, \quad \forall \lambda \in I_\delta^+, \\ \|T_{n,A_-}(\lambda)\|_1 &\leq C_0 (|\lambda|^{-1} + |\lambda|^{-2/M}), \quad \forall \lambda \in I_\delta^-, \end{aligned} \quad (57)$$

$$\|\tilde{T}_n^\pm(\lambda)\|_2 \leq C_0, \quad \forall \lambda \in I_\delta^\pm. \quad (58)$$

Proof. Let us start with $\text{Re } T(\overline{\mathcal{E}}_n - \lambda + i0)$. From Proposition 5.2 and (55) we can write

$$\text{Re } T(\overline{\mathcal{E}}_n - \lambda + i0) = T_{n,A_-}(\lambda) + \tilde{T}_n^-(\lambda), \quad (59)$$

with

$$\begin{aligned}\tilde{T}_n^-(\lambda) &= \int_{A_-}^{\overline{\mathcal{E}}_n - \underline{\mathcal{E}}_n} \frac{G_n(\varrho_n(s))^* G_n(\varrho_n(s))}{s-\lambda} \varrho_n'(s) ds \\ &\quad + \sum_{l \in L_n, l \neq n} \text{p.v.} \int_0^{\overline{\mathcal{E}}_l - \underline{\mathcal{E}}_l} \frac{G_l(\varrho_l(s))^* G_l(\varrho_l(s))}{s - \overline{\mathcal{E}}_l + \overline{\mathcal{E}}_n - \lambda} \varrho_l'(s) ds \\ &\quad + V^{1/2}((H_0 - \overline{\mathcal{E}}_n + \lambda)P_n)^{-1} V^{1/2},\end{aligned} \quad (60)$$

where L_n was defined in (50).

Now, to prove (58), if λ is in I_δ^- , the $\mathfrak{S}_1$ -norm of the first term of $\tilde{T}_n^-(\lambda)$ can be estimated as follows:

$$\begin{aligned}\left\| \int_{A_-}^{\overline{\mathcal{E}}_n - \underline{\mathcal{E}}_n} \frac{G_n(\varrho_n(s))^* G_n(\varrho_n(s))}{s-\lambda} \varrho_n'(s) ds \right\|_2 &\leq \sup_{s \in [A_-, \overline{\mathcal{E}}_n - \underline{\mathcal{E}}_n]} \frac{1}{|s-\lambda|} \int_{-\infty}^{\varrho_n(A_-)} \|G_n(k)^* G_n(k)\|_2 dk \\ &\leq ((1-\delta)A_-)^{-1} \int_{-\infty}^{\infty} \|G_n(k)^* G_n(k)\|_2 dk.\end{aligned}$$

Therefore, using [Lemma 5.1](#), we can see that this term is uniformly bounded for λ in I_δ^- . A similar procedure can be performed to find a uniform bound for the second term of [\(60\)](#).

For the third term in [\(60\)](#) we use

$$\|V^{1/2}((H_0 - \bar{\mathcal{E}}_n + \lambda)P_n)^{-1}V^{1/2}\|_2 \leq \|V^{1/2}H_0^{-1}\|_2 \|H_0((H_0 - \bar{\mathcal{E}}_n + \lambda)P_n)^{-1}V^{1/2}\|,$$

and the uniform boundedness of $\|H_0((H_0 - \bar{\mathcal{E}}_n + \lambda)P_n)^{-1}\|$ for $\lambda \in I_\delta^-$.

Finally, we write

$$\begin{aligned} T_{n,A-}(\lambda) = & \text{p.v.} \int_{\lambda/2}^{3\lambda/2} \frac{G_n(\varrho_n(s))^* G_n(\varrho_n(s))}{s - \lambda} \varrho'_n(s) ds \\ & + \int_{\varrho_n(A-)}^{\varrho_n(3\lambda/2)} \frac{G_n(k)^* G_n(k)}{E_n(k) - \bar{\mathcal{E}}_n + \lambda} dk + \int_{\varrho_n(\lambda/2)}^{\infty} \frac{G_n(k)^* G_n(k)}{E_n(k) - \bar{\mathcal{E}}_n + \lambda} dk. \end{aligned} \quad (61)$$

Using the monotonicity of $E_n(k) - \bar{\mathcal{E}}_n$, we get $E_n(k) - \bar{\mathcal{E}}_n + \lambda > \frac{\lambda}{2}$ for all $k \in (\varrho_n(\lambda/2), +\infty)$ and $E_n(k) - \bar{\mathcal{E}}_n + \lambda < -\frac{\lambda}{2}$ for all $k \in (\varrho_n(A-), \varrho_n(3\lambda/2))$. By the integrability of $\|G_n(k)^* G_n(k)\|$ we conclude that the $\mathfrak{S}_1$ -norms of the last two terms in [\(61\)](#) are controlled by λ^{-1} . The first term in [\(61\)](#) can be bounded applying [Lemma 5.3](#) with $\epsilon_\lambda = \frac{\lambda}{2}$, obtaining that it is controlled by $\lambda^{-2/M}$. This concludes the proof of [\(57\)](#). The case $\text{Re} T(\bar{\mathcal{E}}_n + \lambda + i0)$ is simpler: we prove in the same way that $\tilde{T}_n^+$ satisfies [\(58\)](#) for all $\lambda \in I_\delta^+$, and we can read directly on [\(55\)](#) that $T_{n,A+}(-\lambda)$ is a well defined integral for $\lambda > 0$, satisfying [\(57\)](#). $\square$

6. Proof of the behavior of the spectral shift function

In this section, we prove [Theorems 4.1, 4.3 and 4.4](#).

6.1. Preliminary

6.1.1. Some results on the counting function of compact operators

For T being a compact self-adjoint operator set

$$n_\pm(r; T) := \text{Rank } \mathbb{P}_{(r, \infty)}(\pm T);$$

thus the functions $n_\pm(\cdot; T)$ are respectively the counting functions of the positive and negative eigenvalues of the operator T . If T is compact but not necessarily self-adjoint we will use the notation

$$n_*(r; T) := n_+(r^2; T^*T), \quad r > 0;$$

thus $n_*(\cdot; T)$ is the counting function of the singular values of T . Evidently,

$$n_*(r; T) = n_*(r; T^*), \quad n_+(r; T^*T) = n_+(r; TT^*), \quad r > 0. \quad (62)$$

Besides, for $r_1, r_2 > 0$, we have the Weyl inequalities

$$n_\pm(r_1 + r_2; T_1 + T_2) \leq n_\pm(r_1; T_1) + n_\pm(r_2; T_2), \quad (63)$$

where T_j , $j = 1, 2$, are linear self-adjoint compact operators (see e.g. [\[2, Theorem 9.2.9\]](#)), as well as the Ky Fan inequality

$$n_*(r_1 + r_2; T_1 + T_2) \leq n_*(r_1; T_1) + n_*(r_2; T_2), \quad (64)$$

for compact but not necessarily self-adjoint T_j , $j = 1, 2$ (see e.g. [2, Subsection 11.1.3]).

Further, since for $T \in \mathfrak{S}_p$, $\|T\|_p = \left(-\int_0^\infty r^p dn_*(r; T)\right)^{1/p}$, the Chebyshev-type estimate

$$n_*(r; T) \leq r^{-p} \|T\|_p^p \quad (65)$$

holds true for any $r > 0$ and $p \in [1, \infty)$.

6.1.2. A useful representation of the SSF

Let us recall the *Pushnitski's representation of the SSF*. For $z \in \mathbb{C}^+$ and $V \geq 0$, we have defined $T(z) = V^{1/2}(H_0 - z)^{-1}V^{1/2}$. As shown in Proposition 5.2, the norm limits $\lim_{\delta \downarrow 0} T(E + i\delta) = T(E + i0)$ exist for every $E \in \mathbb{R} \setminus \mathcal{T}_{H_0}$, provided that V satisfies (37). Then, by [26, Theorem 1.2]: For almost all $E \in \mathbb{R}$, the SSF $\xi(E; H_\pm, H_0)$ admits the representation

$$\xi(E; H_\pm, H_0) = \pm \frac{1}{\pi} \int_{\mathbb{R}} n_\mp(1; \operatorname{Re} T(E + i0) + t \operatorname{Im} T(E + i0)) \frac{dt}{1 + t^2}. \quad (66)$$

6.2. Boundedness and continuity of the SSF

Thanks to representation (66) and (64), for any $\epsilon \in (0, 1)$

$$|\xi(\bar{\mathcal{E}}_n - \lambda; H_\pm, H_0)| \leq n_*(1 - \epsilon; \operatorname{Re} T(\bar{\mathcal{E}}_n - \lambda)) + \frac{1}{\pi} \int_{\mathbb{R}} n_*(\epsilon; t \operatorname{Im} T(\bar{\mathcal{E}}_n - \lambda)) \frac{dt}{1 + t^2}. \quad (67)$$

Due to Proposition 5.2, the second term is uniformly bounded for λ in $(0, \bar{\mathcal{E}}_n - \gamma_n^-)$. Let I_δ^- be an interval as in Proposition 5.4. Using (56) and the Ky Fan inequality (64) we get:

$$\forall \epsilon \in (0, 1), \quad n_*(1 - \epsilon; \operatorname{Re} T(\bar{\mathcal{E}}_n - \lambda)) \leq n_*\left(\frac{1 - \epsilon}{2}; T_{n,A}(\lambda)\right) + n_*\left(\frac{1 - \epsilon}{2}; \tilde{T}_n^-(\lambda)\right). \quad (68)$$

Combining (65), (58) and (68) we can see that for all $\epsilon \in (0, 1)$ and all $\lambda \in I_\delta^-$

$$n_*(1 - \epsilon; \operatorname{Re} T(\bar{\mathcal{E}}_n - \lambda)) \leq \frac{4}{(1 - \epsilon)^2} \|T_{n,A}(\lambda)\|_1 + C_0.$$

By Proposition 5.4, if K is any compact set contained in I_δ^- , $\|T_{n,A}(\lambda)\|_1$ is uniformly bounded for λ in K . Moreover, using (57), we get

$$n_*(1 - \epsilon; \operatorname{Re} T(\bar{\mathcal{E}}_n - \lambda)) \leq C_0(1 + \lambda^{-1} + \lambda^{-2/M}), \quad \forall \lambda \in I_\delta^-,$$

implying the first part of (41). The second part is similar.

Likewise, from (66)

$$\forall \epsilon \in (0, 1), \quad |\xi(\bar{\mathcal{E}}_n + \lambda; H_-, H_0)| \leq n_+\left(\frac{1 - \epsilon}{2}; T_{n,A}(-\lambda)\right) + O(1),$$

and

$$T_{n,A}(-\lambda) = - \int_{\varrho_n(A)}^{\infty} \frac{G_n(k)^* G_n(k)}{\bar{\mathcal{E}}_n - E_n(k) - \lambda},$$

which is clearly non-positive self adjoint operator, see (55). Therefore we have $n_+(\frac{1-\epsilon}{2}, T_{n,A}(-\lambda)) = 0$, and we get (42).

The continuity of the SSF follows from [26, Corollary 2.6] together with Proposition 5.2, and is similar to the proof of [6, Proposition 2.5].

6.3. Precise behavior at thresholds

For $\lambda \in \mathbb{R}$ and $I \subset (0, \bar{\mathcal{E}}_n - \underline{\mathcal{E}}_n) \setminus \{\lambda\}$ let us introduce $S_V[I] : L^2(\varrho_n(I)) \rightarrow L^2(\mathbb{R}^2)$ as the integral operator with kernel

$$(2\pi)^{-1/2} V(x, y)^{1/2} e^{iky} u_n(x; k) |E_n(k) - \bar{\mathcal{E}}_n + \lambda|^{-1/2},$$

that is:

$$(S_V[I]f)(x, y) = (2\pi)^{-1/2} \int_{k \in I} V(x, y)^{1/2} e^{iky} u_n(x; k) |E_n(k) - \bar{\mathcal{E}}_n + \lambda|^{-1/2} f(k) dk.$$

The next lemma link the behavior of the SSF at the threshold with $S_V[I]$.

Lemma 6.1. *Let $\lambda \in (0, \bar{\mathcal{E}}_n - \underline{\mathcal{E}}_n)$, $\epsilon_\lambda \in (0, \lambda)$ and $A_- \in (0, \bar{\mathcal{E}}_n - \gamma_n^-)$. Then, for all $\delta \in (0, 1)$:*

$$\begin{aligned} & n_+(1 + \delta, S_V S_V^*[(0, \lambda - \epsilon_\lambda)]) - n_+(\delta/2, S_V S_V^*[(\lambda + \epsilon_\lambda, A)]) + O\left(\frac{\epsilon_\lambda}{\lambda^{2/M+1}}\right) \\ & \leq -\xi(\bar{\mathcal{E}}_n - \lambda; H_-, H_0) \end{aligned} \quad (69)$$

$$\begin{aligned} & \leq n_+(1 - \delta, S_V S_V^*[(0, \lambda - \epsilon_\lambda)]) + O\left(\frac{\epsilon_\lambda}{\lambda^{2/M+1}}\right), \\ & n_+(1 + \delta, S_V S_V^*[(\lambda + \epsilon_\lambda, A)]) - n_+(\delta/2, S_V S_V^*[(0, \lambda - \epsilon_\lambda)]) + O\left(\frac{\epsilon_\lambda}{\lambda^{2/M+1}}\right) \\ & \leq \xi(\bar{\mathcal{E}}_n - \lambda; H_+, H_0) \end{aligned} \quad (70)$$

$$\leq n_+(1 - \delta, S_V S_V^*[(\lambda + \epsilon_\lambda, A)]) + O\left(\frac{\epsilon_\lambda}{\lambda^{2/M+1}}\right),$$

and

$$\begin{aligned} & n_+(1 + \delta, S_V S_V^*[(0, A)]) \\ & \leq \xi(\bar{\mathcal{E}}_n + \lambda; H_+, H_0) \\ & \leq n_+(1 - \delta, S_V S_V^*[(0, A)]), \end{aligned} \quad (71)$$

where the remainders are uniform with respect to λ and ϵ_λ .

Proof. The proof of (69), (70) and (71) follows the same lines. Let us prove (69). Proposition 5.4 together with the representation (66) and the inequality (63) implies that for all $r_1 < 1 < r_2$, as $\lambda \downarrow 0$:

$$n_+(r_2; T_{n,A}(\lambda)) + O(1) \leq -\xi(\bar{\mathcal{E}}_n - \lambda; H_-, H_0) \leq n_+(r_1; T_{n,A}(\lambda)) + O(1). \quad (72)$$

Note the decomposition

$$T_{n,A}(\lambda) = \text{p.v.} \int_{\lambda - \epsilon_\lambda}^{\lambda + \epsilon_\lambda} \frac{G_n(\varrho_n(s))^* G_n(\varrho_n(s))}{\lambda - s} \varrho'_n(s) ds - S_V S_V^*[(\lambda + \epsilon_\lambda, A)] + S_V S_V^*[(0, \lambda - \epsilon_\lambda)].$$

Then, the Weyl inequalities (63) together with Lemma 5.3 imply that for any $\eta \in (0, 1)$:

$$n_+(r_1, T_{n,A}(\lambda)) \leq n_+(r_1(1 - \eta), S_V S_V^*[(0, \lambda - \epsilon_\lambda)]) + O\left(\frac{\epsilon_\lambda}{\lambda^{2/M+1}}\right).$$

Now, for $\delta \in (0, 1)$ being given, we settle the parameters so that $1 - \delta = r_1(1 - \eta)$ and we get the upper bound by using (72).

In the same way, we have for any $\eta \in (0, 1)$ and $a \in (0, 1)$:

$$n_+(r_2, T_{n,A}(\lambda)) \geq n_+(r_2(1 + \eta), S_V S_V^*[(0, \lambda - \epsilon_\lambda)]) - n_+(r_2 a \eta, S_V S_V^*[(\lambda + \epsilon_\lambda, A)]) + O\left(\frac{\epsilon_\lambda}{\lambda^{2/M+1}}\right),$$

and we get the lower bound by choosing the parameters so that $1 + \delta = r_2(1 + \eta)$ and $\frac{\delta}{2} = r_2 a \eta$ and using (72). $\square$

6.3.1. Proof of Theorem 4.3

From (65), (48) and (37)

$$\begin{aligned} n_+(r, S_V S_V^*[\lambda, (0, \lambda - \epsilon_\lambda)]) &= n_+(r, S_V^* S_V[(0, \lambda - \epsilon_\lambda)]) \leq \frac{1}{r^2} \|S_V[(0, \lambda - \epsilon_\lambda)]\|_2^2 \\ &= \frac{1}{2\pi r^2} \int_{\varrho_n(0, \lambda - \epsilon_\lambda)}^\infty \int_{\mathbb{R}^2} |E_n(k) - \bar{\mathcal{E}}_n + \lambda|^{-1} u_n(x; k)^2 V(x, y) dx dy dk \\ &\leq C \int_{(\varrho_n(\lambda - \epsilon_\lambda))}^\infty \int_{\mathbb{R}^2} |E_n(k) - \bar{\mathcal{E}}_n + \lambda|^{-1} e^{-\tau(x - x_k)^2} \langle x, y \rangle^{-m} dx dy dk \end{aligned} \quad (73)$$

Making the change of variables $k = \varrho_n(s)$ and using (30) we can control the last integral as follows:

$$\begin{aligned} &= \int_0^{\lambda - \epsilon_\lambda} \int_{\mathbb{R}^2} |s - \lambda|^{-1} |\varrho'_n(s)| \langle x, y \rangle^{-m} e^{-\tau(x - a^{-1}(\varrho_n(s)))^2} dx dy ds \\ &= C \int_0^{\lambda - \epsilon_\lambda} |s - \lambda|^{-1} |\varrho'_n(s)| \int_{\mathbb{R}} \langle x \rangle^{1-m} e^{-\tau(x - a^{-1}(\varrho_n(s)))^2} dx ds \\ &\leq C \int_0^{\lambda - \epsilon_\lambda} |s - \lambda|^{-1} |\varrho'_n(s)| \langle a^{-1}(\varrho_n(s)) \rangle^{1-m} ds \\ &\leq C \int_0^{\lambda - \epsilon_\lambda} |s - \lambda|^{-1} s^{-1-1/M} s^{(-1+m)/M} ds. \end{aligned}$$

Making $s = \lambda u$ and using that $\frac{m-2-M}{M} > 0$:

$$\int_0^{\lambda - \epsilon_\lambda} |s - \lambda|^{-1} s^{-1-1/M} s^{(-1+m)/M} ds = \lambda^{\frac{m-2-M}{M}} \int_0^{1 - \frac{\epsilon_\lambda}{\lambda}} (1 - u)^{-1} u^{\frac{m-2-M}{M}} du \leq C \log\left(\frac{\epsilon_\lambda}{\lambda}\right) \lambda^{\frac{m-2-M}{M}}$$

Setting $\epsilon_\lambda = \lambda^\theta$, we have proved:

$$n_+(r, S_V^* S_V[(0, \lambda - \epsilon_\lambda)]) \leq C \log(\lambda) \lambda^{\frac{m-2-M}{M}},$$

and if $\theta > \frac{2}{M} + 1$, using [Lemma 6.1](#), we conclude the proof of [Theorem 4.3](#) for the behavior of $\xi(\mathcal{E}_n - \lambda, H_-, H_0)$ as $\lambda \downarrow 0$.

For the cases $\xi(\mathcal{E}_n - \lambda, H_+, H_0)$ and $\xi(\mathcal{E}_n + \lambda, H_+, H_0)$ we use [\(70\)](#) and [\(71\)](#), respectively, where each different term is controlled exactly in the same way as above.

Remark 6.2. From the above inequalities we can see that the proof is still true if we only assume [\(37\)](#) valid only for $x > 0$.

6.3.2. Proof of [Theorem 4.4](#)

In what follows we will present the proof of [Theorem 4.4](#) for the operator pair (H_-, H_0) . The case (H_+, H_0) is similar and simpler, since the corresponding effective Hamiltonian $S_V S_V^*[(0, A)]$ does not contain a principal value term (see [\(71\)](#)).

Consider the class of symbols $\mathcal{S}_p^q$ defined as the set of smooth functions f such that for any $(\alpha, \beta) \in \mathbb{Z}_+^2$ the quantity

$$n_{\alpha, \beta}^{p, q}(f) := \sup_{(x, y) \in \mathbb{R}^2} |\langle x \rangle^p \langle x, y \rangle^{q+\alpha} \partial_y^\alpha \partial_x^\beta f(x, y)| \quad (74)$$

is finite.

For $f \in \mathcal{S}_p^q$ we define the operator $Op^W(f)$ using the Weyl quantization

$$(Op^W(f)u)(x) := \frac{1}{2\pi} \int_{\mathbb{R}^2} f\left(\frac{x+k}{2}, y\right) e^{-i(x-k)y} u(k) dk dy,$$

where u in the Schwartz space $\mathcal{S}(\mathbb{R})$.

Define $Q_V : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}^2)$ as the integral operator with kernel

$$(2\pi)^{-1/2} V(x, y)^{1/2} e^{iky} u_n(x, k), \quad (75)$$

in the sense that

$$(Q_V f)(x, y) = (2\pi)^{-1/2} \int_{k \in \mathbb{R}} V(x, y)^{1/2} e^{iky} u_n(x; k) f(k) dk.$$

Notice that for any $I \subset (0, \bar{\mathcal{E}}_n - \underline{\mathcal{E}}_n) \setminus \{\lambda\}$,

$$S_V[I] = Q_V \mathbb{1}_I(k) |E_n(k) - \bar{\mathcal{E}}_n + \lambda|^{-1/2}. \quad (76)$$

Since $V \in \mathcal{S}_0^m$, we know from [\[31, Lemma 5.1\]](#) that for any $n \in \mathbb{N}$, the operator $Q_V^* Q_V : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ satisfies

$$Q_V^* Q_V = Op^W(w_n), \quad \text{with } w_n \in \mathcal{S}_0^m. \quad (77)$$

Moreover the symbol w_n satisfies

$$w_n(x, y) = V(a^{-1}(x), -y) + R_1(x, y), \quad \text{with } R_1 \in \mathcal{S}_0^{m+1}. \quad (78)$$

In the two following lemmas, we will control the terms appearing in [Lemma 6.1](#), [\(69\)](#). The core argument is to approximate $S_V S_V^*[(0, \lambda - \epsilon_\lambda)]$ by pseudodifferential operators, whose symbols involve both V and a .

Lemma 6.3. For any $r > 0$, $\theta > 1$ and $p > m$

$$n_+(r, S_V S_V^*[(\lambda + \lambda^\theta, A)]) = O(\lambda^{-(p/M+\theta)/m}), \quad \lambda \downarrow 0.$$

Proof. Due to $\min_{k \in \varrho(\lambda + \lambda^\theta, A)} |\bar{\mathcal{E}}_n - E_n(k) - \lambda| = \lambda^\theta$, and using (76), we have the inequality $S_V S_V^*[(\lambda + \lambda^\theta, A)] \leq (\lambda^\theta)^{-1} Q_V \mathbb{1}_{\varrho(\lambda + \lambda^\theta, A)}(k) Q_V^*$. Therefore,

$$\begin{aligned} n(r, S_V S_V^*[(\lambda + \lambda^\theta, A)]) &\leq n(r\lambda^\theta, Q_V \mathbb{1}_{\varrho(\lambda + \lambda^\theta, A)}(k) Q_V^*) \\ &\leq n(r\rho_\lambda \lambda^\theta, Q_V \langle k \rangle^{-p} Q_V^*) \\ &= n(r\rho_\lambda \lambda^\theta, \langle k \rangle^{-p/2} Q_V^* Q_V \langle k \rangle^{-p/2}), \end{aligned} \quad (79)$$

where $\rho_\lambda = (\sup_{k \in \varrho(\lambda + \lambda^\theta, A)} \langle k \rangle^p)^{-1}$. Next, (77) together with the composition formula, imply that $\langle k \rangle^{-p/2} Q_V^* Q_V \langle k \rangle^{-p/2}$ is a pseudodifferential operator whose Weyl symbol belongs to S_p^m . Since for any $f \in S_p^m$ holds true that $n_+(\lambda, Op^W(f)) = O(\lambda^{-1/m})$ if $p > m$ (see Lemma 4.3 in [4]), having in mind (30) we will obtain

$$\begin{aligned} n(\rho_\lambda \lambda^\theta, \langle \cdot \rangle^{-p/2} Q_V^* Q_V \langle \cdot \rangle^{-p/2}) &= O((\rho_\lambda \lambda^\theta)^{-1/m}) \\ &= O(\lambda^{-(p/M+\theta)/m}), \quad \lambda \downarrow 0. \end{aligned} \quad (80)$$

Gathering (79) with (80) we get the result. $\square$

Let $\varepsilon > 0$ and take a smooth function η_ε with bounded derivatives such that $0 \leq \eta_\varepsilon(x) \leq 1$ for all $x \in \mathbb{R}$, $\eta_\varepsilon(x) = 0$ for $x \leq -2\varepsilon$ and $\eta_\varepsilon(x) = 1$ for $x \geq -\varepsilon$. We define then

$$V_\eta(x, y) := \eta_\varepsilon(x) V(x, y).$$

Lemma 6.4. Assume that $V \geq 0$ satisfies (43) with $m > 2$, $N_0(\lambda, V)$ satisfies (44), (45) and b satisfies (3), (10). Then, for any $r > 0$, $\delta \in (0, 1)$, $\theta > 1$, $p > m$

$$\begin{aligned} &n_+((1 + \delta)r\lambda, Op^W(V_\eta(a^{-1}(x), -y))) + O(\lambda^{-(p/M+\theta)/m}) + o(\lambda^{-2/m}) \\ &\leq n_+(r, S_V S_V^*[(0, \lambda - \lambda^\theta)]) \\ &\leq n_+((1 - \delta)r\lambda, Op^W(V_\eta(a^{-1}(x), -y))) + O(\lambda^{-(p/M+\theta)/m}) + o(\lambda^{-2/m}), \quad \lambda \downarrow 0. \end{aligned}$$

Proof. First, note that using (75) we can define the operator Q_{V_η} , and similarly to (77), $Q_{V_\eta}^* Q_{V_\eta}$ has a symbol $w_{\eta,n}$ in S_0^m which satisfies (78), with V_η instead of V . Therefore we will be allowed to use arguments similar to those of Lemma 6.3.

It is clear that $S_V^* S_V - S_{V_\eta}^* S_{V_\eta} = S_{V-V_\eta}^* S_{V-V_\eta}$, and the function $V - V_\eta$ is equal to zero for $x \geq -\varepsilon$. Then, using Remark 6.2, the proof of Theorem 4.3 and the Weyl's inequalities, we conclude that for any $r > 0$, $\delta \in (0, 1)$ and $\theta > 1$

$$\begin{aligned} &n_+(r(1 + \delta); S_{V_\eta} S_{V_\eta}^*[(0, \lambda - \lambda^\theta)]) \\ &\leq n_+(r; S_V S_V^*[(0, \lambda - \lambda^\theta)]) \\ &\leq n_+(r(1 - \delta); S_{V_\eta} S_{V_\eta}^*[(0, \lambda - \lambda^\theta)]), \quad \lambda \downarrow 0. \end{aligned} \quad (81)$$

Now, for $\delta \in (0, 1)$ write

$$S_{V_\eta} S_{V_\eta}^*[(0, \lambda - \lambda^\theta)] = S_{V_\eta} S_{V_\eta}^*[(0, \delta\lambda)] + S_{V_\eta} S_{V_\eta}^*[(\delta\lambda, \lambda - \lambda^\theta)]. \quad (82)$$

Here, arguing as in the proof of [Lemma 6.3](#) we obtain

$$n_+(r, S_{V_\eta} S_{V_\eta}^*[(\delta\lambda, \lambda - \lambda^\theta)]) = O(\lambda^{-(p/M+\theta)/m}). \quad (83)$$

Combining (81), (82), (83) and using (62), we get that for all $r > 0$ and $\delta \in (0, 1)$,

$$\begin{aligned} & n_+(r(1+\delta); S_{V_\eta} S_{V_\eta}^*[(0, \delta\lambda)]) + O(\lambda^{-(p/M+\theta)/m}) \\ & \leq n_+(r; S_V S_V^*[(0, \lambda - \lambda^\theta)]) \\ & \leq n_+(r(1-\delta); S_{V_\eta} S_{V_\eta}^*[(0, \delta\lambda)]) + O(\lambda^{-(p/M+\theta)/m}), \quad \lambda \downarrow 0. \end{aligned} \quad (84)$$

Next, to estimate $n_+(r, S_{V_\eta} S_{V_\eta}^*[(0, \delta\lambda)])$ we use (76):

$$\lambda^{-1} \mathbb{1}_{\varrho(0, \delta\lambda)} Q_{V_\eta} Q_{V_\eta}^* \mathbb{1}_{\varrho(0, \delta\lambda)} \leq S_{V_\eta} S_{V_\eta}^*[(0, \delta\lambda)] \leq ((1-\delta)\lambda)^{-1} Q_{V_\eta} Q_{V_\eta}^*. \quad (85)$$

To get the lower bound, since $Q_{V_\eta} \mathbb{1}_{\varrho(0, \delta\lambda)} Q_{V_\eta}^* = Q_{V_\eta} Q_{V_\eta}^* + Q_{V_\eta} (1 - \mathbb{1}_{\varrho(0, \delta\lambda)}) Q_{V_\eta}^*$, by (63)

$$\begin{aligned} & n_+(r; \mathbb{1}_{\varrho(0, \delta\lambda)} Q_{V_\eta} Q_{V_\eta}^* \mathbb{1}_{\varrho(0, \delta\lambda)}) = n_+(r; Q_{V_\eta} \mathbb{1}_{\varrho(0, \delta\lambda)} Q_{V_\eta}^*) \\ & \geq n_+(r(1+\delta); Q_{V_\eta} Q_{V_\eta}^*) + n_+(r\delta; Q_{V_\eta} (1 - \mathbb{1}_{\varrho(0, \delta\lambda)}) Q_{V_\eta}^*) \\ & = n_+(r(1+\delta); Q_{V_\eta} Q_{V_\eta}^*) + O(\lambda^{-(p/M+\theta)/m}), \end{aligned} \quad (86)$$

where the last estimate is obtained arguing again as in the proof of [Lemma 6.3](#).

Then, (85), (86) and the min-max principle yield

$$\begin{aligned} & n_+(r(1+\delta)\lambda; Q_{V_\eta}^* Q_{V_\eta}) + O(\lambda^{-(p/M+\theta)/m}) \\ & \leq n_+(r; S_V S_V^*[(0, \delta\lambda)]) \\ & \leq n_+(r(1-\delta)\lambda; Q_{V_\eta}^* Q_{V_\eta}) + O(\lambda^{-(p/M+\theta)/m}), \quad \lambda \downarrow 0. \end{aligned} \quad (87)$$

Using (78), we apply Lemma 4.5 in [7] and we have $n_+(\lambda, Q_{V_\eta}^* Q_{V_\eta}) = n_+(\lambda, Op^W(V_\eta(a^{-1}(x), -y))) + o(\lambda^{-2/m})$. We combine this with (84) and (87) to conclude. $\square$

Putting together [Lemmas 6.1, 6.3 and 6.4](#) we obtain that for any $r > 0$ and $\delta \in (0, 1)$

$$\begin{aligned} & n_+((1+\delta)r\lambda, Op^W(V_\eta(a^{-1}(x), -y))) + O(\lambda^{-(p/M+\theta)/m}) + O(\lambda^{\theta-1-2/M}) + o(\lambda^{-2/m}) \\ & \leq -\xi(\bar{\mathcal{E}}_n - \lambda, H_-, H_0) \\ & \leq n_+((1-\delta)r\lambda, Op^W(V_\eta(a^{-1}(x), -y))) + O(\lambda^{-(p/M+\theta)/m}) + O(\lambda^{\theta-1-2/M}) + o(\lambda^{-2/m}). \end{aligned}$$

Then, if $m < M$, $m < p$ and $\theta > 1$, the biggest term between $\lambda^{\theta-1-2/M}$ and $\lambda^{-(p/M+\theta)/m}$ is the last one. Minimizing both p and θ , this term becomes $\lambda^{-(1/m+1/M)}$. Therefore, we obtain that for all $\delta \in (0, 1)$

$$\begin{aligned} & n_+((1+\delta)\lambda; Op^W(V_\eta(a^{-1}(x), -y))) + o(\lambda^{-2/m}) \\ & \leq -\xi(\bar{\mathcal{E}}_n - \lambda; H_-, H_0) \\ & \leq n_+((1-\delta)\lambda; Op^W(V_\eta(a^{-1}(x), -y))) + o(\lambda^{-2/m}), \quad \lambda \downarrow 0. \end{aligned} \quad (88)$$

Next, it can be proved that $V_\eta(a^{-1}(x), -y) \in S_0^m$ and that $N_0(\lambda, V)$ satisfying (44)-(45) implies the same for $N(\lambda, V_\eta)$, where $N(\lambda, V_\eta) := \frac{1}{2\pi} vol\{(x, y) \in \mathbb{R}^2; V_\eta(x, y) > \lambda\}$ ([24, [Lemmas 3.7 and 3.8](#)]). Then, by [7, [Theorem 1.3](#)], we know that there exists $\nu > 0$ such that

$$n_+(\lambda, Op^W(V_\eta(a^{-1}(x), -y))) = N(\lambda, V_\eta(a^{-1}(x), -y))(1 + O(\lambda^\nu)), \quad \lambda \downarrow 0. \quad (89)$$

Easy computations ([24, Lemma 3.8]) show that

$$\lim_{\lambda \downarrow 0} \frac{N(\lambda, V_\eta(a^{-1}(x), -y))}{b_+ N_0(\lambda, V)} = 1;$$

$$\lim_{\delta \downarrow 0} \liminf_{\lambda \downarrow 0} \frac{N_0(\lambda(1 + \delta), V)}{N_0(\lambda, V)} = \lim_{\delta \downarrow 0} \limsup_{\lambda \downarrow 0} \frac{N_0(\lambda(1 - \delta), V)}{N_0(\lambda, V)} = 1. \quad (90)$$

As a consequence, (88), (89) and (90) altogether imply (46).

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