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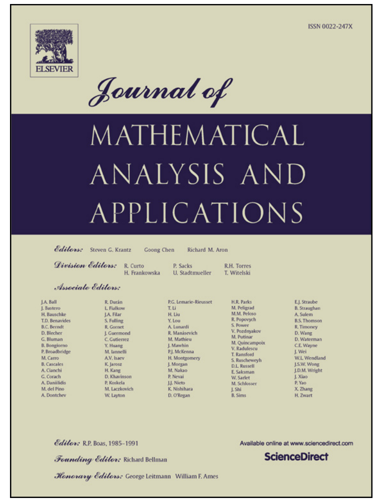

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CAUCHY TRANSFORMS ARISING FROM HOMOMORPHIC CONDITIONAL EXPECTATIONS PARAMETRIZE NONCOMMUTATIVE PICK FUNCTIONS

J. E. PASCOE† AND RYAN TULLY-DOYLE

ABSTRACT. Nevanlinna showed that Cauchy transforms of probability measures parametrize all functions from the upper half plane into itself satisfying a certain asymptotic condition at infinity. We show that the correspondence fails in general for the unbounded case for somewhat trivial reasons; however, we show that in a setting of "homomorphic" operator valued free probability that Cauchy transforms of homomorphic conditional expectations parametrize noncommutative Pick functions.

CONTENTS

1. INTRODUCTION

Classically, R. Nevanlinna proved the following result.

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Theorem 1.1 (Nevanlinna [6])**.** *Let Π denote the upper half plane. Let $f : \Pi \to \mathbb{C}$. The function f is analytic, maps Π to $\overline{\Pi}$ and satisfies*

$$\liminf_{s\to\infty} sf(sz) = -z^{-1},$$

for all $z \in \Pi$, if and only if there exists a probability measure μ on $\mathbb{R}$ such that

$$f(z) = \int_{\mathbb{R}} \frac{1}{t-z} \mathrm{d}\mu(t).$$

Thus, functions with positive imaginary part satisfying good asymptotics are parametrized by probability measures on the real line.

The quantity

$$f(z) = \int_{\mathbb{R}} \frac{1}{t-z} \mathrm{d}\mu(t),$$

occurring in Nevanlinna's theorem is often referred to as the **Cauchy transform**. Recent work by Anshelevich and Williams [3, 11, 12] has explored the connection between distribution and function theory in free probability in terms of the noncommutative Cauchy transform and the related R-transform. The Cauchy transform and the R-transform have served as a vibrant part of free probability, which is evidenced by the large amount of recent work on the subject.

We resolve the correspondence between Cauchy transforms and the class of functions on the upper half plane in the noncommutative context of operator-valued free probability and free analysis.

1.1. **The noncommutative context.** Let B be a C^*-algebra. In this manuscript, all C^*-algebras will be assumed to be unital. The **noncommutative space over** B, denoted $\mathcal{M}(B)$, is the set of square matrices over B, that is

$$\mathcal{M}(B) = \bigcup_{n=1}^{\infty} M_n(B).$$

Next, the **upper half plane over** B**,** denoted $\Pi(B)$, is given by

$$\Pi(B) = \{X \in \mathcal{M}(B) | \operatorname{Im} X > 0\}.$$

Here, we say a self-adjoint operator $A > 0$ if its spectrum is contained in the positive reals and $A \geq 0$ if A has spectrum contained in the non-negative reals. Similarly, the **closed upper half plane over** B**,** denoted $\overline{\Pi}(B)$, is

$$\overline{\Pi}(B) = \{X \in \mathcal{M}(B) | \operatorname{Im} X \geq 0\}.$$

For any $\mathcal{D} \subset \mathcal{M}(B_1)$, a **noncommutative function** $f : \mathcal{D} \to \mathcal{M}(B_2)$ is graded and respects intertwining maps. That is, f takes an $n \times n$

matrix over B_1 to an $n \times n$ matrix over B_2, and if $\Gamma X = Y\Gamma$ for some rectangular matrix Γ of scalars, then $\Gamma f(X) = f(Y)\Gamma$. We denote the set of noncommutative functions from $\mathcal{D}$ to $\mathcal{R}$ by Free$(\mathcal{D}, \mathcal{R})$. (For more elaborate exposition regarding free analysis, see e.g. the comprehensive presentation in [5]. Occasionally, the noncommutative space over B has been referred to as *the matrix universe over* B and noncommutative functions have sometimes been referred to as *free functions* by other authors.)

We have adopted a *vertical tensor notation* to save space: $\begin{smallmatrix} A \\ \otimes \\ B \end{smallmatrix}$ represents the same object as $A \otimes B$.

In this noncommutative context, a **noncommutative Pick function** is just a noncommutative function $f : \Pi(B_1) \to \overline{\Pi}(B_2)$.

Given:

(1) A C^*-algebra B,
(2) A C^*-algebra M unitally containing B, (Here by *unitally containing* we mean that B is a subalgebra of M and the identity in B is equal to the identity in M,)
(3) An unbounded self-adjoint operator A affiliated to M, that is, if M is a von Neumann algebra, an operator so that each of its spectral projections are contained in M, otherwise, an operator affiliated to some weak closure of M such that $\left(\begin{smallmatrix} A \\ \otimes \\ I \end{smallmatrix} - Z \right)^{-1} \in \mathcal{M}(M)$ for all $z \in \Pi(B)$,
(4) A noncommutative conditional expectation $E : M \to B$, that is, E is a completely positive unital map satisfying $E(b_1 m b_2) = b_1 E(m) b_2$ for all $b_1, b_2 \in B$ and $m \in M$,

we define the **noncommutative Cauchy transform** of A to be the noncommutative function $f : \Pi(B) \to \Pi(B)$ given by the equation

$$f(Z) = \begin{smallmatrix} E \\ \otimes \\ \mathrm{id} \end{smallmatrix} \left(\left(\begin{smallmatrix} A \\ \otimes \\ I \end{smallmatrix} - Z \right)^{-1} \right),$$

where id denotes the identity map on matrices.

The obvious analogue of Nevanlinna's theorem would be that any noncommutative function $f : \Pi(B) \to \Pi(B)$ satisfying

$$\lim_{\substack{s \to +\infty \\ s \in \mathbb{R}}} s f(sZ) = -Z^{-1}$$

for all $Z \in \Pi(B)$ would be given by a noncommutative Cauchy transform arising from some M, E and A which could be constructed from f.

The obvious analogue of Nevanlinna's theorem is shown to be false in Subsection 1.3, and thus the ability to reconstruct an algebra, a conditional expectation and an unbounded operator from a noncommutative function $f : \Pi(B) \to \Pi(B)$ is resolved in the negative.

However, in an expanded "homomorphic" notion of conditional expectation, we show that self maps of the noncommutative upper half plane satisfying good asymptotic conditions are parametrized by Cauchy transforms.

1.2. Main result.

Definition 1.2. Let B, M be C^*-algebras. Let $\hat{B}$ be a unital subalgebra of M. We define a **homomorphic conditional expectation** to be a completely positive unital map $E : M \to B$ such that $E|_{\hat{B}}$ is a homomorphism.

The name homomorphic conditional expectation is justified by the following analogue of Tomiyama's theorem [10].

Proposition 1.3 (Homomorphic Tomiyama's theorem)**.** *If $E : M \to B$ is a homomorphic conditional expectation over B, then for all $b_1, b_2 \in \hat{B}$,*

$$E(b_1 m b_2) = E(b_1)E(m)E(b_2).$$

We prove the above proposition in Section 3

Definition 1.4. Let B, $\hat{B}$ be C^*-algebras. We define a **symmetric dilation** to be a completely positive map $\psi : B \to \hat{B}$ so that there exists a $*$-homomorphism $E : \hat{B} \to B$ such that $E \circ \psi$ is the identity.

Our main result is as follows.

Theorem 1.5. *Let $f : \Pi(B) \to \overline{\Pi}(B)$ be a noncommutative function. The following are equivalent*

(1) *For all $Z \in \Pi(B)$,*

$$\lim_{\substack{s \to +\infty \\ s \in \mathbb{R}}} sf(sZ) = -Z^{-1}.$$

(2) *There exist:*
  (a) *A C^*-algebra M,*
  (b) *A unital subalgebra of $\hat{B} \subseteq M$,*
  (c) *An unbounded self-adjoint operator A affiliated to M,*
  (d) *A homomorphic conditional expectation $E : M \to B$,*
  (e) *A symmetric dilation $\psi : B \to \hat{B}$ such that $E \circ \psi$ is the identity,*

so that the function f can be written as

$$f(Z) = \underset{\mathrm{id}_n}{\overset{E}{\otimes}} \left[\left(\underset{I_n}{\overset{A}{\otimes}} - \underset{\mathrm{id}_n}{\overset{\psi}{\otimes}} (Z) \right)^{-1} \right].$$

We note that Williams showed that the above theorem holds when E is a conditional expectation and ψ is an identity map if we assume additionally that f has some large analytic continuation at infinity corresponding to the classical compactly supported case [11]. Our result also generalizes previous results in [9, Section 5]. In the language of this paper, the representations established in the earlier setting held for $B = \mathbb{C}^m$.

We emphatically take the viewpoint that homomorphic conditional expectations are what makes the Nevanlinna theorem work for noncommutative Cauchy transforms– we leave to the reader whether or not they generate any deeply interesting analogue of operator-valued free probability. However, we view that our results suggest that either (**1**) free noncommutative function theory is an incomplete method for understanding free probability or (**2**) that theorems in free probability should extend somewhat trivially to "homomorphic" operator valued free probability.

1.3. **Failure of the main result in the usual free probabilistic case.** We note that we cannot always reduce to the case where the symmetric dilation ψ is the identity map and E is a *bona fide* conditional expectation.

Take $B = \mathbb{C}^2$, $\hat{B} = \mathbb{C}^3$, and $M = \mathbb{C} \oplus M_2(\mathbb{C})$, where $\hat{B}$ is naturally included in M by the map $(w_1, w_2, w_3) \to \left(w_1, \left[\begin{smallmatrix} w_2 & 0 \\ 0 & w_3 \end{smallmatrix}\right]\right)$. Define $\psi(z_1, z_2) = (z_1, z_2, \frac{1}{2}(z_1 + z_2))$. Define $E\left(w_1, \left[\begin{smallmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{smallmatrix}\right]\right) = (w_1, v_{11})$. Now define A to be $A = \left(0, \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]\right)$.

Consider

$$f(Z) = \underset{\mathrm{id}_n}{\overset{E}{\otimes}} \left[\left(\underset{I_n}{\overset{A}{\otimes}} - \underset{\mathrm{id}_n}{\overset{\psi}{\otimes}} (Z) \right)^{-1} \right].$$

More concretely, including $\mathbb{C}^2$ as diagonal matrices in $M_2(\mathbb{C})$ and $\mathbb{C}^3$ and $\mathbb{C} \oplus M_2(\mathbb{C})$ as similarly naturally included in $M_3(\mathbb{C})$, let

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, E(m) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}^* m \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$\psi(z_1, z_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} z_1 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} z_2.$$

Then f is given by the formula

$$\begin{aligned} f(Z) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}^* \begin{bmatrix} -Z_1 & 0 & 0 \\ 0 & -Z_2 & 1 \\ 0 & 1 & -\frac{Z_1+Z_2}{2} \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -Z_1^{-1} & 0 \\ 0 & -\left(Z_2 - \left(\frac{Z_1+Z_2}{2}\right)^{-1}\right)^{-1} \end{bmatrix}. \end{aligned}$$

One can show, using the above concrete matricial representations, that for $(z_1, z_2) \in \mathbb{C}^2$,

$$f(z_1, z_2) = (-z_1^{-1}, -z_2^{-1}(1 - 2(z_1 + z_2)^{-1} z_2^{-1})^{-1}).$$

Now we observe that

$$f(z_1, z_2) = (-z_1^{-1}, 0) + \sum_k (0, -z_2^{-1} 2^k [z_2(z_1 + z_2)]^{-k}). \tag{1.1}$$

If we could choose $\tilde{E}$ a conditional expectation and $\tilde{\psi}$ to be the identity, the homogeneous terms in the above expansion would be polynomials in z_1^{-1} and z_2^{-1} but, evidently, they are not.

Suppose we had such a realization for f,

$$f(Z) = \overset{\tilde{E}}{\underset{\mathrm{id}_n}{\otimes}} \left[\left(\overset{\tilde{A}}{\underset{I_n}{\otimes}} - Z \right)^{-1} \right].$$

In the case where $\tilde{A}$ is bounded, we have the following asymptotic expansion at infinity,

$$f(Z) = \sum_k \overset{\tilde{E}}{\underset{\mathrm{id}_n}{\otimes}} \left[Z^{-1} \left(\overset{\tilde{A}}{\underset{I_n}{\otimes}} Z^{-1} \right)^k \right]. \tag{1.2}$$

Noting that

$$Z = \overset{P}{\underset{Z_1}{\otimes}} + \overset{1-P}{\underset{Z_2}{\otimes}}$$

where P is the projection $(1, 0) \in B = \mathbb{C}^2$, we see

$$Z^{-1} = \overset{P}{\underset{Z_1^{-1}}{\otimes}} + \overset{1-P}{\underset{Z_2^{-1}}{\otimes}}.$$

Therefore, the terms is Equation (1.2) must be homogeneous polynomials in Z_1^{-1} and Z_2^{-1}, contradicting the expansion given in Equation (1.1).

In the case where $\tilde{A}$ is unbounded, it a perhaps somewhat involved exercise to show that the homogenous terms in Equation (1.2) are well defined, but the same logic applies. (Essentially, one needs to show that a certain map $\hat{E}(x) = E(AxA)$ is well defined.) The exercise fits into the so-called extended HVMS calculus developed in [7], and thus we now give a more formal verification of our counterexample based on that. Restricting f to scalars, representing all C^*-algebras involved as operators, and E as conjugation by an isometry, when we consider the second coordinate of f, call it g, it is of the form

$$g(z_1, z_2) = \left\langle (\tilde{A} - z_P)\alpha, \alpha \right\rangle$$

where P is a projection, $z_P = z_1 P + z_2(1 - P)$ and α is a vector. However, by Equation (1.1) we know that

$$g(z_1, z_2) = z_2^{-1} + -z_2^{-2}\left(\frac{z_1 + z_2}{2}\right)^{-1} + o(\|(z_1^{-1}, z_2^{-1})\|^3).$$

Having an expansion to order 3 of this form implies $z_P^{-1} A z_P^{-1}\alpha$ is a well-defined vector valued function when z_1 and z_2 have positive imaginary part by the extended HVMS calculus [7, Theorem 2.3]. (That is, $z_P^{-1}\alpha$ is in the domain of A for all z where z_P^{-1} is defined. In fact, a careful calculation gives that $z_P^{-1}\alpha = z_2^{-1}\alpha$.) Noting that P is a projection gives that $z_P^{-1} = z_1^{-1}P + z_2^{-1}(1 - P)$. So, in fact, $z_P^{-1} A z_P^{-1}\alpha$ is a vector valued polynomial, which would imply that $-z_2^{-2}(\frac{z_1+z_2}{2})^{-1}$ is a homogeneous polynomial in z_1^{-1} and z_2^{-1} by the extended HVMS calculus [7, Theorem 2.4], which is untrue. (In the language of [7], the veracity of our counterexample follows from the fact that the function is in the intermediate Löwner class $\mathcal{L}^{2-}$ but not in the Löwner class $\mathcal{L}^2$. In some sense, our current example is constructed by appending the example in [7, Section 4] to some other function.)

2. Proof of the main result

We now prove our main theorem, Theorem 1.5.

The **ball over** B**,** denoted $\mathrm{Ball}(B)$, is the set of contractive matrices over B, that is,

$$\mathrm{Ball}(B) = \{X \in \mathcal{M}(B) | \, \|X\| < 1\}.$$

Similarly, the **right half plane over** B**,** denoted $\mathrm{RHP}(B)$, is

$$\mathrm{RHP}(B) = \{X \in \mathcal{M}(B) | \, \mathrm{Re}\ X \geq 0\}.$$

In [8], the following was proved.

Theorem 2.1 ([8])**.** *Let $h : \mathrm{Ball}(B_1) \to \mathrm{RHP}(B_2)$ be a noncommutative function. Then there exists:*

(1) *A C^*-algebra M unitally containing B_1,*
(2) *A completely positive linear (not necessarily unital) map $R : M \to B_2$,*
(3) *A unitary $U \in M$,*
(4) *A bounded self-adjoint operator T,*

such that

$$h(X) = \underset{I_n}{\overset{iT}{\otimes}} + \underset{\mathrm{id}_n}{\overset{R}{\otimes}} \left[\left(I + \underset{I_n}{\overset{U}{\otimes}} X\right)\left(I - \underset{I_n}{\overset{U}{\otimes}} X\right)^{-1}\right]. \tag{2.1}$$

We note that although the statement in [8, Corollary 3.6] assumes an exactness hypothesis on B_1, recent advances in Agler model theory by Ball, Marx and Vinnikov in the preprint [4, Corollary 3.2] give the full result by [8, Lemma 3.3].

We use Theorem 2.1 to show the following Nevanlinna representation via a Hilbert space geometric derivation.

Theorem 2.2. *Let $f : \Pi(B_1) \to \overline{\Pi}(B_2)$ be a noncommutative function. The following are equivalent*

(1)
$$\liminf_{\substack{s \to +\infty \\ s \in \mathbb{R}}} |isf(is)| < \infty.$$

(2) *There exists:*
   (a) *A C^*-algebra M,*
   (b) *An unbounded self-adjoint operator A affiliated to M,*
   (c) *A completely positive unital map $\psi : B_1 \to M$,*
   (d) *A completely positive map $R : M \to B_2$,*

   so that the function f can be written as
$$f(Z) = \underset{\mathrm{id}_n}{\overset{R}{\otimes}} \left[\left(\underset{I_n}{\overset{A}{\otimes}} - \underset{\mathrm{id}_n}{\overset{\psi}{\otimes}} (Z)\right)^{-1}\right].$$

Proof. Without loss of generality, all C^*-algebras will be assumed to be given as unitally contained subalgebras of bounded operators on a Hilbert space for the duration of the proof.

We adopt the technique used in the proof of a general Nevanlinna types theorem as in [2, 9].

Let f be as in the statement of the Theorem. By concretely realizing Theorem 2.1, we can instantiate a Herglotz function h which satisfies $ih((Z+i)^{-1}(Z-i)) = f(Z) - T$ for some self-adjoint T. By Theorem 2.1, h can be written concretely as

$$h(\Lambda) = \underset{I}{\overset{V^*}{\otimes}} \left(\underset{I}{\overset{L}{\otimes}} - \Lambda\right)^{-1} \left(\underset{I}{\overset{L}{\otimes}} + \Lambda\right) \underset{I}{\overset{V}{\otimes}} .$$

(Here we have concretely written $R(x) = V^*xV$ and used a resolvent of the form $(L-X)^{-1}(L+X)$ instead of $(1-UX)^{-1}(1+UX)$ to agree with [2, 1]. However, L is still a unitary. If fact, the algebra will show that $L = U^*$. Here, we use the word concrete to emphasize that we are treating everything as an operator.)

Let

$$f(Z)-\overset{T}{\underset{I}{\otimes}} = i\overset{V^*}{\underset{I}{\otimes}}\left(\overset{L}{\underset{I}{\otimes}} - (Z+i)^{-1}(Z-i)\right)^{-1}\left(\overset{L}{\underset{I}{\otimes}} + (Z+i)^{-1}(Z-i)\right)\overset{V}{\underset{I}{\otimes}}.$$

One can show as an elementary exercise in the spectral theorem that every vector of the form Vw is in the domain of the normal inverse $(1-L)^{-1}$. That is, $1-L$ is a normal operator, so its inverse is well-defined on the orthogonal complement of its kernel.) Notably, this reduces to an exercise in measure theory and manipulation of classical Herglotz integrals.

Lemma 2.3. *Any vector of the form Vw is in the domain of the normal inverse of $(1-L)^{-1}$. Namely, the range of V is perpendicular to the kernel of $1-L$.*

Proof. Consider our function

$$f(Z)-\overset{T}{\underset{I}{\otimes}} = i\overset{V^*}{\underset{I}{\otimes}}\left(\overset{L}{\underset{I}{\otimes}} - (Z+i)^{-1}(Z-i)\right)^{-1}\left(\overset{L}{\underset{I}{\otimes}} + (Z+i)^{-1}(Z-i)\right)\overset{V}{\underset{I}{\otimes}}.$$

Evaluate at $Z = is$.

$$f(is) - T = iV^*\left(L - (is+i)^{-1}(is-i)\right)^{-1}\left(L + (is+i)^{-1}(is-i)\right)V.$$

So, since L is unitary and thus normal, evaluating $w^*(f(is)-T)w$ gives, via the the spectral theorem,

$$\begin{aligned} w^*(f(is)-T)w &= iw^*V^*\left(L - (is+i)^{-1}(is-i)\right)^{-1}\left(L + (is+i)^{-1}(is-i)\right)Vw \\ &= i\int_{\mathbb{T}} \frac{\omega + (is+i)^{-1}(is-i)}{\omega - (is+i)^{-1}(is-i)} d\mu_{Vw}(\omega) \\ &= i\int_{\mathbb{T}} \frac{\omega(s+1) + (s-1)}{\omega(s+1) - (s-1)} d\mu_{Vw}(\omega). \end{aligned}$$

Note that the condition

$$\liminf_{\substack{s\to+\infty \\ s\in\mathbb{R}}} |isf(is)| < \infty.$$

implies *a fortiori* that

$$\liminf_{\substack{s\to+\infty \\ s\in\mathbb{R}}} s\mathrm{Im}\ f(is) < \infty.$$

So, consider

$$\begin{aligned} s\mathrm{Im}\ w^* f(is)w &= s\mathrm{Im}\ w^*(f(is) - T)w \\ &= s\mathrm{Im}\ i \int_{\mathbb{T}} \frac{\omega(s+1) + (s-1)}{\omega(s+1) - (s-1)} d\mu_{Vw}(\omega) \\ &= \int_{\mathbb{T}} \frac{s^2}{s^2 + 1 - (s^2-1)\mathrm{Re}\ \omega} d\mu_{Vw}(\omega). \end{aligned}$$

As s goes to infinity, noting that the integrand is monotone increasing in s, by monotone convergence theorem

$$\int_{\mathbb{T}} \frac{1}{1 - \mathrm{Re}\ \omega} d\mu_{Vw}(\omega) = \liminf_{s\to\infty} s\mathrm{Im}\ w^* f(is)w < \infty.$$

Since

$$\int_{\mathbb{T}} \frac{1}{1 - \mathrm{Re}\ \omega} d\mu_{Vw}(\omega) = \int_{\mathbb{T}} \frac{2}{|1-\omega|^2} d\mu_{Vw}(\omega),$$

we are done, because Vw is the domain of $f(L)$ if and only if $|f|^2$ is integrable with respect to $d\mu_{Vw}$. $\square$

Straightforward algebra gives

$$\begin{aligned} f(Z) - \overset{T}{\underset{I}{\otimes}} &= i \overset{V^*}{\underset{I}{\otimes}} \left(\overset{L}{\underset{I}{\otimes}} - (Z+i)^{-1}(Z-i)\right)^{-1} \left(\overset{L}{\underset{I}{\otimes}} + (Z+i)^{-1}(Z-i)\right) \overset{V}{\underset{I}{\otimes}} \\ &= i \overset{V^*}{\underset{I}{\otimes}} \left((Z+i) \overset{L}{\underset{I}{\otimes}} - (Z-i)\right)^{-1} \left((Z+i) \overset{L}{\underset{I}{\otimes}} + (Z-i)\right) \overset{V}{\underset{I}{\otimes}} \\ &= i \overset{V^*}{\underset{I}{\otimes}} \left(Z \overset{L-I}{\underset{I}{\otimes}} + i \overset{L+I}{\underset{I}{\otimes}} \right)^{-1} \left(Z \overset{L+I}{\underset{I}{\otimes}} - i \overset{L-I}{\underset{I}{\otimes}} \right) \overset{V}{\underset{I}{\otimes}}. \end{aligned}$$

Decompose L into blocks acting on $\ker 1 - L$ and $\ker(1-L)^{\perp}$ as

$$L = \left[\begin{smallmatrix} 1 & 0 \\ 0 & L_0 \end{smallmatrix}\right]$$

so that $\ker 1 - L_0$ is trivial. Multiply through on the left by $I = \left[\begin{smallmatrix} 1 & 0 \\ 0 & (1-L_0)^{-1} \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1-L_0 \end{smallmatrix}\right]$. We get

$$
\begin{aligned}
& i \overset{V^*}{\underset{I}{\otimes}} \overset{\left[\begin{smallmatrix} 1 & 0 \\ 0 & (1-L_0)^{-1} \end{smallmatrix}\right]}{\underset{I}{\otimes}} \overset{\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1-L_0 \end{smallmatrix}\right]}{\underset{I}{\otimes}} \left(Z \overset{(L-I)}{\underset{I}{\otimes}} + \overset{i(L+I)}{\underset{I}{\otimes}} \right)^{-1} \left(Z \overset{(L+I)}{\underset{I}{\otimes}} - \overset{i(L-i)}{\underset{I}{\otimes}} \right) \overset{V}{\underset{I}{\otimes}} \\
&= i \overset{V^*}{\underset{I}{\otimes}} \overset{\left[\begin{smallmatrix} 1 & 0 \\ 0 & (1-L_0)^{-1} \end{smallmatrix}\right]}{\underset{I}{\otimes}} \left(Z \overset{(L-I)}{\underset{I}{\otimes}} \overset{\left[\begin{smallmatrix} 1 & 0 \\ 0 & (1-L_0)^{-1} \end{smallmatrix}\right]}{\underset{I}{\otimes}} + \overset{i(L+I)}{\underset{I}{\otimes}} \overset{\left[\begin{smallmatrix} 1 & 0 \\ 0 & (1-L_0)^{-1} \end{smallmatrix}\right]}{\underset{I}{\otimes}} \right)^{-1} \\
&\qquad \times \left(Z \overset{(L+I)}{\underset{I}{\otimes}} - \overset{i(L-i)}{\underset{I}{\otimes}} \right) \overset{V}{\underset{I}{\otimes}} \\
&= i \overset{V^*}{\underset{I}{\otimes}} \overset{\left[\begin{smallmatrix} 1 & 0 \\ 0 & (1-L_0)^{-1} \end{smallmatrix}\right]}{\underset{I}{\otimes}} \left(Z \overset{\left[\begin{smallmatrix} 0 & 0 \\ 0 & -1 \end{smallmatrix}\right]}{\underset{I}{\otimes}} + \overset{i\left[\begin{smallmatrix} 2 & 0 \\ 0 & (1+L_0)(1-L_0)^{-1} \end{smallmatrix}\right]}{\underset{I}{\otimes}} \right)^{-1} \left(Z \overset{(L+I)}{\underset{I}{\otimes}} - \overset{i(L-I)}{\underset{I}{\otimes}} \right) \overset{V}{\underset{I}{\otimes}} \\
&= i \overset{V^*}{\underset{I}{\otimes}} \overset{\left[\begin{smallmatrix} 1 & 0 \\ 0 & (1-L_0)^{-1} \end{smallmatrix}\right]}{\underset{I}{\otimes}} \left(Z \overset{\left[\begin{smallmatrix} 0 & 0 \\ 0 & -1 \end{smallmatrix}\right]}{\underset{I}{\otimes}} + \overset{i\left[\begin{smallmatrix} 2 & 0 \\ 0 & (1+L_0)(1-L_0)^{-1} \end{smallmatrix}\right]}{\underset{I}{\otimes}} \right)^{-1} \\
&\qquad \times \left(Z \overset{\left[\begin{smallmatrix} 2 & 0 \\ 0 & L_0+1 \end{smallmatrix}\right]}{\underset{I}{\otimes}} - \overset{i\left[\begin{smallmatrix} 0 & 0 \\ 0 & L_0-I \end{smallmatrix}\right]}{\underset{I}{\otimes}} \right) \overset{V}{\underset{I}{\otimes}}
\end{aligned}
$$

The operator

$$A = i \frac{1 + L_0}{1 - L_0}$$

is a densely defined self-adjoint unbounded operator since L_0 has no kernel.

Since we are only interested in $\overset{V^*}{\underset{I}{\otimes}} M(Z) \overset{V}{\underset{I}{\otimes}}$, (where $M(Z)$ is the apparently unwieldy quantity between $\overset{V^*}{\underset{I}{\otimes}}$ and $\overset{V}{\underset{I}{\otimes}}$) the upper triangular form of

$$\left(Z \overset{\left[\begin{smallmatrix} 0 & 0 \\ 0 & -1 \end{smallmatrix}\right]}{\underset{I}{\otimes}} + \overset{i\left[\begin{smallmatrix} 2 & 0 \\ 0 & (1+L_0)(1-L_0)^{-1} \end{smallmatrix}\right]}{\underset{I}{\otimes}} \right)^{-1}$$

and the structure of V, namely that V is perpendicular to the kernel of $1 - L$, gives that the relevant operator is the (2,2) block. Then compress Z to

$$\overset{\psi}{\underset{\mathrm{id}}{\otimes}} (Z) = Z_\psi = \overset{P}{\underset{I}{\otimes}} Z \overset{P^*}{\underset{I}{\otimes}}$$

where P is the projection onto the perp of the kernel of $I - L$. Then our resolvent has the form

$$
\begin{aligned}
f(Z) - \underset{I}{\overset{T}{\otimes}} &= \underset{I}{\overset{V^*(I-L_0)^{-1}}{\otimes}} \left(\underset{I}{\overset{A}{\otimes}} - Z_\psi \right)^{-1} \left(iZ_\psi \underset{I}{\overset{(L_0+I)}{\otimes}} + \underset{I}{\overset{(L_0-I)}{\otimes}} \right) \underset{I}{\overset{V}{\otimes}} \\
&= \underset{I}{\overset{V^*(I-L_0)^{-1}}{\otimes}} \left(\underset{I}{\overset{A}{\otimes}} - Z_\psi \right)^{-1} \left(iZ_\psi \underset{I}{\overset{(L_0+I)(I-L_0)^{-1}}{\otimes}} + I \right) \underset{I}{\overset{(I-L_0)V}{\otimes}} \\
&= \underset{I}{\overset{V^*(I-L_0)^{-1}}{\otimes}} \left(\underset{I}{\overset{A}{\otimes}} - Z_\psi \right)^{-1} \left(Z_\psi \underset{I}{\overset{A}{\otimes}} + I \right) \underset{I}{\overset{(I-L_0)V}{\otimes}} \\
&= \underset{I}{\overset{V^*(I-L_0)^{-1}}{\otimes}} \left(\underset{I}{\overset{A}{\otimes}} - Z_\psi \right)^{-1} \left(Z_\psi \underset{I}{\overset{A}{\otimes}} - \underset{I}{\overset{A^2}{\otimes}} + \underset{I}{\overset{A^2}{\otimes}} + I \right) \underset{I}{\overset{(I-L_0)V}{\otimes}} \\
&= \underset{I}{\overset{V^*(I-L_0)^{-1}}{\otimes}} \left(\underset{I}{\overset{A}{\otimes}} - Z_\psi \right)^{-1} \left[\left(Z_\psi - \underset{I}{\overset{A}{\otimes}} \right) \underset{I}{\overset{A}{\otimes}} + \left(\underset{I}{\overset{A^2}{\otimes}} + I \right) \right] \underset{I}{\overset{(I-L_0)V}{\otimes}} \\
&= \underset{I}{\overset{V^*(I-L_0)^{-1}A(I-L_0)V}{\otimes}} + \underset{I}{\overset{V(I-L_0)^{-1}}{\otimes}} \left(\underset{I}{\overset{A}{\otimes}} - Z_\psi \right)^{-1} \left(\underset{I}{\overset{A^2}{\otimes}} + I \right) \underset{I}{\overset{(I-L_0)V}{\otimes}} \\
&= \underset{I}{\overset{V^*AV}{\otimes}} + \underset{I}{\overset{V^*(I-L_0)^{-1}}{\otimes}} \left(\underset{I}{\overset{A}{\otimes}} - Z_\psi \right)^{-1} \left(\underset{I}{\overset{A^2}{\otimes}} + I \right) \underset{I}{\overset{(I-L_0)V}{\otimes}} \\
&= \underset{I}{\overset{V^*AV}{\otimes}} + \underset{I}{\overset{V^*(I-L_0)^{-1}}{\otimes}} \left(\underset{I}{\overset{A}{\otimes}} - Z_\psi \right)^{-1} \underset{I}{\overset{(I-L_0^*)^{-1}V}{\otimes}} .
\end{aligned}
$$

Now, the asymptotic condition implies that the constant terms must vanish, so

$$f(Z) = \underset{I}{\overset{V^*(I-L)^{-1}}{\otimes}} \left(\underset{I}{\overset{A}{\otimes}} - Z_\psi \right)^{-1} \underset{I}{\overset{(I-L^*)^{-1}V}{\otimes}} .$$

Defining a new $R(x) = V^*(I-L)^{-1}x(I-L^*)^{-1}V$ and ψ to be as above, we are done with our construction.

□

The main result Theorem 1.5 now follows by noting that

$$E(-\psi(Z)^{-1}) = \lim_{\substack{s \to +\infty \\ s \in \mathbb{R}}} sf(sZ) = -Z^{-1}.$$

So we see that

$$E(\psi(Z)^{-1}) = Z^{-1}.$$

One can show that

$$E(\psi(H_1)\ldots\psi(H_k)) = H_1 \ldots H_k,$$

by taking $Z = I_{k+1} - H$, where H has $H_1, \ldots, H_k$ on the upper diagonal.

Lemma 2.4.

$$E(\psi(H_1)\ldots\psi(H_k)) = H_1 \ldots H_k.$$

Proof. Note

$$(I-H)^{-1} = \sum_{i=0}^{\infty} H^i,$$

and

$$\underset{\mathrm{id}}{\overset{E}{\otimes}} \left(\underset{\mathrm{id}}{\overset{\psi}{\otimes}} (I-H)^{-1} \right) = \sum_{i=0}^{\infty} \underset{\mathrm{id}}{\overset{E}{\otimes}} \left(\left[\underset{\mathrm{id}}{\overset{\psi}{\otimes}} (H) \right]^i \right).$$

So we obtain that $\underset{\mathrm{id}}{\overset{E}{\otimes}} \left(\left[\underset{\mathrm{id}}{\overset{\psi}{\otimes}} (H) \right]^k \right) = H^k$. Evaluating at

$$H = \begin{pmatrix} 0 & H_1 & & \\ & \ddots & \ddots & \\ & & 0 & H_k \\ & & & 0 \end{pmatrix}$$

and looking at the block $(1, k+1)$ entry gives the claim. $\square$

Now, we obtain the necessary homomorphic properties by letting $\hat{B}$ be the algebra generated the range of ψ, so we are done.

3. Proof of the homomorphic Tomiyama's theorem

We now prove our analog of Tomiyama's theorem for homomorphic conditional expectations. We restate the homomorphic Tomiyama's theorem here for clarity.

Proposition 3.1 (Homomorphic Tomiyama's theorem)**.** *If $E : M \to B$ is a homomorphic conditional expectation over B, then for all $b_1, b_2 \in \hat{B}$,*

$$E(b_1 m b_2) = E(b_1)E(m)E(b_2).$$

Proof. Our proof follows Tomiyama's original method in [10].

Suppose E is a homomorphic conditional expectation. Without loss of generality, assume all C^*-algebras involved are weakly closed. (That is, we can extend everything with the Stinespring theorem.) It is sufficient to show that for any projection e in $\hat{B}$ we have that $E(em) = E(e)E(m)$.

Let e be a projection in $\hat{B}$. Let x be a positive element of M. Note that

$$E(exe) \leq E(e\|x\|e) = \|x\|E(e).$$

So, since $E(e)$ is a projection by the homomorphic property,

$$E(exe) = E(e)E(exe)E(e).$$

Now with a general element $m \in M$,

$$0 \leq \begin{pmatrix} 1 & 0 \\ 0 & 1-E(e) \end{pmatrix} E \begin{pmatrix} 1 & me \\ em^* & emm^*e \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1-E(e) \end{pmatrix} = \begin{pmatrix} 1 & E(m^*e)(1-E(e)) \\ (1-E(e))E(em) & 0 \end{pmatrix},$$

and so

$$(1 - E(e))E(em) = 0.$$

Thus, $E(em) = E(e)E(em) = E(e)E(em) + E(e)E((1-e)m) = E(e)E(m)$ and we are done. $\square$

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