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A comparison theorem for two divided differences and applications to special functions

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ABSTRACT

In this paper, we present a general comparison theorem for two divided differences of a three times differentiable function. This gives a unified treatment for (logarithmically) complete monotonicity, monotonicity and inequalities involving some special functions including gamma, psi and polygamma functions. As their consequences, we not only refine and generalize some important results, but also present simple and interesting alternative proofs of certain earlier results.

1. Introduction

The Euler's gamma and psi (digamma) functions are defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)},$$

respectively. The derivatives ψ', ψ'', ψ''', ... are known as polygamma functions.

Denote by $\psi_n = (-1)^{n-1} \psi^{(n)}$ for $n \in \mathbb{N}$ and $\psi_{-1} = \ln \Gamma$, $\psi_0 = -\psi$. Then ψ_n has some simple properties:

(P1) $\psi_n' = -\psi_{n+1} < 0$ for $n \ge 0$.
(P2) ψ' is strictly completely monotonic on $(0, \infty)$, and so is ψ_n for $n \ge 0$.
(P3) The sequence $\{\psi_{n+1}/\psi_n\}_{n \in \mathbb{N}}$ is strictly increasing and concave.

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(P4) ψ_n/ψ_{n+1} for $n \in \mathbb{N}$ is strictly increasing and convex on $(0, \infty)$.
(P5) ψ_n for $n \in \mathbb{N}$ is log-convex on $(0, \infty)$.
(P6) $x\psi_{n+1}/\psi_n$ for $n \in \mathbb{N}$ is strictly decreasing from $(0, \infty)$ onto $(n, n + 1)$.
(P7) $\psi_{n+1}^2/(\psi_n\psi_{n+2})$ is strictly decreasing from $(0, \infty)$ onto $(n/(n+1), (n+1)/(n+2))$.

Properties (P3)–(P5) were proved in [41,47], (P6) is due to Alzer [4,5], while (P7) was showed in [41]. More properties of polygamma functions can be found in [2–6,41], [8, Theorem 2.7], [21,33,38,40].

Let $f : I \to \mathbb{R}$ be (strictly) monotonic and $a, b \in I$. Then the so-called integral f-mean of a and b is defined as [16]

$$\mathcal{I}_f(a, b) = f^{-1}\left(\frac{\int_a^b f(x)\,dx}{b-a}\right) \text{ if } a \neq b \text{ and } \mathcal{I}_f(a, a) = a.$$

Elezović and Pečarić [16, Theorem 6] proved that for $a, b > 0$, $\mathcal{I}_{\psi'}(a, b) \le \mathcal{I}_\psi(a, b)$ and the function $x \mapsto \mathcal{I}_\psi(x + a, x + b) - x$ is increasing concave with

$$\lim_{x\to\infty}(\mathcal{I}_\psi(x + a, x + b) - x) = \frac{a+b}{2}.$$

And therefore, for $a, b > 0$ the double inequality

$$x + \mathcal{I}_\psi(a, b) < \mathcal{I}_\psi(x + a, x + b) < x + \frac{a+b}{2}$$

holds for $x \ge 0$. Very recently, Yang and Zheng [47] further showed that for $a, b > 0$ with $a \neq b$, the sequence $\{\mathcal{I}_{\psi_n}(a, b)\}_{n\ge0}$ is strictly decreasing with $\lim_{n\to\infty}\mathcal{I}_{\psi_n}(a, b) = \min(a, b)$, and the function $x \mapsto A_{\psi_n}(x) = \mathcal{I}_{\psi_n}(x + a, x + b) - x$ is strictly increasing from $(-\min(a, b), \infty)$ onto $(\min(a, b), (a + b)/2)$. And consequently, the double inequality

$$x + \min(a, b) < \mathcal{I}_{\psi_n}(x + a, x + b) < x + \frac{a+b}{2}$$

holds for all $x > -\min(a, b)$.

In [3, Theorem 2] Alzer established that for an integer $n \ge 0$ and a real number $s \in (0, 1)$, the double inequality

$$\frac{n!}{(x + \alpha_n(s))^{n+1}} < \frac{\psi_n(x + s) - \psi_n(x + 1)}{1 - s} < \frac{n!}{(x + \beta_n(s))^{n+1}} \tag{1.1}$$

holds for all real numbers $x > 0$ with the best possible constants

$$\alpha_n(s) = \left(\frac{\psi_n(s) - \psi_n(1)}{n!\,(1-s)}\right)^{-1/(n+1)} \quad \text{and} \quad \beta_n(s) = \frac{s}{2}.$$

If let $s \to 1$ then inequality (1.1) becomes as

$$\frac{n!}{[x + \alpha_n(1)]^{n+1}} < \left|\psi^{(n)}(x + 1)\right| < \frac{n!}{[x + \beta_n(1)]^{n+1}}, \tag{1.2}$$

where $\alpha_n(1) = \left(n!/\left|\psi^{(n+1)}(1)\right|\right)^{1/(n+1)}$ and $\beta_n(1) = 1/2$ are the best constants, which was proved in [21, Theorem 1] by Guo, Qi, Zhao and Luo.

Inspired by these results, we put forward a problem: how to compare $\mathcal{I}_{\psi_n}(x+p, x+q)$ with $\mathcal{I}_{\psi_n}(x+r, x+s)$ for fixed $p, q, r, s \in \mathbb{R}$ and any $x > -\min(p, q, r, s)$? In other words, what are the conditions such that the comparison inequality

$$\mathcal{I}_{\psi_n}(x+p, x+q) \leq \mathcal{I}_{\psi_n}(x+r, x+s) \tag{1.3}$$

holds for all $x > -\min(p, q, r, s)$?

We note that the comparison inequality (1.3), for $(p-q)(r-s) \neq 0$ and $n \geq 0$, is equivalent to

$$\frac{\psi_{n-1}(x+p) - \psi_{n-1}(x+q)}{p-q} \leq \frac{\psi_{n-1}(x+r) - \psi_{n-1}(x+s)}{r-s},$$

where $\psi_{-1} := \ln\Gamma$ and $\psi_0 = -\psi$. Now let $\phi : (a, b) \to \mathbb{R}$ be a differentiable function. For fixed $p, q \in \mathbb{R}$ with $\max(p, q) - \min(p, q) < b - a$, let $\Phi_{p,q}$ be defined on $(a - \min(p, q), b - \max(p, q))$ by

$$\Phi_{p,q}(x) = \begin{cases} \dfrac{\phi(x+p) - \phi(x+q)}{p-q} & \text{if } p \neq q, \\ \phi'(x+p) & \text{if } p = q. \end{cases} \tag{1.4}$$

Then such comparison problem mentioned above can be boiled down to determine those parameters p, q, r, s such that the comparison inequality

$$\Phi_{p,q}(x) \leq \Phi_{r,s}(x)$$

holds for all $x \in (a - \rho, b - \sigma)$, where $\rho = \min(p, q, r, s)$ and $\sigma = \max(p, q, r, s)$. The main purpose of this paper is to answer such comparison problem. Our main result is stated as follows.

Theorem 1. *Let $p, q, r, s, a, b \in \mathbb{R}$ with $b - a > \sigma - \rho$, where $\sigma = \max(p, q, r, s)$ and $\rho = \min(p, q, r, s)$. Assume that $\phi : (a, b) \to \mathbb{R}$ is a three times differentiable function satisfying that $\phi''(x) > (<)0$ and $\eta(x) = \phi'''(x)/\phi''(x)$ is strictly monotonic for all $x \in (a, b)$. Then for real $c_1, c_2 \in (a - \rho, b - \sigma)$ with $c_2 > c_1$, the comparison inequality*

$$\Phi_{p,q}(x) \leq \Phi_{r,s}(x)$$

holds for all $x \in [c_1, c_2]$ if and only if

$$\Phi_{p,q}(c_1) \leq \Phi_{r,s}(c_1) \quad \text{and} \quad \Phi_{p,q}(c_2) \leq \Phi_{r,s}(c_2). \tag{1.5}$$

Letting $f = \phi'$. Then Theorem 1 can be equivalently stated in the following form.

Theorem 2. *Let $p, q, r, s, a, b \in \mathbb{R}$ with $b - a > \sigma - \rho$, where $\sigma = \max(p, q, r, s)$ and $\rho = \min(p, q, r, s)$. Assume that $f : (a, b) \to \mathbb{R}$ is a two times differentiable function satisfying that both f and f''/f' are strictly monotonic for all $x \in (a, b)$. Then for real $c_1, c_2 \in (a - \rho, b - \sigma)$ with $c_2 > c_1$, the comparison inequality*

$$\mathcal{I}_f(x+p, x+q) \leq \mathcal{I}_f(x+r, x+s)$$

holds for all $x \in [c_1, c_2]$ if and only if

$$\begin{cases} \mathcal{I}_f(c_1+p, c_1+q) \leq \mathcal{I}_f(c_1+r, c_1+s), \\ \mathcal{I}_f(c_2+p, c_2+q) \leq \mathcal{I}_f(c_2+r, c_2+s). \end{cases}$$

Our second aim of this paper is to deal with the comparison inequality for psi and polygamma functions (1.3) by applying Theorem 1 or 2. We will prove the following

Theorem 3. Let $p, q, r, s \in \mathbb{R}$ *and* $n \geq 0$ *be an integer.*

(i) The comparison inequality (1.3) holds for all $x > -\min(p, q, r, s)$ *if and only if* $\min(p, q) \leq \min(r, s)$ *and* $p + q \leq r + s$.
(ii) The comparison inequality (1.3) holds for all $x > x_0 \geq -\min(p, q, r, s)$ *if and only if* $\mathcal{I}_{\psi_n}(x_0 + p, x_0 + q) \leq \mathcal{I}_{\psi_n}(x_0 + r, x_0 + s)$ *and* $p + q \leq r + s$.
(iii) The comparison inequality (1.3) holds for all $-\min(p, q, r, s) < x < x_0$ *if and only if* $\min(p, q) \leq \min(r, s)$ *and* $\mathcal{I}_{\psi_n}(x_0 + p, x_0 + q) \leq \mathcal{I}_{\psi_n}(x_0 + r, x_0 + s)$.

Corollary 1. *Let* $p, q, r, s \in \mathbb{R}$ *and* $n \geq 0$ *be an integer. Then the equality*

$$\mathcal{I}_{\psi_n}(x + p, x + q) = \mathcal{I}_{\psi_n}(x + r, x + s)$$

holds for all $x > -\min(p, q, r, s)$ *if and only if* $\min(p, q) = \min(r, s)$ *and* $\max(p, q) = \max(r, s)$.

The rest of this paper is organized as follows. In the next section, we prove Theorems 1 and 3 by means of three lemmas. In the last section, as a consequence of Theorem 3, a necessary and sufficient condition for a ratio of gamma functions to be logarithmically completely monotonic is presented, which unifies many known results. Moreover, by making use of Theorem 3, some monotonicity and inequalities involving gamma, psi and polygamma functions are reproved and extended.

2. Proofs of Theorems 1 and 3

2.1. Lemmas

To prove Theorem 1, we need the following lemmas.

Lemma 1. *Let* $x_1, x_2, x_3 \in [a, b]$ *and* ϕ *be two times differentiable on* $[a, b]$. *Define that*

$$[x_1, x_2; \phi] := \begin{cases} \dfrac{\phi(x_1) - \phi(x_2)}{x_1 - x_2} & \text{if } x_1 \neq x_2, \\ \phi'(x_2) & \text{if } x_1 = x_2, \end{cases} \tag{2.1}$$

$$[x_1, x_2, x_3; \phi] := \begin{cases} \dfrac{[x_1, x_2; \phi] - [x_2, x_3; \phi]}{x_1 - x_3} & \text{if } x_1 \neq x_3, \\ \left.\dfrac{\partial [x_1, x_2; \phi]}{\partial x_1}\right|_{x_1 = x_3} & \text{if } x_1 = x_3. \end{cases} \tag{2.2}$$

Then we have

(i) $[x_1, x_2, x_3; \phi]$ *are symmetric with respect to* x_1, x_2 *and* x_3, *that is,*

$$\begin{aligned} [x_1, x_2, x_3; \phi] &= [x_1, x_3, x_2; \phi] = [x_2, x_3, x_1; \phi] \\ &= [x_2, x_1, x_3; \phi] = [x_3, x_1, x_2; \phi] = [x_3, x_2, x_1; \phi]; \end{aligned}$$

(ii) $[x_1, x_2, x_3; \phi] \geq (\leq) 0$ *if and only if* ϕ *is convex (concave) on* $[a, b]$;

(iii) (Mean value theorem) if ϕ is two times differentiable on $[a,b]$ and $x_1, x_2, x_3 \in [a,b]$, then there is a ξ between the smallest and the largest x_i such that

$$[x_1, x_2, x_3; \phi] = \frac{\phi''(\xi)}{2!}. \tag{2.3}$$

The following lemma will play an important role in the proof of Theorem 1.

Lemma 2. *Suppose that (i) $p, q, r, s \in \mathbb{R}$ with $p \geq q, r \geq s$ and $(s-p)(q-r)(s-q)(r-p) \neq 0$; (ii) $a, b \in \mathbb{R}$ with $b-a > \sigma - \rho$, where $\sigma = \max(p,r)$ and $\rho = \min(q,s)$; (iii) $\phi : (a,b) \to \mathbb{R}$ is a three times differentiable function satisfying that $\phi''(x) > (<)0$ and $\eta(x) = \phi'''(x)/\phi''(x)$ is strictly monotonic for all $x \in (a,b)$. Then the function*

$$x \mapsto Q(x) = \frac{\Phi_{r,s}(x) - \Phi_{r,p}(x)}{s-p} \Big/ \frac{\Phi_{p,q}(x) - \Phi_{p,r}(x)}{q-r} \tag{2.4}$$

is strictly monotonic on $I_0 := (a-\rho, b-\sigma)$, where $\Phi_{p,q}(x)$ is defined by (1.4). More precisely, we have $\mathrm{sgn} Q'(x) = \mathrm{sgn}(s-q)$ *for $x \in I_0$.*

Proof. Without loss of generality, we assume that $\eta(x) = \phi'''(x)/\phi''(x)$ is strictly monotonic increasing on (a,b). We prove this lemma stepwise.

Firstly, we use (2.2) to obtain

$$\begin{aligned}
\frac{\Phi_{r,s}(x) - \Phi_{r,p}(x)}{s-p} &= [r+x, s+x, p+x; \phi] \\
&= \begin{cases} \dfrac{[s+x, p+x; \phi] - [r+x, p+x; \phi]}{s-r} & \text{if } s \neq r, \\ \left.\dfrac{\partial [s+x, p+x; \phi]}{\partial s}\right|_{s=r} & \text{if } s = r \end{cases} \\
&= \begin{cases} \dfrac{\Phi_{s,p}(x) - \Phi_{r,p}(x)}{s-r} & \text{if } s \neq r, \\ \lim_{s \to r} \dfrac{\Phi_{s,p}(x) - \Phi_{r,p}(x)}{s-r} & \text{if } s = r. \end{cases}
\end{aligned}$$

Then $Q(x)$ can be expressed as

$$Q(x) = \frac{[r+x, s+x, p+x; \phi]}{[p+x, q+x, r+x; \phi]} \tag{2.5}$$

$$= \begin{cases} \dfrac{\Phi_{s,p}(x) - \Phi_{r,p}(x)}{s-r} \Big/ \dfrac{\Phi_{q,p}(x) - \Phi_{r,p}(x)}{q-r} & \text{if } s \neq r, \\ \lim\limits_{s \to r} \left[\dfrac{\Phi_{s,p}(x) - \Phi_{r,p}(x)}{s-r} \Big/ \dfrac{\Phi_{q,p}(x) - \Phi_{r,p}(x)}{q-r} \right] & \text{if } s = r. \end{cases} \tag{2.6}$$

From (2.5) we easily see that $Q(x)$ is symmetric with respect to p and r, and so we can assume that $p > r$. Then $\sigma = \max(p,r) = p$.

Secondly, we prove that

$$\frac{Q'(x)}{Q(x)} = (s-q) \cdot [s+x, q+x, p+x; \phi] \cdot [h(s), h(r), h(q); g \circ h^{-1}], \tag{2.7}$$

where h and g are defined on $[\min(q,s), p]$ by

$$h(t) := \Phi_{t,p}(x) = \begin{cases} \dfrac{\phi(t+x) - \phi(p+x)}{t-p} & \text{if } t \neq p, \\ \phi'(p+x) & \text{if } t = p, \end{cases} \tag{2.8}$$

$$g(t) := \Phi'_{t,p}(x) = \begin{cases} \dfrac{\phi'(t+x) - \phi'(p+x)}{t-p} & \text{if } t \neq p, \\ \phi''(p+x) & \text{if } t = p, \end{cases} \tag{2.9}$$

respectively.

Indeed, for $s \neq r$, applying logarithmic derivative yields

$$\begin{aligned} \frac{Q'(x)}{Q(x)} &= \frac{\Phi'_{s,p}(x) - \Phi'_{r,p}(x)}{\Phi_{s,p}(x) - \Phi_{r,p}(x)} - \frac{\Phi'_{q,p}(x) - \Phi'_{r,p}(x)}{\Phi_{q,p}(x) - \Phi_{r,p}(x)} \\ &= \frac{g(s) - g(r)}{h(s) - h(r)} - \frac{g(q) - g(r)}{h(q) - h(r)} \\ &= [h(s) - h(q)] \cdot \frac{1}{h(s) - h(q)} \left(\frac{g(s) - g(r)}{h(s) - h(r)} - \frac{g(q) - g(r)}{h(q) - h(r)} \right). \end{aligned}$$

Using notations given by (2.1) and (2.2) give

$$\begin{aligned} h(s) - h(q) = \Phi_{s,p}(x) - \Phi_{q,p}(x) &= [s+x, p+x; \phi] - [q+x, p+x; \phi] \\ &= (s-q) \frac{[s+x, p+x; \phi] - [q+x, p+x; \phi]}{(s+x) - (q+x)} \\ &= (s-q) \left[s+x, q+x, p+x; \phi\right]. \end{aligned}$$

On the other hand, in view of $\phi''(x) > (<)0$ we have that

$$h'(x) = \frac{\int_p^x (u-p)\, \phi''(u+x) du}{(x-p)^2} > (<)0 \tag{2.10}$$

for $x \in (\min(q, s), p)$, which implies that $h(x)$ is strictly increasing (decreasing), and so h^{-1} exists and is strictly increasing (decreasing). Then, we get

$$\frac{1}{h(s) - h(q)} \left[\frac{g(s) - g(r)}{h(s) - h(r)} - \frac{g(q) - g(r)}{h(q) - h(r)} \right] = [h(s), h(r), h(q); g \circ h^{-1}],$$

which proves (2.7).

It is easy to verify that (2.7) is also true in the case of $s = r$.

Thirdly, we prove $\operatorname{sgn}(Q'(x)) = \operatorname{sgn}(s-q)$ for $x \in I_0$.

From (ii) of Lemma 1 and assumption $\phi''(x) > (<)0$ it follows that

$$Q(x) = \frac{[r+x, s+x, p+x; \phi]}{[p+x, q+x, r+x; \phi]} > 0 \text{ and } [s+x, q+x, p+x; \phi] > (<)0. \tag{2.11}$$

If we prove that $Q_1(x) := [h(s), h(r), h(q); g \circ h^{-1}] > (<)0$ for all $x \in I_0$, then by (2.7) we have $\operatorname{sgn}(Q'(x)) = \operatorname{sgn}(s-q)$, and the proof is complete.

Now, by (iii) of Lemma 1, there is a $\xi \in (\alpha, \beta)$ such that

$$Q_1(x) = [h(s), h(r), h(q); g \circ h^{-1}] = \frac{1}{2} (g(h^{-1}(y)))'' \Big|_{y=\xi}, \tag{2.12}$$

where

$$\alpha = \min\left[h\left(q\right), h\left(r\right), h\left(s\right)\right] \quad \text{and} \quad \beta = \max\left[h\left(q\right), h\left(r\right), h\left(s\right)\right]. \tag{2.13}$$

We claim that $F(y) := g(h^{-1}(y))$ is convex (concave) on (α, β) if $\phi'' > (<)\, 0$. In fact, differentiation gives

$$F'(y) = \frac{g'(t)}{h'(t)}, \quad F''(y) = \left(\frac{g'(t)}{h'(t)}\right)' \frac{1}{h'(t)}, \tag{2.14}$$

where $t = h^{-1}(y)$. Note that for $t \in (\min\left(q, s\right), p)$,

$$g'(t) = \frac{\int_p^t \left(u - p\right) \phi'''(u + x) du}{(t - p)^2},$$

which together with (2.10) yields

$$\frac{g'(t)}{h'(t)} = \frac{\int_p^t \left(u - p\right) \phi'''(u + x) du}{\int_p^t \left(u - p\right) \phi''(u + x) du}. \tag{2.15}$$

Then for $t \in (\min\left(q, s\right), p)$, we obtain

$$\begin{aligned}\left(\frac{g'(t)}{h'(t)}\right)' &= \frac{(t-p)\,\phi'''(t+x)\int_p^t (u-p)\,\phi''(u+x)du - (t-p)\,\phi''(t+x)\int_p^t (u-p)\,\phi'''(u+x)du}{\left(\int_p^t (u-p)\,\phi''(u+x)du\right)^2} \\ &= \int_p^t (t-p)\,(t-u)\,(u-p)\,\phi''(t+x)\phi''(u+x)\frac{\phi'''(t+x)/\phi''(t+x) - \phi'''(u+x)/\phi''(u+x)}{(t+x)-(u+x)}du.\end{aligned}$$

From $\phi''(t) > (<)0$ and assumption that $\eta(t) = \phi'''(t)/\phi''(t)$ is strictly monotonic increasing on (a, b) it follows that

$$\left(\frac{g'(t)}{h'(t)}\right)' > 0 \text{ for all } t \in \left(\min\left(q, s\right), p\right). \tag{2.16}$$

This in combination with $h'(t) > (<)0$ leads us to

$$F''(y) = \left(\frac{g'(t)}{h'(t)}\right)' \frac{1}{h'(t)} > (<)0 \text{ for all } t \in \left(\min\left(q, s\right), p\right), \tag{2.17}$$

where $t = h^{-1}(y)$.

We also have to show that $h^{-1}\left(\xi\right) \in \left(\min\left(q, s\right), p\right)$. Since h^{-1} is strictly increasing (decreasing) if $\phi'' > (<)\, 0$, $\xi \in (\alpha, \beta)$ together with (2.13) implies that

$$h^{-1}\left(\xi\right) \in \left(\min\left(q, r, s\right), \max\left(q, r, s\right)\right) = \left(\min\left(q, s\right), \max\left(q, r\right)\right) \subset \left(\min\left(q, s\right), p\right).$$

Then from (2.12) we get that

$$Q_1(x) = \frac{1}{2}(g(h^{-1}(y)))''\Big|_{y=\xi} = \left(\frac{g'(t)}{h'(t)}\right)' \frac{1}{h'(t)}\Big|_{t=h^{-1}(\xi)} > (<)0, \tag{2.18}$$

which completes the proof of this lemma. □

2.2. Proof of Theorem 1

Now we are in a position to prove Theorem 1.

Proof of Theorem 1. The necessity is obvious, and we only need to prove the sufficiency.

Since p, q is symmetric, and so is r, s, we assume that $p \geq q$, $r \geq s$.

(i) In the case of $(s-p)(q-r)(s-q)(r-p) = 0$. For instance, if $p = r, q \neq s$ then

$$\begin{aligned}\Phi_{p,q}(x) - \Phi_{r,s}(x) &= \Phi_{p,q}(x) - \Phi_{p,s}(x) \\ &= (q-s)\frac{\dfrac{\phi(p+x)-\phi(q+x)}{p+x-(q+x)} - \dfrac{\phi(p+x)-\phi(s+x)}{p+x-(s+x)}}{q+x-(s+x)} \\ &= (q-s)[p+x, q+x, s+x; \phi].\end{aligned}$$

Suppose that $\Phi_{p,q}(c_1) \leq \Phi_{r,s}(c_1)$ and $\Phi_{p,q}(c_2) \leq \Phi_{r,s}(c_2)$. From $[p+x, q+x, s+x; \phi] > (<)0$ due to $\phi''(x) > (<)0$ and

$$\Phi_{p,q}(c_1) - \Phi_{r,s}(c_1) = (q-s)[p+c_1, q+c_1, s+c_1; \phi] \leq 0$$

it follows that $q - s \leq (\geq)\, 0$. This yields

$$\Phi_{p,q}(x) - \Phi_{r,s}(x) = (q-s)[p+x, q+x, s+x; \phi] \leq 0$$

for all $x \in [c_1, c_2]$.

In a similar way, the desired result is also true in other cases.

(ii) In the case of $(s-p)(q-r)(s-q)(r-p) \neq 0$. We have

$$\begin{aligned}\Phi_{p,q}(x) - \Phi_{r,s}(x) &= [\Phi_{p,q}(x) - \Phi_{p,r}(x)] - [\Phi_{r,s}(x) - \Phi_{r,p}(x)] \\ &= \frac{\Phi_{p,q}(x) - \Phi_{p,r}(x)}{q-r}\left(q-r-(s-p)\frac{\dfrac{\Phi_{r,s}(x)-\Phi_{r,p}(x)}{s-p}}{\dfrac{\Phi_{p,q}(x)-\Phi_{p,r}(x)}{q-r}}\right) \\ &:= [p+x, q+x, r+x; \phi]\, P(x), \end{aligned} \tag{2.19}$$

where

$$P(x) = (q-r) - (s-p)Q(x), \tag{2.20}$$

here $Q(x)$ is defined by (2.4).

Suppose that $\Phi_{p,q}(c_1) \leq \Phi_{r,s}(c_1)$ and $\Phi_{p,q}(c_2) \leq \Phi_{r,s}(c_2)$. Due to $\phi''(x) > (<)0$, so is $[p+x, q+x, r+x; \phi] > (<)0$, we have $P(c_1) \leq (\geq)0$ and $P(c_2) \leq (\geq)0$. As shown in Lemma 2, P is strictly monotonic on I_0, then we arrive at $P(x) \leq (\geq)0$ for all $x \in [c_1, c_2]$. From this it follows that

$$\Phi_{p,q}(x) - \Phi_{r,s}(x) = [p+x, q+x, r+x; \phi]\, P(x) \leq 0$$

for all $x \in [c_1, c_2]$, which completes the proof. □

2.3. Proof of Theorem 3

Proof of Theorem 3. (i) By properties of ψ_n (P1) and (P4), for each integer $n \geq 0$, we have that $\psi_n' = -\psi_{n+1} < 0$ and $\psi_n''/\psi_n' = -\psi_{n+2}/\psi_{n+1}$ is strictly increasing on $(0, \infty)$. Then, by Theorem 2, to prove comparison inequality (1.3) holds for all $x > -\rho = -\min(p, q, r, s)$, it suffices to check that $D_n(-\rho^+) \leq 0$ and $D_n(\infty) \leq 0$ if and only if $\min(p, q) \leq \min(r, s)$ and $p + q \leq r + s$, where

$$D_n(x) = \mathcal{I}_{\psi_n}(x + p, x + q) - \mathcal{I}_{\psi_n}(x + r, x + s).$$

It has been shown in [47, Theorem 1.3],

$$\lim_{x \to -\min(p,q)} \left(\mathcal{I}_{\psi_n}(x + p, x + q) - x\right) = \min(p, q),$$

$$\lim_{x \to \infty} \left(\mathcal{I}_{\psi_n}(x + p, x + q) - x\right) = \frac{p + q}{2},$$

so we have

$$\begin{aligned}
\lim_{x \to -\rho} D_n(x) &= \lim_{x \to -\rho} \left(\mathcal{I}_{\psi_n}(x + p, x + q) - x\right) - \lim_{x \to -\rho} \left(\mathcal{I}_{\psi_n}(x + r, x + s) - x\right) \\
&= \begin{cases} -\mathcal{I}_{\psi_n}(r - \rho, s - \rho) < 0 & \text{if } \rho = \min(p, q), \\ \mathcal{I}_{\psi_n}(p - \rho, q - \rho) > 0 & \text{if } \rho = \min(r, s), \end{cases} \\
\lim_{x \to \infty} D_n(x) &= \lim_{x \to \infty} \left(\mathcal{I}_{\psi_n}(x + p, x + q) - x\right) - \lim_{x \to \infty} \left(\mathcal{I}_{\psi_n}(x + r, x + s) - x\right) \\
&= \frac{p + q}{2} - \frac{r + s}{2},
\end{aligned}$$

which prove the first assertion of this theorem.

(ii) Similarly, the necessary and sufficient conditions for the comparison inequality (1.3) to hold for all $x > x_0 > -\rho$ are $D_n(x_0) \leq 0$ and $D_n(\infty) \leq 0$, that is, $\mathcal{I}_{\psi_n}(x_0 + p, x_0 + q) \leq \mathcal{I}_{\psi_n}(x_0 + r, x_0 + s)$ and $p + q \leq r + s$.

(iii) In the same way, the comparison inequality (1.3) holds for all $-\min(p, q, r, s) < x < x_0$ if and only if $D_n(-\rho^+) \leq 0$ and $D_n(x_0) \leq 0$, which in turn if and only if $\min(p, q) \leq \min(r, s)$ and $\mathcal{I}_{\psi_n}(x_0 + p, x_0 + q) \leq \mathcal{I}_{\psi_n}(x_0 + r, x_0 + s)$.

The proof is done. □

3. Consequences and remarks

3.1. Complete monotonicity of ratios of gamma functions

Recall that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and $(-1)^n (f(x))^{(n)} \geq 0$ for $x \in I$ and $n \geq 0$ (see [11,39]). A positive function f is called logarithmically completely monotonic on an interval I if f has derivatives of all orders on I and its logarithm $\ln f$ satisfies $(-1)^n (\ln f(x))^{(n)} \geq 0$ for all $n \in \mathbb{N}$ on I (see [7,30]). For convenience, we denote respectively the sets of the completely monotonic functions and the logarithmically completely monotonic functions on I by $\mathcal{C}[I]$ and $\mathcal{L}[I]$.

There is one classical paper [23] involving (logarithmically) complete monotonicity of the ratio of two gamma functions, which was presented by Ismail, Lorch and Muldoon in 1986. Since then, this topic has attracted the attention and interest of many scholars, who published a number of results including improvements, generalizations by different ideals and methods (see [12,20,23,24,28,29,31–35,37,43,44,47]).

In this subsection, we will use Theorem 3 to establish necessary and sufficient conditions for the ratio of two gamma functions to be completely monotonic, which unify and generalize known results given in [33, Theorem 1], [12,23], [32, Theorem 1], [43].

Proposition 1. *For $p, q, r, s \in \mathbb{R}$, $\rho = \min(p, q, r, s)$, let the function $g_{p,q}$ be defined on $(-\rho, \infty)$ by*

$$g_{p,q}(x) = \left(\frac{\Gamma(x+p)}{\Gamma(x+q)}\right)^{1/(p-q)} \quad \text{if } p \neq q \text{ and } g_{p,p}(x) = \exp\psi(x+p). \tag{3.1}$$

Then $\ln(g_{p,q}/g_{r,s})$ is completely monotonic on $(-\rho, \infty)$ if and only if $\min(r,s) \leq \min(p,q)$ and $r+s \leq p+q$.

Proof. For $(p-q)(r-s) \neq 0$, we have

$$\begin{aligned}
\ln\frac{g_{p,q}(x)}{g_{r,s}(x)} &= \frac{\ln\Gamma(x+p) - \ln\Gamma(x+q)}{p-q} - \frac{\ln\Gamma(x+r) - \ln\Gamma(x+s)}{r-s} \\
&= \frac{\int_q^p \psi(x+t)\,dt}{p-q} - \frac{\int_s^r \psi(x+t)\,dt}{r-s}.
\end{aligned}$$

Differentiation yields that for $n \geq 0$,

$$\begin{aligned}
(-1)^n\left(\ln\frac{g_{p,q}(x)}{g_{r,s}(x)}\right)^{(n)} &= (-1)^n\left(\frac{\int_q^p \psi^{(n)}(x+t)\,dt}{p-q} - \frac{\int_s^r \psi^{(n)}(x+t)\,dt}{r-s}\right) \\
&= -\frac{\int_q^p \psi_n(x+t)\,dt}{p-q} + \frac{\int_s^r \psi_n(x+t)\,dt}{r-s} \\
&= -\psi_n\left[\mathcal{I}_{\psi_n}(x+p, x+q)\right] + \psi_n\left[\mathcal{I}_{\psi_n}(x+r, x+s)\right],
\end{aligned}$$

which is obviously true for $(p-q)(r-s) = 0$.

Due to ψ_n is strictly decreasing on $(0, \infty)$, the inequality

$$-\psi_n(\mathcal{I}_{\psi_n}(x+p, x+q)) + \psi_n(\mathcal{I}_{\psi_n}(x+r, x+s)) \geq 0$$

holds for $x > -\rho$ is equivalent to

$$\mathcal{I}_{\psi_n}(x+r, x+s) \leq \mathcal{I}_{\psi_n}(x+p, x+q) \text{ for } x > -\rho,$$

which, by Theorem 3, is in turn equivalent to $\min(r,s) \leq \min(p,q)$ and $r+s \leq p+q$, that is, the desired assertion holds true.

This completes the proof. □

Let us define

$$H_{p,q,r}(x) := \left(\frac{g_{p,q}(x)}{g_{r,r+1}(x)}\right)^{p-q} = (x+r)^{q-p}\frac{\Gamma(x+p)}{\Gamma(x+q)}.$$

Then

$$\frac{1}{H_{p,q,r}(x)} = \left(\frac{g_{r,r+1}(x)}{g_{p,q}(x)}\right)^{p-q} = (x+r)^{p-q}\frac{\Gamma(x+q)}{\Gamma(x+p)}.$$

By Proposition 1, we have immediately

Corollary 2. *Let $p, q, r \in \mathbb{R}$ with $p > q$ and $\rho = \min(q, r)$. Then*

(i) $\ln H_{p,q,r} \in \mathcal{C}[(-\rho, \infty)]$ *if and only if* $r \leq \min((p+q-1)/2, q)$*;*
(ii) $-\ln H_{p,q,r} \in \mathcal{C}[(-\rho, \infty)]$ *if and only if* $r \geq \max((p+q-1)/2, q)$*.*

Remark 1. Recently, Yang and Chu [43] showed that $\ln(g_{p,q}/g_{r,r+1}) \in \mathcal{C}[(-\rho, \infty)]$ if and only if $2r \leq p+q-\max(|p-q|, 1)$, while $\ln(g_{r,r+1}/g_{p,q}) \in \mathcal{C}[(-\rho, \infty)]$ if and only if $2r \geq p+q-\min(|p-q|, 1)$. It is easy to check that

$$\{r \leq \min[(p+q-1)/2, \min(p,q)]\} = \{2r \leq p+q-\max(|p-q|, 1)\},$$
$$\{r \geq \max[(p+q-1)/2, \min(p,q)]\} = \{2r \geq p+q-\min(|p-q|, 1)\}.$$

Also, these are equivalent to Qi's Theorem 1 in [32], [33], and generalize Ismail, Lorch and Muldoon's Theorem 1 in [23] and Bustoz and Ismail's Theorem 3 in [12, Theorem 3].

Remark 2. Recently, Mortici, Cristea and Luin [27, Theorems 1 and 2] proved that

$$-\ln\left[x^2\left(\frac{\Gamma(x+1/3)}{\Gamma(x+1)}\right)^3\right] \in \mathcal{C}[(0,\infty)] \quad \text{and} \quad -\ln\left[x\left(\frac{\Gamma(x+2/3)}{\Gamma(x+1)}\right)^3\right] \in \mathcal{C}[(0,\infty)].$$

Chen [13, Theorem 1] showed that for $a, b > 0$,

$$\ln\frac{\Gamma(x+1)}{(x+a)^{2/3}\Gamma(x+1/3)} \in \mathcal{C}[(0,\infty)] \text{ if and only if } a \leq \frac{1}{6},$$
$$\ln\frac{\Gamma(x+1)}{(x+b)^{1/3}\Gamma(x+2/3)} \in \mathcal{C}[(0,\infty)] \text{ if and only if } b \leq \frac{1}{3}.$$

Clearly, Mortici et al.'s results are special cases of Theorem 1 given in [23] by Ismail, Lorch and Muldoon. While Chen's results are immediate consequences of Theorem 3 in [12] proved by Bustoz and Ismail.

For $s = r$, the function $g_{p,q}/g_{r,r}$ can be expressed as

$$\frac{g_{p,q}(x)}{g_{r,r}(x)} = \begin{cases} \dfrac{1}{\exp[\psi(x+r)]}\left(\dfrac{\Gamma(x+p)}{\Gamma(x+q)}\right)^{1/(p-q)} & \text{if } p \neq q, \\ e^{\psi(x+p)-\psi(x+r)} & \text{if } p = q. \end{cases}$$

As a direct consequence of Proposition 1 we immediately get the following

Corollary 3. *Let $p, q, r \in \mathbb{R}$ and $\rho = \min(p, q, r)$. Then* $\ln(g_{p,q}/g_{r,r}) \in \mathcal{C}[(-\rho, \infty)]$ *if and only if* $r \leq \min(p, q)$*, while* $\ln(g_{r,r}/g_{p,q}) \in \mathcal{C}[(-\rho, \infty)]$ *if and only if* $r \geq (p+q)/2$*.*

Remark 3. The above corollary slightly improves Qi and Guo's Theorem 1 in [34]. An alternative proof of this corollary can see [47].

Putting $(p, q) = (a+b, a)$ and $(r, s) = (b+c, c)$ yield

$$\frac{g_{a+b,a}(x)}{g_{b+c,c}(x)} = \left[\frac{\Gamma(x+a+b)\Gamma(x+c)}{\Gamma(x+a)\Gamma(x+b+c)}\right]^{1/b}.$$

Application of Proposition 1 gives a generalization of Theorem 6 in [12] (see also [24, Lemma 1]).

Corollary 4. *Let $a, c \in \mathbb{R}$ and $b \geq 0$. Then the function*

$$x \mapsto \frac{\Gamma(x+a+b)\Gamma(x+c)}{\Gamma(x+a)\Gamma(x+b+c)} \in \mathcal{L}[(-\min(a,c),\infty)] \tag{3.2}$$

if and only if $a \geq c$. In particular, if $a \geq c = 0$, then the function

$$x \mapsto \frac{\Gamma(x)\Gamma(x+a+b)}{\Gamma(x+a)\Gamma(x+b)} \in \mathcal{L}[(0,\infty)].$$

Remark 4. The function (3.2) is a generalization of Gurland's ratio defined by

$$T(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma((x+y)/2)^2}, \quad x, y > 0, \tag{3.3}$$

which appeared in Gurland's paper [22] in 1956. See [12], [18], [24], for more information on Gurland's ratio.

3.2. Monotonicity and inequalities for gamma, psi and polygamma functions

There are various monotonicity results and bounds for gamma, psi and polygamma functions, see for example, Alzer [2], [5], Batir [9], [10], Chen [14], [15], Guo and Qi [19], [21], [36], Mortici [25], [26], Yang and Chu [42], [48], Yang and Tian [45], [46]. The aim of this subsection is to utilize Theorem 3 to prove or reprove some monotonicity and inequalities for gamma, psi and polygamma functions.

Corollary 5. *Let $p, q \in \mathbb{R}$ with $p > q$. Then for $n \in \mathbb{N}$ the double inequalities*

$$r_1 \leq \left(\frac{\Gamma(x+p)}{\Gamma(x+q)}\right)^{1/(p-q)} - x \leq r_2, \tag{3.4}$$

$$r_1 \leq \left[\frac{(-1)^{n-1}(p-q)(n-1)!}{\psi^{(n-1)}(x+p) - \psi^{(n-1)}(x+q)}\right]^{-1/n} - x \leq r_2 \tag{3.5}$$

hold for all $x > -q$ if and only if $r_1 \leq \min((p+q-1)/2, q)$ and $r_2 \geq \max((p+q-1)/2, q)$.

Proof. Taking $(r,s) = (r_1, r_1+1)$ in Theorem 3, we see that for $n \geq 0$, the comparison inequality

$$\mathcal{I}_{\psi_n}(x+p, x+q) \geq \mathcal{I}_{\psi_n}(x+r_1, x+r_1+1),$$

or equivalently,

$$-\frac{\ln\Gamma(x+p) - \ln\Gamma(x+q)}{p-q} \leq -\frac{\ln\Gamma(x+r_1+1) - \ln\Gamma(x+r_1)}{r_1+1-r_1}, \tag{3.6}$$

$$(-1)^{n-1}\frac{\psi^{(n-1)}(x+p) - \psi^{(n-1)}(x+q)}{p-q} \leq (-1)^{n-1}\frac{\psi^{(n-1)}(x+r_1+1) - \psi^{(n-1)}(x+r_1)}{r_1+1-r_1} \tag{3.7}$$

hold for all $x > -\min(q, r_1)$ if and only if $p + q \geq 2r_1 + 1$ and $\min(p,q) \geq \min(r_1, r_1+1)$, i.e. $r_1 \leq \min((p+q-1)/2, q)$. By the recurrence formulas $\Gamma(x+1) = x\Gamma(x)$ and

$$\psi^{(n-1)}(x+1) - \psi^{(n-1)}(x) = \frac{(-1)^{n-1}(n-1)!}{x^n}$$

[1, p. 260], inequalities (3.6) and (3.7) are in turn equivalent to the first ones of (3.4) and (3.5), respectively.

The second ones of (3.4) and (3.5) can be proved in a similar way. □

Remark 5. Inequalities (3.4) and (3.5) are in fact a consequence of Theorem 3 in [12]. In [17, Theorem 1], Elezović, Giordano and Pečarić further showed that the function

$$G_{p,q}(x) = \left(\frac{\Gamma(x+p)}{\Gamma(x+q)}\right)^{1/(p-q)} - x \text{ if } p \neq q \text{ and } G_{p,p}(x) = \exp[\psi(x+p)] - x \tag{3.8}$$

is either convex and decreasing for $|p-q| < 1$ or concave and increasing for $|p-q| < 1$. Alzer [3, Theorem 2] proved that the function

$$P_{p,q}^{[n]}(x) = \begin{cases} \left(\dfrac{(-1)^{n-1}(p-q)(n-1)!}{\psi^{(n-1)}(x+p) - \psi^{(n-1)}(x+q)}\right)^{1/n} - x & \text{if } p \neq q, \\ \left(\dfrac{(-1)^{n-1}(n-1)!}{\psi^{(n)}(x+p)}\right)^{1/n} - x & \text{if } p = q \end{cases} \tag{3.9}$$

is decreasing on $(0,\infty)$ in the case of $(p,q) = (1,s)$ with $s \in (0,1)$. In fact, using Theorem 3 we can give a simple and interesting proof of the monotonicity of $G_{p,q}$, and extend Alzer's result.

Proposition 2. *Let $p, q \in \mathbb{R}$ with $p > q$ and $n \in \mathbb{N}$. Then both the functions $x \mapsto G_{p,q}(x)$, $P_{p,q}^{[n]}(x)$ are increasing from $(-q,\infty)$ onto $(q,(p+q-1)/2)$ if $p-q > 1$, and decreasing from $(-q,\infty)$ onto $((p+q-1)/2, q)$ if $p-q < 1$.*

Proof. By asymptotic formulas [1, pp. 257–260, (6.141), (6.147), (6.3.18), (6.4.11)], we easily verify that

$$G_{p,q}\left(-q^+\right) = P_{p,q}^{[n]}\left(-q^+\right) = q \text{ and } G_{p,q}(\infty) = P_{p,q}^{[n]}(\infty) = \frac{p+q-1}{2},$$

and hence

$$\sup_{x>-q} G_{p,q}(x) = \sup_{x>-q} P_{p,q}^{[n]}(x) = \begin{cases} \dfrac{p+q-1}{2} & \text{if } p-q > 1, \\ q & \text{if } p-q < 1, \end{cases}$$

$$\inf_{x>-q} G_{p,q}(x) = \inf_{x>-q} P_{p,q}^{[n]}(x) = \begin{cases} q & \text{if } p-q > 1, \\ \dfrac{p+q-1}{2} & \text{if } p-q < 1. \end{cases}$$

Now, suppose that $p-q > 1$. By Theorem 3 inequality (3.6), or equivalently,

$$r_1 \leq \exp\frac{\ln\Gamma(x+p) - \ln\Gamma(x+q)}{p-q} - x = G_{p,q}(x)$$

holds for $x > x_0 > -\min(p,q,r_1)$ if and only if

$$\begin{aligned} &(p,q,r_1) \in \{p+q \geq 2r_1+1, \mathcal{I}_{\psi_0}(x+p,x+q) \geq \mathcal{I}_{\psi_0}(x+r_1,x+r_1+1)\} \\ &= \left\{r_1 \leq \frac{p+q-1}{2}, r_1 \leq \exp\frac{\ln\Gamma(x_0+p) - \ln\Gamma(x_0+q)}{p-q} - x_0 = G_{p,q}(x_0)\right\}. \end{aligned}$$

Due to $\sup_{x>-q} G_{p,q}(x) = (p+q-1)/2$ if $p-q > 1$, Therefore, the inequality $r_1 \leq G_{p,q}(x)$ holds for all $x > x_0 > -\min(p,q,r_1)$ if and only if $(p,q,r_1) \in \{r_1 \leq G_{p,q}(x_0)\}$. When $r_1 = G_{p,q}(x_0)$, we have $G_{p,q}(x_0) \leq G_{p,q}(x)$, which, in view of arbitrariness of x_0, indicates that $x \mapsto G_{p,q}$ is increasing on $(-q,\infty)$ if $p-q > 1$. Similarly, we can show that $G_{p,q}$ is decreasing on $(-q,\infty)$ if $p-q < 1$.

In the same way, the function $x \mapsto P_{p,q}^{[n]}(x)$ is increasing (decreasing) on $(-q, \infty)$ if $p - q > (< 1)$. Thus we complete the proof. $\square$

Application of the monotonicity of $x \mapsto P_{p,q}^{[n]}(x)$ on $(-\min(p,q), \infty)$ gives a generalization of Alzer's inequality (1.1).

Corollary 6. *Let $p, q > 0$ with $p > q$ and $n \in \mathbb{N}$. Then the double inequality*

$$\frac{(n-1)!}{(x+r_2)^n} < (-1)^{n-1} \frac{\psi^{(n-1)}(x+p) - \psi^{(n-1)}(x+q)}{p-q} < \frac{(n-1)!}{(x+r_1)^n} \tag{3.10}$$

holds for $x > 0$ if $p - q > 1$ with the best constants $r_1 = c_n(p,q)$ and $r_2 = (p+q-1)/2$, where

$$c_n(p,q) = \left(\frac{(-1)^{n-1}(p-q)(n-1)!}{\psi^{(n-1)}(p) - \psi^{(n-1)}(q)} \right)^{1/n} \text{ if } p \neq q \text{ and } c_n(q,q) = \left(\frac{(-1)^{n-1}(n-1)!}{\psi^{(n)}(q)} \right)^{1/n}.$$

Inequality (3.10) reverse for $x > -\min(0, (p+q-1)/2)$ if $p - q < 1$.

Using Theorem 3, we can also give an alternative proof of the increasing property of $x \mapsto A_{\psi_n}(x) = I_{\psi_n}(x+p, x+q) - x$.

Proposition 3. *Let $p, q \in \mathbb{R}$ with $p > q$ and $n \geq 0$ be an integer. Then the double inequality*

$$r_1 \leq A_{\psi_n}(x) = I_{\psi_n}(x+p, x+q) - x \leq r_2, \tag{3.11}$$

holds for all $x > -\min(p,q)$ if and only if $r_1 \leq \min(p,q)$ and $r_2 \geq (p+q)/2$. Furthermore, the function $x \mapsto A_{\psi_n}(x)$ is increasing from $(-\min(p,q), \infty)$ onto $(\min(p,q), (p+q)/2)$.

Proof. Taking $(r,s) = (r_1, r_1)$ in Theorem 3, we see that for $n \geq 0$, the comparison inequality

$$\mathcal{I}_{\psi_n}(x+p, x+q) \geq \mathcal{I}_{\psi_n}(x+r_1, x+r_1) = x + r_1, \tag{3.12}$$

or equivalently,

$$r_1 \leq I_{\psi_n}(x+p, x+q) - x = A_{\psi_n}(x)$$

holds for all $x > -\min(p,q,r_1)$ if and only if $p + q \geq 2r_1$ and $\min(p,q) \geq \min(r_1, r_1)$, i.e. $r_1 \leq \min(p,q)$. Analogously, the right hand side inequality in (3.11) holds for all $x > -\min(p,q,r_2)$ if and only if $r_2 \geq (p+q)/2$.

These together with $A_{\psi_n}(-\rho^+) = \min(p,q)$ and $A_{\psi_n}(\infty) = (p+q)/2$, proved in [47, Theorem 1.3], yield $\inf_{x>-\rho} A_{\psi_n}(x) = \min(p,q)$ and $\sup_{x>-\rho} A_{\psi_n}(x) = (p+q)/2$, where $\rho = \min(p,q)$.

On the other hand, by Theorem 3 inequality (3.12), or equivalently, $r_1 \leq A_{\psi_n}(x)$ holds for $x > x_0 > -\min(p,q,r_1)$ if and only if

$$(p,q,r_1) \in \{p+q \geq 2r_1 \text{ and } r_1 \leq A_{\psi_n}(x_0)\} = \{r_1 \leq A_{\psi_n}(x_0)\}.$$

Thus when $r_1 = A_{\psi_n}(x_0)$ we have $A_{\psi_n}(x_0) \leq A_{\psi_n}(x)$ for $x > x_0$. This shows that $x \mapsto A_{\psi_n}(x)$ is increasing on $(-\min(p,q), \infty)$, which completes the proof. $\square$

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