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Second main theorem for meromorphic mappings with moving hypersurfaces in subgeneral position

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ABSTRACT

Let $Q_1, ..., Q_q$ be q slowly moving hypersurfaces in $\mathbf{P}^n(\mathbf{C})$ of degree d_i which are located in N-subgeneral position. Let f be a meromorphic mapping from $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ which is algebraically nondegenerate over the field generated by Q_i's. In this paper, we will prove that, for every $\epsilon > 0$, there exists a positive integer M such that

$$|| \ (q - (N - n + 1)(n + 1) - \epsilon)T_f(r) \leq \sum_{i=1}^{q} \frac{1}{d_i} N^{[M]}(r, f^*Q_i) + o(T_f(r)).$$

Moreover, an explicit estimate for M is given. Our result is an extension of the previous second main theorems for meromorphic mappings and moving hypersurfaces.

1. Introduction

Let f be a meromorphic mapping from $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ with a reduced representation $\tilde{f} = (f_0, ..., f_n)$. For each meromorphic mapping a from $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})^*$, which is usually called a moving hyperplane, with a reduced representation $\tilde{a} = (a_0, ..., a_n)$ such that $(\tilde{f}, \tilde{a}) = \sum_{i=0}^{n} a_i f_i \not\equiv 0$, we denote by f^*a the zero divisor of $(\tilde{f}, \tilde{a})$. We see that f^*a is defined independently from the choices of $\tilde{f}$ and $\tilde{a}$, and is called the intersecting divisor of f with a. We denote by $N^{[M]}(r, f^*a)$ or $N^{[M]}_{(f,a)}(r)$ the counting function of f^*a (see Section 2 for the definitions). As usual, we denote by $T_f(r)$ the characteristic function of f with respect to the hyperplane line bundle of $\mathbf{P}^n(\mathbf{C})$. The moving hyperplane a is said to be slow with respect to f if $T_a(r) = o(T_f(r))$ as $r \to +\infty$ excluding a finite Borel measures subset of $[0; +\infty)$.

Let $\{a_i\}_{i=1}^q$ be moving hyperplanes of $\mathbf{P}^n(\mathbf{C})$ with reduce representations $\tilde{a}_i = (a_{i0}, \dots, a_{in})$. Let $N \geq n$ and $q \geq N+1$. We say that the family $\{a_i\}_{i=1}^q$ is in N-*subgeneral position* if for every subset $R \subset \{1, 2, \cdots, q\}$

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with the cardinality $|R| = N + 1$,

$$\mathrm{rank}_{\mathcal{M}}\{\tilde{a}_i \mid i \in R\} = n + 1,$$

where $\mathcal{M}$ denotes the field consisting of all meromorphic functions on $\mathbf{C}^m$. If they are in n-subgeneral position, we simply say that they are in *general position*. We also denote by $\mathcal{K}_{\{a_i\}_{i=1}^q}$ the smallest subfield of $\mathcal{M}$, which contains $\mathbf{C}$ and all $\frac{a_{ij}}{a_{ik}}$ for $a_{ik} \not\equiv 0$.

In 1991, W. Stoll and M. Ru [12,13] proved the following second main theorem.

Theorem A *(Cf. [12,13]). Let $f : \mathbf{C}^m \longrightarrow \mathbf{P}^n(\mathbf{C})$ be a nonconstant meromorphic mapping. Let $\{a_i\}_{i=1}^q$ be meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})^*$ in general position such that a_i are slow with respect to f and f is linearly nondegenerate over $\mathcal{K}_{\{a_i\}_{i=1}^q}$. Then for every $\epsilon > 0$,*

$$|| \quad (q - n - 1 - \epsilon)T_f(r) \le \sum_{i=1}^{q} N(r, f^*a_i) + o(T_f(r)).$$

Here, by the notation "$\|P$" we mean that the assertion P holds for all $r \in [0, \infty)$ excluding a Borel subset E of the interval $[0, \infty)$ with $\int_E dr < \infty$.

After that the above result of W. Stoll and M. Ru was reproved by M. Shirosaki [14] with a simpler proof. This second main theorem plays an important role in Nevanlinna theory, with many applications to Algebraic or Analytic geometry. We note that in the above result, the mapping f is assumed to be linearly nondegenerate over the field $\mathcal{K}_{\{a_i\}_{i=1}^q}$. To treat the case where f may be degenerate, we need consider the case where the hyperplanes may be not in general position, but in subgeneral position. Thanks the notion of Nochka weights introduced by Nochka [5], D.D. Thai and S.D. Quang [15] gave the following second main theorem for the case where the family of hyperplanes is in subgeneral position.

Theorem B *(Cf. [15]). Let $f : \mathbf{C}^m \longrightarrow \mathbf{P}^n(\mathbf{C})$ be a nonconstant meromorphic mapping. Let $\{a_i\}_{i=1}^q$ be meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})^*$ in N-subgeneral position such that a_i are slow with respect to f and f is linearly nondegenerate over $\mathcal{K}_{\{a_i\}_{i=1}^q}$. Then for an arbitrary $\epsilon > 0$,*

$$|| \quad (q - 2N + n - 1 - \epsilon)T_f(r) \le \sum_{i=1}^{q} N^{[M]}(r, f^*a_i) + o(T_f(r)),$$

where M is a positive integer (explicitly estimated).

A natural question here is *"how to generalize these results to the case where hyperplanes are replaced by hypersurfaces"*. By proposing a new technique (using a result of Corvaja and Zannier [2] on the dimension of spaces of homogeneous polynomials), in 2004, M. Ru [11] proved a second main theorem for algebraically nondegenerate meromorphic mappings into $\mathbb{P}^n(\mathbb{C})$ intersecting hypersurfaces in general position in $\mathbb{P}^n(\mathbb{C})$. He proved the following.

Theorem C *(Cf. [11]). Let $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ be an algebraically nondegenerate meromorphic mapping and let $Q_1, ..., Q_q$ be q hypersurfaces in $\mathbf{P}^n(\mathbf{C})$ of degree d_i, in general position. Then, for every $\epsilon > 0$,*

$$|| \ (q - n - 1 - \epsilon)T_f(r) \le \sum_{i=1}^{q} \frac{1}{d_i} N(r, f^*Q_i) + o(T_f(r)).$$

With the same assumptions, T.T.H. An and H.T. Phuong [1] improved the result of M. Ru by giving an explicit truncation level for counting functions. Recently, in [9] we have generalized the results of M. Ru and T.T.H. An–H.T. Phuong to the following.

Theorem D *(Cf. [9]). Let $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ be an algebraically nondegenerate meromorphic mapping and let $Q_1, ..., Q_q$ be hypersurfaces in $\mathbf{P}^n(\mathbf{C})$ of degree d_i, in N-subgeneral position. Then, for every $\epsilon > 0$,*

$$|| \ (q - (N - n + 1)(n + 1) - \epsilon)T_f(r) \leq \sum_{i=1}^{q} \frac{1}{d_i} N^{[M_0-1]}_{Q_i(f)}(r) + o(T_f(r)),$$

where M_0 is positive integer (explicitly estimated).

For the case of slowly moving hypersurfaces (see Section 2 for the definition), recently G. Dethloff and T.V. Tan [3] generalized the second main theorem of M. Ru to the following.

Theorem E *(Dethloff–Tan [3]). Let f be a nonconstant meromorphic map of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$. Let $\{Q_i\}_{i=1}^q$ be a set of slowly (with respect to f) moving hypersurfaces in weakly general position with $\deg Q_j = d_j$ $(1 \leq i \leq q)$. Assume that f is algebraically nondegenerate over $\mathcal{K}_{\{Q_i\}_{i=1}^q}$. Then for any $\epsilon > 0$ there exist positive integers L_j $(j = 1, ..., q)$, depending only on n, ϵ and d_j $(j = 1, ..., q)$ in an explicit way such that*

$$|| \ (q - n - 1 - \epsilon)T_f(r) \leq \sum_{i=1}^{q} \frac{1}{d_i} N^{[L_j]}_{Q_i(f)}(r) + o(T_f(r)).$$

Here, $\mathcal{K}_{\{Q_i\}}$ denotes the field generated by $\{Q_i\}_{i=1}^q$ (see Section 2 for the definition).

Our purpose in this paper is to generalize all these above mentioned results to the case of moving hypersurfaces in subgeneral position. We will prove a second main theorem for meromorphic mappings into $\mathbf{P}^n(\mathbf{C})$ intersecting a family of moving hypersurfaces in subgeneral position with truncated counting functions. Namely, we will prove the following.

Theorem 1.1. *Let f be a nonconstant meromorphic map of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$. Let $\{Q_i\}_{i=1}^q$ be a family of slowly (with respect to f) moving hypersurfaces in weakly N-subgeneral position with $\deg Q_i = d_i$ $(1 \leq i \leq q)$. Assume that f is algebraically nondegenerate over $\mathcal{K}_{\{Q_i\}_{i=1}^q}$. Then for any $\epsilon > 0$, we have*

$$|| \ (q - (N - n + 1)(n + 1) - \epsilon)T_f(r) \leq \sum_{i=1}^{q} \frac{1}{d_i} N^{[L_j]}_{Q_i(f)}(r) + o(T_f(r)),$$

where $L_j = \frac{1}{d_j} L_0$ and L_0 is a positive number which is defined by:

$$\begin{aligned} L_0 &:= \binom{L+n}{n} p_0^{\binom{L+n}{n}\left(\binom{L+n}{n}-1\right)\binom{q}{n}-2} - 1 \\ \textit{with} \ \ L &:= (n+1)d + 2(N-n+1)(n+1)^3 I(\epsilon^{-1})d, \\ d &:= lcm(d_1, \ldots, d_q) (\textit{the least common multiple of all } d_i\textit{'s}), \\ \textit{and} \ \ p_0 &:= \left[\frac{\binom{L+n}{n}\left(\binom{L+n}{n} - 1\right)\binom{q}{n} - 1}{\log(1 + \frac{\epsilon}{3(n+1)(N-n+1)})}\right]^2. \end{aligned}$$

Here, by $I(x)$ we denote the smallest integer which is not less than x. We see that, if the family of moving hypersurfaces is in general position, i.e., $N = n$, then our result will imply the second main theorem of G. Dethloff and T.V. Tan. Our idea to avoid using the Nochka weights here is that from every $N+1$ arbitrary moving hypersurfaces in weakly N-subgeneral position we will construct $n+1$ new moving hypersurfaces in weakly general position (see Lemma 3.1).

Let Q be a moving hypersurface of $\mathbf{P}^n(\mathbf{C})$. We define the truncated defect of f with respect to Q by

$$\delta_f^{[L]}(D) = 1 - \liminf_{r\to+\infty} \frac{N^{[M]}(r, f^*Q)}{dT_f(r)}.$$

From the above theorem, we have the following defect relation for meromorphic mappings with a family of slowly moving hypersurfaces as follows.

Corollary 1.2. *Let f be a nonconstant meromorphic map of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$. Let $\{Q_i\}_{i=1}^q$ be a family of slowly (with respect to f) moving hypersurfaces in weakly N-subgeneral position with $\deg Q_j = d_j$ $(1 \le i \le q)$. Assume that f is algebraically nondegenerate over $\mathcal{K}_{\{Q_i\}_{i=1}^q}$. Then we have*

$$\sum_{i=1}^q \delta_f^{[L_0]}(D) \le (N-n+1)(n+1).$$

2. Basic notions and auxiliary results from Nevanlinna theory

2.1. The first main theorem in Nevanlinna theory

We set $\|z\| = \big(|z_1|^2 + \cdots + |z_m|^2\big)^{1/2}$ for $z = (z_1, \ldots, z_m) \in \mathbf{C}^m$ and define

$$B(r) := \{z \in \mathbf{C}^m : \|z\| < r\}, \quad S(r) := \{z \in \mathbf{C}^m : \|z\| = r\} \ (0 < r < \infty).$$

Define

$$v_{m-1}(z) := \big(dd^c\|z\|^2\big)^{m-1} \quad \text{and}$$
$$\sigma_m(z) := d^c\log\|z\|^2 \wedge \big(dd^c\log\|z\|^2\big)^{m-1} \quad \text{on} \quad \mathbf{C}^m \setminus \{0\}.$$

Let F be a nonzero meromorphic function on a domain Ω in $\mathbf{C}^m$. For a set $\alpha = (\alpha_1, ..., \alpha_m)$ of nonnegative integers, we set $|\alpha| = \alpha_1 + ... + \alpha_m$ and

$$\mathcal{D}^\alpha F = \frac{\partial^{|\alpha|} F}{\partial^{\alpha_1} z_1 ... \partial^{\alpha_m} z_m}.$$

We denote by ν_F^0, ν_F^∞ and ν_F the zero divisor, the pole divisor, and the divisor of the meromorphic function F respectively.

For a divisor ν on $\mathbf{C}^m$ and for a positive integer M or $M = \infty$, we set

$$\nu^{[M]}(z) = \min\,\{M, \nu(z)\},$$

$$n(t) = \begin{cases} \displaystyle\int_{|\nu|\,\cap B(t)} \nu(z) v_{m-1} & \text{if } m \ge 2, \\ \displaystyle\sum_{|z|\le t} \nu(z) & \text{if } m = 1. \end{cases}$$

The counting function of ν is defined by

$$N(r,\nu)=\int\limits_1^r \frac{n(t)}{t^{2m-1}}dt \quad (1<r<\infty).$$

Similarly, we define $N(r,\nu^{[M]})$ and denote it by $N^{[M]}(r,\nu)$.

Let $\varphi : \mathbf{C}^m \longrightarrow \mathbf{C}$ be a meromorphic function. Define

$$N_\varphi(r)=N(r,\nu^0_\varphi),\ N^{[M]}_\varphi(r)=N^{[M]}(r,\nu^0_\varphi).$$

For brevity we will omit the character $^{[M]}$ if $M=\infty$.

Let $f : \mathbf{C}^m \longrightarrow \mathbf{P}^n(\mathbf{C})$ be a meromorphic mapping. For arbitrarily fixed homogeneous coordinates $(w_0 : \cdots : w_n)$ on $\mathbf{P}^n(\mathbf{C})$, we take a reduced representation $\tilde f=(f_0,\ldots,f_n)$, which means that each f_i is a holomorphic function on $\mathbf{C}^m$ and $f(z)=\big(f_0(z):\cdots:f_n(z)\big)$ outside the analytic set $I(f)=\{f_0=\cdots=f_n=0\}$ of codimension ≥ 2. Set $\|\tilde f\|=\big(|f_0|^2+\cdots+|f_n|^2\big)^{1/2}$. The characteristic function of f is defined by

$$T_f(r)=\int\limits_{S(r)}\log\|\tilde f\|\sigma_m-\int\limits_{S(1)}\log\|\tilde f\|\sigma_m.$$

Let φ be a nonzero meromorphic function on $\mathbf{C}^m$, which are occasionally regarded as a meromorphic map into $\mathbf{P}^1(\mathbf{C})$. The proximity function of φ is defined by

$$m(r,\varphi):=\int\limits_{S(r)}\log\max\ (|\varphi|,1)\sigma_m.$$

The Nevanlinna's characteristic function of φ is defined as follows

$$T(r,\varphi):=N_{\frac{1}{\varphi}}(r)+m(r,\varphi).$$

Then

$$T_\varphi(r)=T(r,\varphi)+O(1).$$

The function φ is said to be small (with respect to f) if $||\ T_\varphi(r)=o(T_f(r))$.

We denote by $\mathcal{M}$ (resp. $\mathcal{K}_f$) the field of all meromorphic functions (resp. small meromorphic functions with respect to f) on $\mathbf{C}^m$.

2.2. Family of moving hypersurfaces

We recall some following from [7,8].

Denote by $\mathcal{H}_{\mathbf{C}^m}$ the ring of all holomorphic functions on $\mathbf{C}^m$. Let Q be a homogeneous polynomial in $\mathcal{H}_{\mathbf{C}^m}[x_0,\ldots,x_n]$ of degree $d\geq 1$. Denote by $Q(z)$ the homogeneous polynomial over $\mathbf{C}$ obtained by substituting a specific point $z\in\mathbf{C}^m$ into the coefficients of Q. We also call a moving hypersurface in $\mathbf{P}^n(\mathbf{C})$ each homogeneous polynomial $Q\in\mathcal{H}_{\mathbf{C}^m}[x_0,\ldots,x_n]$ such that the common zero set of all coefficients of Q has codimension at least two.

Let Q be a moving hypersurface in $\mathbf{P}^n(\mathbf{C})$ of degree $d\geq 1$ given by

$$Q(z)=\sum_{I\in\mathcal{T}_d}a_I\omega^I,$$

where $\mathcal{T}_d = \{(i_0, ..., i_n) \in \mathbf{N}_0^{n+1} \ ; \ i_0 + \cdots + i_n = d\}$, $a_I \in \mathcal{H}_{\mathbf{C}^m}$ and $\omega^I = \omega_0^{i_0} \cdots \omega_n^{i_n}$. We consider the meromorphic mapping $Q' : \mathbf{C}^m \to \mathbf{P}^N(\mathbf{C})$, where $N = \binom{n+d}{n}$, given by

$$Q'(z) = (a_{I_0}(z) : \cdots : a_{I_N}(z)) \ (\mathcal{T}_d = \{I_0, ..., I_N\}).$$

Here $I_0 < \cdots < I_N$ in the lexicographic ordering. By changing the homogeneous coordinates of $\mathbf{P}^n(\mathbf{C})$ if necessary, we may assume that for each given moving hypersurface as above, $a_{I_0} \not\equiv 0$ (note that $I_0 = (0, \ldots, 0, d)$ and a_{I_0} is the coefficient of ω_n^d). We set

$$\tilde{Q} = \sum_{j=0}^{N} \frac{a_{I_j}}{a_{I_0}} \omega^{I_j}.$$

The moving hypersurfaces Q is said to be "slow" (with respect to f) if $||\ T_{Q'}(r) = o(T_f(r))$. This is equivalent to $||T_{\frac{a_{I_j}}{a_{I_0}}}(r) = o(T_f(r))$ $(\forall 1 \le j \le N)$, i.e., $\frac{a_{I_j}}{a_{I_0}} \in \mathcal{K}_f$.

Let $\{Q_i\}_{i=1}^q$ be a family of moving hypersurfaces in $\mathbf{P}^n(\mathbf{C})$, $\deg Q_i = d_i$. Assume that

$$Q_i = \sum_{I \in \mathcal{T}_{d_i}} a_{iI} \omega^I.$$

We denote by $\mathcal{K}_{\{Q_i\}_{i=1}^q}$ the smallest subfield of $\mathcal{M}$ which contains $\mathbf{C}$ and all $\frac{a_{iI}}{a_{iJ}}$ with $a_{iJ} \not\equiv 0$. We say that $\{Q_i\}_{i=1}^q$ are in weakly N-subgeneral position $(N \ge n)$ if there exists $z \in \mathbf{C}^m$ such that all a_{iI} $(1 \le i \le q$, $I \in \mathcal{I})$ are holomorphic at z and for any $1 \le i_0 < \cdots < i_N \le q$ the system of equations

$$\begin{cases} Q_{i_j}(z)(w_0, \ldots, w_n) = 0 \\ 0 \le j \le N \end{cases}$$

has only the trivial solution $w = (0, \ldots, 0)$ in $\mathbf{C}^{n+1}$. If $\{Q_i\}_{i=1}^q$ is in weakly n-subgeneral position then we say that it is in weakly general position.

2.3. Some theorems and lemmas

Let f be a nonconstant meromorphic map of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$. Denote by $\mathcal{C}_f$ the set of all non-negative functions $h : \mathbf{C}^m \setminus A \longrightarrow [0, +\infty] \subset \overline{\mathbf{R}}$, which are of the form

$$h = \frac{|g_1| + \cdots + |g_l|}{|g_{l+1}| + \cdots + |g_{l+k}|},$$

where $k, l \in \mathbf{N},\ g_1, ..., g_{l+k} \in \mathcal{K}_f \setminus \{0\}$ and $A \subset \mathbf{C}^m$, which may depend on $g_1, ..., g_{l+k}$, is an analytic subset of codimension at least two. Then, for $h \in \mathcal{C}_f$ we have

$$\int\limits_{S(r)} \log h \sigma_m = o(T_f(r)).$$

Lemma 2.1 *(See [3]). Let $\{Q_i\}_{i=0}^n$ be a set of homogeneous polynomials of degree d in $\mathcal{K}_f[x_0, ..., x_n]$. Then there exists a function $h_1 \in \mathcal{C}_f$ such that, outside an analytic set of $\mathbf{C}^m$ of codimension at least two,*

$$\max_{i \in \{0, \ldots, n\}} |Q_i(f_0, ..., f_n)| \le h_1 \|f\|^d.$$

If, moreover, this set of homogeneous polynomials is in weakly general position, then there exists a nonzero function $h_2 \in \mathcal{C}_f$ such that, outside an analytic set of $\mathbf{C}^m$ of codimension at least two,

$$h_2\|f\|^d \leq \max_{i\in\{0,\dots,n\}} |Q_i(f_0,...,f_n)|.$$

Lemma 2.2 *(Lemma on logarithmic derivative, see [6]). Let f be a nonzero meromorphic function on $\mathbf{C}^m$. Then*

$$\Big|\Big| \quad m\left(r, \frac{\mathcal{D}^\alpha(f)}{f}\right) = O(\log^+ T(r,f)) \ (\alpha \in \mathbf{Z}^m_+).$$

Repeating the argument in (Prop. 4.5 [4]), we have the following.

Proposition 2.3 *(See [4, Prop. 4.5]). Let $\Phi_1, ..., \Phi_k$ be meromorphic functions on $\mathbf{C}^m$ such that $\{\Phi_1, ..., \Phi_k\}$ are linearly independent over $\mathbf{C}$. Then there exists an admissible set*

$$\{\alpha_i = (\alpha_{i1}, ..., \alpha_{im})\}_{i=1}^k \subset \mathbf{Z}^m_+$$

with $|\alpha_i| = \sum_{j=1}^m |\alpha_{ij}| \leq i-1$ $(1 \leq i \leq k)$ such that the following are satisfied:

(i) $\{\mathcal{D}^{\alpha_i}\Phi_1, ..., \mathcal{D}^{\alpha_i}\Phi_k\}_{i=1}^k$ *is linearly independent over $\mathcal{M}$, i.e.,* $\det(\mathcal{D}^{\alpha_i}\Phi_j) \not\equiv 0$,
(ii) $\det\big(\mathcal{D}^{\alpha_i}(h\Phi_j)\big) = h^k \cdot \det\big(\mathcal{D}^{\alpha_i}\Phi_j\big)$ *for any nonzero meromorphic function h on $\mathbf{C}^m$.*

The next general form of second main theorem for hyperplanes is due to M. Ru [10].

Theorem 2.4 *(See [10, Theorem 2.3]). Let f be a linearly nondegenerate meromorphic mapping of $\mathbf{C}^m$ in $\mathbf{P}^n(\mathbf{C})$ with a reduced representation $\tilde{f} = (f_0, ..., f_n)$ and let $H_1, .., H_q$ be q arbitrary hyperplanes in $\mathbf{P}^n(\mathbf{C})$. Then we have*

$$|| \int\limits_{S(r)} \max_K \log\left(\prod_{j\in K} \frac{\|\tilde{f}\|.\|H_j\|}{|H_j(\tilde{f})|}\sigma_m\right) \leq (n+1)T_f(r) - N_{W^\alpha(f_i)}(r) + o(T_f(r)),$$

where α is an admissible set with respect to $\tilde{f}$ (as in Proposition 2.3) and the maximum is taken over all subsets $K \subset \{1, ..., q\}$ such that $\{H_j\ ;\ j \in K\}$ is linearly independent.

We note that the original theorem of M. Ru states only for the case of holomorphic curves from $\mathbf{C}$. However its proof also is valid for the case of meromorphic mappings from $\mathbf{C}^m$ with a slight modification.

We have some following algebraic lemmas from [2,3]

Lemma 2.5 *(See [2, Lemma 2.2]). Let A be a commutative ring and let $\{\phi_1, \dots, \phi_p\}$ be a regular sequence in A, i.e., for $i = 1, \dots, p$, ϕ_i is not a zero divisor of $A/(\phi_1, \dots, \phi_{i-1})$. Denote by I the ideal in A generated by $\phi_1, \dots, \phi_p$. Suppose that for some $q, q_1, \dots, q_h \in A$, we have an equation*

$$\phi_1^{i_1} \cdots \phi_p^{i_p} \cdot q = \sum_{r=1}^h \phi_1^{j_1(r)} \cdots \phi_p^{j_p(r)} \cdot q_r,$$

where $(j_1(r), \dots, j_p(r)) > (i_1, \dots, i_p)$ for $r = 1, \dots, h$. Then $q \in I$.

Here, as throughout this paper, we use the lexicographic order on $\mathbf{N}_0^p$. Namely,

$$(j_1, \dots, j_p) > (i_1, \dots, i_p)$$

iff for some $s \in \{1, \dots, p\}$ we have $j_l = i_l$ for $l < s$ and $j_s > i_s$.

Lemma 2.6 *(See [3, Lemma 3.2]). Let $\{Q_i\}_{i=1}^q$ $(q \geq n+1)$ be a set of homogeneous polynomials of common degree $d \geq 1$ in $\mathcal{K}_f[x_0, \dots, x_n]$ in weakly general position. Then for any pairwise different $1 \leq j_0, \dots, j_n \leq q$ the sequence $\{Q_{j_0}, \dots, Q_{j_n}\}$ of elements in $\mathcal{K}_{\{Q_i\}}[x_0, \dots, x_n]$ is a regular sequence, as well as all its subsequences.*

3. Second main theorems for moving hypersurfaces

We first prove the following lemma.

Lemma 3.1. *Let $Q_1, ..., Q_{N+1}$ be homogeneous polynomials in $\mathcal{K}_f[x_0, ..., x_n]$ of the same degree $d \geq 1$, in weakly N-subgeneral position. Then there exist n homogeneous polynomials $P_2, ..., P_{n+1}$ in $\mathcal{K}_f[x_0, ..., x_n]$ of the forms*

$$P_t = \sum_{j=2}^{N-n+t} c_{tj} Q_j,\ c_{tj} \in \mathbf{C},\ t = 2, ..., n+1,$$

such that the family $\{P_1, ..., P_{n+1}\}$ is in weakly general position, where $P_1 = Q_1$.

Proof. We assume that Q_i $(1 \leq i \leq N+1)$ has the following form

$$Q_i = \sum_{I \in \mathcal{T}_d} a_{iI} \omega^I.$$

By the definition of the weakly subgeneral position, there exists a point $z_0 \in \mathbf{C}^m$ such that a_{iI} is holomorphic at z_0 for all i and I, and the following system of equations

$$Q_i(z_0)(\omega_0, ..., \omega_n) = 0, 1 \leq i \leq N+1,$$

has only trivial solution $(0, ..., 0)$. We may assume that $Q_i(z_0) \not\equiv 0$ for all $1 \leq i \leq N+1$.

For each homogeneous polynomials $Q \in \mathbf{C}[x_0, ..., x_n]$, we will denote by Q^* the fixed hypersurface in $\mathbf{P}^n(\mathbf{C})$ defined by Q, i.e.,

$$Q^* = \{(\omega_0 : \cdots : \omega_n) \in \mathbf{P}^n(\mathbf{C}) \mid Q(\omega_0, ..., \omega_n) = 0\}.$$

Setting $P_1 = Q_1$, we will show that

$$\dim\left(\bigcap_{i=1}^{t} Q_i^*(z_0)\right) \leq N - t,\ t = N - n + 2, ..., N+1, \tag{3.2}$$

where $\dim \emptyset = -\infty$. In fact, suppose that (3.2) does not hold. Then there exists an index $t \in \{N - n + 2, ..., N+1\}$ such that $\dim\left(\bigcap_{i=1}^{t} Q_i^*(z_0)\right) \geq N - t + 1$. This implies that

$$\dim\left(\bigcap_{i=1}^{N+1} Q_i^*(z_0)\right) \geq N - t + 1 - (N + 1 - t) = 0.$$

This contradicts that $\left(\bigcap_{i=1}^{N+1} Q_i^*(z_0)\right) = \emptyset$. Hence the inequality (3.2) must be hold.

Step 1. We will construct P_2 as follows. For each irreducible component Γ of dimension $n-1$ of $Q_1^*(z_0)$, we put

$$V_{1\Gamma} = \{c = (c_2, ..., c_{N-n+2}) \in \mathbf{C}^{N-n+1} \ ; \ \Gamma \subset Q_c^*(z_0), \text{ where } Q_c = \sum_{j=2}^{N-n+2} c_j Q_j\}.$$

Then $V_{1\Gamma}$ is a linear subspace of $\mathbf{C}^{N-n+1}$. Since $\dim\left(\bigcap_{i=1}^{N-n+2} Q_i^*(z_0)\right) \le n-2$, there exists $i \in \{2, ..., N-n+2\}$ such that $\Gamma \not\subset Q_i^*(z_0)$. This implies that $V_{1\Gamma}$ is a proper linear subspace of $\mathbf{C}^{N-n+1}$. Since the set of irreducible components of dimension $n-1$ of $Q_1^*(z_0)$ is at most countable,

$$\mathbf{C}^{N-n+1} \setminus \bigcup_{\Gamma} V_{1\Gamma} \neq \emptyset.$$

Hence, there exists $(c_{12}, ..., c_{1(N-n+2)}) \in \mathbf{C}^{N-n+1}$ such that

$$\Gamma \not\subset P_2^*(z_0)$$

for all irreducible components of dimension $n-1$ of $Q_1^*(z_0)$, where $P_2 = \sum_{j=2}^{N-n+2} c_{1j} Q_j$. This clearly implies that $\dim(P_1^*(z_0) \cap P_2^*(z_0)) \le n-2$.

Step 2. For each irreducible component Γ' of dimension $n-2$ of $(P_1^*(z_0) \cap P_2^*(z_0))$, put

$$V_{2\Gamma'} = \{c = (c_2, ..., c_{N-n+3}) \in \mathbf{C}^{N-n+2} \ ; \ \Gamma \subset Q_c^*(z_0), \text{ where } Q_c = \sum_{j=2}^{N-n+3} c_j Q_j\}.$$

Hence, $V_{2\Gamma'}$ is a linear subspace of $\mathbf{C}^{N-n+2}$. Since $\dim\left(\bigcap_{i=1}^{N-n+3} Q_i^*(z_0)\right) \le n-3$, there exists $i, (2 \le i \le N-n+3)$ such that $\Gamma' \not\subset Q_i^*(z_0)$. This implies that $V_{2\Gamma'}$ is a proper linear subspace of $\mathbf{C}^{N-n+2}$. Since the set of irreducible components of dimension $n-2$ of $(P_1^*(z_0) \cap P_2^*(z_0))$ is at most countable,

$$\mathbf{C}^{N-n+2} \setminus \bigcup_{\Gamma'} V_{2\Gamma'} \neq \emptyset.$$

Then, there exists $(c_{22}, ..., c_{2(N-n+3)}) \in \mathbf{C}^{N-n+2}$ such that

$$\Gamma' \not\subset P_3^*(z_0)$$

for all irreducible components of dimension $n-2$ of $P_1^*(z_0) \cap P_2^*(z_0)$, where $P_3 = \sum_{j=2}^{N-n+3} c_{2j} Q_j$. It is clear that $\dim(P_1^*(z_0) \cap P_2^*(z_0) \cap P_3^*(z_0)) \le n-3$.

Repeating again the above step, after the n-th step we get the hypersurfaces $P_2, ..., P_{n+1}$ satisfying that

$$\dim\left(\bigcap_{j=1}^{t} P_j^*(z_0)\right) \le n-t, \ t = 2, \dots, n+1.$$

In particular, $\left(\bigcap_{j=1}^{n+1} P_j^*(z_0)\right) = \emptyset$. This yields that $P_1, ..., P_{n+1}$ are in weakly general position. We complete the proof of the lemma. □

Proof of Theorem 1.1. Replacing Q_i by Q_i^{d/d_i} if necessary with the note that

$$\frac{1}{d} N^{[L_0]}(r, f^* Q_i^{d/d_i}) \le \frac{1}{d_i} N^{[L_j]}(r, f^* Q_i),$$

we may assume that all hypersurfaces Q_i $(1 \le i \le q)$ are of the same degree d. We may also assume that $q > (N - n + 1)(n + 1)$.

Consider a reduced representation $\tilde{f} = (f_0, \dots, f_n) : \mathbf{C} \to \mathbf{C}^{n+1}$ of f. We also note that

$$N^{[L_0]}_{Q_i(\tilde{f})}(r) = N^{[L_0]}_{\tilde{Q}(\tilde{f})}(r) + o(T_f(r)).$$

Then without loss of generality we may assume that $Q_i \in \mathcal{K}_f[x_0, ..., x_n]$.

We set

$$\mathcal{I} = \{(i_1, ..., i_{N+1})\ ; 1 \le i_j \le q, i_j \ne i_t\ \forall j \ne t\}.$$

For each $I = (i_1, ..., i_{N+1}) \in \mathcal{I}$, we denote by $P_{I1}, ..., P_{I(n+1)}$ the hypersurfaces obtained in Lemma 3.1 with respect to the family of hypersurfaces $\{Q_{i_1}, ..., Q_{i_{N+1}}\}$. It is easy to see that there exists a positive function $h \in \mathcal{C}_f$ such that

$$|P_{It}(\omega)| \le h \max_{1 \le j \le N+1-n+t} |Q_{i_j}(\omega)|,$$

for all $I \in \mathcal{I}$ and $\omega = (\omega_0, ..., \omega_n) \in \mathbf{C}^{n+1}$.

For a fixed point $z \in \mathbf{C}^m \setminus \bigcup_{i=1}^q Q_i(\tilde{f})^{-1}(\{0, \infty\})$. We may assume that

$$|Q_{i_1}(\tilde{f})(z)| \le |Q_{i_2}(\tilde{f})(z)| \le \cdots \le |Q_{i_q}(\tilde{f})(z)|.$$

Let $I = (i_1, ..., i_{N+1})$. Since $P_{I1}, \dots, P_{I(n+1)}$ are in weakly general position, there exist functions $g_0, g \in \mathcal{C}_f$, which may be chosen independent of I and z, such that

$$\|\tilde{f}(z)\|^d \le g_0(z) \max_{1 \le j \le n+1} |P_{Ij}(\tilde{f})(z)| \le g(z)|Q_{i_{N+1}}(\tilde{f})(z)|.$$

Therefore, we have

$$\begin{aligned}
\prod_{i=1}^q \frac{\|\tilde{f}(z)\|^d}{|Q_i(\tilde{f})(z)|} &\le g^{q-N}(z) \prod_{j=1}^N \frac{\|\tilde{f}(z)\|^d}{|Q_{i_j}(\tilde{f})(z)|} \\
&\le g^{q-N}(z)h^{n-1}(z) \frac{\|\tilde{f}(z)\|^{Nd}}{\left(\prod_{j=2}^{N-n+1} |Q_{i_j}(\tilde{f})(z)|\right) \cdot \prod_{j=1}^n |P_{Ij}(\tilde{f})(z)|} \\
&\le g^{q-N}(z)h^{n-1}(z) \frac{\|\tilde{f}(z)\|^{Nd}}{|P_{I1}(\tilde{f})(z)|^{N-n+1} \cdot \prod_{j=2}^n |P_{Ij}(\tilde{f})(z)|} \\
&\le g^{q-N}(z)h^{n-1}(z)\zeta^{(N-n)(n-1)}(z) \frac{\|\tilde{f}(z)\|^{Nd+(N-n)(n-1)d}}{\prod_{j=1}^n |P_{Ij}(\tilde{f})(z)|^{N-n+1}},
\end{aligned}$$

where $I = (i_1, ..., i_{N+1})$ and ζ is a function in $\mathcal{C}_f$, which is chosen common for all $I \in \mathcal{I}$, such that

$$|P_{Ij}(z)(\omega)| \le \zeta(z)\|\omega\|^d,\ \forall \omega = (\omega_0, ..., \omega_n) \in \mathbf{C}^{n+1}.$$

The above inequality implies that

$$\log \prod_{i=1}^q \frac{\|\tilde{f}(z)\|^d}{|Q_i(\tilde{f})(z)|} \le \log(g^{q-N}h^{n-1}\zeta^{(N-n)(n-1)})(z) + (N - n + 1) \log \frac{\|\tilde{f}(z)\|^{nd}}{\prod_{j=1}^n |P_{Ij}(\tilde{f})(z)|}. \tag{3.3}$$

Now, for each non-negative integer L, we denote by V_L the vector space (over $\mathcal{K}_{\{Q_i\}}$) consisting of all homogeneous polynomials of degree L in $\mathcal{K}_{\{Q_i\}}[x_0, \ldots, x_n]$ and the zero polynomial. Denote by $(P_{I1}, \ldots, P_{In})$ the ideal in $\mathcal{K}_{\{Q_i\}}[x_0, \ldots, x_n]$ generated by $P_{I1}, \ldots, P_{In}$.

Lemma 3.4 *(See [1, Lemma 5], [3, Proposition 3.3]). Let $\{P_i\}_{i=1}^q$ $(q \geq n+1)$ be a set of homogeneous polynomials of common degree $d \geq 1$ in $\mathcal{K}_f[x_0, \ldots, x_n]$ in weakly general position. Then for any nonnegative integer N and for any $J := \{j_1, \ldots, j_n\} \subset \{1, \ldots, q\}$, the dimension of the vector space $\frac{V_L}{(P_{j_1}, \ldots, P_{j_n}) \cap V_L}$ is equal to the number of n-tuples $(s_1, \ldots, s_n) \in \mathbf{N}_0^n$ such that $s_1 + \cdots + s_n \leq L$ and $0 \leq s_1, \ldots, s_n \leq d-1$. In particular, for all $L \geq n(d-1)$, we have*

$$\dim \frac{V_L}{(P_{j_1}, \ldots, P_{j_n}) \cap V_L} = d^n.$$

For each positive integer L divisible by d and for each $(\mathbf{i}) = (i_1, \ldots, i_n) \in \mathbf{N}_0^n$ with $\sigma(\mathbf{i}) = \sum_{s=1}^n i_s \leq \frac{L}{d}$, we set

$$W_{(\mathbf{i})}^I = \sum_{(\mathbf{j})=(j_1,\ldots,j_n) \geq (\mathbf{i})} P_{I1}^{j_1} \cdots P_{In}^{j_n} \cdot V_{L-d\sigma(\mathbf{j})}.$$

It is clear that $W_{(0,\ldots,0)}^I = V_L$ and $W_{(\mathbf{i})}^I \supset W_{(\mathbf{j})}^I$ if $(\mathbf{i}) < (\mathbf{j})$ in the lexicographic ordering. Hence, $W_{(\mathbf{i})}^I$ is a filtration of V_L.

Let $(\mathbf{i}) = (i_1, \ldots, i_n), (\mathbf{i}') = (i'_1, \ldots, i'_n) \in \mathbf{N}_0^n$. Suppose that $(\mathbf{i}')$ follows $(\mathbf{i})$ in the lexicographic ordering. We consider the following vector space homomorphism

$$\varphi : \gamma \in V_{L-d\sigma(\mathbf{i})} \mapsto [P_{I1}^{i_1} \cdots P_{In}^{i_n}\gamma] \in \frac{W_{(\mathbf{i})}^I}{W_{(\mathbf{i}')}^I},$$

where $[P_{I1}^{i_1} \cdots P_{In}^{i_n}\gamma]$ is the equivalent class in $\frac{W_{(\mathbf{i})}^I}{W_{(\mathbf{i}')}^I}$ containing $P_{I1}^{i_1} \cdots P_{In}^{i_n}\gamma$. We see that φ is surjective. We will show that $\ker\varphi$ is equal to $(P_{I1}, \ldots, P_{In}) \cap V_{L-d\sigma(\mathbf{i})}$.

In fact, for any $\gamma \in \ker\varphi$, we have

$$\begin{aligned} P_{I1}^{i_1} \cdots P_{In}^{i_n}\gamma &= \sum_{(\mathbf{j})=(j_1,\ldots,j_n) \geq (\mathbf{i}')} P_{I1}^{j_1} \cdots P_{In}^{j_n}\gamma_{\mathbf{j}} \\ &= \sum_{(\mathbf{j})=(j_1,\ldots,j_n) > (\mathbf{i})} P_{I1}^{j_1} \cdots P_{In}^{j_n}\gamma_{\mathbf{j}}, \end{aligned}$$

where $\gamma_{\mathbf{j}} \in V_{L-d\sigma(\mathbf{j})}$. By Lemma 2.5 and Lemma 2.6, we have $\gamma \in (P_{I1}, \ldots, P_{In})$. Then

$$\ker\varphi \subset (P_{I1}, \ldots, P_{In}) \cap V_{L-d\sigma(\mathbf{i})}.$$

Conversely, for any $\gamma \in (P_{I1}, \ldots, P_{In}) \cap V_{L-d\sigma(\mathbf{i})}$, $(\gamma \neq 0)$, we have

$$\gamma = \sum_{s=1}^n P_{Is}h_s, \quad h_s \in V_{L-d(\sigma(\mathbf{i})+1)}.$$

It implies that

$$\varphi(\gamma) = \sum_{s=1}^n [P_{I1}^{i_1} \cdots P_{Is-1}^{i_s-1} P_{Is}^{i_s+1} P_{Is+1}^{i_s+1} \cdots P_{In}^{i_n} h_s].$$

It is clear that $P_{I1}^{i_1}\cdots P_{Is-1}^{i_s-1}P_{Is}^{i_s+1}P_{Is+1}^{i_s+1}\cdots P_{In}^{i_n}h_s \in W^I_{(\mathbf{i}')}$, and hence $\varphi(\gamma)=0$, i.e., $\gamma\in\ker\varphi$. Therefore, we have

$$\ker\varphi = (P_{I1},\dots,P_{In})\cap V_{L-d\sigma(\mathbf{i})}.$$

This yields that

$$\dim\frac{W^I_{(\mathbf{i})}}{W^I_{(\mathbf{i}')}} = \dim\frac{V_{L-d\sigma(\mathbf{i})}}{(P_{I1},\dots,P_{In})\cap V_{L-d\sigma(\mathbf{i})}}. \tag{3.5}$$

Fix a number L large enough (chosen later). Set $u=u_L:=\dim V_L=\binom{L+n}{n}$. We assume that

$$V_L = W^I_{(\mathbf{i}_1)}\supset W^I_{(\mathbf{i}_2)}\supset\cdots\supset W^I_{(\mathbf{i}_K)},$$

where $W^I_{(\mathbf{i}_{s+1})}$ follows $W^I_{(\mathbf{i}_s)}$ in the ordering and $(\mathbf{i}_K)=(\frac{L}{d},0,\dots,0)$. It is easy to see that K is the number of n-tuples $(i_1,\dots,i_n)$ with $i_j\ge 0$ and $i_1+\cdots+i_n\le\frac{L}{d}$. Then we have

$$K=\binom{\frac{L}{d}+n}{n}.$$

For each $k\in\{1,\dots,K-1\}$ we set $m_k^I=\dim\frac{W^I_{(\mathbf{i}_k)}}{W^I_{(\mathbf{i}_{k+1})}}$, and set $m_K^I=1$. Then by Lemma 3.6, m_k^I does not depend on $\{P_{I1},\dots,P_{In}\}$ and k, but on $\sigma(\mathbf{i}_k)$. Hence, we set $m_k=m_k^I$. We also note that

$$m_k=d^n \tag{3.6}$$

for all k with $L-d\sigma(\mathbf{i}_k)\ge nd$ (it is equivalent to $\sigma(\mathbf{i}_k)\le\dfrac{L}{d}-n$).

From the above filtration, we may choose a basis $\{\psi_1^I,\cdots,\psi_u^I\}$ of V_L such that

$$\{\psi_{u-(m_s+\cdots+m_K)+1},\dots,\psi_u^I\}$$

is a basis of $W^I_{(\mathbf{i}_s)}$. For each $k\in\{1,\dots,K\}$ and $l\in\{u-(m_k+\cdots+m_K)+1,\dots,u-(m_{k+1}+\cdots+m_K)\}$, we may write

$$\psi_l^I=P_{I1}^{i_{1k}}\cdots P_{In}^{i_{nk}}h_l,\quad\text{where }(i_{1k},\dots,i_{nk})=(\mathbf{i}_k),h_l\in W^I_{L-d\sigma(\mathbf{i}_k)}.$$

Then we have

$$\begin{aligned}|\psi_l^I(\tilde f)(z)| &\le |P_{I1}(\tilde f)(z)|^{i_{1k}}\cdots|P_{In}(\tilde f)(z)|^{i_{nk}}|h_l(\tilde f)(z)|\\ &\le c_l|P_{I1}(\tilde f)(z)|^{i_{1k}}\cdots|P_{In}(\tilde f)(z)|^{i_{nk}}\|\tilde f(z)\|^{L-d\sigma(\mathbf{i}_k)}\\ &= c_l\left(\frac{|P_{I1}(\tilde f)(z)|}{\|\tilde f(z)\|^d}\right)^{i_{1k}}\cdots\left(\frac{|P_{In}(\tilde f)(z)|}{\|\tilde f(z)\|^d}\right)^{i_{nk}}\|\tilde f(z)\|^L,\end{aligned}$$

where $c_l\in\mathcal{C}_f$, which does not depend on f and z. Taking the product the both sides of the above inequalities over all l and then taking logarithms, we obtain

$$\begin{aligned}\log\prod_{l=1}^u|\psi_l^I(\tilde f)(z)| \le \sum_{k=1}^K m_k&\left(i_{1k}\log\frac{|P_{I1}(\tilde f)(z)|}{\|\tilde f(z)\|^d}+\cdots+i_{nk}\log\frac{|P_{In}(\tilde f)(z)|}{\|\tilde f(z)\|^d}\right)\\ &+uL\log\|\tilde f(z)\|+\log c_I(z),\end{aligned} \tag{3.7}$$

where $c_I=\prod_{l=1}^q c_l\in\mathcal{C}_f$, which does not depend on f and z.

For each integer l ($0 \le l \le \frac{L}{d}$), we set $m(l) = m_k$, where k is an index such that $\sigma(\mathbf{i}_k) = l$. Since m_k only depends on $\sigma(\mathbf{i}_k)$, the above definition of $m(l)$ is well defined. We see that

$$\sum_{k=1}^{K} m_k i_{sk} = \sum_{l=0}^{\frac{L}{d}} \sum_{k|\sigma(\mathbf{i}_k)=l} m(l) i_{sk} = \sum_{l=0}^{\frac{L}{d}} m(l) \sum_{k|\sigma(\mathbf{i}_k)=l} i_{sk}.$$

Note that, by the symmetry $(i_1, \ldots, i_n) \to (i_{\sigma(1)}, \ldots, i_{\sigma(n)})$ with $\sigma \in S(n)$, $\sum_{k|\sigma(\mathbf{i}_k)=l} i_{sk}$ does not depend on s. We set

$$A := \sum_{k=1}^{K} m_k i_{sk}, \quad \text{which is independent of } s \text{ and } I.$$

Hence, (3.7) gives

$$\log \prod_{l=1}^{u} |\psi_l^I(\tilde{f})(z)| \le A \left(\log \prod_{i=1}^{n} \frac{|P_{Ii}(\tilde{f})(z)|}{\|\tilde{f}(z)\|^d} \right) + uL \log \|\tilde{f}(z)\| + \log c_I(z),$$

i.e.,

$$A \left(\log \prod_{i=1}^{n} \frac{\|\tilde{f}(z)\|^d}{|P_{Ii}(\tilde{f})(z)|} \right) \le \log \prod_{l=1}^{u} \frac{\|\tilde{f}(z)\|^L}{|\psi_l^I(\tilde{f})(z)|} + \log c_I(z).$$

Set $c_0 = g^{q-N} h^{n-1} \zeta^{(N-n)(n-1)} \prod_I (1 + c_I^{(N-n+1)/A}) \in \mathcal{C}_f$. Combining the above inequality with (3.3), we obtain that

$$\log \prod_{i=1}^{q} \frac{\|\tilde{f}(z)\|^d}{|Q_i(\tilde{f})(z)|} \le \frac{N-n+1}{A} \log \prod_{l=1}^{u} \frac{\|\tilde{f}(z)\|^L}{|\psi_l^I(\tilde{f})(z)|} + \log c_0. \tag{3.8}$$

We now write

$$\psi_l^I = \sum_{J \in \mathcal{T}_L} c_{lJ}^I x^J \in V_L, \quad c_{lJ}^I \in \mathcal{K}_{\{Q_i\}},$$

where $\mathcal{T}_L$ is the set of all $(n+1)$-tuples $J = (i_0, \ldots, i_n)$ with $\sum_{s=0}^n j_s = L$, $x^J = x_0^{j_0} \cdots x_n^{j_n}$ and $l \in \{1, \ldots, u\}$. For each l, we fix an index $J_l^I \in J$ such that $c_{lJ_l^I}^I \not\equiv 0$. Define

$$\mu_{lJ}^I = \frac{c_{lJ}^I}{c_{lJ_l^I}^I}, \ J \in \mathcal{T}_L.$$

Set $\Phi = \{\mu_{lJ}^I; I \subset \{1, \ldots, q\}, \sharp I = n, 1 \le l \le M, J \in \mathcal{T}_L\}$. Note that $1 \in \Phi$. Let $B = \sharp\Phi$. We see that $B \le u\binom{q}{n}(\binom{L+n}{n} - 1) = \binom{L+n}{n}(\binom{L+n}{n} - 1)\binom{q}{n}$. For each positive integer l, we denote by $\mathcal{L}(\Phi(l))$ the linear span over $\mathbf{C}$ of the set

$$\Phi(l) = \{\gamma_1 \cdots \gamma_l; \gamma_i \in \Phi\}.$$

It is easy to see that

$$\dim \mathcal{L}(\Phi(l)) \le \sharp\Phi(l) \le \binom{B+l-1}{B-1}.$$

We may choose a positive integer p such that

$$p \le p_0 := [\frac{B-1}{\log(1+\frac{\epsilon}{3(n+1)(N-n+1)})}]^2 \text{ and } \frac{\dim \mathcal{L}(\Phi(p+1))}{\dim \mathcal{L}(\Phi(p))} \le 1 + \frac{\epsilon}{3(n+1)(N-n+1)}.$$

Indeed, if $\dfrac{\dim \mathcal{L}(\Phi(p+1))}{\dim \mathcal{L}(\Phi(p))} > 1 + \dfrac{\epsilon}{3(n+1)(N-n+1)}$ for all $p \le p_0$, we have

$$\dim \mathcal{L}(\Phi(p_0+1)) \ge (1 + \frac{\epsilon}{3(n+1)(N-n+1)})^{p_0}.$$

Therefore, we have

$$\begin{aligned}
\log(1+\frac{\epsilon}{3(n+1)(N-n+1)}) &\le \frac{\log \dim \mathcal{L}(\Phi(p_0+1))}{p_0} \le \frac{\log \binom{B+p_0}{B-1}}{p_0} \\
&= \frac{1}{p_0} \log \prod_{i=1}^{B-1} \frac{p_0+i+1}{i} < \frac{(B-1)\log(p_0+2)}{p_0} \\
&\le \frac{B-1}{\sqrt{p_0}} \le \frac{(B-1)\log(1+\frac{\epsilon}{3(n+1)(N-n+1)})}{B-1} \\
&= \log(1+\frac{\epsilon}{3(n+1)(N-n+1)}).
\end{aligned}$$

This is a contradiction.

We fix a positive integer p satisfying the above condition. Put $s = \dim \mathcal{L}(\Phi(p))$ and $t = \dim \mathcal{L}(\Phi(p+1))$. Let $\{b_1, \dots, b_t\}$ be an $\mathbf{C}$-basis of $\mathcal{L}(\Phi(p+1))$ such that $\{b_1, \dots, b_s\}$ be a $\mathbf{C}$-basis of $\mathcal{L}(\Phi(p))$.

For each $l \in \{1, \dots, u\}$, we set

$$\tilde{\psi}_l^I = \sum_{J \in \mathcal{T}_L} \mu_{lJ}^I x^I.$$

For each $J \in \mathcal{T}_L$, we consider homogeneous polynomials $\phi_J(x_0, \dots, x_n) = x^J$. Let F be a meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^{tu-1}(\mathbf{C})$ with a reduced representation $\tilde{F} = (hb_i\phi_J(\tilde{f}))_{1 \le i \le t, J \in \mathcal{T}_L}$, where h is a nonzero meromorphic function on $\mathbf{C}^m$. We see that

$$|| \ N_h(r) + N_{1/h}(r) = o(T_f(r)).$$

Since f is assumed to be algebraically nondegenerate over $\mathcal{K}_{\{Q_i\}}$, F is linearly nondegenerate over $\mathbf{C}$. We see that there exist nonzero functions $c_1, c_2 \in \mathcal{C}_f$ such that

$$c_1|h|.\|\tilde{f}\|^L \le \|\tilde{F}\| \le c_2|h|.\|\tilde{f}\|^L.$$

For each $l \in \{1, \dots, u\}, 1 \le i \le s$, we consider the linear form L_{il}^I in x^J such that

$$hb_i\tilde{\psi}_l^I(\tilde{f}) = L_{il}^I(\tilde{F}).$$

Since f is algebraically nondegenerate over $\mathcal{K}_{\{Q_i\}}$, it is easy to see that $\{b_i\tilde{\psi}_l^I(\tilde{f}); 1 \le i \le s, 1 \le l \le M\}$ is linearly independent over $\mathbf{C}$, and so is $\{L_{il}^I(\tilde{F}); 1 \le i \le s, 1 \le l \le u\}$. This yields that $\{L_{il}^I; 1 \le i \le s, 1 \le l \le u\}$ is linearly independent over $\mathbf{C}$.

For every point z which is not neither zero nor pole of any $hb_i\psi_l^I(\tilde f)$, we also see that

$$
\begin{aligned}
s\log\prod_{l=1}^{u}\frac{\|\tilde f(z)\|^L}{|\psi_l^I(\tilde f)(z)|} &= \log\prod_{\substack{1\le l\le u\\1\le i\le s}}\frac{\|\tilde F(z)\|}{|hb_i\psi_l^I(\tilde f)(z)|}+\log c_3(z)\\
&= \log\prod_{\substack{1\le l\le u\\1\le i\le s}}\frac{\|\tilde F(z)\|\cdot\|L_{il}^I\|}{|L_{il}^I(\tilde F)(z)|}+\log c_4(z),
\end{aligned}
$$

where c_3, c_4 are nonzero functions in $\mathcal{C}_f$, not depend on f and I, but on $\{Q_i\}_{i=1}^q$. Combining this inequality and (3.8), we obtain that

$$
\log\prod_{i=1}^{q}\frac{\|\tilde f(z)\|^d}{|Q_i(\tilde f)(z)|}\le\frac{N-n+1}{sA}\left(\max_I\log\prod_{\substack{1\le l\le u\\1\le i\le s}}\frac{\|\tilde F(z)\|\cdot\|L_{il}^I\|}{|L_{il}^I(\tilde F)(z)|}+\log c_4(z)\right)+\log c_0(z), \tag{3.9}
$$

for all z outside an analytic subset of $\mathbf{C}^m$.

Since $\tilde F$ is linearly nondegenerate over $\mathbf{C}$, there exists an admissible set $\alpha=(\alpha_{iJ})_{\substack{1\le i\le t\\J\in\mathcal{T}_L}}$ with $\alpha_{iJ}\in\mathbf{Z}_+^m$, $\|\alpha_{iJ}\|\le tu-1$, such that

$$
W^\alpha(hb_i\tilde\phi_J(\tilde f))=\det\left(\mathcal{D}^{\alpha_{i'J'}}(hb_i\tilde\phi_J(\tilde f))\right)\not\equiv 0.
$$

By Theorem 2.4, we have

$$
\Big|\Big|\int\limits_{S(r)}\max_I\left\{\log\prod_{\substack{1\le l\le u\\1\le i\le s}}\frac{\|\tilde F(z)\|\cdot\|L_{il}^I\|}{|L_{il}^I(\tilde F)(z)|}\right\}\le tuT_F(r)-N_{W^\alpha(hb_i\tilde\phi_J(\tilde f))}(r)+o(T_F(r)). \tag{3.10}
$$

Integrating both sides of (3.9) and using (3.10), we obtain that

$$
\begin{aligned}
qdT_f(r)-\sum_{i=1}^{q}N(r,f^*Q_i)\le\frac{tu(N-n+1)}{sA}T_F(r)-\frac{N-n+1}{sA}N_{W^\alpha(hb_i\tilde\phi_J(\tilde f))}(r)\\
+o(T_F(r)+T_f(r)).
\end{aligned} \tag{3.11}
$$

We now estimate the quantity $\sum_{i=1}^q N(r,f^*Q_i)-\frac{N-n+1}{sA}N_{W^\alpha(hb_i\tilde\phi_J(\tilde f))}(r)$. Fix a point $z_0\in\mathbf{C}^m\setminus I(f)$. Without lose of generality, we may assume that

$$
\nu_{Q_1(\tilde f)}(z_0)\ge\cdots\ge\nu_{Q_N(\tilde f)}(z_0)\ge\cdots\ge\nu_{Q_q(\tilde f)}(z_0).
$$

First, we recall that

$$
Q_i(x)=\sum_{J\in\mathcal{T}_d}a_{iJ}x^I\in\mathcal{K}_{\{Q_i\}}[x_0,\dots,x_n].
$$

Let $T=(\cdots,t_{kJ},\cdots)$ $(k\in\{1,\dots,q\}, J\in\mathcal{T}_d)$ be a family of variables and

$$
Q_i^T=\sum_{J\in\mathcal{T}_d}T_{iJ}x^I\in\mathbf{Z}[T,x],\quad i=1,\dots,q.
$$

For each ordered subset $I = (i_1, ..., i_{N+1}) \subset \{1, \dots, q\}$, we denote by $\tilde{R}_I \in \mathbf{Z}[T]$ the resultant of $\{Q_i^T\}_{i \in I}$. Then there exist a positive integer λ (common for all I) and polynomials $\tilde{b}_{ij}^I$ $(0 \le i \le n, j \in I)$ in $\mathbf{Z}[T, x]$, which are zero or homogeneous in x with degree of $\lambda - d$ such that

$$x_i^\lambda \cdot \tilde{R}_I = \sum_{j \in I} \tilde{b}_{ij}^I Q_j^T \text{ for all } i \in \{1, \dots, q\},$$

and $R_I = \tilde{R}_I(\dots, a_{kJ}, \dots) \not\equiv 0$. We see that $R_I \in \mathcal{K}_f$. Set

$$b_{ij}^I = \tilde{b}_{ij}^I((\dots, a_{jJ}, \dots), (x_0, \dots, x_n)).$$

Then we have

$$f_i^\lambda \cdot R_I = \sum_{j \in I} b_{ij}^I(\tilde{f}) Q_j(\tilde{f}) \text{ for all } i \in \{0, \dots, n\}.$$

This implies that

$$\nu_{R_I} \ge \min_{j \in I} \nu_{Q_j(\tilde{f})} + \min_{0 \le i \le n, j \in I} \nu_{b_{ij}^I(\tilde{f})}.$$

We set $R = \prod_{I \subset \{1, \dots, q\}} R_I \in \mathcal{K}_{\{Q_i\}}$. It is easy to see that

$$\nu_{b_{ij}^I(\tilde{f})} \ge O(\min_{k,J} \nu_{a_{kJ}}),$$

and the left hand side of this inequality is only depend on $\{Q_i\}$. Then it implies that there exists a constant c, which depends only on $\{Q_i\}$, such that

$$\min_{j \in I} \nu_{Q_j(f)} \le \nu_R - c \min_{k,J} \nu_{a_{kJ}},$$

for each ordered subset $I \subset \{1, \dots, q\}$ with $\sharp I = N + 1$.

Now, we let $I = \{1, ..., N + 1\} \subset \{1, \dots, q\}$. Then

$$\nu_{Q_j(f)}(z_0) \le \nu_R(z_0) - c \min_{k,J} \nu_{a_{kJ}}(z_0), \ j = N + 1, ..., q.$$

Also it is easy to see that

$$\nu_{Q_{N-n+i}(\tilde{f})}(z_0) \le \nu_{P_{Ii}}(z_0),$$

and hence

$$\nu_{Q_{N-n+i}(\tilde{f})}(z_0) - \nu_{Q_{N-n+i}(\tilde{f})}^{[tu-1]}(z_0) \le \nu_{P_{Ii}}(z_0) - \nu_{P_{Ii}}^{[tu-1]}(z_0), \ i = 2, ..., n.$$

Therefore,

$$\begin{aligned}\sum_{i=1}^q (\nu_{Q_i(\tilde{f})}(z_0) - \nu_{Q_i(\tilde{f})}^{[tu-1]}(z_0)) &\le (N - n + 1)(\nu_{Q_1(\tilde{f})}(z_0) - \nu_{Q_1(\tilde{f})}^{[tu-1]}(z_0)) \\ &\quad + \sum_{i=2}^n (\nu_{P_{Ii}(\tilde{f})}(z_0) - \nu_{P_{Ii}(\tilde{f})}^{[tu-1]}(z_0)) + (q - N)\nu_{Q_{N+1}(\tilde{f})}(z_0)\end{aligned}$$

$$\leq (N-n+1)\sum_{i=1}^{n}(\nu_{P_{Ii}(\tilde f)}(z_0)-\nu^{[tu-1]}_{P_{Ii}(\tilde f)}(z_0)) \tag{3.12}$$
$$+(q-N)(\nu_R(z_0)-c\min_{k,J}\nu_{a_{kJ}}(z_0)).$$

Take linear forms h^I_{il} in x^J, $1\leq l\leq u, s+1\leq i\leq t, J\in\mathcal{T}_L$ such that $\{L^I_{il}; 1\leq l\leq u, 1\leq i\leq s\}\cup\{h^I_{il}; 1\leq l\leq u, s+1\leq i\leq t\}$ is linearly independent over $\mathbf{C}$. Moreover, we easily see that

$$\begin{aligned}\nu_{W^\alpha(hb_i\tilde\phi_J(\tilde f))}(z_0) &= \nu_{W^\alpha(L^I_{il}(\tilde F),\dots,h^I_{il}(\tilde F))}(z_0)\\ &\geq \sum_{\substack{1\leq l\leq u\\ 1\leq i\leq s}}\left(\nu_{L^I_{il}(\tilde F)}(z_0)-\nu^{[tu-1]}_{L^I_{il}(\tilde F)}(z_0)\right)\\ &\geq \sum_{\substack{1\leq l\leq u\\ 1\leq i\leq s}}\left(\nu_{hb_i\tilde\psi^I_{il}(\tilde f)}(z_0)-\nu^{[tu-1]}_{hb_i\tilde\psi^I_{il}(\tilde f)}(z_0)\right)\\ &\geq \sum_{\substack{1\leq l\leq u\\ 1\leq i\leq s}}\left(\nu_{\tilde\psi^I_{il}(\tilde f)}(z_0)-\nu^{[tu-1]}_{\tilde\psi^I_{il}(\tilde f)}(z_0)\right)-C\max_{1\leq i\leq s}\nu^\infty_{hb_i}(z_0),\end{aligned} \tag{3.13}$$

where C is a positive constant, which is chosen independently of I, since there are only finite ordered subset I.

Now for integers x, y, we easily see that

$$\max\{0,x+y-L\}\geq\max\{0,x-L\}+\min\{0,y\}.$$

It yields that

$$\nu_{\varphi_1\varphi_2}(z)-\nu^{[L]}_{\varphi_1\varphi_2}(z)\geq\nu_{\varphi_1}(z)-\nu^{[L]}_{\varphi_1}(z)-\nu^\infty_{\varphi_2}(z), \tag{3.14}$$

for every nonzero meromorphic functions φ_1,φ_2. If let $x_1,\dots,x_{k_1}$ be k_1 non-negative integers and let $y_1,\dots,y_{k_2}$ be k_2 negative integers, then we have the following estimate

$$\begin{aligned}&\max\{0,x_1+\dots+x_{k_1}+y_1+\dots+y_{k_2}-L\}\\ &\geq\sum_{i=1}^{k_1}\max\{0,x_i-L\}+\sum_{i=1}^{k_2}\min\{0,y_i\}\\ &=\sum_{i=1}^{k_1}(\max\{0,x_i-L\}+\min\{0,x_i\})+\sum_{i=1}^{k_2}(\max\{0,y_i-L\}+\min\{0,y_i\}).\end{aligned}$$

This yields that

$$\nu_{\prod_{i=1}^k\varphi_i}(z)-\nu^{[L]}_{\prod_{i=1}^k\varphi_i}(z)\geq\sum_{i=1}^{k}(\nu_{\varphi_i}(z)-\nu^{[L]}_{\varphi_i}(z)-\nu^\infty_{\varphi_i}(z)), \tag{3.15}$$

for any meromorphic functions φ_i $(1\leq i\leq k)$.

For each $1\leq l\leq u, 1\leq i\leq s$ we have

$$\tilde\psi^I_l(\tilde f)=\frac{1}{c^I_{lJ^I_l}}\prod_{j=1}^{n}P^{i_{jk}}_{Ij}(\tilde f)h_l(\tilde f),$$

where $(i_{1k},\dots,i_{nk})=I_k, h_l\in V_{L-d\sigma(\mathbf{i}_k)}$ and h_l is independent of f. Now using (3.14) and (3.15), we have

$$\begin{aligned}\nu_{\tilde\psi^I_{il}(\tilde f)}(z_0)-\nu^{[tu-1]}_{\tilde\psi^I_{il}(\tilde f)}(z_0)&\ge \nu_{\prod_{j=1}^n P_{Ij}^{i_{jk}}(\tilde f)}(z_0)-\nu^{[tu-1]}_{\prod_{j=1}^n P_{Ij}^{i_{jk}}(\tilde f)}(z_0)-\nu_{c^I_{lJ^I_l}}(z_0)\\&\ge\sum_{j=1}^n i_{jk}(\nu_{P_{Ij}(\tilde f)}(z_0)-\nu^{[tu-1]}_{P_{Ij}(\tilde f)}(z_0))-c_1\max_{j,J}\nu_{a_{jJ}}(z_0),\end{aligned}$$

where c_1 is a constant, which depends only on $\{Q_i\},t$ and L. Summing-up both sides of the above inequalities over all $1\le i\le u, 1\le l\le s$, we get

$$\begin{aligned}\sum_{\substack{1\le i\le u\\1\le l\le s}}(\nu_{\tilde\psi^I_{il}(\tilde f)}(z_0)-\nu^{[tu-1]}_{\tilde\psi^I_{il}(\tilde f)}(z_0))&\ge\sum_{j=1}^n s\sum_{k=1}^K m^I_k i_{jk}(\nu_{P_{Ij}(\tilde f)}(z_0)-\nu^{[tu-1]}_{P_{Ij}(\tilde f)}(z_0))-c_2\max_{j,J}\nu_{a_{jJ}}(z_0)\\&=As\sum_{j=1}^n(\nu_{P_{Ij}(\tilde f)}(z_0)-\nu^{[tu-1]}_{P_{Ij}(\tilde f)}(z_0))-c_2\max_{j,J}\nu_{a_{jJ}}(z_0),\end{aligned}\tag{3.16}$$

where c_2 is a constant, which depends only on $\{Q_i\},t$ and L.

Combining (3.13) and (3.16), we get

$$\nu_{W^\alpha(hb_i\tilde\phi_J(\tilde f))}(z_0)\ge As\sum_{j=1}^n(\nu_{P_{Ij}(\tilde f)}(z_0)-\nu^{[tu-1]}_{P_{Ij}(\tilde f)}(z_0))-c_2\max_{j,J}\nu_{a_{jJ}}(z_0)-C\max_{1\le i\le s}\nu^\infty_{hb_i}(z_0).$$

Combining (3.12) and this inequality, we obtain

$$\begin{aligned}&\frac{N-n+1}{As}\nu_{W^\alpha(b_i\tilde\phi_J(\tilde f))}(z_0)\\&\ge\sum_{i=1}^q(\nu_{Q_i(\tilde f)}(z_0)-\nu^{[tu-1]}_{Q_i(\tilde f)}(z_0))-(N-n+1)(q-N)(\nu_R(z_0)\\&\quad+c\max_{1\le k\le q}\nu_{a_kJ_k}(z_0))-\frac{N-n+1}{As}(c_2\max_{j,J}\nu_{a_{jJ}}(z_0)+C\max_{1\le i\le s}\nu^\infty_{hb_i}(z_0)).\end{aligned}$$

Integrating both sides of the above inequality, we obtain that

$$||\ \frac{N-n+1}{As}N_{W^\alpha(hb_i\tilde\phi_J(\tilde f))}(r)\ge\sum_{i=1}^q(N_{Q_i(\tilde f)}(r)-N^{[tu-1]}_{Q_i(\tilde f)}(r))+o(T_f(r)).$$

From this inequality and (3.11) with a note that $T_F(r)=LT_f(r)+o(T_f(r))$, we have

$$||\ (q-\frac{tuL(N-n+1)}{dAs})T_f(r)\le\sum_{i=1}^q\frac{1}{d}N^{[tu-1]}(r,f^*Q_i)+o(T_f(r)).\tag{3.17}$$

Now we give some estimates for A, t and s. For each $I_k=(i_{1k},\dots,i_{nk})$ with $\sigma(\mathbf{i}_k)\le\frac{L}{d}-n$, we set

$$i_{(n+1)k}=\frac{L}{d}-n-\sum_{s=1}^n i_s.$$

Since the number of nonnegative integer p-tuples with summation $\le T$ is equal to the number of nonnegative integer $(p+1)$-tuples with summation exactly equal to $T\in\mathbf{Z}$, which is $\binom{T+n}{n}$, and since the sum below is independent of s, we have

$$A=\sum_{\sigma(\mathbf{i}_k)\le\frac{L}{d}} m_k^I i_{sk} \ge \sum_{\sigma(\mathbf{i}_k)\le\frac{L}{d}-n} m_k^I i_{sk} = \frac{d^n}{n+1}\sum_{\sigma(\mathbf{i}_k)\le\frac{L}{d}-n}\sum_{s=1}^{n+1} i_{sk}$$
$$=\frac{d^n}{n+1}\cdot\binom{\frac{L}{d}}{n}\cdot\left(\frac{L}{d}-n\right)=d^n\binom{\frac{L}{d}}{n+1}.$$

Now, for every positive number $x\in[0,\frac{1}{(n+1)^2}]$, we have

$$\begin{aligned}(1+x)^n&=1+nx+\sum_{i=2}^n\binom{n}{i}x^i\le 1+nx+\sum_{i=2}^n\frac{n^i}{i!(n+1)^{2i-2}}x\\&\le 1+nx+\sum_{i=2}^n\frac{1}{i!}x\le 1+(n+1)x.\end{aligned}\tag{3.18}$$

We chose $L=(n+1)d+2(N-n+1)(n+1)^3I(\epsilon^{-1})d$. Then L is divisible by d and we have

$$\frac{(n+1)d}{L-(n+1)d}=\frac{(n+1)d}{2(N-n+1)(n+1)^3I(\epsilon^{-1})d}\le\frac{1}{2(n+1)^2}.\tag{3.19}$$

Therefore, using (3.18) and (3.19) we have

$$\begin{aligned}\frac{uL}{dA}&\le\frac{\binom{L+n}{n}L}{d^{n+1}\binom{\frac{L}{d}}{n+1}}=\frac{L\cdot(L+1)\cdots(L+n)}{1\cdot 2\cdots n}\Big/\frac{(L-nd)\cdot(L-(n-1)d)\cdots L}{1\cdot 2\cdots(n+1)}\\&=(n+1)\prod_{i=1}^n\frac{L+i}{L-(n-i+1)d}<(n+1)\Big(\frac{L}{L-(n+1)d}\Big)^n\\&=(n+1)\left(1+\frac{(n+1)d}{L-(n+1)d}\right)^n<(n+1)\left(1+\frac{(n+1)^2d}{2(N-n+1)(n+1)^3I(\epsilon^{-1})d}\right)\\&\le(n+1)+\frac{(n+1)^3d}{2(n+1)^3(N-n+1)\epsilon^{-1}}\le n+1+\frac{\epsilon}{2(N-n+1)}.\end{aligned}$$

Then we have

$$\begin{aligned}\frac{tuL}{dAs}&\le\Big(1+\frac{\epsilon}{3(n+1)(N-n+1)}\Big)\Big(n+1+\frac{\epsilon}{2(N-n+1)}\Big)\\&\le n+1+\frac{\epsilon}{2(N-n+1)}+\frac{\epsilon}{3(N-n+1)}+\frac{\epsilon}{6(N-n+1)}\\&=n+1+\frac{\epsilon}{N-n+1}.\end{aligned}\tag{3.20}$$

Combining (3.17) and (3.20), we get

$$(q-(N-n+1)(n+1)-\epsilon)\,T_f(r)\le\sum_{i=1}^q\frac{1}{d}N^{[tu-1]}(r,f^*Q_i)+o(T_f(r)).\tag{3.21}$$

Here we note that:

- $L:=(n+1)d+2(N-n+1)(n+1)^3I(\epsilon^{-1})d,$
- $p_0:=\left[\dfrac{B-1}{\log(1+\frac{\epsilon}{3(n+1)(N-n+1)})}\right]^2\le\left[\dfrac{\binom{L+n}{n}(\binom{L+n}{n}-1)\binom{q}{n}-1}{\log(1+\frac{\epsilon}{3(n+1)(N-n+1)})}\right]^2,$
- $tu-1\le\binom{L+n}{n}\binom{B+p}{B-1}-1\le\binom{L+n}{n}p^{B-1}-1\le\binom{L+n}{n}p_0^{\binom{L+n}{n}(\binom{L+n}{n}-1)\binom{q}{n}-2}-1=L_0.$

By these estimates and by (3.21), we obtain

$$|| \ (q - (N - n + 1)(n + 1) - \epsilon)T_f(r) \leq \sum_{i=1}^{q} \frac{1}{d} N^{[L_0]}(r, f^*Q_i) + o(T_f(r)).$$

The theorem is proved. □

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