



ABSOLUTELY NORM ATTAINING PARANORMAL OPERATORS

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ABSTRACT. A bounded linear operator $T : H_1 \rightarrow H_2$, where H_1, H_2 are Hilbert spaces is said to be norm attaining if there exists a unit vector $x \in H_1$ such that $\|Tx\| = \|T\|$. If for any closed subspace M of H_1 , the restriction $T|_M : M \rightarrow H_2$ of T to M is norm attaining, then T is called an absolutely norm attaining operator or \mathcal{AN} -operator. We prove the following characterization theorem:

A positive operator T defined on an infinite dimensional Hilbert space H is an \mathcal{AN} -operator if and only if the essential spectrum of T is a single point and $[m(T), m_e(T))$ contains atmost finitely many points. Here $m(T)$ and $m_e(T)$ are the minimum modulus and essential minimum modulus of T .

As a consequence we obtain a sufficient condition under which the \mathcal{AN} -property of an operator implies \mathcal{AN} -property of its adjoint.

We also study the structure of paranormal \mathcal{AN} -operators and give a necessary and sufficient condition under which a paranormal \mathcal{AN} -operator is normal.

1. INTRODUCTION

In this article we continue the study of absolutely norm attaining operators of the earlier work from [7]. The class of absolutely norm attaining operators is introduced in [3] and further the detailed study of these operators is appeared in [6, 10, 7].

In the present article first we prove a characterization theorem for positive \mathcal{AN} -operators. In general if T is an \mathcal{AN} -operator, it may not be true that T^* is also an \mathcal{AN} -operator (See [10, Example 6.3] for more details). We give a sufficient condition under which this result holds true.

Next, we study the structure of absolutely norm attaining paranormal operators. Specifically, we show that if T is a paranormal \mathcal{AN} -operator, then there exists pairs (H_α, U_α) , where H_α is a reducing subspace of T and U_α is an isometry on H_α such that

Date: 08:09 Wednesday 9th May, 2018.

1991 Mathematics Subject Classification. Primary 47A15; Secondary 47B07, 47B20, 47B40.

Key words and phrases. Compact operator, norm attaining operator, \mathcal{AN} -operator, Weyl's theorem, paranormal operator, reducing subspace.

$$H = \bigoplus_{\beta \in \sigma(|T|)} H_\beta \quad \text{and} \quad T = \bigoplus_{\beta \in \sigma(|T|)} \beta U_\beta.$$

Here $\sigma(|T|)$ is the spectrum of $|T|$, the modulus of T .

Among all H_α 's atmost one can have infinite dimension. That means atmost of the U_β 's can be an isometry and the remaining must be unitary.

We describe a necessary and sufficient condition which ensure the normality of a paranormal \mathcal{AN} -operator.

We organize the article as follows: In the remaining part of this section we describe basic definitions and notations needed to prove our main results. In the second section, we prove a characterization of positive \mathcal{AN} -operators and as a consequence we give a sufficient condition which guarantees the \mathcal{AN} -property of the adjoint operator when the operator has \mathcal{AN} -property. In the final section, we discuss the structure of paranormal \mathcal{AN} -operators. As a consequence we show that every such operator has countably many finite dimensional reducing subspaces. Another important consequence is that a paranormal compact operator is normal.

Throughout we consider complex Hilbert spaces which will be denoted by H, H_1, H_2 etc. The inner product and the induced norm on H are designated by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. If M is a subspace of H , then $S_M := \{x \in M : \|x\| = 1\}$ is the unit sphere of M and the orthogonal complement of M in H is denoted by M^\perp . If M is a closed subspace of M , then the orthogonal projection onto M is denoted by P_M .

We denote the space of all bounded linear operators between H_1 and H_2 by $\mathcal{B}(H_1, H_2)$. In case if $H_1 = H_2 = H$, then we denote $\mathcal{B}(H_1, H_2)$ by $\mathcal{B}(H)$. For $T \in \mathcal{B}(H_1, H_2)$, there exists a unique operator denoted by $T^* : H_2 \rightarrow H_1$ satisfying

$$\langle Tx, y \rangle = \langle x, T^*y \rangle \text{ for all } x \in H_1 \text{ and for all } y \in H_2.$$

This operator T^* is called the adjoint of T .

The null space and the range spaces of T are denoted by $N(T)$ and $R(T)$, respectively.

Let $T \in \mathcal{B}(H)$. Then T is said to be *normal* if $T^*T = TT^*$, *self-adjoint* if $T = T^*$. If T is self-adjoint and $\langle Tx, x \rangle \geq 0$ for all $x \in H$, then T is called *positive*. It is well known that for a positive operator T , there exists a unique positive operator $S \in \mathcal{B}(H)$ such that $S^2 = T$. The operator S is called as the *positive square root* of T and is denoted by $T^{\frac{1}{2}}$.

If $S, T \in \mathcal{B}(H)$ are self-adjoint and $S - T \geq 0$, then we write this by $S \geq T$.

If $P \in \mathcal{B}(H)$ is such that $P^2 = P$, then P is called a *projection*. If $N(P)$ and $R(P)$ are orthogonal to each other, then P is called an *orthogonal projection*. A projection P is an orthogonal projection if and only if it is self-adjoint if and only if it is normal.

We call an operator $V \in \mathcal{B}(H_1, H_2)$ to be an *isometry* if $\|Vx\| = \|x\|$ for each $x \in H_1$. An operator $V \in \mathcal{B}(H_1, H_2)$ is said to be a *partial isometry*

if $V|_{N(V)^\perp}$ is an isometry. That is $\|Vx\| = \|x\|$ for all $x \in N(V)^\perp$. If $V \in \mathcal{B}(H)$ is isometry and onto, then V is said to be a *unitary operator*.

If $T \in \mathcal{B}(H_1, H_2)$, then $T^*T \in \mathcal{B}(H_1)$ is positive and $|T| := (T^*T)^{\frac{1}{2}}$ is called the *modulus* of T . In fact, there exists a unique partial isometry $V \in \mathcal{B}(H_1, H_2)$ such that $T = V|T|$ and $N(V) = N(T)$. This factorization is called the *polar decomposition* of T .

A closed subspace M of H is said to be *invariant* under $T \in \mathcal{B}(H)$ if $TM \subseteq M$ and *reducing* if both M and M^\perp are invariant under T .

For $T \in \mathcal{B}(H)$, the set

$$\rho(T) := \{\lambda \in \mathbb{C} : T - \lambda I : H \rightarrow H \text{ is invertible and } (T - \lambda I)^{-1} \in \mathcal{B}(H)\}$$

is called the *resolvent set* and the complement $\sigma(T) = \mathbb{C} \setminus \rho(T)$ is called the *spectrum* of T . It is well known that $\sigma(T)$ is a non empty compact subset of \mathbb{C} . The point spectrum of T is defined by

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not one-to-one}\}.$$

Note that $\sigma_p(T) \subseteq \sigma(T)$.

If $T \in \mathcal{B}(H_1, H_2)$, then T is said to be *compact* if for every bounded set S of H_1 , the set $T(S)$ is pre-compact in H_2 . Equivalently, for every bounded sequence (x_n) of H_1 , (Tx_n) has a convergent subsequence in H_2 . We denote the set of all compact operators between H_1 and H_2 by $\mathcal{K}(H_1, H_2)$. In case if $H_1 = H_2 = H$, then $\mathcal{K}(H_1, H_2)$ is denoted by $\mathcal{K}(H)$.

A bounded linear operator $T : H_1 \rightarrow H_2$ is called *finite rank* if $R(T)$ is finite dimensional. The space of all finite rank operators between H_1 and H_2 is denoted by $\mathcal{F}(H_1, H_2)$ and we write $\mathcal{F}(H, H) = \mathcal{F}(H)$.

All the above mentioned basics of operator theory can be found in [12, 4, 2, 11].

For $T \in \mathcal{B}(H_1, H_2)$, the quantity

$$m(T) := \inf \{\|Tx\| : x \in S_{H_1}\}$$

is called the *minimum modulus* of T . If $H_1 = H_2 = H$ and $T^{-1} \in \mathcal{B}(H)$, then $m(T) = \frac{1}{\|T^{-1}\|}$ (see [1, Theorem 1] for details).

The following definition is available in [9] for densely defined closed operators (not necessarily bounded) in a Hilbert space, and this holds true automatically for bounded operators.

Definition 1.1. [9, Definition 8.3 page 178] Let $T = T^* \in \mathcal{B}(H)$. The *discrete spectrum* $\sigma_d(T)$ of T is defined as the set of all eigenvalues of T with finite multiplicities which are isolated points of the spectrum $\sigma(T)$ of T . The complement set $\sigma_{ess}(T) = \sigma(T) \setminus \sigma_d(T)$ is called the *essential spectrum* of T .

By the Weyl's theorem we can assert that if $T = T^*$ and $K = K^* \in \mathcal{K}(H)$, then $\sigma_{ess}(T + K) = \sigma_{ess}(T)$ (see [9, Corollary 8.16, page 182] for details). If H is a separable Hilbert space, the *essential minimum modulus* of T is

defined to be $m_e(T) := \inf \{ \lambda : \lambda \in \sigma_{ess}(|T|) \}$ (see [1] for details). The same result in the general case is dealt in [8, Proposition 2.1].

Remark 1.2. Let $T \in \mathcal{B}(H)$ be self-adjoint. Since $\sigma_{ess}(T) \subseteq \sigma(T)$, then by [6, Proposition 2.1] we have

$$m(T) = \inf \{ |\lambda| : \lambda \in \sigma(T) \} \leq \inf \{ |\mu| : \mu \in \sigma_{ess}(T) \} = m_e(T).$$

Let $H = H_1 \oplus H_2$ and $T \in \mathcal{B}(H)$. Let $P_j : H \rightarrow H$ be an orthogonal projection onto H_j for $j = 1, 2$. Then $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$, where $T_{ij} : H_j \rightarrow H_i$ is the operator given by $T_{ij} = P_i T P_j|_{H_j}$. In particular, $T(H_1) \subseteq H_1$ if and only if $T_{12} = 0$. Also, H_1 reduces T if and only if $T_{12} = 0 = T_{21}$ (for details see [11, 4]).

2. POSITIVE \mathcal{AN} -OPERATORS

In this section we describe the structure of operators which are positive and satisfy the \mathcal{AN} -property. First, we recall a few results which are necessary for proving our results.

An operator $T \in \mathcal{B}(H_1, H_2)$ is said to be *norm attaining* if there exists a $x \in S_{H_1}$ such that $\|Tx\| = \|T\|$. We denote the class of norm attaining operators by $\mathcal{N}(H_1, H_2)$. It is known that $\mathcal{N}(H_1, H_2)$ is dense in $\mathcal{B}(H_1, H_2)$ with respect to the operator norm of $\mathcal{B}(H_1, H_2)$. We refer [5] for a simple proof this fact.

Recall that $T \in \mathcal{B}(H_1, H_2)$ is said to be *absolutely norm attaining* or \mathcal{AN} -operator (shortly), if $T|_M$, the restriction of T to M , is norm attaining for every non zero closed subspace M of H_1 . That is $T|_M \in \mathcal{N}(M, H_2)$ for every non zero closed subspace M of H_1 . This class contains $\mathcal{K}(H_1, H_2)$, and the class of partial isometries with finite dimensional null space or finite dimensional range space (See [3] for more details).

We have the following characterization of norm attaining operators:

Proposition 2.1. [3, Proposition 2.4] *Let $T \in \mathcal{B}(H)$ be self-adjoint. Then*

- (1) *$T \in \mathcal{N}(H)$ if and only if either $\|T\| \in \sigma_p(T)$ or $-\|T\| \in \sigma_p(T)$*
- (2) *if $T \geq 0$, then $T \in \mathcal{N}(H)$ if and only if $\|T\| \in \sigma_p(T)$.*

We recall a characterization of positive \mathcal{AN} -operators that we need later.

Theorem 2.2. [10, Theorem 5.1] *Let H be a complex Hilbert space of arbitrary dimension and let P be a positive operator on H . Then P is an \mathcal{AN} -operator iff P is of the form $P = \alpha I + K + F$, where $\alpha \geq 0$, K is a positive compact operator and F is self-adjoint finite rank operator.*

A slight improved version of the above result is proved in [7].

Theorem 2.3. [7] *Let H be an infinite dimensional Hilbert space and $T \in \mathcal{AN}(H)$. Then the following are equivalent:*

- (1) *$T \geq 0$ and $T \in \mathcal{AN}(H)$*

- (2) *there exists a unique triple (K, F, α) where $K \in \mathcal{K}(H)$ is positive, $F \in \mathcal{F}(H)$ is positive with $KF = 0$, and $\alpha \geq 0$, $K \leq \alpha I$ such that $T = K - F + \alpha I$.*

Next, we prove a new characterization of positive \mathcal{AN} -operators in terms of the essential spectrum of the operator.

Theorem 2.4. *Let H be an infinite dimensional Hilbert space H and $T \in \mathcal{B}(H)$ be positive. Then $T \in \mathcal{AN}(H)$ if and only $\sigma_{ess}(T)$ is a singleton set and $[m(T), m_e(T))$ contains only finitely many points.*

Proof. If $T \in \mathcal{AN}(H)$, then there exists a unique triple (K, F, α) as in Theorem 2.3. Observe that $\sigma_{ess}(T) = \{\alpha\}$ and $m_e(T) = \alpha$. If $[m(T), m_e(T))$ contains infinitely many points say (β_n) , then (β_n) has monotone, subsequence, say (β_{n_k}) . If (β_{n_k}) is decreasing, then (β_{n_k}) converges, say $\beta_{n_k} \rightarrow \beta$. This shows that $\beta \in \sigma_{ess}(T)$, which means $m_e(T) = \beta$. This is a contradiction. If (β_{n_k}) increases to β , then $\beta = m_e(T)$. But this contradicts the \mathcal{AN} property of T , which can be seen by [10, Proposition 3.4].

To prove the converse, assume that $\sigma_{ess}(T)$ is a singleton set and $[m(T), m_e(T))$ contains only finitely many points. If $\sigma_{ess}(T) = \{\alpha\}$, then $m_e(T) = \alpha$. Thus $\sigma(T) \setminus \{m_e(T)\} = \sigma_d(T)$. As $\sigma_d(T)$ contains only isolated points of $\sigma(T)$ with finite multiplicities, it must be countable.

First, let us consider the case when $\sigma_d(T)$ is finite. Suppose $\sigma_d(T) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$. Then clearly in this case $m_e(T)$ cannot be the limit of $\sigma_d(T)$, it can be an eigenvalue with infinite multiplicity. Let $H_1 = \bigoplus_{j=1}^n (T - \lambda_j I)$ and $H_2 = N(T - m_e(T)I)$. Then $H = H_1 \oplus H_2$. Define T_1 on H_1 by $T_1 = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$, the diagonal operator with diagonal entries $\lambda_1, \lambda_2, \dots, \lambda_n$. Define T_2 on H_2 by $T_2 = m_e(T)I_{H_2}$. Then clearly, $T = T_1 \oplus T_2$. Thus

$$T = F + m_e(T)I,$$

where $F = \text{diag}(\lambda_1 - m_e(T), \lambda_2 - m_e(T), \dots, \lambda_n - m_e(T)) \oplus 0$ is a finite rank, self-adjoint operator. Now by Theorem 2.2, we can conclude that $T \in \mathcal{AN}(H)$.

Next, we consider the case when $\sigma_d(T)$ is countably infinite. Let $\sigma_d(T) = \{\lambda_1, \lambda_2, \lambda_3, \dots\}$ and $H_1 = \bigoplus_{n=1}^{\infty} H_n$, where $H_n := N(T - \lambda_n I)$. Since $\sigma_d(T)$ is countable, it follows that $\sigma(T)$ is countable, if the multiplicity of α is ignored (this case occurs when α is an eigenvalue with infinite multiplicity). Without loss of generality assume that $\lambda_1 = \sup \{\lambda : \lambda \in \sigma_d(T)\}$. Then $\lambda_1 = \|T\|$. Let $T_1 = \text{diag}(\lambda_1, \lambda_2, \dots)$ be the diagonal operator. Then $T_1 \in \mathcal{B}(H_1)$ and $\sigma(T_1) = \sigma_d(T)$.

Let $\{\beta_1, \beta_2, \dots, \beta_m\}$ be the set of points in $[m(T), m_e(T))$. Define $H_2 = \bigoplus_{k=1}^m \tilde{H}_k$, with $\tilde{H}_k = N(T - \beta_k I)$ and $T_2 := \text{diag}(\beta_1, \beta_2, \dots, \beta_m)$. Then $T = T_1 \oplus T_2$ with $\sigma(T_1) = \{\lambda_1, \lambda_2, \dots\} \cup \{m_e(T)\}$ and $\sigma(T_2) = \{\beta_1, \beta_2, \dots, \beta_m\}$. Note that $T_1 - m_e(T)I_{H_1} =: K_1$, a positive compact operator. Also, $T_2 -$

$m_e(T)I_{H_2} = -F_1$, where F_1 is positive $m \times m$ diagonal matrix with diagonal $m_e(T) - \beta_1, \dots, m_e(T) - \beta_m$. Thus T can be written as

$$\begin{aligned} T &= \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \\ &= \begin{pmatrix} K + m_e(T)I_{H_1} & \\ 0 & -F + m_e(T)I_{H_2} \end{pmatrix} \\ &= \begin{pmatrix} K_1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & F_1 \end{pmatrix} + m_e(T) \begin{pmatrix} I_{H_1} & 0 \\ 0 & I_{H_2} \end{pmatrix}. \end{aligned}$$

Write $K := \begin{pmatrix} K_1 & 0 \\ 0 & 0 \end{pmatrix}$, $F := \begin{pmatrix} 0 & 0 \\ 0 & F_1 \end{pmatrix}$. Then T can be written as $T = K - F + m_e(T)I$, which is an \mathcal{AN} -operator, by Theorem 2.3. \square

In general if $T \in \mathcal{AN}(H_1, H_2)$ it need not imply that $T^* \in \mathcal{AN}(H_2, H_1)$. An example supporting this fact can be found in [10, Example 6.3]. Here we give a sufficient condition for such a result to hold true.

The following lemma will be used in our next result.

Lemma 2.5. [18, Lemma 3] *Let $T \in \mathcal{B}(H)$. Then $\lambda \in \sigma_{ess}(|T|)$ if and only if $\lambda^2 \in \sigma_{ess}(T^*T)$.*

Theorem 2.6. [7, Corollary 2.10] *Let $T \in \mathcal{B}(H_1, H_2)$. Then $T \in \mathcal{AN}(H_1, H_2)$ if and only if $T^*T \in \mathcal{AN}(H_1)$.*

Theorem 2.7. *Let $T \in \mathcal{B}(H_1, H_2)$. Suppose that $\sigma_{ess}(T^*T) = \sigma_{ess}(TT^*)$. Then the following holds.*

- (1) $T \in \mathcal{AN}(H_1, H_2)$ if and only if $T^* \in \mathcal{AN}(H_2, H_1)$
- (2) $T^*T \in \mathcal{AN}(H_1)$ if and only if $TT^* \in \mathcal{AN}(H_2)$.

Proof. Proof of (1): Suppose $T \in \mathcal{AN}(H_1, H_2)$. Then $T^*T \in \mathcal{AN}(H_1)$, by Theorem 2.6. Thus by Theorem 2.3, we have $T^*T = K - F + \alpha I$, where $K \in \mathcal{K}(H_1)$ is positive, $F \in \mathcal{F}(H_1)$ is positive and $\alpha \geq 0$ has the property that $F \leq \alpha I$. It is clear that $\sigma_{ess}(T^*T) = \{\alpha\}$ and $m_e(T^*T) = \alpha$. Note that if $\alpha = 0$, then clearly $T \in \mathcal{K}(H_1, H_2)$ and hence $T^* \in \mathcal{K}(H_2, H_1)$. So $T^* \in \mathcal{AN}(H_2, H_1)$. Thus in this case the result is proved.

Now, assume that $\alpha > 0$. By the hypothesis, we have $\sigma_{ess}(TT^*) = \{\alpha\} = \sigma_{ess}(T^*T)$. Since $\sigma(T^*T) \setminus \{0\} = \sigma(TT^*) \setminus \{0\}$, $[\alpha, \|T^*T\|]$ and $[\alpha, \|TT^*\|]$ contains same points. Also, $[m(TT^*), \alpha)$ contains finitely many points as the non zero points of $[m(TT^*), \alpha)$ and $[m(T^*T), \alpha)$ are the same. We can conclude by Theorem 2.4 that $TT^* \in \mathcal{AN}(H_2)$. Thus $T^* \in \mathcal{AN}(H_2, H_1)$.

The other implication follows by replacing T by T^* in the above argument.

Proof of (2): If $T^*T \in \mathcal{AN}(H_1)$, then $T \in \mathcal{AN}(H_1, H_2)$ by Theorem 2.6. Now by (1), it is clear that $T^* \in \mathcal{AN}(H_2, H_1)$. Now by Theorem 2.6 again, we can conclude that $TT^* \in \mathcal{AN}(H_2)$. Applying these arguments for T^* , we get the reverse argument. \square

Remark 2.8. Let $T \in \mathcal{B}(H_1, H_2)$ such that $\sigma_{ess}(T^*T) = \sigma_{ess}(TT^*)$. Then by Lemma 2.5, we can easily show that $m_e(T) = m_e(T^*)$.

Theorem 2.9. *Let $T \in \mathcal{B}(H)$. Suppose that*

- (1) $m_e(T) = m_e(T^*)$
- (2) both T and $T^* \in \mathcal{AN}(H)$

*Then $\sigma_{ess}(T^*T) = \sigma_{ess}(TT^*)$.*

Proof. Since both T and T^* are \mathcal{AN} -operators, we have $\sigma_{ess}(T^*T) = \{\alpha\}$ and $\sigma_{ess}(TT^*) = \{\beta\}$. Clearly, $m_e(T^*T) = \alpha$ and $m_e(TT^*) = \beta$. Note that by Lemma 2.5 and the hypothesis, we have $m_e(|T|) = m_e(T) = m_e(T^*) = m_e(|T^*|)$, it follows that $\alpha = \beta$. Thus $\sigma_{ess}(T^*T) = \sigma_{ess}(TT^*)$. \square

3. PARANORMAL \mathcal{AN} -OPERATORS

In this section we describe the structure of paranormal \mathcal{AN} -operators. We give sufficient conditions under which a paranormal operator will be \mathcal{AN} -operator. The structure of self-adjoint and normal \mathcal{AN} -operators is discussed in [7].

First, we prove a few elementary results which will be useful in proving main theorems.

The following Lemma is proved in [16, Lemma 1]. We write the same in a format that is useful to us.

Lemma 3.1. *Let $T \in \mathcal{B}(H)$ and $M = \{x \in H : \|Tx\| = \|T\|\|x\|\}$. Then $M = N(\|T\|^2 I - T^*T) = N(|T| - \|T\|I)$.*

Proof. If $x \in M$, then $\|Tx\|^2 = \|T\|^2\|x\|^2$. This is equivalent to the condition; $\langle (\|T\|^2 I - T^*T)x, x \rangle = 0$. Since $\|T\|^2 I - T^*T \geq 0$, it follows that $T^*Tx = \|T\|^2x$. Thus $x \in N(\|T\|^2 I - T^*T)$.

On the other hand if, $x \in N(\|T\|^2 I - T^*T)$, then $(\|T\|^2 I - |T|^2)x = 0$. Hence $(\|T\|I + |T|)(\|T\|I - |T|)x = 0$. Since $(\|T\|I + |T|)^{-1} \in \mathcal{B}(H)$, we can conclude that $|T|x = \|T\|x$. That is $x \in N(|T| - \|T\|I)$. Moreover, $\|Tx\| = \|T\|\|x\|$, we obtain that $x \in M$. Hence $M = N(\|T\|^2 I - T^*T)$.

If $x \in N(|T| - \|T\|I)$, then $|T|x = \|T\|x$ and $|T|^2x = \|T\|^2x$. That is $T^*Tx = \|T\|^2x$, concluding $N(|T| - \|T\|I) \subseteq N(\|T\|^2 I - T^*T)$. \square

Corollary 3.2. *Let $T \in \mathcal{B}(H)$. Then $T \in \mathcal{N}(H)$ if and only if $M \neq \{0\}$.*

Definition 3.3. Let $T \in \mathcal{B}(H)$. Then T is said to be

- (1) Hyponormal if $T^*T - TT^* \geq 0$ or $\|Tx\|^2 \geq \|T^*x\|^2$ for all $x \in H$.
- (2) Paranormal if $\|T^2x\| \geq \|Tx\|^2$ for all $x \in S_H$. In other words, T is paranormal if $\|Tu\|^2 \leq \|T^2u\|\|u\|$ for all $u \in H$.

A hyponormal operator is paranormal. We refer [13, 14, 15] and [17] for more details on these class of operators.

Lemma 3.4. [16, Theorem 3] *Suppose $T \in \mathcal{B}(H)$ be paranormal, norm attaining. Then $N(\|T\|^2 I - T^*T)$ is invariant under T . Moreover, $\frac{T}{\|T\|}$ is isometry on M .*

Lemma 3.5. [13, Lemma 2.3] *Let $T \in \mathcal{B}(H)$ be paranormal with $\|T\| = 1$. Then*

$$M_{T^*} = \{x \in H : TT^*x = x\} = N(I - TT^*)$$

is invariant under T .

Remark 3.6. Let $T \in \mathcal{B}(H)$. Then $M_{T^*} = N(\|T\|^2 I - TT^*)$ is invariant under T .

By combining the above two lemmas we can prove the following result which is a key point in the structure theorem of paranormal \mathcal{AN} -operators.

Lemma 3.7. *Let $T \in \mathcal{B}(H)$ be paranormal. Then $M_T = N(\|T\|^2 I - T^*T)$ is a reducing subspace for T . In addition if $T \in \mathcal{N}(H)$, then $\|T\| \in \sigma_p(|T|)$.*

Proof. By Lemma 3.4, M_T is invariant under T . To prove the Lemma it is enough to prove that M_T is invariant under T^* . This follows by applying Lemma 3.5 to T^* and the fact that $T^{**} = T$.

If $T \in \mathcal{N}(H)$, then $M_T \neq \{0\}$ and by Lemma 3.1, we have $\|T\| \in \sigma_p(|T|)$. \square

Lemma 3.8. *Let $T \in \mathcal{B}(H)$ be paranormal. If M is a non zero invariant subspace, then $T|_M$ is also paranormal.*

Proof. Let $T_M := T|_M$. For $x \in M$, we have

$$\|T_M x\|^2 = \|Tx\|^2 \leq \|T^2 x\| \|x\| = \|T_M(Tx)\| \|x\| = \|T_M^2 x\| \|x\|.$$

Thus T_M is paranormal. \square

Theorem 3.9. *Let $T \in \mathcal{AN}(H)$ be paranormal and $\Lambda = \sigma(|T|)$. Then there exists $(H_\beta, U_\beta)_{\beta \in \Lambda}$, where*

- (1) H_β is a reducing subspace for T
- (2) $U_\beta \in \mathcal{B}(H_\beta)$ is an isometry

such that

$$(a) \quad H = \bigoplus_{\beta \in \Lambda} H_\beta$$

$$(b) \quad T = \bigoplus_{\beta \in \Lambda} \beta U_\beta$$

$$(c) \quad \sigma(T) \subseteq \bigoplus_{\beta \in \Lambda} \beta \mathbb{T}, \text{ where } \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}.$$

Proof. Since $T \in \mathcal{AN}(H)$, $H_1 := N(T^*T - \|T\|^2 I) = N(|T| - \|T\|I)$ is a reducing subspace for T . Let $T_1 := T|_{H_1}$. Then $T_1 \in \mathcal{AN}(H_1)$. Note that for all $x \in H_1$, we have $T^*Tx = \|T\|^2 x$. In fact, $T_1^* = T^*|_{H_1}$, since H_1 reduces T . If $H_1 = H$, then $T^*T = \|T\|^2 I$. Write $\alpha_1 = \|T\|$. Then $\alpha_1 \in \sigma_p(|T|)$ and $T^*T = \alpha_1^2 I$. That is $\frac{1}{\alpha_1} T$ is an isometric operator, call it U_1 . Then $T_1 = \alpha_1 U_1$.

If $H_1 \subset H$, then $H = H_1 \oplus^\perp H_1^\perp$. We can write $T = T_1 \oplus T_2$, where $T_2 := T|_{H_1^\perp}$. In fact, T has the following representation;

$$(3.1) \quad T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 U_1 & 0 \\ 0 & T_2 \end{pmatrix}.$$

Write $H_2 = H_1^\perp$. Note that $T_2 \in \mathcal{AN}(H_2)$. As T_2 is paranormal by Lemma 3.8, by proceeding as above, either $T_2 = \alpha_2 U_2$ or $T_2 = \alpha_2 U_2 \oplus T_3$, where $\alpha_2 = \|T_2\| \in \sigma(|T|)$. Thus T has the following representations;

$$(3.2) \quad T = \begin{pmatrix} \alpha_1 U_1 & 0 \\ 0 & \alpha_2 U_2 \end{pmatrix}$$

or

$$(3.3) \quad T = \begin{pmatrix} \alpha_1 U_1 & 0 & 0 \\ 0 & \alpha_2 U_2 & 0 \\ 0 & 0 & T_3 \end{pmatrix}.$$

Proceeding as above we end up with the following two cases:

Case 1: The process stops after n number of steps; In this case, we have that $H = \bigoplus_{k=1}^n H_k$ and $T = \alpha_1 U_1 \oplus \alpha_2 U_2 \oplus \cdots \oplus \alpha_n U_n$. Note that $\alpha_1 < \alpha_2 < \cdots < \alpha_n$ and each $\alpha_i \in \sigma_p(|T|)$ for $i = 1, 2, \dots, n$. Clearly, $|T| = \bigoplus_{k=1}^n I_k$, where $I_k : H_k \rightarrow H_k$ is the identity operator.

If all the eigenvalue α_k has finite multiplicity, then each H_k is finite dimensional and each U_k is a unitary. In this case

$$(3.4) \quad H = \bigoplus_{k=1}^n H_k \quad \text{and} \quad T = \bigoplus_{i=1}^n \alpha_i U_i.$$

Clearly H is a finite dimensional Hilbert space and each of U_j is a unitary.

Since $|T| \in \mathcal{AN}(H)$ it can have atmost one eigenvalue with infinite multiplicity, say α_k , ($1 \leq k \leq n$). That is H_k is infinite dimensional. Thus

$$(3.5) \quad H = \bigoplus_{j=1, j \neq k}^n H_j \oplus H_k \quad \text{and} \quad T = \left(\bigoplus_{j=1, j \neq k}^n \alpha_j U_j \right) \oplus \alpha_k U_k.$$

Note that each U_j ($j \neq k$) is an unitary and U_k is an isometry. Clearly, we have $\sigma(T) \subset \bigoplus_{j=1}^n \alpha_j \mathbb{T}$.

Case 2: The process does not stop after finite steps;

By proceeding as above we get a sequence (α_n) of positive numbers such that $\alpha_n < \alpha_{n+1}$, subspaces H_n such that $T_n = T|_{H_n} = \alpha_n U_n$ for each $n \in \mathbb{N}$.

Since $\alpha_n \in \sigma_p(|T|)$ for each $n \in \mathbb{N}$ and $\alpha_n \geq m(T)$, α_n converges to α , say. Then by [7, Remark 2.5] we can conclude that $\alpha = m_e(T)$, the essential minimum modulus of T . Also, there may exists atmost finitely many spectral values, $\beta_1, \beta_2, \dots, \beta_m$ between $m(T)$ and $m_e(T)$, by the structure of positive \mathcal{AN} -operators (Theorem 2.4). Let \tilde{H}_j , ($j = 1, 2, \dots, m$) be the eigenspaces and \tilde{U}_j , ($j = 1, 2, \dots, m$) be the corresponding unitaries associated with

$\beta_1, \beta_2, \dots, \beta_m$, respectively. Since by [10, Theorem 3.1], the eigenvectors of $|T|$ spans a dense subspace of H , we must have $H = \bigoplus_{k=1}^{\infty} H_k \oplus \bigoplus_{j=1}^m \tilde{H}_k$. Also note that $\sigma(|T|) = \{\alpha_n\}_{n=1}^{\infty} \cup \{\alpha\} \cup \{\beta_j\}_{j=1}^m$.

Now the representation of T can be written as;

$$(3.6) \quad T = \left(\bigoplus_{n=1}^{\infty} \alpha_n U_n \right) \oplus \left(\bigoplus_{j=1}^m \beta_j \tilde{U}_j \right) = \bigoplus_{\beta \in \Lambda} \beta U_{\beta}.$$

Now it is easy to compute the spectrum. Clearly,

$$(3.7) \quad \sigma(T) = \bigsqcup_{n=1}^{\infty} \sigma(\alpha_n U_n) \oplus \bigsqcup_{k=1}^n \sigma(\beta_k U_k) \subseteq \bigsqcup_{n=1}^{\infty} \alpha_n \mathbb{T} \oplus \bigsqcup_{k=1}^n \beta_k \mathbb{T} = \bigoplus_{\beta \in \Lambda} \beta \mathbb{T},$$

where \bigsqcup denote the disjoint union.

If β is an eigenvalue of $|T|$ with infinite multiplicity, then by [10, Theorem 3.8(iv)] $\alpha = \beta$. That is α is both an eigenvalue with infinite multiplicity as well as the unique limit point of $\sigma(|T|)$. In this case the representation of T can be written as;

$$(3.8) \quad T = \left(\bigoplus_{\beta \in \Lambda, \beta \neq \alpha} \beta U_{\beta} \right) \oplus \alpha U_{\alpha}. \quad \square$$

Corollary 3.10. *If $T \in \mathcal{AN}(H)$ paranormal, then T has infinitely many finite dimensional reducing subspaces.*

Proof. The subspaces $\{H_{\alpha}\}_{\alpha \in \Lambda}$ is a reducing subspace of T . Since Λ is countable, the conclusion holds. \square

Remark 3.11. Unlike compact operators, a paranormal \mathcal{AN} -operator need not be normal. For example, the right shift operator on ℓ^2 is paranormal, \mathcal{AN} -operator, but it is not normal. Hence we have the following question.

Question 3.12. When can a paranormal \mathcal{AN} -operator is normal?

Here we answer the above question.

Theorem 3.13. *Let $T \in \mathcal{B}(H)$ be paranormal and $T \in \mathcal{AN}(H)$. We have the following;*

- (1) *if $|T|$ has no eigenvalue with infinite multiplicity, then T is normal.*
- (2) *If $|T|$ has an eigenvalue with infinite multiplicity, say β , and $H_{\beta} = N(|T| - \beta I)$, then T is normal if and only if $T|_{H_{\beta}}$ is normal.*

Proof. Proof of (1): If $|T|$ has no eigenvalue with infinite multiplicity, then T is represented as in Equation 3.4 or Equation 3.6. In both cases, since each U_j is finite dimensional isometry, it must be unitary. Thus T must be normal.

Proof of (2): We have $H_{\beta} = N(T^*T - \beta^2 I)$. If T is normal, as H_{β} is a reducing subspace for T , it is clear that $T|_{H_{\beta}}$ is normal.

On the other hand, if $T|_{H_\beta}$ is normal, then $U_\beta = \frac{1}{\beta}T|_{H_\beta}$ is unitary. Now the normality of T follows by Equations 3.5 and 3.8, depending on the case. \square

Remark 3.14. If $T \in \mathcal{K}(H)$ is paranormal, then by (1) of Theorem 3.13, T must be normal.

REFERENCES

- [1] R. Bouldin, The essential minimum modulus, *Indiana Univ. Math. J.* **30** (1981), no. 4, 513–517. MR0620264 (82i:47001)
- [2] Paul Richard Halmos, *A Hilbert space problem book*, volume 19 of *Graduate Texts in Mathematics*, second edition, Encyclopedia of Mathematics and its Applications, 17 (Springer-Verlag, New York, 1982).
- [3] X. Carvajal and W. Neves, Operators that achieve the norm, *Integral Equations Operator Theory* **72** (2012), no. 2, 179–195. MR2872473 (2012k:47044)
- [4] John B. Conway, *A course in functional analysis*, volume 96 of *Graduate Texts in Mathematics*, second edition (Springer-Verlag, New York, 1990).
- [5] P. Enflo, J. Kover and L. Smithies, Denseness for norm attaining operator-valued functions, *Linear Algebra Appl.* **338** (2001), 139–144. MR1861118 (2002g:47148)
- [6] G. Ramesh, Structure theorem for \mathcal{AN} -operators, *J. Aust. Math. Soc.* **96** (2014), no. 3, 386–395. MR3217722
- [7] G. Ramesh and D. Venkunaidu, On Absolutely norm attaining operators, Preprint, 2016(<https://arxiv.org/abs/1801.02432>).
- [8] I. S. Feshchenko, On the essential spectrum of the sum of self-adjoint operators and the closedness of the sum of operator ranges, *Banach J. Math. Anal.* **8** (2014), no. 1, 55–63. MR3161682
- [9] Schmüdgen, Konrad, Unbounded self-adjoint operators on Hilbert space, *Graduate Texts in Mathematics*, 265, Springer, Dordrecht, 2012, xx+432, MR2953553
- [10] S. K. Pandey and V. I. Paulsen, A spectral characterization of \mathcal{AN} operators, *J. Aust. Math. Soc.* **102** (2017), no. 3, 369–391. MR3650963
- [11] A. E. Taylor and D. C. Lay, *Introduction to functional analysis*, second edition, John Wiley & Sons, New York, 1980. MR0564653 (81b:46001)
- [12] W. Rudin, *Functional analysis*, McGraw-Hill, New York, 1973. MR0365062 (51 #1315)
- [13] V. Istrăţescu, On some hyponormal operators, *Pacific J. Math.* **22** (1967), 413–417. MR0213893
- [14] S. K. Berberian, A note on hyponormal operators, *Pacific J. Math.* **12** (1962), 1171–1175. MR0149281
- [15] J. G. Stampfli, Hyponormal operators, *Pacific J. Math.* **12** (1962), 1453–1458. MR0149282
- [16] Lee, Jun Ik, On the norm attaining operators, *The Korean Journal of Mathematics* **20**(2012)
- [17] T. Furuta, *Invitation to linear operators*, Taylor & Francis (2001). MR1978629
- [18] J. Kover, Compact perturbations and norm attaining operators, *Quaest. Math.* **28** (2005)no. 4. MR2182451

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