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A quasilinear fully parabolic chemotaxis system with indirect signal production and logistic source ☆

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ABSTRACT

In this paper we study the quasilinear fully parabolic chemotaxis system with indirect signal production and logistic source: $u_t = \nabla \cdot (D(u)\nabla u - S(u)\nabla v) + f(u)$, $v_t = \Delta v - a_1 v + b_1 w$, $w_t = \Delta w - a_2 w + b_2 u$, under homogeneous Neumann boundary conditions in a bounded and smooth domain $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$, where $a_i, b_i > 0$ $(i = 1, 2)$, $D, S \in C^2([0, \infty))$ and $f : \mathbb{R} \to \mathbb{R}$ is a smooth function generalizing the logistic source $f(s) = b - \mu s^r$ for all $s \geq 0$ with $b \geq 0$, $\mu > 0$ and $r \geq 1$. We obtain the global boundedness of solutions in four cases: (i) the self-diffusion dominates the cross-diffusion; (ii) the logistic source suppresses the cross-diffusion; (iii) the logistic dampening balances the cross-diffusion with $\mu > 0$ suitably large; (iv) the self-diffusion and the logistic source both balance the cross-diffusion to some extent with $\mu > 0$ arbitrary. As corollaries, we also consider the global boundedness of solutions for the quasilinear attraction-repulsion chemotaxis model with logistic source: $\tilde{u}_t = \nabla \cdot (D(\tilde{u})\nabla \tilde{u}) - \chi \nabla \cdot (\tilde{u}\nabla z) + \xi \nabla \cdot (\tilde{u}\nabla \tilde{w}) + f(\tilde{u})$, $z_t = \Delta z - \rho z + \eta \tilde{u}$, $\tilde{w}_t = \Delta \tilde{w} - \delta \tilde{w} + \gamma \tilde{u}$, where $\chi, \eta, \xi, \gamma, \rho, \delta > 0$.

1. Introduction

This paper is concerned with the quasilinear fully parabolic chemotaxis system with indirect signal production and logistic source:

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u - S(u)\nabla v) + f(u), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - a_1 v + b_1 w, & x \in \Omega,\ t > 0, \\ w_t = \Delta w - a_2 w + b_2 u, & x \in \Omega,\ t > 0, \\ \partial_\nu u = \partial_\nu v = \partial_\nu w = 0, & x \in \partial\Omega,\ t > 0, \\ (u(x,0), v(x,0), w(x,0)) = (u_0(x), v_0(x), w_0(x)), & x \in \Omega \end{cases} \tag{1.1}$$

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in a bounded and smooth domain $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$, where a_i and b_i $(i = 1, 2)$ are positive constants, ∂_ν denotes the outer normal derivative on $\partial\Omega$, and the initial data $(u_0, v_0, w_0) \in C^\omega(\bar{\Omega}) \times [W^{1,\infty}(\Omega)]^2$ with $0 < \omega < 1$ is nonnegative. The system (1.1) describes a biological process, known as chemotaxis, in which cells (with density u) migrate towards higher concentrations of a chemical signal v. This kind of aggregation of cells is reflected by the chemoattractive cross-diffusion term $-\nabla \cdot (S(u)\nabla v)$ with the density-dependent sensitivity function $S(u)$. But unlike the classical Keller-Segel model [12]

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + f(u), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0 \end{cases} \tag{1.2}$$

with $\chi > 0$ and $f \equiv 0$, the signal production mechanism in (1.1) is indirect [6,8,26,27], that is, the chemoattractant v is not produced by cells directly, but is governed by the quantity w arising from u. In addition, the term $\nabla \cdot (D(u)\nabla u)$ in (1.1) stands for the random self-diffusion of cells, and the inhomogeneity $f(u)$ represents the cell kinetic mechanism.

With regard to the chemotaxis model, the properties of solutions (e.g. the global existence or the finite-time blow-up) depend on which of the cross-diffusion and the self-diffusion plays a dominated role in the system. In (1.2) with $f \equiv 0$, this can be characterized by the spatial dimension. More precisely, the solutions are always globally bounded for $n = 1$ [22]; when $n = 2$, a critical mass phenomenon occurs in the radially symmetric setting, meaning that the solutions remain globally bounded if $\int_\Omega u_0 < 8\pi/\chi$ [20] and there exist smooth initial data with $\int_\Omega u_0 > 8\pi/\chi$ such that the corresponding solutions blow up in finite time [7,19]; whereas in higher-dimensional balls, the finite-time blow-up solutions are constructed with any small mass [35]. Lately, a novel type of critical mass phenomenon for *infinite-time* blow-up of solutions has been identified for a parabolic-elliptic-ODE system with indirect signal production [26]. This makes us be aware that just the indirect signal production mechanism causes some unusual features of solutions concerning the blow-up or global existence, and can even lead to a distinct competition between the self-diffusion and the cross-diffusion. As expected, Fujie and Senba [6] proved for (1.1) with $D \equiv 1$, $S(s) = \chi s$, $f \equiv 0$ and $a_i = b_i = 1$ $(i = 1, 2)$ that the solutions are globally bounded if $n \leq 3$, or $\int_\Omega u_0 < (8\pi)^2/\chi$ in the four-dimensional and radially symmetric case, whence it is plausible to think that here $n = 4$ is a threshold for distinguishing the blow-up or global existence of solutions. Furthermore, for the general quasilinear system (1.1) without growth source, when $D, S \in C^2([0, \infty))$ generalize the prototypes $D(s) = (1 + s)^{-\alpha}$ and $S(s) = s(1 + s)^{\beta - 1}$ with $\alpha, \beta \in \mathbb{R}$, the global boundedness of solutions was determined under the condition $\alpha + \beta < \min\{1 + 2/n, 4/n\}$ [4]; whereas for the direct signal production system

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u - S(u)\nabla v) + f(u), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0 \end{cases} \tag{1.3}$$

with D and S prescribed as above and $f \equiv 0$, it was proved in [25,9] that the solutions are globally bounded if $\alpha + \beta < 2/n$.

The indirect signal production mechanism can also give rise to different interactions of the cross-diffusion and the logistic source. It is well-known that an appropriate logistic dampening can prevent blow-up of solutions. Indeed, for $f(s) = \mu s(1 - s)$ with $\mu > 0$, it was shown in [23] that the solutions of (1.2) are globally bounded in dimension $n = 2$ regardless of the size of μ. Furthermore, if $n \geq 3$ and Ω is convex, Winkler [33] indicated the global boundedness of solutions for general f satisfying $f(s) \leq b - \mu s^2$ for all $s \geq 0$ with $b \geq 0$ and $\mu > 0$ properly large. Also, Zheng [37] obtained the global boundedness of solutions for $f(s) = bs - \mu s^r$ with $b \geq 0$, $\mu > 0$ and $r > 2$. Different from these results, it was proved in [8] for the indirect signal production chemotaxis system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\nabla v) + \mu u(1-u), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + w, & x \in \Omega,\ t > 0, \\ \varepsilon w_t + \delta w = u, & x \in \Omega,\ t > 0 \end{cases}$$

that the solutions are always globally bounded in dimension $n = 3$ for arbitrary $\mu > 0$, and for (1.1) with $D \equiv 1$ and $S(s) = \chi s$, equivalent to an attraction-repulsion model, it can be found from [29] that it is possible to ensure the global boundedness of solutions even though the logistic exponent is smaller than two. Now turn to the quasilinear problem with logistic source. For the quasilinear fully parabolic Keller-Segel system (1.3), considerable efforts have been made to investigate the global boundedness of solutions due to the inhibition of self-diffusion and logistic dampening to the cross-diffusion. See e.g. [3,37,38,30] and references therein. Assume for simplicity that $D(s) = (1+s)^{-\alpha}$, $S(s) = s(1+s)^{\beta-1}$ for $s \geq 0$ with $\alpha, \beta \in \mathbb{R}$, and $f(s) = bs - \mu s^r$ for $s \geq 0$ with $b \geq 0$, $\mu > 0$ and $r > 1$. Then the solutions of (1.3) are globally bounded if

- $\alpha + \beta < 1 + \min\left\{\frac{(2r-n-2)_+}{nr}, \frac{1}{n}\right\} - \min\left\{\frac{(n-2)_+}{n}, \frac{(n+2-2r)_+}{n+2}\right\}$ [37,38,30], or (1.4)
- $r > \frac{n+2}{n+4}(2\beta+\alpha+1) \geq \frac{n+2}{n+4}(\beta + 2 + \frac{2}{n} - \frac{n+2}{nr})$ and $\frac{n+2}{2} \leq r < n+2$ [30], or
  $r > 2\beta + \alpha - 1 \geq \beta + \frac{1}{n}$ and $r \geq n+2$ [30], or $r > \beta + 1$ [3,37], or (1.5)
- $r = \beta + 1$ with $\mu > 0$ large enough [37]. (1.6)

The aim of this paper is to extend the study for (1.3) to the indirect signal production system (1.1). Our results on global boundedness of solutions, compared with those for (1.3), actually exhibit different interactions between the self-diffusion and the cross-diffusion/between the logistic source and the cross-diffusion (see Remark 1 (ii) below). And the existing methods for the direct signal production system don't seem to be applicable to (1.1). Therefore, the proof adopted in the present paper involves some new ideas and techniques.

Main results. To precisely formulate the main results of this paper, we suppose that the diffusivity $D \in C^2([0,\infty))$ and the density-dependent sensitivity $S \in C^2([0,\infty))$ with $S(0) = 0$ satisfy

$$D(s) \geq a_0(s+1)^{-\alpha}, \quad 0 \leq S(s) \leq b_0(s+1)^{\beta} \quad \text{for all } s \geq 0, \tag{1.7}$$

where $a_0, b_0 > 0$ and $\alpha, \beta \in \mathbb{R}$ are constants. Also, the logistic source $f : \mathbb{R} \to \mathbb{R}$ is smooth and fulfills $f(0) \geq 0$ as well as

$$f(s) \leq b - \mu s^r \quad \text{for all } s \geq 0 \text{ with some } b \geq 0,\ \mu > 0 \text{ and } r \geq 1. \tag{1.8}$$

Under these hypotheses, we shall firstly assert the global boundedness of solutions when the cross-diffusion is dominated by the self-diffusion and the logistic source, respectively.

Theorem 1. *Let f be given as in (1.8). If D and S satisfy (1.7) with*

$$\alpha + \beta < 1 + \frac{2}{n} - \min\left\{\frac{(n-2)_+}{n}, \frac{(n+2-2r)_+}{n+2}\right\}, \tag{1.9}$$

then for any nonnegative $(u_0, v_0, w_0) \in C^{\omega}(\bar{\Omega}) \times [W^{1,\infty}(\Omega)]^2$ with $0 < \omega < 1$, the problem (1.1) possesses a globally bounded and classical solution (u, v, w).

Theorem 2. *Assume that D and S fulfill (1.7). If (1.8) is valid for f with*

$$r > \beta + \min\left\{\frac{(n-2)_+}{n}, \frac{(n+2-2\beta)_+}{n+4}\right\}, \tag{1.10}$$

then the solution (u, v, w) of (1.1), corresponding to the nonnegative initial data $(u_0, v_0, w_0) \in C^\omega(\bar{\Omega}) \times [W^{1,\infty}(\Omega)]^2$ with $0 < \omega < 1$, is global and remains bounded in time.

Remark 1. (i) The conditions (1.9) and (1.10) can be specifically described as

$$\alpha + \beta < \begin{cases} 1 + \dfrac{2}{n} & \text{when } n \le 2, \text{ or } n \ge 3 \text{ and } r \ge \dfrac{n+2}{2}, \\ \dfrac{4}{n} & \text{when } n \ge 3 \text{ and } 1 \le r \le \dfrac{n+2}{n}, \\ \dfrac{2}{n} + \dfrac{2r}{n+2} & \text{when } \dfrac{n+2}{n} \le r < \dfrac{n+2}{2} \end{cases} \tag{1.11}$$

and

$$r > \begin{cases} \beta & \text{when } n \le 2, \text{ or } n \ge 3 \text{ and } \beta \ge \dfrac{n+2}{2}, \\ \beta + \dfrac{n-2}{n} & \text{when } n \ge 3 \text{ and } \beta \le \dfrac{4}{n}, \\ \dfrac{(\beta+1)(n+2)}{n+4} & \text{when } \dfrac{4}{n} \le \beta < \dfrac{n+2}{2}. \end{cases} \tag{1.12}$$

Recall that the solutions of the quasilinear system (1.1) without logistic source are globally bounded if $\alpha + \beta < \min\{1 + 2/n, 4/n\}$ [4]. Also, it is clear that $2/n + 2r/(n+2) > 4/n$ when $(n+2)/n < r \le (\beta+1)(n+2)/(n+4) < (n+2)/2$. Therefore, due to the contribution of logistic kinetics, a larger range for $\alpha + \beta$ is allowed in (1.1) to ensure the global boundedness of solutions.

(ii) The restrictions of the self-diffusion and the logistic source in (1.1), in contrast to (1.3), are weakened since (1.4) and (1.5) are stronger than (1.9) and (1.10), respectively. Indeed, the former is trivial, while the latter (or equivalently, (1.5) $\Rightarrow$ (1.12)) can be easily derived by just noticing that if

$$r > \frac{n+2}{n+4}\left(\beta + 2 + \frac{2}{n} - \frac{n+2}{nr}\right) \quad \text{and} \quad \frac{n+2}{2} \le r < n+2,$$

then $\beta < n+2$, and so

$$r > \frac{n+2}{n+4}\left(\beta + 2 + \frac{2}{n} - \frac{n+2}{nr}\right) > \beta.$$

Secondly, we will see that if the logistic dampening balances the cross-diffusion with the coefficient $\mu > 0$ suitably large, then the solutions are globally bounded.

Theorem 3. *Suppose that D and S satisfy (1.7), and that (1.8) holds for f with*

$$r = \beta + \min\left\{\frac{(n-2)_+}{n}, \frac{(n+2-2\beta)_+}{n+4}\right\} \quad \text{and} \quad \frac{4}{n} \le \beta < \frac{n+2}{2}. \tag{1.13}$$

Then for any $m_* > 0$, there exists $\mu_* > 0$, relying on m_*, b_0, a_i, b_i $(i = 1, 2)$, b, r, n and Ω, such that for any nonnegative $(u_0, v_0, w_0) \in C^\omega(\bar{\Omega}) \times [W^{1,\infty}(\Omega)]^2$ $(0 < \omega < 1)$ fulfilling

(a) $\|u_0\|_{L^r(\Omega)} \le m_*$ *and* $\|w_0\|_{W^{1,\infty}(\Omega)} \le m_*$ *with* $r \le 2$, *or*
(b) $\|u_0\|_{L^1(\Omega)} \le m_*$ *and* $\|w_0\|_{W^{2,r}(\Omega)} \le m_*$ *with* $r > 2$, $w_0 \in W^{2,r}(\Omega)$ *and* $\partial_\nu w_0|_{\partial\Omega} = 0$,

the solution of (1.1) *is globally bounded provided that* $\mu > \mu_*$.

Remark 2. Note that $4/n \le \beta < (n+2)/2$ implies here $r = (\beta+1)(n+2)/(n+4) \in [(n+2)/n, (n+2)/2)$ with $n \ge 3$.

In the case that the self-diffusion and the logistic source both balance the cross-diffusion, viz.

$$\alpha + \beta = 1 + \frac{2}{n} - \min\left\{\frac{(n-2)_+}{n}, \frac{(n+2-2r)_+}{n+2}\right\} \quad \text{and}$$
$$r = \beta + \min\left\{\frac{(n-2)_+}{n}, \frac{(n+2-2\beta)_+}{n+4}\right\},$$

we find that with the additional restriction $\beta = 4/n = 1$ (and so $r = 3/2$ and $\alpha = 0$), the solutions of (1.1) are globally bounded regardless of the size of parameter $\mu > 0$. This reads as follows.

Theorem 4. *Let* $n = 4$. *Assume that* D *and* S *obey* (1.7) *with* $\alpha = 0$ *and* $\beta = 1$, *and that* f *satisfies* (1.8) *with* $r = 3/2$. *Then for any nonnegative initial data,* (1.1) *admits a globally bounded solution.*

Attraction-repulsion problem. As a by-product of Theorems 1–4, we can further consider the following quasilinear fully parabolic attraction-repulsion chemotaxis model [18,24]

$$\begin{cases} \tilde{u}_t = \nabla\cdot(D(\tilde{u})\nabla\tilde{u}) - \chi\nabla\cdot(\tilde{u}\nabla z) + \xi\nabla\cdot(\tilde{u}\nabla\tilde{w}) + f(\tilde{u}), & x \in \Omega,\ t > 0, \\ z_t = \Delta z - \rho z + \eta\tilde{u}, & x \in \Omega,\ t > 0, \\ \tilde{w}_t = \Delta\tilde{w} - \delta\tilde{w} + \gamma\tilde{u}, & x \in \Omega,\ t > 0, \\ \partial_\nu\tilde{u} = \partial_\nu z = \partial_\nu\tilde{w} = 0, & x \in \partial\Omega,\ t > 0, \\ (\tilde{u}(x,0), z(x,0), \tilde{w}(x,0)) = (\tilde{u}_0(x), z_0(x), \tilde{w}_0(x)), & x \in \Omega \end{cases} \tag{1.14}$$

in the case that the repulsion cancels the attraction (i.e. $\chi\eta = \xi\gamma$), where χ, η, ξ, γ, ρ and δ are positive constants. Indeed, define

$$\tilde{v}(x,t) := \chi z(x,t) - \xi\tilde{w}(x,t).$$

Then $(\tilde{u}, \tilde{v}, \tilde{w})$ solves the system like (1.1)

$$\begin{cases} \tilde{u}_t = \nabla\cdot(D(\tilde{u})\nabla\tilde{u} - \tilde{u}\nabla\tilde{v}) + f(\tilde{u}), & x \in \Omega,\ t > 0, \\ \tilde{v}_t = \Delta\tilde{v} - \rho\tilde{v} + \xi(\delta-\rho)\tilde{w}, & x \in \Omega,\ t > 0, \\ \tilde{w}_t = \Delta\tilde{w} - \delta\tilde{w} + \gamma\tilde{u}, & x \in \Omega,\ t > 0 \end{cases} \tag{1.15}$$

since $\chi\eta = \xi\gamma$.

A striking feature of the attraction-repulsion chemotaxis model is the positive effect of repulsion on the global boundedness of solutions when the repulsion prevails over (i.e. $\xi\gamma - \chi\eta > 0$) or cancels (viz. $\xi\gamma - \chi\eta = 0$) the attraction, and related research can be found in [10,11,15–17]. For instance, under the case that $f \equiv 0$, for any initial data, the semilinear version of (1.14) (that is, $D \equiv 1$) has a unique globally bounded and classical solution if $n = 1$ [11,17], or $n = 2$ and $\xi\gamma \ge \chi\eta$ [10,15,16], or $n = 3$ with $\xi\gamma = \chi\eta$ [15], and thus any blow-up is excluded with the help of repulsion even if $n = 2$ or 3. Moreover, when $\xi\gamma = \chi\eta$, there is a critical mass phenomenon in dimension $n = 4$ [6], as the one of the two-dimensional Keller-Segel system (1.2) with $f \equiv 0$. So far, few works genuinely concern the contribution of repulsion in the general quasilinear system (1.14). In [4], assuming that $\xi\gamma = \chi\eta$, $f \equiv 0$ and $D(s) \geq c_0(s+1)^{M-1}$ for all $s \geq 0$ with $c_0 > 0$ and $M \in \mathbb{R}$, Ding and Wang established the boundedness for (1.14) under the subcritical condition $M > \max\{2 - 4/n, 1 - 2/n\}$. When the repulsion is absent from (1.14), it is known that the solutions are globally bounded if $M > 2 - 2/n$ (cf. [25,9]). Accordingly, the result of [4] shows that the repulsion in (1.14) actually benefits the boundedness of solutions with the restrictions to the random self-diffusion weakened.

The repulsion mechanism can also relax the growth restrictions of logistic type to ensure the global boundedness of solutions. On this point, very recently, Wang et al. [29] proved, for the problem (1.14) with $D \equiv 1$ and f satisfying (1.8), that when $\xi\gamma = \chi\eta$, the solution is globally bounded if

$$n \leq 3, \quad \text{or } r > r_n := \min\left\{\frac{n+2}{4}, \frac{n\sqrt{n^2+6n+17} - n^2 - 3n + 4}{4}\right\} \text{ with } n \geq 2. \tag{1.16}$$

Hence, owing to the effect of repulsion, the exponent r is allowed to take values less than 2 such that the solution remains uniformly bounded in time. The similar discussions for the semilinear parabolic-elliptic counterpart of (1.14) can be available [13]. Observe that in (1.16), $r > r_4 = 3/2$ when $n = 4$. By analogy with the boundedness results of (1.2) with logistic source, it can be expected for the semilinear version of (1.14) that when the repulsion cancels the attraction and f fulfills (1.8) with $r = 3/2$, the solutions are always globally bounded in dimension $n = 4$, just as the assertion given in the classical two-dimensional Keller-Segel model (1.2) with logistic source bearing quadratic degradation. This conjecture has been confirmed in [14] for the parabolic-elliptic problem. Unfortunately, for the fully parabolic system, it has to be left open there.

As the proof of Theorems 1–4 in Sections 3 and 4 will show, the results among them can easily be carried over to (1.15) even if the coefficient $\xi(\delta - \rho)$ in the second equation is nonpositive. Therefore, we immediately obtain the following global boundedness of solutions for the quasilinear system (1.14) with logistic source.

Corollary 1. *Let $\chi\eta = \xi\gamma$, and assume that D and f are restricted as in (1.7) and (1.8), respectively. If*

$$\alpha < \frac{2}{n} - \min\left\{\frac{(n-2)_+}{n}, \frac{(n+2-2r)_+}{n+2}\right\}, \quad \text{or } r > 1 + \min\left\{\frac{(n-2)_+}{n}, \frac{n}{n+4}\right\},$$

then for any nonnegative $(\tilde{u}_0, z_0, \tilde{w}_0) \in C^\omega(\bar{\Omega}) \times [W^{1,\infty}(\Omega)]^2$ with $0 < \omega < 1$, the problem (1.14) possesses a globally bounded and classical solution $(\tilde{u}, z, \tilde{w})$.

Remark 3. From this corollary, we get, for (1.14) with $D \equiv 1$ and f satisfying (1.8), that when $\chi\eta = \xi\gamma$, the solutions are globally bounded if $n \leq 3$, or $r > 2(n+2)/(n+4)$ and $n \geq 4$. This improves the result obtained in [29].

Corollary 2. *Under the conditions of Corollary 1, if*

$$r = \frac{2(n+2)}{n+4} \quad \text{with } n \geq 4,$$

then for any $m_* > 0$, there exists $\mu_* = \mu_*(m_*, \xi, \gamma, \rho, \delta, b, n, \Omega) > 0$ such that for any nonnegative $(\tilde{u}_0, z_0, \tilde{w}_0) \in C^\omega(\bar{\Omega}) \times [W^{1,\infty}(\Omega)]^2$ $(0 < \omega < 1)$ with $\|\tilde{u}_0\|_{L^r(\Omega)} \leq m_*$ and $\|\tilde{w}_0\|_{W^{1,\infty}(\Omega)} \leq m_*$, the solution of (1.14) is globally bounded provided that $\mu > \mu_*$.

Corollary 3. *Under the conditions of Corollary 1, if $\alpha = 0$ and $r = 3/2$ with $n = 4$, then for any nonnegative initial data, (1.14) has a globally bounded solution.*

Remark 4. This corollary entails for the semilinear version of (1.14) (i.e. $D \equiv 1$) that when $\chi\eta = \xi\gamma$ and f fulfills (1.8) with $r = 3/2$, the solutions are always globally bounded in dimension $n = 4$, and thereby solves the open problem left in [14].

Ideas of proof for Theorems 1–4. In order to obtain the global boundedness of solutions to a chemotaxis system, based upon the Moser-type iteration, it is sufficient to establish the L^p-boundedness of the density u for large p. An effective analysis used in the derivation for the L^p-bound of u is tracking the time evolution for the coupled energy of the density and the signal (cf. e.g. [25,33]). Unfortunately, such approach is limited in the present problem (1.1). Alternatively, we estimate the integral $\int_\Omega u^p$ separately by constructing an absorptive differential inequality for $t \mapsto \int_\Omega u^p$. Roughly, testing the first equation in (1.1) with $p(u+1)^{p-1}$ and integrating by parts, we get (Lemma 4.2)

$$\begin{aligned}\frac{d}{dt}\int_\Omega (u+1)^p \le -p(p-1)a_0\int_\Omega (u+1)^{p-\alpha-2}|\nabla u|^2 + \frac{p(p-1)b_0}{p+\beta-1}\int_\Omega (u+1)^{p+\beta-1}|\Delta v|\\ + p(b+\mu)\int_\Omega (u+1)^{p-1} - 2^{1-r}p\mu\int_\Omega (u+1)^{p+r-1}\\ =: -I_{s\text{-}d} + I_{c\text{-}d} + I_{l\text{-}s+} - I_{l\text{-}s-},\end{aligned}$$

where the integral involving the cross-diffusive term has been integrated by parts twice with all derivatives transferred onto v. Clearly, the ill-signed term $I_{l\text{-}s+}$ can be absorbed by $-I_{l\text{-}s-}$. As for $I_{c\text{-}d}$, it can be properly controlled by $-I_{s\text{-}d}$ when the self-diffusion prevails over the cross-diffusion (Lemma 4.3); by $-I_{l\text{-}s-}$ in the case that the logistic source suppresses or balances the cross-diffusion (Lemma 4.4 and Subsection 4.3); and by $-I_{s\text{-}d}$ if the self-diffusion and the logistic dampening both balance the cross-diffusion (Subsection 4.4). However, in these processes, an additional term related to the integral $\int_\Omega |\Delta v|^{p'}$ for some $p' > 1$ has occurred. To deal with it, the maximal Sobolev regularity for abstract differential equations with nonzero initial data (Lemma 2.2) plays a crucial role. For the non-flux linear parabolic problem $\zeta_t = \Delta\zeta - a\zeta + g$, we use the maximal Sobolev regularity to elaborately establish two kinds of spatio-temporal integral estimates for the principal part (Proposition 2.2 (i) (iii)), and spatial integral estimates of solutions (Proposition 2.2 (ii)). These preparations enable us on the one hand to show the L^q-boundedness of w with suitable $q \ge 1$ (Lemma 3.3), and on the other hand to establish the proper integral relations between Δv and u (Lemmas 3.4 and 3.5). Accordingly, the appropriate differential inequalities for $\int_\Omega (u+1)^p(\cdot,t)$ with the boundedness property (Lemmas 2.4 and 2.5) are obtained. We mention that the estimates in Proposition 2.2 (iii) have already been used to investigate the global boundedness of solutions resulting from the logistic dampening [3,36,37]. One of the novelty of our proof is that in considering the inhibition of self-diffusion to the cross-diffusion, we have developed an effective estimate as in Proposition 2.2 (i) to serve the current treatment. This method is universal, and is certainly applicable to the direct signal production system. Also, Proposition 2.2 (i) guarantees the boundedness of $\|\Delta v(\cdot,t)\|_{L^6(\Omega)}$ on some temporal average in dimension $n = 4$ (Lemma 3.5), a key for the proof of Theorem 4, which however cannot be achieved by a test procedure. In addition, with regard to Proposition 2.2 (ii), it provides the integral estimates of w that can reach the borderline case (Lemma 3.3 (ii)). This is essentially required by Theorem 3.

This paper is organized as follows. In Section 2, we provide some preliminary material, especially the regularity properties of solutions for the Neumann problem on the linear parabolic equations in (1.1), which are derived by the smoothing estimates of the Neumann heat semigroup and the maximal Sobolev regularity with nonzero initial data belonging to some real interpolation spaces. Besides, as a potential preparation, the boundedness features of solutions for some ordinary differential inequalities are also given in this section. Indeed, Section 2 is the cornerstone of our work. And then some crucial estimates for the proof of main results, such as the L^q-boundedness of w with some $q \geq 1$ and the proper integral relations between Δv and u, are established in Section 3. Finally, in Section 4, we prove the main results of this paper (Theorems 1–4).

2. Preliminaries

In this section, we will resort the L^p-L^q estimates of the Neumann heat semigroup [32, Theorem 1.3] and the maximal Sobolev regularity [2, Section 4 of Chapter III] to establish some *a priori* estimates of solutions for the Neumann problem related to the latter two parabolic equations in (1.1).

Firstly, the following regularity properties of solutions (see e.g. [34, Lemma 2.4]) can be derived by a straightforward application of the smoothing estimates for the Neumann heat semigroup. For clarifying the dependence of the constants therein, we prove it in detail.

Proposition 2.1. *Let $a > 0$, $\zeta_0 \in W^{1,\infty}(\Omega)$ and $g \in C^0(\bar{\Omega} \times [0,T))$ with $T \in (0,\infty]$. Suppose that $\zeta \in C^{2,1}(\bar{\Omega} \times (0,T)) \cap C^0(\bar{\Omega} \times [0,T))$ solves*

$$\begin{cases} \zeta_t = \Delta\zeta - a\zeta + g, & (x,t) \in \Omega \times (0,T), \\ \partial_\nu \zeta = 0, & (x,t) \in \partial\Omega \times (0,T), \\ \zeta(x,0) = \zeta_0(x), & x \in \Omega. \end{cases} \tag{2.1}$$

(i) *If $(n/2)(1/p - 1/q) < 1$ with $1 \leq p, q \leq \infty$, then there exists $C > 0$, relying on a, p, q, n and Ω, such that*

$$\|\zeta(\cdot,t)\|_{L^q(\Omega)} \leq C\big(\sup_{s\in(0,t)} \|g(\cdot,s)\|_{L^p(\Omega)} + \|\zeta_0\|_{L^\infty(\Omega)}\big) \quad \textit{for each } t \in (0,T).$$

(ii) *Assume that $(1/2) + (n/2)(1/p - 1/q) < 1$ and $1 \leq p, q \leq \infty$. Then*

$$\|\nabla\zeta(\cdot,t)\|_{L^q(\Omega)} \leq C\big(\sup_{s\in(0,t)} \|g(\cdot,s)\|_{L^p(\Omega)} + \|\nabla\zeta_0\|_{L^\infty(\Omega)}\big) \quad \textit{for any } t \in (0,T)$$

with some $C = C(a,p,q,n,\Omega) > 0$.

Proof. (i) First, we assume that $q \geq p$. Invoking the variation-of-constants formula for ζ yields

$$\zeta(\cdot,t) = \mathrm{e}^{t(\Delta-a)}\zeta_0(\cdot) + \int_0^t \mathrm{e}^{(t-s)(\Delta-a)} g(\cdot,s)ds \quad \text{for any } t \in (0,T). \tag{2.2}$$

Therefore, it follows from [32, Lemma 1.3 (i)] and the maximum principle that

$$\begin{aligned} &\|\zeta(\cdot,t)\|_{L^q(\Omega)} \\ &\leq \|\mathrm{e}^{t(\Delta-a)}\zeta_0(\cdot)\|_{L^q(\Omega)} + \int_0^t \|\mathrm{e}^{(t-s)(\Delta-a)}(g(\cdot,s) - \bar{g}(s))\|_{L^q(\Omega)} + \|\mathrm{e}^{(t-s)(\Delta-a)}\bar{g}(s)\|_{L^q(\Omega)}ds \\ &= \|\mathrm{e}^{t(\Delta-a)}\zeta_0(\cdot)\|_{L^q(\Omega)} + \int_0^t \mathrm{e}^{-a(t-s)}\|\mathrm{e}^{(t-s)\Delta}(g(\cdot,s) - \bar{g}(s))\|_{L^q(\Omega)} + \mathrm{e}^{-a(t-s)}\|\bar{g}(s)\|_{L^q(\Omega)}ds \\ &\leq |\Omega|^{\frac{1}{q}}\|\mathrm{e}^{t(\Delta-a)}\zeta_0(\cdot)\|_{L^\infty(\Omega)} \end{aligned}$$

$$+\int_0^t \Big[C_1\big(1+(t-s)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\big)\|g(\cdot,s)-\bar g(s)\|_{L^p(\Omega)}+\|\bar g(s)\|_{L^q(\Omega)}\Big]\mathrm{e}^{-a(t-s)}ds$$

$$\le |\Omega|^{\frac{1}{q}}\|\zeta_0\|_{L^\infty(\Omega)}+\int_0^t \Big[2C_1\big(1+(t-s)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\big)+|\Omega|^{\frac{1}{q}-\frac{1}{p}}\Big]\mathrm{e}^{-a(t-s)}\|g(\cdot,s)\|_{L^p(\Omega)}ds$$

$$\le |\Omega|^{\frac{1}{q}}\|\zeta_0\|_{L^\infty(\Omega)}+\sup_{s\in(0,t)}\|g(\cdot,s)\|_{L^p(\Omega)}\int_0^\infty \Big[2C_1\big(1+s^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\big)+|\Omega|^{\frac{1}{q}-\frac{1}{p}}\Big]\mathrm{e}^{-as}ds$$

for all $t\in(0,T)$, where $\bar g(s)=\frac{1}{|\Omega|}\int_\Omega g(\cdot,s)$, and $C_1>0$ depends on Ω only. Noticing that

$$\int_0^\infty \Big[2C_1\big(1+s^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\big)+|\Omega|^{\frac{1}{q}-\frac{1}{p}}\Big]\mathrm{e}^{-as}ds<\infty$$

for $(n/2)(1/p-1/q)<1$, we arrive at the claimed estimate. When $q<p$, the assertion results from the Hölder inequality and the L^p-bound of ζ.

(ii) Likewise, we may assume that $q\ge p$. Now with ∇ applied to both sides of (2.2), we deduce by [32, Lemma 1.3 (ii) and (iii)] that

$$\begin{aligned}
&\|\nabla\zeta(\cdot,t)\|_{L^q(\Omega)}\\
&\le \|\nabla\mathrm{e}^{t(\Delta-a)}\zeta_0(\cdot)\|_{L^q(\Omega)}+\int_0^t \|\nabla\mathrm{e}^{(t-s)(\Delta-a)}g(\cdot,s)\|_{L^q(\Omega)}\\
&\le \|\nabla\mathrm{e}^{t\Delta}\zeta_0(\cdot)\|_{L^q(\Omega)}+\int_0^t \mathrm{e}^{-a(t-s)}\|\nabla\mathrm{e}^{(t-s)\Delta}g(\cdot,s)\|_{L^q(\Omega)}ds\\
&\le |\Omega|^{\frac{1}{q}}\|\nabla\mathrm{e}^{t\Delta}\zeta_0(\cdot)\|_{L^\infty(\Omega)}+\int_0^t C_2\big(1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\big)\mathrm{e}^{-a(t-s)}\|g(\cdot,s)\|_{L^p(\Omega)}ds\\
&\le C_3|\Omega|^{\frac{1}{q}}\|\nabla\zeta_0\|_{L^\infty(\Omega)}+\sup_{s\in(0,t)}\|g(\cdot,s)\|_{L^p(\Omega)}\int_0^\infty C_2\big(1+s^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\big)\mathrm{e}^{-as}ds
\end{aligned}$$

for all $t\in(0,T)$ with $C_i=C_i(\Omega)>0$ $(i=2,3)$. Because of the convergence of the integral $\int_0^\infty C_2(1+s^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})})\mathrm{e}^{-as}ds$, the desired estimate is established. The proof is complete. □

As needed later, we next introduce two notations.

(1) For $\lambda>0$ and $p\in(1,\infty)$, we let $A_{\lambda,p}$ denote the realization of $-\Delta+\lambda$ in $L^p(\Omega)$ under the homogeneous Neumann boundary conditions, defined by

$$A_{\lambda,p}\varphi:=-\Delta\varphi+\lambda\varphi\quad\text{for }\varphi\in D(A_{\lambda,p}):=W^{2,p}_N:=\big\{\varphi\in W^{2,p}(\Omega):\partial_\nu\varphi|_{\partial\Omega}=0\big\}.$$

If no confusion is likely, we sometimes abbreviate $A_{\lambda,p}$ to A_λ.

(2) For $p,q\in(1,\infty)$, write

$$L_{p,q}:=(L^p(\Omega),W^{2,p}_N)_{1-1/q,q}\quad\text{with the norm }\|\cdot\|_{p,q},$$

where $(\cdot,\cdot)_{1-1/q,q}$ represents the real interpolation functor of exponent $1-1/q$ and parameter q [2, Example 2.4.1]. That is to say, $L_{p,q}$ denotes the Banach space of the functions φ in $L^p(\Omega)+W^{2,p}_N$ such that

$$\|\varphi\|_{p,q} := \Big(\int_0^\infty \tau^{-q} K^q(\tau,\varphi)d\tau\Big)^{\frac{1}{q}} < \infty$$

with

$$K(\tau,\varphi) := \inf\{\|\varphi_1\|_{L^p(\Omega)} + \tau\|\varphi_2\|_{W^{2,p}(\Omega)} : \varphi = \varphi_1+\varphi_2\} \quad \text{for } \tau>0.$$

Clearly, the embedding $W^{2,p}_N \hookrightarrow L_{p,q}$ is valid by definition.

Some basic properties on the space $L_{p,p}$ with $p\in(1,\infty)$ are summed in the following lemma.

Lemma 2.1.

(i) *It holds that*

$$\begin{cases} L_{p,p} \simeq W^{2(1-1/p),p}(\Omega) & \textit{for } 1<p<3,\\ L_{p,p} \hookrightarrow W^{2(1-1/p),p}(\Omega) & \textit{for } p=3,\\ L_{p,p} \simeq \{u\in W^{2(1-1/p),p}(\Omega) : \partial_\nu u|_{\partial\Omega}=0\} & \textit{for } p>3, \end{cases} \tag{2.3}$$

where $W^{2(1-1/p),p}(\Omega)$ with $p\neq 2$ denotes the Sobolev-Slobodeckij space (see e.g. [1, Section 11]), and $\simeq$ indicates that the corresponding norms are equivalent.

(ii) *Moreover, we have*

$$L_{p,p} \hookrightarrow W^{\tilde{s},\tilde{p}}(\Omega) \quad \textit{if } 2-(n+2)/p \geq \tilde{s}-n/\tilde{p} \textit{ with } \tilde{s}\geq 0 \textit{ and } 1<p\leq\tilde{p}<\infty, \tag{2.4}$$

and

$$W^{1,p}(\Omega) \hookrightarrow L_{p,p} \quad \textit{for } 1<p\leq 2. \tag{2.5}$$

Proof. (i) The assertions for $p\neq 2,3$ come from [1, Theorem 13.3]. Due to [28, Theorem 1.18.10] and again by [1, Theorem 13.3], we have

$$L_{2,2} = (L^2(\Omega), D(A_{1,2}))_{1/2,2} \simeq [L^2(\Omega), D(A_{1,2})]_{1/2} \simeq W^{1,2}(\Omega)$$

with $[\cdot,\cdot]_{1/2}$ denoting the complex interpolation functor of exponent $1/2$ [2, Example 2.4.2]. As for $p=3$, it can be readily checked with [1, Theorem 11.6] that

$$L_{3,3} = (L^3(\Omega), W^{2,3}_N)_{2/3,3} \hookrightarrow (L^3(\Omega), W^{2,3}(\Omega))_{2/3,3} \simeq W^{4/3,3}(\Omega).$$

(ii) This is immediate from (2.3) and the Sobolev embedding theorem [1, Theorem 11.5]. □

Now we give the maximal Sobolev regularity for abstract differential equations with nonzero initial data.

Lemma 2.2. *([2, Theorems 4.10.2 and 4.10.7, and Remark 4.10.9 (c)]) Let $p, q \in (1, \infty)$ and $\lambda > 0$. Then for any $\mathbf{g} \in L^q((0,\infty); L^p(\Omega))$ and every $\varphi \in L_{p,q}$, the initial value problem*

$$\begin{cases} \mathbf{u}'(s) + A_\lambda \mathbf{u}(s) = \mathbf{g}(s), \quad s > 0, \\ \mathbf{u}(0) = \varphi \end{cases} \tag{2.6}$$

possesses a unique solution $\mathbf{u} \in L^q((0,\infty); W^{2,p}_N) \cap W^{1,q}((0,\infty); L^p(\Omega))$ satisfying

$$c \sup_{s\in(0,\infty)} \|\mathbf{u}(s)\|^q_{p,q} \le \int_0^\infty \|\mathbf{u}(s)\|^q_{L^p(\Omega)} ds + \int_0^\infty \|\mathbf{u}'(s)\|^q_{L^p(\Omega)} ds + \int_0^\infty \|A_\lambda \mathbf{u}(s)\|^q_{L^p(\Omega)} ds$$
$$\le C\Big(\int_0^\infty \|\mathbf{g}(s)\|^q_{L^p(\Omega)} ds + \|\varphi\|^q_{p,q}\Big)$$

for some positive constants c and C depending on λ, p, q, n and Ω.

The foregoing preparations allow us to further establish integral estimates of solutions for the Neumann problem related to the latter two parabolic equations in (1.1).

Proposition 2.2. *Let $a > 0$ and $0 \le t_0 < T \le \infty$. Assume that $\zeta \in C^{2,1}(\bar\Omega \times [t_0, T))$ satisfies*

$$\begin{cases} \zeta_t = \Delta\zeta - a\zeta + g, & (x,t) \in \Omega \times (t_0, T), \\ \partial_\nu \zeta = 0, & (x,t) \in \partial\Omega \times (t_0, T). \end{cases}$$

(i) *For any $p, q \in (1,\infty)$, there exist positive constants C and $\hat{C}$, relying on a, p, q, n and Ω, such that*

$$\|\zeta(\cdot,t)\|^q_{p,q} \le C\Big(\sup_{s\in(t_0,t]} \int_{t_0+(s-1-t_0)_+}^{s} \|g(\cdot,\tau)\|^q_{L^p(\Omega)} d\tau + \|\zeta(\cdot,t_0)\|^q_{p,q}\Big) \tag{2.7}$$

for each $t \in (t_0, T)$, and

$$\int_{t_0+(t-1-t_0)_+}^{t} \|A_a\zeta(\cdot,\tau)\|^q_{L^p(\Omega)} d\tau$$
$$\le \hat{C}\Big(\sup_{s\in(t_0,t]} \int_{t_0+(s-1-t_0)_+}^{s} \|g(\cdot,\tau)\|^q_{L^p(\Omega)} d\tau + \|\zeta(\cdot,t_0)\|^q_{p,q}\Big) \tag{2.8}$$

for any $t \in (t_0, T)$.

(ii) *Suppose that $2 - (n+2)/p \ge -n/\tilde{p}$ with $1 < p < \infty$ and $1 \le \tilde{p} < \infty$. Then there exists $C = C(a, p, n, \Omega, \tilde{p}) > 0$ such that*

$$\|\zeta(\cdot,t)\|^p_{L^{\tilde{p}}(\Omega)} \le C\Big(\sup_{s\in(t_0,t]} \int_{t_0+(s-1-t_0)_+}^{s} \|g(\cdot,\tau)\|^p_{L^p(\Omega)} d\tau + \|\zeta(\cdot,t_0)\|^p_{p,p}\Big)$$

for each $t \in (t_0, T)$.

(iii) *For any $p, q \in (1,\infty)$ and every $\underline{a} \in [0,a)$, there exists $C > 0$, depending on $a - \underline{a}$, p, q, n and Ω, such that*

$$\int_{t_0}^{t} \mathrm{e}^{\underline{a}q\tau} \|A_{a-\underline{a}}\zeta(\cdot,\tau)\|^q_{L^p(\Omega)} d\tau \le C\Big(\int_{t_0}^{t} \mathrm{e}^{\underline{a}q\tau} \|g(\cdot,\tau)\|^q_{L^p(\Omega)} d\tau + \mathrm{e}^{\underline{a}qt_0} \|\zeta(\cdot,t_0)\|^q_{p,q}\Big)$$

for any $t \in (t_0, T)$.

Proof. Let $t_0 \le t_1 < t_2 < T$ and $0 \le \underline{a} < a$. Set

$$\mathbf{u}(s) = \mathbf{u}(s; t_1, t_2, \underline{a}) := \begin{cases} \mathrm{e}^{\underline{a}(s+t_1)}\zeta(\cdot, s+t_1) & \text{if } 0 \le s \le t_2 - t_1, \\ \upsilon(\cdot, s - t_2 + t_1) & \text{if } s > t_2 - t_1 \end{cases}$$

with $\upsilon(x,s) = \upsilon(x,s;t_2,\underline{a})$ solving

$$\begin{cases} \upsilon_s = \Delta\upsilon - (a - \underline{a})\upsilon, & x \in \Omega,\ s > 0, \\ \partial_\nu \upsilon = 0, & x \in \partial\Omega,\ s > 0, \\ \upsilon(x,0) = \mathrm{e}^{\underline{a}t_2}\zeta(x,t_2), & x \in \Omega. \end{cases}$$

Then for any $p, q \in (1,\infty)$, $\mathbf{u} \in L^q((0,\infty); W^{2,p}_N) \cap W^{1,q}((0,\infty); L^p(\Omega))$ is the solution of (2.6) with

$$\lambda = \lambda(\underline{a}) := a - \underline{a},$$

$$\mathbf{g}(s) = \mathbf{g}(s; t_1, t_2, \underline{a}) := \begin{cases} \mathrm{e}^{\underline{a}(s+t_1)} g(\cdot, s+t_1), & 0 \le s < t_2 - t_1, \\ 0, & s > t_2 - t_1, \end{cases} \quad \text{and}$$

$$\varphi(x) = \varphi(x; t_1, \underline{a}) := \mathrm{e}^{\underline{a}t_1}\zeta(x, t_1).$$

(i) Given $p, q \in (1,\infty)$, we at first prove (2.7). In view of Lemma 2.2, there exists $C_1 = C_1(a,p,q,n,\Omega)$ such that

$$\|\mathbf{u}(t - t_0; t_0, t, a/2)\|^q_{p,q} \le C_1\Big(\int_0^\infty \|\mathbf{g}(\tau; t_0, t, a/2)\|^q_{L^p(\Omega)} d\tau + \|\varphi(\cdot; t_0, a/2)\|^q_{p,q}\Big)$$

for all $t \in (t_0, T)$, which entails

$$\begin{aligned} \|\zeta(\cdot,t)\|^q_{p,q} &\le C_1\Big(\int_0^{t-t_0} \mathrm{e}^{-\frac{aq}{2}(t-\tau-t_0)} \|g(\cdot,\tau+t_0)\|^q_{L^p(\Omega)} d\tau + \mathrm{e}^{-\frac{aq}{2}(t-t_0)} \|\zeta(\cdot,t_0)\|^q_{p,q}\Big) \\ &\le C_1\Big(\int_{t_0}^{t} \mathrm{e}^{-\frac{aq}{2}(t-\tau)} \|g(\cdot,\tau)\|^q_{L^p(\Omega)} d\tau + \|\zeta(\cdot,t_0)\|^q_{p,q}\Big) \quad \text{for all } t \in (t_0, T) \end{aligned}$$

with the integral on the right-hand side estimated as

$$\int_{t_0}^{t} \mathrm{e}^{-\frac{aq}{2}(t-\tau)} \|g(\cdot,\tau)\|^q_{L^p(\Omega)} d\tau$$

$$
\begin{aligned}
&= \int_0^{t-t_0} e^{-\frac{aq}{2}\tau} \|g(\cdot,t-\tau)\|^q_{L^p(\Omega)} d\tau \\
&= \sum_{k=0}^{[t-t_0]-1} \int_k^{k+1} e^{-\frac{aq}{2}\tau} \|g(\cdot,t-\tau)\|^q_{L^p(\Omega)} d\tau + \int_{[t-t_0]}^{t-t_0} e^{-\frac{aq}{2}\tau} \|g(\cdot,t-\tau)\|^q_{L^p(\Omega)} d\tau \\
&\le \sum_{k=0}^{[t-t_0]-1} e^{-\frac{aq}{2}k} \int_{t-k-1}^{t-k} \|g(\cdot,\tau)\|^q_{L^p(\Omega)} d\tau + \int_{t_0}^{t-[t-t_0]} \|g(\cdot,\tau)\|^q_{L^p(\Omega)} d\tau \\
&\le \Big(\sum_{k=0}^{\infty} e^{-\frac{aq}{2}k} + 1 \Big) \sup_{s\in(t_0,t]} \int_{t_0+(s-1-t_0)_+}^{s} \|g(\cdot,\tau)\|^q_{L^p(\Omega)} d\tau \\
&= \Big(\frac{1}{1-e^{-aq/2}} + 1 \Big) \sup_{s\in(t_0,t]} \int_{t_0+(s-1-t_0)_+}^{s} \|g(\cdot,\tau)\|^q_{L^p(\Omega)} d\tau \quad \text{for all } t\in(t_0,T).
\end{aligned}
$$

This proves (2.7) by choosing $C := C_1(\frac{1}{1-e^{-aq/2}} + 1)$.

We proceed to verify (2.8). Again by Lemma 2.2, we can find $C_2 > 0$, determined by a, p, q, n and Ω, satisfying

$$
\begin{aligned}
&\int_0^{\infty} \big\|A_{\lambda(0)}\mathbf{u}\big(\tau;t_0+(t-1-t_0)_+,t,0\big)\big\|^q_{L^p(\Omega)} d\tau \\
&\le C_2\Big(\int_0^{\infty} \big\|\mathbf{g}\big(\tau;t_0+(t-1-t_0)_+,t,0\big)\big\|^q_{L^p(\Omega)} d\tau + \big\|\varphi\big(\cdot;t_0+(t-1-t_0)_+,0\big)\big\|^q_{p,q} \Big)
\end{aligned}
$$

for all $t\in(t_0,T)$, from which we further derive that

$$
\begin{aligned}
&\int_{t_0+(t-1-t_0)_+}^{t} \|A_a\zeta(\cdot,\tau)\|^q_{L^p(\Omega)} d\tau \\
&= \int_0^{t-[t_0+(t-1-t_0)_+]} \big\|A_a\zeta\big(\cdot,\tau+t_0+(t-1-t_0)_+\big)\big\|^q_{L^p(\Omega)} d\tau \\
&\le C_2\Big(\int_0^{t-[t_0+(t-1-t_0)_+]} \big\|g\big(\cdot,\tau+t_0+(t-1-t_0)_+\big)\big\|^q_{L^p(\Omega)} d\tau \\
&\quad + \big\|\zeta\big(\cdot,t_0+(t-1-t_0)_+\big)\big\|^q_{p,q} \Big) \\
&= C_2\Big(\int_{t_0+(t-1-t_0)_+}^{t} \|g(\cdot,\tau)\|^q_{L^p(\Omega)} d\tau + \big\|\zeta\big(\cdot,t_0+(t-1-t_0)_+\big)\big\|^q_{p,q} \Big)
\end{aligned} \tag{2.9}
$$

for all $t\in(t_0,T)$. In addition, we have

$$
\big\|\zeta\big(\cdot,t_0+(t-1-t_0)_+\big)\big\|^q_{p,q} = \|\zeta(\cdot,t_0)\|^q_{p,q} \quad \text{if } t\in(t_0,T) \text{ and } t\le 1+t_0, \tag{2.10}
$$

or

$$\begin{aligned}
&\|\zeta(\cdot, t_0 + (t-1-t_0)_+)\|_{p,q}^q \\
&\leq C\Big(\sup_{s\in(t_0,t_0+(t-1-t_0)_+]} \int\limits_{t_0+(s-1-t_0)_+}^{s} \|g(\cdot,\tau)\|_{L^p(\Omega)}^q d\tau + \|\zeta(\cdot,t_0)\|_{p,q}^q\Big) \\
&\leq C\Big(\sup_{s\in(t_0,t-1]} \int\limits_{t_0+(s-1-t_0)_+}^{s} \|g(\cdot,\tau)\|_{L^p(\Omega)}^q d\tau + \|\zeta(\cdot,t_0)\|_{p,q}^q\Big) \qquad (2.11)
\end{aligned}$$

if $t \in (t_0, T)$ and $t > 1 + t_0$ due to (2.7). Therefore, one can easily see from (2.9)–(2.11) that (2.8) is valid with $\hat{C} := C_2(1 + C)$.

(ii) It is obviously true if $2 - (n+2)/p \geq -n/\tilde{p}$ and $1 < p \leq \tilde{p} < \infty$ by (2.7) and (2.4). When $2 - (n+2)/p \geq -n/\tilde{p}$ and $1 \leq \tilde{p} < p$, since $2 - (n+2)/p \geq -n/p$, the assertion results from the Hölder inequality and the corresponding estimate for $\|\zeta(\cdot,t)\|_{L^p(\Omega)}^p$.

(iii) Applying Lemma 2.2 to $\mathbf{u}(\cdot; t_0, t, \underline{a})$, we obtain

$$\int\limits_0^\infty \|A_{\lambda(\underline{a})}\mathbf{u}(\tau; t_0, t, \underline{a})\|_{L^p(\Omega)}^q d\tau \leq C_3\Big(\int\limits_0^\infty \|\mathbf{g}(\tau; t_0, t, \underline{a})\|_{L^p(\Omega)}^q d\tau + \|\varphi(\cdot; t_0, \underline{a})\|_{p,q}^q\Big)$$

for all $t \in (t_0, T)$ with $C_3 > 0$ relying on $a - \underline{a}$, p, q, n and Ω, whence

$$\begin{aligned}
&\int\limits_0^{t-t_0} \mathrm{e}^{\underline{a}q(\tau+t_0)} \|A_{a-\underline{a}}\zeta(\cdot, \tau+t_0)\|_{L^p(\Omega)}^q d\tau \\
&\qquad \leq C_3\Big(\int\limits_0^{t-t_0} \mathrm{e}^{\underline{a}q(\tau+t_0)} \|g(\cdot,\tau+t_0)\|_{L^p(\Omega)}^q d\tau + \mathrm{e}^{\underline{a}qt_0}\|\zeta(\cdot,t_0)\|_{p,q}^q\Big)
\end{aligned}$$

for all $t \in (t_0, T)$, as desired. □

Proposition 2.3. *Assume that $2 - (n+2)/p \geq -n/\tilde{p}$ with $1 < p < \infty$ and $1 \leq \tilde{p} < \infty$. Then under the conditions of Proposition 2.1, there exists $C = C(a, p, n, \Omega, \tilde{p}) > 0$ such that*

$$\|\zeta(\cdot,t)\|_{L^{\tilde{p}}(\Omega)}^p \leq C\Big(\sup_{s\in(0,t]} \int\limits_{(s-1)_+}^{s} \|g(\cdot,\tau)\|_{L^p(\Omega)}^p d\tau + \|g(\cdot,0)\|_{L^p(\Omega)}^p + \|\zeta_0\|_{W^{1,\infty}(\Omega)}^p + 1\Big)$$

for all $t \in (0,T)$ if $1 < p \leq 2$, or

$$\|\zeta(\cdot,t)\|_{L^{\tilde{p}}(\Omega)}^p \leq C\Big(\sup_{s\in(0,t]} \int\limits_{(s-1)_+}^{s} \|g(\cdot,\tau)\|_{L^p(\Omega)}^p d\tau + \|\zeta_0\|_{p,p}^p\Big)$$

for each $t \in (0,T)$ if $2 < p < \infty$ and further $\zeta_0 \in L_{p,p}$.

Proof. Let us first treat the case $1 < p \leq 2$. By Proposition 2.1 and the continuity of g, there exist $C_1 = C_1(a, p, n, \Omega) > 0$ and a small $t_* > 0$ such that

$$\sup_{t\in(0,t_*)} \|\zeta(\cdot,t)\|_{W^{1,p}(\Omega)} \le C_1\big(\sup_{t\in(0,t_*)} \|g(\cdot,t)\|_{L^p(\Omega)} + \|\zeta_0\|_{W^{1,\infty}(\Omega)}\big)$$
$$\le C_1\big(1 + \|g(\cdot,0)\|_{L^p(\Omega)} + \|\zeta_0\|_{W^{1,\infty}(\Omega)}\big). \tag{2.12}$$

For any $t \in (0,T)$, we take $0 < t_1 < \min\{t,t_*\}$, and then gain by Proposition 2.2 (ii) and (2.5) that

$$\|\zeta(\cdot,t)\|^p_{L^{\tilde p}(\Omega)} \le C_2\Big(\sup_{s\in(t_1,t]} \int_{t_1+(s-1-t_1)_+}^{s} \|g(\cdot,\tau)\|^p_{L^p(\Omega)}d\tau + \|\zeta(\cdot,t_1)\|^p_{p,p}\Big)$$
$$\le C_2\Big(\sup_{s\in(0,t]} \int_{(s-1)_+}^{s} \|g(\cdot,\tau)\|^p_{L^p(\Omega)}d\tau + C_3\|\zeta(\cdot,t_1)\|^p_{W^{1,p}(\Omega)}\Big)$$

with $C_2 = C_2(a,p,n,\Omega,\tilde p) > 0$ and $C_3 = C_3(p,n,\Omega) > 0$. This along with (2.12) yields the claimed estimate.

Now, assume that $2 < p < \infty$ and in addition that $\zeta_0 \in L_{p,p}$. For any fixed $0 < T' < T$, let $\mathbf{u} \in L^p((0,\infty);W^{2,p}_N) \cap W^{1,p}((0,\infty);L^p(\Omega))$ be the solution of (2.6) with

$$\lambda = a, \quad \mathbf{g}(s) = \begin{cases} g(\cdot,s), & 0 \le s < T', \\ 0, & s > T', \end{cases} \quad \text{and} \quad \varphi(x) = \zeta_0(x).$$

Note that $2 < p < \infty$. Therefore, $\mathbf{u} \in L^2((0,T');H^1(\Omega))$ with $\mathbf{u}' \in L^2((0,T');L^2(\Omega))$ is a weak solution of (2.1) in $\Omega \times (0,T')$, that is

$$\int_0^{T'} (\mathbf{u}',\mathbf{v}) = -\int_0^{T'} (\nabla\mathbf{u},\nabla\mathbf{v}) - a\int_0^{T'} (\mathbf{u},\mathbf{v}) + \int_0^{T'} (\mathbf{g},\mathbf{v})$$

for all $\mathbf{v} \in C_0^\infty((0,T');H^1(\Omega))$, and $\mathbf{u}(0) = \zeta_0$, where the pairing $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$. Define

$$\tilde{\mathbf{u}} := \tilde{\mathbf{u}}(t) := \zeta(\cdot,t), \quad t \in (0,T').$$

Then $\tilde{\mathbf{u}} \in L^\infty((0,T');H^1(\Omega))$ due to Proposition 2.1, and furthermore testing the first equation in (2.1) by ζ_t gives $\tilde{\mathbf{u}}' \in L^2((0,T');L^2(\Omega))$. Clearly, $\tilde{\mathbf{u}}$ is also a weak solution of (2.1) in $\Omega \times (0,T')$. By uniqueness, $\tilde{\mathbf{u}} = \mathbf{u} \in L^p((0,T');W^{2,p}_N) \cap W^{1,p}((0,T');L^p(\Omega))$, and hence $\zeta \in L^p_{\mathrm{loc}}([0,T);W^{2,p}_N) \cap W^{1,p}_{\mathrm{loc}}([0,T);L^p(\Omega))$ for the arbitrariness of T'. Just because of this, we can follow the proof of (2.7) to get with some $C_4 = C_4(a,p,n,\Omega) > 0$ that

$$\|\zeta(\cdot,t)\|^p_{p,p} \le C_4\Big(\sup_{s\in(0,t]} \int_{(s-1)_+}^{s} \|g(\cdot,\tau)\|^p_{L^p(\Omega)}d\tau + \|\zeta_0\|^p_{p,p}\Big)$$

for all $t \in (0,T)$, which together with (2.4) leads to the desired estimate. $\square$

In the following, we need an extended version of the classical Gagliardo-Nirenberg inequality (see e.g. [5,21,31]).

Lemma 2.3. *Assume that $q, r \in (0,\infty]$ are such that*

$$\iota := \frac{\frac{1}{r} - \frac{1}{q}}{\frac{1}{r} + \frac{1}{n} - \frac{1}{2}} \in (0,1).$$

Then

$$\|\phi\|_{L^q(\Omega)} \le c\big(\|\nabla\phi\|_{L^2(\Omega)}^{\iota}\|\phi\|_{L^r(\Omega)}^{1-\iota} + \|\phi\|_{L^r(\Omega)}\big), \quad \forall\ \phi \in W^{1,2}(\Omega)\cap L^r(\Omega)$$

with $c > 0$ depending on n, r, ι and Ω.

We end this section with two lemmas that assert the boundedness of solutions for some ordinary differential inequalities and play a key role in the proof of main results.

Lemma 2.4.

(i) *Let $0 \le t_0 < T \le \infty$ and $0 < \ell < \kappa \le 1$. Suppose that nonnegative functions $y \in C^1((t_0,T))\cap C^0([t_0,T))$ and $h \in C^0([t_0,T))$ satisfy*

$$y'(t) + c_1 y^{\kappa}(t) \le c_2 h(t) + c_3, \quad t \in (t_0,T) \tag{2.13}$$

with $c_i > 0$ $(i = 1,2,3)$, and

$$\int_{t_0+(t-1-t_0)_+}^{t} h(\tau)d\tau \le c_4\big(Y^{\ell}(t) + 1\big), \quad t \in (t_0,T)$$

for some $c_4 > 0$ with $Y(t) := \sup_{t_0<\tau<t} y(\tau)$, $t \in (t_0,T)$. Then y is bounded in (t_0,T).

(ii) *Let $0 \le t_0 < T \le \infty$ and $0 < \ell < 1 < \kappa$. If nonnegative functions $y \in C^1((t_0,T))\cap C^0([t_0,T))$ and $h \in C^0([t_0,T))$ fulfill*

$$y'(t) + c_1 y^{\kappa}(t) \le c_2 h(t) + c_3, \quad t \in (t_0,T)$$

with $c_i > 0$ $(i = 1,2,3)$, and

$$\int_{t_0}^{t} \mathrm{e}^{\Lambda\tau} h(\tau)d\tau \le c_4\Big(\int_{t_0}^{t} \mathrm{e}^{\Lambda\tau} y^{\ell}(\tau)d\tau + 1\Big), \quad t \in (t_0,T)$$

for some $c_4, \Lambda > 0$, then y is bounded in (t_0,T).

Proof. (i) Without loss of generality, we may assume that $y > 0$ on $[t_0,T)$. Multiply (2.13) by $\mathrm{e}^{\int_{t_0}^{t} c_1 y^{\kappa-1}(s)ds}$ and integrate to get

$$\begin{aligned} y(t) - y(t_0) &\le \int_{t_0}^{t} \mathrm{e}^{-\int_{\tau}^{t} c_1 y^{\kappa-1}(s)ds}(c_2 h(\tau) + c_3)d\tau \\ &\le \int_{t_0}^{t} \mathrm{e}^{-c_1 Y^{\kappa-1}(t)(t-\tau)}(c_2 h(\tau) + c_3)d\tau \\ &= \int_{0}^{t-t_0} \mathrm{e}^{-c_1 Y^{\kappa-1}(t)\tau}(c_2 h(t-\tau) + c_3)d\tau \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{[t-t_0]-1} \int_k^{k+1} \mathrm{e}^{-c_1 Y^{\kappa-1}(t)\tau}(c_2 h(t-\tau)+c_3)d\tau \\
&\quad + \int_{[t-t_0]}^{t-t_0} \mathrm{e}^{-c_1 Y^{\kappa-1}(t)\tau}(c_2 h(t-\tau)+c_3)d\tau \\
&\le \sum_{k=0}^{[t-t_0]-1} \mathrm{e}^{-c_1 Y^{\kappa-1}(t)k} \int_{t-k-1}^{t-k} (c_2 h(\tau)+c_3)d\tau + \int_{t_0}^{t-[t-t_0]} (c_2 h(\tau)+c_3)d\tau \\
&\le \Big(\sum_{k=0}^{\infty} \mathrm{e}^{-c_1 Y^{\kappa-1}(t)k}+1\Big)\cdot\Big(c_2 c_4\big(Y^{\ell}(t)+1\big)+c_3\Big) \\
&\le \frac{2c_2c_4(Y^{\ell}(t)+1)+2c_3}{1-\mathrm{e}^{-c_1 Y^{\kappa-1}(t)}}
\end{aligned}
$$

for all $t \in (t_0, T)$, where we use the fact that $0 < \kappa \le 1$, the assumption on h and the nondecreasing monotonicity of Y. Consequently,

$$
Y(t) \le y(t_0) + \frac{2c_2c_4(Y^{\ell}(t)+1)+2c_3}{1-\mathrm{e}^{-c_1 Y^{\kappa-1}(t)}} \quad \text{for all } t \in (t_0, T),
$$

whence

$$
Y^{1-\ell}(t)\big(1-\mathrm{e}^{-c_1 Y^{\kappa-1}(t)}\big) \le y^{1-\ell}(t_0) + 2c_2c_4\big(1+y^{-\ell}(t_0)\big) + 2c_3 y^{-\ell}(t_0)
$$

for all $t \in (t_0, T)$, which implies that Y (and thereby y) is bounded in (t_0, T), as otherwise $Y^{1-\ell}(t)(1-\mathrm{e}^{-c_1 Y^{\kappa-1}(t)})$ is unbounded in (t_0, T) since

$$
\lim_{\varsigma\to\infty} \varsigma^{1-\ell}\big(1-\mathrm{e}^{-c_1\varsigma^{\kappa-1}}\big) = \infty
$$

for $0 < \ell < \kappa \le 1$.

(ii) Since $\kappa > 1$, we have by the Young inequality that

$$
\Lambda y(t) \le c_1 y^{\kappa}(t) + c_5, \quad t \in (t_0, T)
$$

with $c_5 := (\Lambda c_1^{-1/\kappa})^{\kappa/(\kappa-1)}$, and thus

$$
y'(t) + \Lambda y(t) \le c_2 h(t) + c_3 + c_5, \quad t \in (t_0, T).
$$

Multiplying this inequality by $\mathrm{e}^{\Lambda t}$, integrating and using the hypothesis on h result in

$$
\begin{aligned}
y(t) &\le y(t_0) + c_2\mathrm{e}^{-\Lambda t}\int_{t_0}^{t} \mathrm{e}^{\Lambda\tau}h(\tau)d\tau + (c_3+c_5)\Lambda^{-1} \\
&\le y(t_0) + c_2c_4\mathrm{e}^{-\Lambda t}\Big(\int_{t_0}^{t} \mathrm{e}^{\Lambda\tau}y^{\ell}(\tau)d\tau + 1\Big) + (c_3+c_5)\Lambda^{-1} \\
&\le y(t_0) + c_2c_4\Lambda^{-1}Y^{\ell}(t) + c_2c_4 + (c_3+c_5)\Lambda^{-1} \quad \text{for all } t \in (t_0, T),
\end{aligned}
$$

where $Y(t) := \sup_{t_0<\tau<t} y(\tau)$, $t \in (t_0, T)$. As in the proof of (i), one can readily deduce with $0 < \ell < 1$ that Y is bounded in (t_0, T). □

Lemma 2.5. *Let $0 \le t_0 < T \le \infty$. If nonnegative functions $y \in C^1((t_0, T)) \cap C^0([t_0, T))$ and $h_1, h_2 \in C^0([t_0, T))$ satisfy*

$$y'(t) \le h_1(t)y(t) + h_2(t), \quad t \in (t_0, T), \tag{2.14}$$

and

$$\int_{t_0+(t-1-t_0)_+}^{t} y(\tau)d\tau \le c_0, \qquad \int_{t_0+(t-1-t_0)_+}^{t} h_1(\tau)d\tau \le c_1, \qquad \int_{t_0+(t-1-t_0)_+}^{t} h_2(\tau)d\tau \le c_2$$

for all $t \in (t_0, T)$ with $c_i > 0$ ($i = 0, 1, 2$), then

$$y \le \big(\max\{y(t_0), c_0\} + c_2\big)\mathrm{e}^{c_1} \quad \textit{in } (t_0, T).$$

Proof. For $t \in (t_0, T)$ with $t > 1 + t_0$, since $\int_{t-1}^{t} y(\tau)d\tau \le c_0$, the mean value theorem for integrals implies that $y(\underline{t}) \le c_0$ for some $\underline{t} \in [t-1, t]$, and thus multiplying (2.14) by $\mathrm{e}^{-\int_{t_0}^{t} h_1(s)ds}$ and integrating over $(\underline{t}, t)$ yield

$$\begin{aligned} y(t) &\le y(\underline{t})\mathrm{e}^{\int_{\underline{t}}^{t} h_1(s)ds} + \int_{\underline{t}}^{t} h_2(\tau)\mathrm{e}^{\int_{\tau}^{t} h_1(s)ds}d\tau \\ &\le y(\underline{t})\mathrm{e}^{\int_{t-1}^{t} h_1(s)ds} + \int_{t-1}^{t} h_2(\tau)\mathrm{e}^{\int_{t-1}^{t} h_1(s)ds}d\tau \\ &\le (c_0 + c_2)\mathrm{e}^{c_1}. \end{aligned}$$

On the other hand, when $t \in (t_0, T)$ and $t \le 1 + t_0$, a similar integration shows

$$\begin{aligned} y(t) &\le y(t_0)\mathrm{e}^{\int_{t_0}^{t} h_1(s)ds} + \int_{t_0}^{t} h_2(\tau)\mathrm{e}^{\int_{\tau}^{t} h_1(s)ds}d\tau \\ &\le y(t_0)\mathrm{e}^{\int_{t_0}^{t} h_1(s)ds} + \int_{t_0}^{t} h_2(\tau)\mathrm{e}^{\int_{t_0}^{t} h_1(s)ds}d\tau \\ &\le (y(t_0) + c_2)\mathrm{e}^{c_1}. \end{aligned}$$

All in all, the proof is complete. □

3. Local existence and crucial estimates

We begin with the local existence of classical solutions to (1.1) that can be asserted by following the proof of [4, Lemma 3.1].

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Assume that $D, S \in C^2([0,\infty))$ satisfy $D(s) > 0$ for $s \geq 0$ and $S(0) = 0$. Also, suppose that $f \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$ complies with $f(0) \geq 0$. Then for any nonnegative $(u_0, v_0, w_0) \in C^{\omega}(\bar{\Omega}) \times [W^{1,\infty}(\Omega)]^2$ with $0 < \omega < 1$, there exist $T_{\max} \in (0,\infty]$ and a triplet (u,v,w) of nonnegative functions from $C^0(\bar{\Omega} \times [0,T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0,T_{\max}))$ solving (1.1) classically in $\Omega \times (0,T_{\max})$. Moreover, if $T_{\max} < \infty$, then*

$$\limsup_{t \to T_{\max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty. \tag{3.1}$$

Henceforth, let (u,v,w) be the classical solution of (1.1) with the maximal existence time $T_{\max} \in (0,\infty]$, and write

$$t_0 := \min\Big\{1, \frac{T_{\max}}{2}\Big\}.$$

Now we immediately have the following simple estimate.

Lemma 3.2. *Assume that f fulfills (1.8). Then*

$$\int_\Omega u(x,t)dx \leq \max\Big\{\int_\Omega u_0, (\frac{b}{\mu})^{\frac{1}{r}}|\Omega|\Big\} =: M_1 \quad \textit{for all } t \in (0,T_{\max}), \tag{3.2}$$

and

$$\int_{(t-1)_+}^{t} \int_\Omega u^r(x,s)dxds \leq \frac{1}{\mu}(b|\Omega| + M_1) =: M_2 \quad \textit{for all } t \in (0,T_{\max}). \tag{3.3}$$

Proof. Integrating Eq. $(1.1)_1$ and using (1.8) give

$$\frac{d}{dt}\int_\Omega u \leq b|\Omega| - \mu\int_\Omega u^r \leq b|\Omega| - \mu|\Omega|^{1-r}\Big(\int_\Omega u\Big)^r \quad \text{for all } t \in (0,T_{\max}),$$

which implies (3.2) and (3.3) clearly. The proof is complete. □

Based upon (3.2) and (3.3), we can use Propositions 2.1–2.3 to show the L^q-boundedness of w with $q \geq 1$ suitable.

Lemma 3.3. *Let f satisfy (1.8).*

(i) *Then for any $q \in [1, n/(n-2)_+) \cup [1, nr/(n+2-2r)_+)$, $w \in L^\infty((0,T_{\max}); L^q(\Omega))$.*
(ii) *In particular, assume that $1 < r \leq 2$ and $n \geq 3$, or $2 < r < (n+2)/2$ and additionally $w_0 \in W^{2,r}_N$. Then there exists $C = C(a_2, b_2, r, n, \Omega, q) > 0$ with $q = nr/(n+2-2r)$ such that*

$$\|w(\cdot,t)\|_{L^q(\Omega)} \leq C(M_2^{\frac{1}{r}} + \|u_0\|_{L^r(\Omega)} + \|w_0\|_{W^{1,\infty}(\Omega)} + 1) \quad \textit{for all } t \in (0,T_{\max})$$

if $1 < r \leq 2$ and $n \geq 3$, or

$$\|w(\cdot,t)\|_{L^q(\Omega)} \leq C(M_2^{\frac{1}{r}} + \|w_0\|_{W^{2,r}(\Omega)}) \quad \textit{for all } t \in (0,T_{\max})$$

if $2 < r < (n+2)/2$ and $w_0 \in W^{2,r}_N$.

Proof. (i) For $q \in [1, n/(n-2)_+)$, this is a consequence of Proposition 2.1 (i) to w and (3.2). When $q \in [1, nr/(n+2-2r)_+)$ with $1 < r < \infty$ (equivalently, $2-(n+2)/r > -n/q$ with $1 < r < \infty$ and $1 \le q < \infty$), in view of Proposition 2.2 (ii) for w and (3.3), we have with some $C_1 = C_1(a_2, r, n, \Omega, q) > 0$ that

$$\begin{aligned}
\|w(\cdot,t)\|^r_{L^q(\Omega)} &\le C_1\Big(\sup_{s\in(t_0,t]} \int\limits_{t_0+(s-1-t_0)_+}^{s} \|b_2 u(\cdot,\tau)\|^r_{L^r(\Omega)} d\tau + \|w(\cdot,t_0)\|^r_{r,r}\Big) \\
&\le C_1\Big(\sup_{s\in(0,t]} \int\limits_{(s-1)_+}^{s} \|b_2 u(\cdot,\tau)\|^r_{L^r(\Omega)} d\tau + \|w(\cdot,t_0)\|^r_{r,r}\Big) \\
&\le C_1(b_2^r M_2 + \|w(\cdot,t_0)\|^r_{r,r}) \quad \text{for each } t \in (t_0, T_{\max}).
\end{aligned}$$

Also, it is evident that $w \in L^\infty((0,t_0); L^q(\Omega))$. So, assertion (i) follows.

(ii) Suppose that $1 < r \le 2$ and $n \ge 3$, and thereby $1 < r < (n+2)/2$. A straightforward application of Proposition 2.3 to w along with (3.3) shows

$$\begin{aligned}
\|w(\cdot,t)\|^r_{L^q(\Omega)} &\le C_2\Big(\sup_{s\in(0,t]} \int\limits_{(s-1)_+}^{s} \|b_2 u(\cdot,\tau)\|^r_{L^r(\Omega)} d\tau + \|b_2 u_0\|^r_{L^r(\Omega)} + \|w_0\|^r_{W^{1,\infty}(\Omega)} + 1\Big) \\
&\le C_2(b_2^r M_2 + b_2^r\|u_0\|^r_{L^r(\Omega)} + \|w_0\|^r_{W^{1,\infty}(\Omega)} + 1) \quad \text{for all } t \in (0, T_{\max})
\end{aligned}$$

with $q = nr/(n+2-2r)$ and $C_2 = C_2(a_2, r, n, \Omega, q) > 0$, as claimed. Likewise, by Proposition 2.3, (3.3) and the embedding $W^{2,r}_N \hookrightarrow L_{r,r}$, we can prove the assertion for the case of $2 < r < (n+2)/2$ and $w_0 \in W^{2,r}_N$. $\square$

In the following lemma, with the norm $\|w\|_{L^\infty((0,T_{\max});L^q(\Omega))}$ for $q \ge 1$ involved, the proper integral relations between Δv and u have been established due to Proposition 2.2.

Lemma 3.4. *Assume that $p, \tilde{p} \in (1,\infty)$ and $q \in [1,\infty)$ satisfy*

$$\tilde{p}\tilde{\iota} \in (1,\infty) \quad \textit{with} \quad \tilde{\iota} := \frac{\frac{1}{q} - \frac{1}{\tilde{p}}}{\frac{1}{q} + \frac{2}{n} - \frac{1}{p}} \in (0,1).$$

(i) *Then*

$$\int\limits_{t_0+(t-1-t_0)_+}^{t} \Big(\int\limits_\Omega |\Delta v(x,\tau)|^{\tilde{p}} dx\Big) d\tau \le C\Big\{\|w\|^{\tilde{p}(1-\tilde{\iota})}_{L^\infty((0,T_{\max});L^q(\Omega))}\Big(\sup_{\tau\in(t_0,t)} \|u(\cdot,\tau)\|^{\tilde{p}\tilde{\iota}}_{L^p(\Omega)} + \|w(\cdot,t_0)\|^{\tilde{p}\tilde{\iota}}_{p,\tilde{p}\tilde{\iota}}\Big) + \|v(\cdot,t_0)\|^{\tilde{p}}_{\tilde{p},\tilde{p}}\Big\} \tag{3.4}$$

for all $t \in (t_0, T_{\max})$ with $C > 0$ relying on a_i, b_i ($i = 1,2$), $\tilde{p}$, p, q, n and Ω.

(ii) *For any $0 \le \Lambda < \min\{\tilde{p}a_1, \tilde{p}\tilde{\iota}a_2\}$, there exists $C > 0$, determined by $a_1 - \Lambda/\tilde{p}$, $a_2 - \Lambda/(\tilde{p}\tilde{\iota})$, b_1, b_2, $\tilde{p}$, p, q, n and Ω, such that*

$$\int\limits_{t_0}^{t} e^{\Lambda\tau}\Big(\int\limits_\Omega |\Delta v(x,\tau)|^{\tilde{p}} dx\Big) d\tau \le C\Big\{\|w\|^{\tilde{p}(1-\tilde{\iota})}_{L^\infty((0,T_{\max});L^q(\Omega))}\Big(\int\limits_{t_0}^{t} e^{\Lambda\tau}\|u(\cdot,\tau)\|^{\tilde{p}\tilde{\iota}}_{L^p(\Omega)} d\tau + e^{\Lambda t_0}\|w(\cdot,t_0)\|^{\tilde{p}\tilde{\iota}}_{p,\tilde{p}\tilde{\iota}}\Big) + e^{\Lambda t_0}\|v(\cdot,t_0)\|^{\tilde{p}}_{\tilde{p},\tilde{p}}\Big\} \tag{3.5}$$

for all $t \in (t_0, T_{\max})$.

Proof. (i) In view of the elliptic regularity estimates [5, Section 19 of Part 1] and (2.8), one can find some $C_1 = C_1(a_1, \tilde{p}, n, \Omega) > 0$ and $C_2 = C_2(a_1, b_1, \tilde{p}, n, \Omega) > 0$ such that

$$\int_{t_0+(t-1-t_0)_+}^{t} \Big(\int_\Omega |\Delta v(x,\tau)|^{\tilde{p}} dx\Big) d\tau \le C_1 \int_{t_0+(t-1-t_0)_+}^{t} \|A_{a_1} v(\cdot,\tau)\|^{\tilde{p}}_{L^{\tilde{p}}(\Omega)} d\tau$$
$$\le C_2\Big(\sup_{s\in(t_0,t]} \int_{t_0+(s-1-t_0)_+}^{s} \|w(\cdot,\tau)\|^{\tilde{p}}_{L^{\tilde{p}}(\Omega)} d\tau + \|v(\cdot,t_0)\|^{\tilde{p}}_{\tilde{p},\tilde{p}}\Big) \tag{3.6}$$

for all $t \in (t_0, T_{\max})$. Furthermore, by means of the Gagliardo-Nirenberg inequality, and using the elliptic regularity estimates along with (2.8) again, we derive

$$\int_{t_0+(s-1-t_0)_+}^{s} \|w(\cdot,\tau)\|^{\tilde{p}}_{L^{\tilde{p}}(\Omega)} d\tau \le \int_{t_0+(s-1-t_0)_+}^{s} \Big(C_3 \|w(\cdot,\tau)\|^{\tilde{\iota}}_{W^{2,p}(\Omega)} \|w(\cdot,\tau)\|^{1-\tilde{\iota}}_{L^q(\Omega)}\Big)^{\tilde{p}} d\tau$$
$$\le (C_3)^{\tilde{p}} \|w\|^{\tilde{p}(1-\tilde{\iota})}_{L^\infty((0,T_{\max});L^q(\Omega))} \int_{t_0+(s-1-t_0)_+}^{s} \|w(\cdot,\tau)\|^{\tilde{p}\tilde{\iota}}_{W^{2,p}(\Omega)} d\tau$$
$$\le (C_3)^{\tilde{p}} \|w\|^{\tilde{p}(1-\tilde{\iota})}_{L^\infty((0,T_{\max});L^q(\Omega))} \cdot C_4 \int_{t_0+(s-1-t_0)_+}^{s} \|A_{a_2} w(\cdot,\tau)\|^{\tilde{p}\tilde{\iota}}_{L^p(\Omega)} d\tau$$
$$\le (C_3)^{\tilde{p}} \|w\|^{\tilde{p}(1-\tilde{\iota})}_{L^\infty((0,T_{\max});L^q(\Omega))}$$
$$\times C_5\Big(\sup_{\varsigma\in(t_0,s]} \int_{t_0+(\varsigma-1-t_0)_+}^{\varsigma} \|u(\cdot,\tau)\|^{\tilde{p}\tilde{\iota}}_{L^p(\Omega)} d\tau + \|w(\cdot,t_0)\|^{\tilde{p}\tilde{\iota}}_{p,\tilde{p}\tilde{\iota}}\Big)$$
$$\le (C_3)^{\tilde{p}} \|w\|^{\tilde{p}(1-\tilde{\iota})}_{L^\infty((0,T_{\max});L^q(\Omega))}$$
$$\times C_5\Big(\sup_{\tau\in(t_0,t)} \|u(\cdot,\tau)\|^{\tilde{p}\tilde{\iota}}_{L^p(\Omega)} + \|w(\cdot,t_0)\|^{\tilde{p}\tilde{\iota}}_{p,\tilde{p}\tilde{\iota}}\Big)$$

for all $s \in (t_0, t]$ with $t \in (t_0, T_{\max})$, where $C_3 = C_3(n, p, q, \tilde{\iota}, \Omega)$, $C_4 = C_4(a_2, p, n, \Omega, \tilde{p}\tilde{\iota})$ and $C_5 = C_5(a_2, b_2, p, n, \Omega, \tilde{p}\tilde{\iota})$ are positive constants. Combining this with (3.6), we arrive at (3.4).

(ii) Let $0 \le \Lambda < \min\{\tilde{p}a_1, \tilde{p}\tilde{\iota}a_2\}$. Then $\Lambda/\tilde{p} < a_1$ and $\Lambda/(\tilde{p}\tilde{\iota}) < a_2$. It follows from the elliptic regularity estimates and Proposition 2.2 (iii) that

$$\int_{t_0}^{t} e^{\Lambda\tau}\Big(\int_\Omega |\Delta v(x,\tau)|^{\tilde{p}} dx\Big) d\tau \le C_6 \int_{t_0}^{t} e^{\Lambda\tau} \|A_{a_1-\Lambda/\tilde{p}} v(\cdot,\tau)\|^{\tilde{p}}_{L^{\tilde{p}}(\Omega)} d\tau$$
$$\le C_7\Big(\int_{t_0}^{t} e^{\Lambda\tau} \|w(\cdot,\tau)\|^{\tilde{p}}_{L^{\tilde{p}}(\Omega)} d\tau + e^{\Lambda t_0} \|v(\cdot,t_0)\|^{\tilde{p}}_{\tilde{p},\tilde{p}}\Big) \tag{3.7}$$

for all $t \in (t_0, T_{\max})$ with $C_6 = C_6(a_1 - \Lambda/\tilde{p}, \tilde{p}, n, \Omega) > 0$ and $C_7 = C_7(a_1 - \Lambda/\tilde{p}, b_1, \tilde{p}, n, \Omega) > 0$. Using the Gagliardo-Nirenberg inequality and again by the elliptic regularity estimates, we further have

$$\|w(\cdot,\tau)\|^{\tilde{p}}_{L^{\tilde{p}}(\Omega)} \le \Big(C_8 \|w(\cdot,\tau)\|^{\tilde{\iota}}_{W^{2,p}(\Omega)} \|w(\cdot,\tau)\|^{1-\tilde{\iota}}_{L^q(\Omega)}\Big)^{\tilde{p}}$$

$$\leq (C_8)^{\tilde{p}} \|w\|^{\tilde{p}(1-\tilde{\iota})}_{L^\infty((0,T_{\max});L^q(\Omega))} \cdot C_9 \|A_{a_2-\Lambda/(\tilde{p}\tilde{\iota})} w(\cdot,\tau)\|^{\tilde{p}\tilde{\iota}}_{L^p(\Omega)}$$

for all $\tau \in (t_0, T_{\max})$ with $C_8 = C_8(p, q, \tilde{\iota}, n, \Omega) > 0$ and $C_9 = C_9(a_2 - \Lambda/(\tilde{p}\tilde{\iota}), p, \tilde{p}\tilde{\iota}, n, \Omega) > 0$, which together with Proposition 2.2 (iii) entails

$$\int_{t_0}^{t} \mathrm{e}^{\Lambda\tau} \|w(\cdot,\tau)\|^{\tilde{p}}_{L^{\tilde{p}}(\Omega)} d\tau \leq (C_8)^{\tilde{p}} \|w\|^{\tilde{p}(1-\tilde{\iota})}_{L^\infty((0,T_{\max});L^q(\Omega))}$$

$$\times C_{10} \Big(\int_{t_0}^{t} \mathrm{e}^{\Lambda\tau} \|u(\cdot,\tau)\|^{\tilde{p}\tilde{\iota}}_{L^p(\Omega)} d\tau + \mathrm{e}^{\Lambda t_0} \|w(\cdot,t_0)\|^{\tilde{p}\tilde{\iota}}_{p,\tilde{p}\tilde{\iota}} \Big)$$

for all $t \in (t_0, T_{\max})$ with $C_{10} = C_{10}(a_2 - \Lambda/(\tilde{p}\tilde{\iota}), b_2, p, \tilde{p}\tilde{\iota}, n, \Omega) > 0$. Inserting this into (3.7) yields (3.5). $\square$

By embedding in the 4-dimensional case, the boundedness of $\|\Delta v(\cdot,t)\|_{L^6(\Omega)}$ in terms of some temporal average can be inferred with the logistic source exponent $r = 3/2$.

Lemma 3.5. *Suppose that $n = 4$, and f satisfies (1.8) with $r = 3/2$. Then there exists $C > 0$, relying on a_i, b_i $(i = 1, 2)$, Ω, M_2, $\|w(\cdot,t_0)\|_{3/2,3/2}$ and $\|v(\cdot,t_0)\|_{6,3/2}$, such that*

$$\int_{t_0+(t-1-t_0)_+}^{t} \|\Delta v(\cdot,\tau)\|^{\frac{3}{2}}_{L^6(\Omega)} d\tau \leq C \quad \textit{for all } t \in (t_0, T_{\max}).$$

Proof. It follows from (2.8) for w and (3.3) that

$$\begin{aligned}
\int_{t_0+(t-1-t_0)_+}^{t} \|A_{a_2} w(\cdot,\tau)\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(\Omega)} d\tau &\leq C_1 \Big(\sup_{s\in(t_0,t]} \int_{t_0+(s-1-t_0)_+}^{s} \|b_2 u(\cdot,\tau)\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(\Omega)} d\tau + \|w(\cdot,t_0)\|^{\frac{3}{2}}_{\frac{3}{2},\frac{3}{2}} \Big) \\
&\leq C_1 \Big(\sup_{s\in(0,t]} \int_{(s-1)_+}^{s} \|b_2 u(\cdot,\tau)\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(\Omega)} d\tau + \|w(\cdot,t_0)\|^{\frac{3}{2}}_{\frac{3}{2},\frac{3}{2}} \Big) \\
&\leq C_1 \Big(b_2^{\frac{3}{2}} M_2 + \|w(\cdot,t_0)\|^{\frac{3}{2}}_{\frac{3}{2},\frac{3}{2}} \Big) \quad \text{for all } t \in (t_0, T_{\max})
\end{aligned}$$

with $C_1 = C_1(a_2, \Omega) > 0$, which together with the Sobolev embedding theorem and the elliptic regularity estimates yields

$$\begin{aligned}
\int_{t_0+(t-1-t_0)_+}^{t} \|w(\cdot,\tau)\|^{\frac{3}{2}}_{L^6(\Omega)} d\tau &\leq C_2 \int_{t_0+(t-1-t_0)_+}^{t} \|w(\cdot,\tau)\|^{\frac{3}{2}}_{W^{2,\frac{3}{2}}(\Omega)} d\tau \\
&\leq C_3 \int_{t_0+(t-1-t_0)_+}^{t} \|A_{a_2} w(\cdot,\tau)\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(\Omega)} d\tau \\
&\leq C_1 C_3 \Big(b_2^{\frac{3}{2}} M_2 + \|w(\cdot,t_0)\|^{\frac{3}{2}}_{\frac{3}{2},\frac{3}{2}} \Big) =: C_4
\end{aligned}$$

for any $t \in (t_0, T_{\max})$ with $C_2 = C_2(\Omega) > 0$ and $C_3 = C_3(a_2, \Omega) > 0$. Combining this with (2.8) for v, we obtain with some $C_5 = C_5(a_1, \Omega) > 0$ that

$$\int_{t_0+(t-1-t_0)_+}^{t} \|A_{a_1} v(\cdot,\tau)\|_{L^6(\Omega)}^{\frac{3}{2}} d\tau \le C_5 \Big(\sup_{s\in(t_0,t]} \int_{t_0+(s-1-t_0)_+}^{s} \|b_1 w(\cdot,\tau)\|_{L^6(\Omega)}^{\frac{3}{2}} d\tau + \|v(\cdot,t_0)\|_{6,\frac{3}{2}}^{\frac{3}{2}} \Big)$$
$$\le C_5 \Big(b_1^{\frac{3}{2}} C_4 + \|v(\cdot,t_0)\|_{6,\frac{3}{2}}^{\frac{3}{2}} \Big) \quad \text{for all } t\in(t_0,T_{\max}),$$

and hence the elliptic regularity estimates warrant for a certain $C_6 = C_6(a_1,\Omega) > 0$ that

$$\int_{t_0+(t-1-t_0)_+}^{t} \|\Delta v(\cdot,\tau)\|_{L^6(\Omega)}^{\frac{3}{2}} d\tau \le C_6 \int_{t_0+(t-1-t_0)_+}^{t} \|A_{a_1} v(\cdot,\tau)\|_{L^6(\Omega)}^{\frac{3}{2}} d\tau$$
$$\le C_5 C_6 \Big(b_1^{\frac{3}{2}} C_4 + \|v(\cdot,t_0)\|_{6,\frac{3}{2}}^{\frac{3}{2}} \Big)$$

for all $t \in (t_0, T_{\max})$, as claimed. □

4. Proof of main results

This section is devoted to the proof of main results (Theorems 1–4), where a crucial ingredient is that the bound of $u(\cdot,t)$ in $L^p(\Omega)$ with any $p>1$ can be turned into the bound in $L^\infty(\Omega)$.

Lemma 4.1. *Let D and S obey (1.7), and assume that (1.8) holds for f. If $u \in L^\infty((0,T_{\max}); L^p(\Omega))$ for any $p>1$, then $u \in L^\infty(\Omega\times(0,T_{\max}))$, and hence the solution is global and remains bounded in time.*

Proof. Since $u \in L^\infty((0,T_{\max}); L^p(\Omega))$ for any $p>1$, applying Propositions 2.1 (i) and (ii) to w and v, respectively, yields $\nabla v \in L^\infty((0,T_{\max}); (L^q(\Omega))^n)$ with any $q>1$. So, we can further infer by [25, Lemma A.1] that $u \in L^\infty(\Omega\times(0,T_{\max}))$. This in conjunction with the extensibility criterion provided by Lemma 3.1 asserts the global boundedness of solutions. □

As a first step towards obtaining the L^p-boundedness of u for any $p>1$, let us establish a preparatory differential inequality for the energy $\int_\Omega (u+1)^{\bar p}$ with $\bar p>1$ large by testing the first equation in (1.1).

Lemma 4.2. *Suppose that D and S satisfy (1.7), and that f fulfills (1.8). Then*

$$\frac{d}{dt}\int_\Omega (u+1)^{\bar p} \le -\bar p(\bar p-1)a_0 \int_\Omega (u+1)^{\bar p-\alpha-2}|\nabla u|^2 + \frac{\bar p(\bar p-1)b_0}{\bar p+\beta-1}\int_\Omega (u+1)^{\bar p+\beta-1}|\Delta v|$$
$$+ \bar p(b+\mu)\int_\Omega (u+1)^{\bar p-1} - 2^{1-r}\bar p\mu \int_\Omega (u+1)^{\bar p+r-1} \tag{4.1}$$

for all $t\in(0,T_{\max})$ with $\bar p > \max\{1, 1-\beta\}$.

Proof. Let $\bar p > \max\{1,1-\beta\}$. Multiply the first equation in (1.1) by $\bar p(u+1)^{\bar p-1}$ and integrate by parts over Ω to get

$$\frac{d}{dt}\int_\Omega (u+1)^{\bar p} = -\bar p(\bar p-1)\int_\Omega D(u)(u+1)^{\bar p-2}|\nabla u|^2$$
$$-\bar p(\bar p-1)\int_\Omega \chi(u)\Delta v + \bar p \int_\Omega f(u)(u+1)^{\bar p-1}$$

for all $t \in (0, T_{\max})$ with $D(u) \geq a_0(u+1)^{-\alpha}$ and

$$0 \leq \chi(u) = \int_0^u S(\varsigma)(\varsigma+1)^{\bar{p}-2} d\varsigma \leq \frac{b_0}{\bar{p}+\beta-1}(u+1)^{\bar{p}+\beta-1}$$

due to (1.7). Also, it is easy to see that

$$\begin{aligned}\bar{p}\int_\Omega f(u)(u+1)^{\bar{p}-1} &\leq \bar{p}\int_\Omega (b-\mu u^r)(u+1)^{\bar{p}-1} \\ &\leq \bar{p}(b+\mu)\int_\Omega (u+1)^{\bar{p}-1} - 2^{1-r}\bar{p}\mu\int_\Omega (u+1)^{\bar{p}+r-1}\end{aligned}$$

by means of (1.8) and the fundamental inequality $(s+1)^r \leq 2^{r-1}(s^r+1)$ for $s \geq 0$. All relations obtained above immediately give (4.1). □

In Subsections 4.1 and 4.2, we consider the global boundedness of solutions with Theorems 1 and 2 proved for the cases that the cross-diffusion is dominated by the self-diffusion and the logistic source, respectively. And then we in Subsection 4.3 prove Theorem 3 that asserts the global boundedness of solutions for the logistic source balancing the cross-diffusion with the coefficient $\mu > 0$ large. The last subsection is devoted to the investigation for the logistic source and the self-diffusion both balancing the cross-diffusion, and we prove with some additional restrictions that the solutions of (1.1) are globally bounded regardless of the size of $\mu > 0$ as claimed in Theorem 4.

4.1. Self-diffusion dominating cross-diffusion. Proof of Theorem 1

This subsection is concerned with the global boundedness of solutions due to the inhibition of the self-diffusion on the cross-diffusion. A key lemma is given as below.

Lemma 4.3. *Assume that D and S satisfy (1.7), and that (1.8) is valid for f. Also, suppose that $w \in L^\infty((0, T_{\max}); L^q(\Omega))$ for some $q \in [1, \infty)$. If*

$$\alpha + \beta < 1 + \frac{2}{n} - \frac{n}{n+2q}, \tag{4.2}$$

then $u \in L^\infty((0, T_{\max}); L^p(\Omega))$ for any $p > 1$.

Proof. Clearly, it suffices to prove that there exists some $p_* > 1$ such that $u \in L^\infty((0, T_{\max}); L^p(\Omega))$ for all $p > p_*$. Take

$$p_0 := \max\Big\{1-\alpha, \big(\frac{n}{2}-1\big)\alpha, \frac{n}{2}-\alpha, \frac{q(\frac{1}{q}+\frac{2}{n}+1)(2+n-\alpha n-\beta n)}{n}\Big\}.$$

It can be readily checked that

$$\begin{aligned}&p+\alpha, \frac{pn+2}{2+n-\alpha n-\beta n} \in (1,\infty), \\ &\bar{\iota} := \bar{\iota}(p) := \frac{\frac{p}{2} - \frac{p}{2(p+\alpha)}}{\frac{p}{2}+\frac{1}{n}-\frac{1}{2}} \in (0,1),\end{aligned}$$

$$\tilde{\iota}_1 := \tilde{\iota}_1(p) := \frac{\frac{1}{q} - \frac{2+n-\alpha n-\beta n}{pn+2}}{\frac{1}{q} + \frac{2}{n} - \frac{1}{p+\alpha}} \in (0,1) \quad \text{and} \quad \frac{(pn+2)\tilde{\iota}_1(p)}{2+n-\alpha n-\beta n} \in (1,\infty)$$

for all $p > p_0$. Now define for $p > p_0$ that

$$\kappa := \kappa(p) := \frac{p}{(p+\alpha)\bar{\iota}(p)} = \frac{pn+2-n}{n(p+\alpha-1)} \quad \text{and}$$

$$\ell := \ell(p) := \frac{(pn+2)\tilde{\iota}_1(p)}{2+n-\alpha n-\beta n} \cdot \frac{1}{p+\alpha} = \frac{\frac{pn+2}{q(2+n-\alpha n-\beta n)} - 1}{(\frac{1}{q} + \frac{2}{n})(p+\alpha) - 1}.$$

Since $\alpha + \beta < 1 + \frac{2}{n} - \frac{n}{n+2q}$, one can choose $p_0^* > p_0$ large enough such that

$$\ell = \ell(p) < \kappa = \kappa(p) \text{ and } \ell = \ell(p) < 1 \quad \text{for all } p > p_0^*. \tag{4.3}$$

Let $p > p_0^*$. With (4.1) applied to $\bar{p} = p + \alpha$, we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega (u+1)^{p+\alpha} \leq{}& -(p+\alpha)(p+\alpha-1)a_0 \int_\Omega (u+1)^{p-2}|\nabla u|^2 \\ &+ \frac{(p+\alpha)(p+\alpha-1)b_0}{p+\alpha+\beta-1}\int_\Omega (u+1)^{p+\alpha+\beta-1}|\Delta v| \\ &+ (p+\alpha)(b+\mu)\int_\Omega (u+1)^{p+\alpha-1} - 2^{1-r}(p+\alpha)\mu \int_\Omega (u+1)^{p+\alpha+r-1}\end{aligned}$$

for all $t \in (0, T_{\max})$. Clearly,

$$(p+\alpha)(b+\mu)(s+1)^{p+\alpha-1} - 2^{1-r}(p+\alpha)\mu(s+1)^{p+\alpha+r-1} \leq C_1$$

for $s \geq 0$ with $C_1 = C_1(p+\alpha, b, \mu, r) > 0$. Therefore,

$$\begin{aligned}\frac{d}{dt}\int_\Omega (u+1)^{p+\alpha} \leq{}& -(p+\alpha)(p+\alpha-1)a_0 \int_\Omega (u+1)^{p-2}|\nabla u|^2 \\ &+ \frac{(p+\alpha)(p+\alpha-1)b_0}{p+\alpha+\beta-1}\int_\Omega (u+1)^{p+\alpha+\beta-1}|\Delta v| + C_1|\Omega|\end{aligned} \tag{4.4}$$

for all $t \in (0, T_{\max})$. Since $\alpha + \beta < 1 + 2/n$, we have by the Young inequality that

$$\frac{(p+\alpha)(p+\alpha-1)b_0}{p+\alpha+\beta-1}\int_\Omega (u+1)^{p+\alpha+\beta-1}|\Delta v| \leq \epsilon \int_\Omega (u+1)^{p+\frac{2}{n}} + c_\epsilon \int_\Omega |\Delta v|^{\frac{pn+2}{2+n-\alpha n-\beta n}} \tag{4.5}$$

for arbitrary $\epsilon > 0$ with

$$c_\epsilon := \epsilon^{-\frac{n(p+\alpha+\beta-1)}{2+n-n\alpha-n\beta}} \left[\frac{(p+\alpha)(p+\alpha-1)b_0}{p+\alpha+\beta-1}\right]^{\frac{np+2}{2+n-n\alpha-n\beta}}.$$

Moreover, it follows from the Gagliardo-Nirenberg inequality (Lemma 2.3) and (3.2) that

$$\begin{aligned}\int_\Omega (u+1)^{p+\frac{2}{n}} &= \|(u+1)^{\frac{p}{2}}\|_{L^{\frac{2(pn+2)}{pn}}(\Omega)}^{\frac{2(pn+2)}{pn}} \\ &\le C_2\Big(\|\nabla(u+1)^{\frac{p}{2}}\|_{L^2(\Omega)}^{\frac{pn}{pn+2}}\|(u+1)^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2}{pn+2}} + \|(u+1)^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}\Big)^{\frac{2(pn+2)}{pn}} \\ &\le C_3\Big(\int_\Omega (u+1)^{p-2}|\nabla u|^2 + 1\Big) \quad \text{for all } t\in(0,T_{\max}) \end{aligned} \tag{4.6}$$

with $C_2 = C_2(n,p,\Omega) > 0$ and $C_3 = C_3(n,p,\Omega,M_1) > 0$. Combining (4.4)–(4.6) and taking $\epsilon = (p+\alpha)(p+\alpha-1)a_0/(2C_3)$, we get

$$\begin{aligned}\frac{d}{dt}\int_\Omega (u+1)^{p+\alpha} \le& -\frac{(p+\alpha)(p+\alpha-1)a_0}{2}\int_\Omega (u+1)^{p-2}|\nabla u|^2 \\ &+ C_4\int_\Omega |\Delta v|^{\frac{pn+2}{2+n-\alpha n-\beta n}} + C_5 \quad \text{for all } t\in(0,T_{\max}),\end{aligned} \tag{4.7}$$

where

$$C_4 = \left[\frac{(p+\alpha)(p+\alpha-1)a_0}{2C_3}\right]^{-\frac{n(p+\alpha+\beta-1)}{2+n-n\alpha-n\beta}}\left[\frac{(p+\alpha)(p+\alpha-1)b_0}{p+\alpha+\beta-1}\right]^{\frac{np+2}{2+n-n\alpha-n\beta}} \quad \text{and}$$

$$C_5 = \frac{(p+\alpha)(p+\alpha-1)a_0}{2} + C_1|\Omega|.$$

Again by the Gagliardo-Nirenberg inequality and (3.2), we have

$$\begin{aligned}\int_\Omega (u+1)^{p+\alpha} &= \|(u+1)^{\frac{p}{2}}\|_{L^{\frac{2(p+\alpha)}{p}}(\Omega)}^{\frac{2(p+\alpha)}{p}} \\ &\le C_6\Big(\|\nabla(u+1)^{\frac{p}{2}}\|_{L^2(\Omega)}^{\bar\iota}\|(u+1)^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{1-\bar\iota} + \|(u+1)^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}\Big)^{\frac{2(p+\alpha)}{p}} \\ &\le C_7\Big(\|\nabla(u+1)^{\frac{p}{2}}\|_{L^2(\Omega)}^{\frac{2}{\kappa}} + 1\Big) \quad \text{for all } t\in(0,T_{\max})\end{aligned}$$

with $C_6 = C_6(p,p+\alpha,n,\Omega) > 0$ and $C_7 = C_7(p,p+\alpha,n,\Omega,M_1) > 0$, whence

$$\begin{aligned}C_8\Big(\int_\Omega (u+1)^{p+\alpha}\Big)^\kappa &\le \frac{2(p+\alpha)(p+\alpha-1)a_0}{p^2}\Big(\|\nabla(u+1)^{\frac{p}{2}}\|_{L^2(\Omega)}^2 + 1\Big) \\ &= \frac{(p+\alpha)(p+\alpha-1)a_0}{2}\int_\Omega (u+1)^{p-2}|\nabla u|^2 + \frac{2(p+\alpha)(p+\alpha-1)a_0}{p^2}\end{aligned}$$

for all $t\in(0,T_{\max})$ with

$$C_8 = \frac{2(p+\alpha)(p+\alpha-1)a_0}{p^2(C_7)^\kappa\max\{2^{\kappa-1},1\}}.$$

This along with (4.7) results in

$$\frac{d}{dt}\int_\Omega (u+1)^{p+\alpha} + C_8\Big(\int_\Omega (u+1)^{p+\alpha}\Big)^\kappa \le C_4\int_\Omega |\Delta v|^{\frac{pn+2}{2+n-\alpha n-\beta n}} + C_9 \tag{4.8}$$

for all $t \in (0, T_{\max})$ with $C_9 = C_5 + 2(p+\alpha)(p+\alpha-1)a_0/p^2$.

As for the integral on the right-hand side of (4.8), it can be properly controlled by $\int_\Omega u^{p+\alpha}$. Indeed, by Lemma 3.4 and the hypothesis that $w \in L^\infty((0,T_{\max}); L^q(\Omega))$, we have

$$\int\limits_{t_0+(t-1-t_0)_+}^{t} \Big(\int\limits_\Omega |\Delta v(x,\tau)|^{\frac{pn+2}{2+n-\alpha n-\beta n}} dx\Big) d\tau \le C_{10}\Big(\sup_{\tau\in(t_0,t)} \|u(\cdot,\tau)\|_{L^{p+\alpha}(\Omega)}^{\frac{(pn+2)\tilde{\iota}_1}{2+n-\alpha n-\beta n}} + 1\Big)$$
$$= C_{10}\Big(\Big(\sup_{\tau\in(t_0,t)} \int\limits_\Omega u^{p+\alpha}(\cdot,\tau)\Big)^{\ell} + 1\Big) \tag{4.9}$$

for all $t \in (t_0, T_{\max})$ with some $C_{10} > 0$ relying on a_i, b_i $(i=1,2)$, $\frac{pn+2}{2+n-\alpha n-\beta n}$, $p+\alpha$, q, n, Ω, $\|w\|_{L^\infty((0,T_{\max});L^q(\Omega))}$, $\|w(\cdot,t_0)\|_{p+\alpha,(p+\alpha)\ell}$ and $\|v(\cdot,t_0)\|_{\frac{pn+2}{2+n-\alpha n-\beta n},\frac{pn+2}{2+n-\alpha n-\beta n}}$, and it also holds that

$$\int\limits_{t_0}^{t} e^{\Lambda\tau}\Big(\int\limits_\Omega |\Delta v(x,\tau)|^{\frac{pn+2}{2+n-\alpha n-\beta n}} dx\Big) d\tau \le C_{11}\Big(\int\limits_{t_0}^{t} e^{\Lambda\tau}\|u(\cdot,\tau)\|_{L^{p+\alpha}(\Omega)}^{\frac{(pn+2)\tilde{\iota}_1}{2+n-\alpha n-\beta n}} d\tau + 1\Big)$$
$$= C_{11}\Big(\int\limits_{t_0}^{t} e^{\Lambda\tau}\Big(\int\limits_\Omega u^{p+\alpha}(x,\tau)dx\Big)^{\ell} d\tau + 1\Big) \tag{4.10}$$

for all $t \in (t_0, T_{\max})$ with

$$\Lambda = \frac{1}{2}\min\Big\{\frac{(pn+2)a_1}{2+n-\alpha n-\beta n}, \frac{(pn+2)\tilde{\iota}_1 a_2}{2+n-\alpha n-\beta n}\Big\}$$

and $C_{11} > 0$ related to a_i, b_i $(i=1,2)$, $\frac{pn+2}{2+n-\alpha n-\beta n}$, $p+\alpha$, q, n, Ω, $\|w\|_{L^\infty((0,T_{\max});L^q(\Omega))}$, t_0, $\|w(\cdot,t_0)\|_{p+\alpha,(p+\alpha)\ell}$ and $\|v(\cdot,t_0)\|_{\frac{pn+2}{2+n-\alpha n-\beta n},\frac{pn+2}{2+n-\alpha n-\beta n}}$.

We next assert that $u \in L^\infty((t_0,T_{\max}); L^{p+\alpha}(\Omega))$. The proof is divided into two cases.

Case 1. If $\alpha \ge \frac{2}{n}$, then $\kappa \le 1$. Combining (4.8) with (4.9) and noticing that $0 < \ell < \kappa$ by (4.3), we know from Lemma 2.4 (i) that $\int_\Omega (u+1)^{p+\alpha}(\cdot,t)$ is bounded in $(t_0, T_{\max})$, and hence $u \in L^\infty((t_0,T_{\max}); L^{p+\alpha}(\Omega))$.

Case 2. Assume that $\alpha < \frac{2}{n}$, and so $\kappa > 1$. According to (4.8) and (4.10), and since $\ell < 1$ for (4.3), Lemma 2.4 (ii) entails the boundedness of $\int_\Omega (u+1)^{p+\alpha}(\cdot,t)$ in $(t_0,T_{\max})$, as desired.

All in all, we gain that $u \in L^\infty((t_0,T_{\max}); L^{p+\alpha}(\Omega))$ for all $p > p_0^*$. By the continuity of u on $\bar{\Omega}\times[0,t_0]$, we thus have $u \in L^\infty((0,T_{\max}); L^{p+\alpha}(\Omega))$ for all $p > p_0^*$, that is, $u \in L^\infty((0,T_{\max}); L^p(\Omega))$ for all $p > p_* := p_0^* + \alpha$. The proof is complete. □

Now, we can easily prove Theorem 1.

Proof of Theorem 1. Note that the condition (1.9) is equivalent to

$$\alpha + \beta < 1 + \frac{2}{n} - \frac{n}{n + 2\max\big\{\frac{n}{(n-2)_+}, \frac{nr}{(n+2-2r)_+}\big\}},$$

which allows us to choose $1 \le q < \max\{n/(n-2)_+, nr/(n+2-2r)_+\}$ such that

$$\alpha + \beta < 1 + \frac{2}{n} - \frac{n}{n+2q}.$$

By Lemma 3.3 (i), we know that $w \in L^\infty((0,T_{\max});L^q(\Omega))$ for q taken above. Therefore, Lemma 4.3 entails that $u \in L^\infty((0,T_{\max});L^p(\Omega))$ for all $p>1$. This results in the global boundedness of solutions due to Lemma 4.1. The proof is complete. $\square$

4.2. Logistic source suppressing cross-diffusion. Proof of Theorem 2

In this subsection, we consider the blow-up prevention by logistic kinetics. To this end, we seek appropriate conditions on the logistic exponent r that warrant the L^p-boundedness of u with $p>1$ arbitrary.

Lemma 4.4. *Assume that D and S satisfy (1.7), and that (1.8) is valid for f. Also, suppose that $w \in L^\infty((0,T_{\max});L^q(\Omega))$ for some $q\in[1,\infty)$. If*

$$r > \beta + \frac{n}{n+2q}, \tag{4.11}$$

then $u \in L^\infty((0,T_{\max});L^p(\Omega))$ for any $p>1$.

Proof. We only need to prove that there exists $p^*>1$ such that for any $p>p^*$, $\|u(\cdot,t)\|_{L^p(\Omega)}$ is bounded in t. A direct computation shows

$$\tilde{\imath}_2 := \tilde{\imath}_2(p) := \frac{\frac{1}{q} - \frac{r-\beta}{p+r-\beta}}{\frac{1}{q}+\frac{2}{n}-\frac{1}{p+r-\beta}} \in (0,1) \quad \text{and} \quad \frac{(p+r-\beta)\tilde{\imath}_2(p)}{r-\beta} \in (1,\infty)$$

when

$$p > p_1 := \max\Big\{|\beta|, \frac{n}{2}-r+\beta, q(\frac{2}{n}+1)(r-\beta)\Big\}.$$

Also, the condition $r>\beta+n/(n+2q)$ guarantees

$$\tilde{\ell} := \tilde{\ell}(p) := \frac{\tilde{\imath}_2(p)}{r-\beta} < 1 \quad \text{for all } p>p_1.$$

Let $p>p_1$. By (4.1), we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega (u+1)^{p-\beta+1} \le{}& \frac{(p-\beta+1)(p-\beta)b_0}{p}\int_\Omega (u+1)^p|\Delta v| \\ &+ (p-\beta+1)(b+\mu)\int_\Omega (u+1)^{p-\beta} - 2^{1-r}(p-\beta+1)\mu\int_\Omega (u+1)^{p+r-\beta}\end{aligned}$$

for all $t\in(0,T_{\max})$, where an application of the Young inequality leads to

$$\begin{aligned}\frac{(p-\beta)b_0}{p}\int_\Omega (u+1)^p|\Delta v| &\le 2b_0\int_\Omega (u+1)^p|\Delta v| \\ &\le 2^{-r-1}\mu\int_\Omega (u+1)^{p+r-\beta} + C_1\int_\Omega |\Delta v|^{\frac{p+r-\beta}{r-\beta}}\end{aligned}$$

with $C_1 = (2b_0)^{\frac{p+r-\beta}{r-\beta}}(2^{-r-1}\mu)^{-\frac{p}{r-\beta}}$, and

$$(b+\mu)\int_\Omega (u+1)^{p-\beta} \le 2^{-r-1}\mu\int_\Omega (u+1)^{p+r-\beta} + C_2$$

with $C_2 = (b+\mu)^{\frac{p+r-\beta}{r}}(2^{-r-1}\mu)^{-\frac{p-\beta}{r}}|\Omega|$. All these therefore yield

$$\frac{d}{dt}\int_\Omega (u+1)^{p-\beta+1} \le (p-\beta+1)\Big(C_1\int_\Omega |\Delta v|^{\frac{p+r-\beta}{r-\beta}} - 2^{-r}\mu\int_\Omega (u+1)^{p+r-\beta} + C_2\Big),$$

and thereby

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (u+1)^{p-\beta+1} + \tilde\Lambda\int_\Omega (u+1)^{p+1-\beta}\\
&\le (p-\beta+1)\Big(C_1\int_\Omega |\Delta v|^{\frac{p+r-\beta}{r-\beta}} - 2^{-r-1}\mu\int_\Omega (u+1)^{p+r-\beta} + C_2\Big)\\
&=: (p-\beta+1)(\tilde h(t) + C_2)
\end{aligned}\tag{4.12}$$

for all $t\in(0,T_{\max})$ with

$$\tilde\Lambda = \frac12\min\Big\{\frac{(p+r-\beta)a_1}{r-\beta}, \frac{(p+r-\beta)\tilde\iota_2 a_2}{r-\beta}, (p-\beta+1)2^{-r}\mu\Big\}.$$

Moreover, we have from Lemma 3.4 (ii) that

$$\begin{aligned}
\int_{t_0}^t e^{\tilde\Lambda\tau}\Big(\int_\Omega |\Delta v(x,\tau)|^{\frac{p+r-\beta}{r-\beta}}dx\Big)d\tau &\le C_3\Big(\int_{t_0}^t e^{\tilde\Lambda\tau}\|u(\cdot,\tau)\|_{L^{p+r-\beta}(\Omega)}^{\frac{(p+r-\beta)\tilde\iota_2}{r-\beta}}d\tau + 1\Big)\\
&\le C_3\Big(\int_{t_0}^t e^{\tilde\Lambda\tau}\Big(\int_\Omega (u+1)^{p+r-\beta}(x,\tau)dx\Big)^{\tilde\ell}d\tau + 1\Big)
\end{aligned}$$

for all $t\in(t_0,T_{\max})$, where $C_3>0$ depends on a_i, b_i $(i=1,2)$, $\tilde\Lambda$, $\frac{p+r-\beta}{r-\beta}$, $p+r-\beta$, q, n, Ω, $\|w\|_{L^\infty((0,T_{\max});L^q(\Omega))}$, t_0, $\|w(\cdot,t_0)\|_{p+r-\beta,(p+r-\beta)\tilde\ell}$ and $\|v(\cdot,t_0)\|_{\frac{p+r-\beta}{r-\beta},\frac{p+r-\beta}{r-\beta}}$. Since $\tilde\ell<1$, this further implies by the Young inequality that

$$\begin{aligned}
&\int_{t_0}^t e^{\tilde\Lambda\tau}\Big(\int_\Omega |\Delta v(x,\tau)|^{\frac{p+r-\beta}{r-\beta}}dx\Big)d\tau\\
&\le \frac{2^{-r-1}\mu}{C_1}\int_{t_0}^t e^{\tilde\Lambda\tau}\Big(\int_\Omega (u+1)^{p+r-\beta}(x,\tau)dx\Big)d\tau + C_4\int_{t_0}^t e^{\tilde\Lambda\tau}d\tau + C_3\\
&\le \frac{2^{-r-1}\mu}{C_1}\int_{t_0}^t e^{\tilde\Lambda\tau}\Big(\int_\Omega (u+1)^{p+r-\beta}(x,\tau)dx\Big)d\tau + C_4\tilde\Lambda^{-1}e^{\tilde\Lambda t} + C_3
\end{aligned}$$

for all $t\in(t_0,T_{\max})$ with $C_4 = C_3^{\frac{1}{1-\tilde\ell}}(2^{-r-1}\mu/C_1)^{-\frac{\tilde\ell}{1-\tilde\ell}}$, whence

$$\int_{t_0}^{t} e^{\tilde{\Lambda}\tau}\tilde{h}(\tau)d\tau \le C_1(C_4\tilde{\Lambda}^{-1}e^{\tilde{\Lambda}t} + C_3) \quad \text{for all } t \in (t_0, T_{\max}).$$

From this we obtain by integrating (4.12) that

$$\begin{aligned}\int_\Omega (u+1)^{p-\beta+1}(\cdot,t) &\le \int_\Omega (u+1)^{p-\beta+1}(\cdot,t_0) + (p-\beta+1)e^{-\tilde{\Lambda}t}\int_{t_0}^{t} e^{\tilde{\Lambda}\tau}(\tilde{h}(\tau)+C_2)d\tau \\ &\le \int_\Omega (u+1)^{p-\beta+1}(\cdot,t_0) + (p-\beta+1)(C_1C_4\tilde{\Lambda}^{-1} + C_1C_3 + C_2\tilde{\Lambda}^{-1})\end{aligned}$$

for all $t \in (t_0, T_{\max})$, and hence $u \in L^\infty((t_0, T_{\max}); L^{p+1-\beta}(\Omega))$. By the continuity of u on $\bar{\Omega} \times [0, t_0]$, we have $u \in L^\infty((0, T_{\max}); L^{p+1-\beta}(\Omega))$. So, it has proved $u \in L^\infty((0, T_{\max}); L^p(\Omega))$ for all $p > p^* := p_1 + 1 - \beta$, as expected. □

Along with Lemma 3.3 (i), we give the proof of Theorem 2.

Proof of Theorem 2. Direct computations show that

$$r > \beta + \frac{n}{n + \frac{2nr}{(n+2-2r)_+}} \Leftrightarrow r > \beta + \frac{(n+2-2\beta)_+}{n+4},$$

from which we can easily check that the condition (1.10) agrees with

$$r > \beta + \frac{n}{n + 2\max\left\{\frac{n}{(n-2)_+}, \frac{nr}{(n+2-2r)_+}\right\}}.$$

This enables us to take $1 \le q < \max\{n/(n-2)_+, nr/(n+2-2r)_+\}$ such that

$$r > \beta + \frac{n}{n+2q}.$$

For such q, $w \in L^\infty((0, T_{\max}); L^q(\Omega))$ by Lemma 3.3 (i). Therefore, $u \in L^\infty((0, T_{\max}); L^p(\Omega))$ for all $p > 1$ in view of Lemma 4.4, which together with Lemma 4.1 proves Theorem 2. □

4.3. Logistic source balancing cross-diffusion. Proof of Theorem 3

When the logistic source balances the cross-diffusion, as asserted in Theorem 3, the solutions of (1.1) remain bounded in time if the coefficient $\mu > 0$ is properly large. Now we prove it.

Proof of Theorem 3. Note that the condition (1.13) entails $r = (\beta+1)(n+2)/(n+4) \in [(n+2)/n, (n+2)/2)$ with $n \ge 3$. Given $m_* > 0$, we know from Lemma 3.3 (ii) that for any nonnegative $(u_0, v_0, w_0) \in C^\omega(\bar{\Omega}) \times [W^{1,\infty}(\Omega)]^2$ $(0 < \omega < 1)$ satisfying

(a) $\|u_0\|_{L^r(\Omega)} \le m_*$ and $\|w_0\|_{W^{1,\infty}(\Omega)} \le m_*$ with $r \le 2$, or
(b) $\|u_0\|_{L^1(\Omega)} \le m_*$ and $\|w_0\|_{W^{2,r}(\Omega)} \le m_*$ with $r > 2$ and $w_0 \in W_N^{2,r}$,

it holds that

$$\|w(\cdot,t)\|_{L^q(\Omega)} \le M_\mu \quad \text{for all } t \in (0, T_{\max}), \tag{4.13}$$

where

$$q = \frac{nr}{n+2-2r},$$

and

$$M_\mu = \begin{cases} L_1\Big(\Big[\frac{1}{\mu}\Big(b|\Omega| + \max\big\{|\Omega|^{1-\frac{1}{r}}m_*, (\frac{b}{\mu})^{\frac{1}{r}}|\Omega|\big\}\Big)\Big]^{\frac{1}{r}} + 2m_* + 1\Big) & \text{for case (a),} \\ L_2\Big(\Big[\frac{1}{\mu}\Big(b|\Omega| + \max\big\{m_*, (\frac{b}{\mu})^{\frac{1}{r}}|\Omega|\big\}\Big)\Big]^{\frac{1}{r}} + m_*\Big) & \text{for case (b)} \end{cases}$$

with the constants $L_1, L_2 > 0$ depending only on a_2, b_2, r, n, Ω and q.

Fix

$$\hat{p} > \frac{n(1-r+\beta)}{2} + \beta - 1,$$

and set

$$\tilde{\imath}_3 := \frac{\frac{1}{q} - \frac{r-\beta}{\hat{p}+r-\beta}}{\frac{1}{q} + \frac{2}{n} - \frac{1}{\hat{p}+r-\beta}}.$$

Since $r = (\beta+1)(n+2)/(n+4)$ and $4/n \le \beta < (n+2)/2$, we have

$$\hat{p} - \beta > \frac{n(1-r+\beta)}{2} - 1 = \frac{n\beta - 4}{n+4} \ge 0 \quad \text{and} \quad \tilde{\imath}_3 = r - \beta = \frac{n+2-2\beta}{n+4} \in \big(0, \frac{n}{n+2}\big).$$

Now, invoking (4.1) with $\bar{p} = \hat{p} - \beta + 1$ yields

$$\begin{aligned} \frac{d}{dt}\int_\Omega (u+1)^{\hat{p}-\beta+1} &\le \frac{(\hat{p}-\beta+1)(\hat{p}-\beta)b_0}{\hat{p}}\int_\Omega (u+1)^{\hat{p}}|\Delta v| \\ &\quad + (\hat{p}-\beta+1)\Big((b+\mu)\int_\Omega (u+1)^{\hat{p}-\beta} - 2^{1-r}\mu\int_\Omega (u+1)^{\hat{p}+r-\beta}\Big) \\ &\le (\hat{p}-\beta+1)\Big(b_0\int_\Omega (u+1)^{\hat{p}}|\Delta v| \\ &\quad + (b+\mu)\int_\Omega (u+1)^{\hat{p}-\beta} - 2^{1-r}\mu\int_\Omega (u+1)^{\hat{p}+r-\beta}\Big) \end{aligned}$$

for all $t \in (0, T_{\max})$. Furthermore, by the same proof as in Lemma 4.4 with the Young inequality used, we gain

$$\begin{aligned} &\frac{d}{dt}\int_\Omega (u+1)^{\hat{p}-\beta+1} \\ &\quad \le (\hat{p}-\beta+1)\Big(b_0^{\frac{\hat{p}+r-\beta}{r-\beta}}(2^{-r-1}\mu)^{-\frac{\hat{p}}{r-\beta}}\int_\Omega |\Delta v|^{\frac{\hat{p}+r-\beta}{r-\beta}} - 2^{-r}\mu\int_\Omega (u+1)^{\hat{p}+r-\beta} + C_1\Big) \end{aligned} \tag{4.14}$$

for all $t \in (0, T_{\max})$ with $C_1 = (b+\mu)^{\frac{\hat{p}+r-\beta}{r}}(2^{-r-1}\mu)^{-\frac{\hat{p}-\beta}{r}}|\Omega|$. In view of the Hölder inequality, we have

$$\int_\Omega (u+1)^{\hat{p}+r-\beta} \ge |\Omega|^{-\frac{r-1}{\hat{p}+1-\beta}}\Big(\int_\Omega (u+1)^{\hat{p}+1-\beta}\Big)^{\hat{\kappa}} \quad \text{for all } t \in (0, T_{\max})$$

with

$$\hat{\kappa} = \frac{\hat{p}+r-\beta}{\hat{p}+1-\beta} > 1.$$

Combining this with (4.14) results in

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (u+1)^{\hat{p}-\beta+1} + c_1\Big(\int_\Omega (u+1)^{\hat{p}+1-\beta}\Big)^{\hat{\kappa}} \\
&\le (\hat{p}-\beta+1)\Big(b_0^{\frac{\hat{p}+r-\beta}{r-\beta}}(2^{-r-1}\mu)^{-\frac{\hat{p}}{r-\beta}}\int_\Omega |\Delta v|^{\frac{\hat{p}+r-\beta}{r-\beta}} - 2^{-r-1}\mu\int_\Omega (u+1)^{\hat{p}+r-\beta} + C_1\Big) \\
&=: (\hat{p}-\beta+1)(\hat{h}(t)+C_1) \qquad (4.15)
\end{aligned}$$

for all $t \in (0, T_{\max})$ with $c_1 = (\hat{p}-\beta+1)2^{-r-1}\mu|\Omega|^{-\frac{r-1}{\hat{p}+1-\beta}}$. We proceed to estimate $\hat{h}(t)$. Lemma 3.4 (ii) shows that there exists $L_3 > 0$, related to a_i, b_i $(i = 1, 2)$, $(\hat{p}+r-\beta)/(r-\beta)$, $\hat{p}+r-\beta$, q, n and Ω, such that

$$\begin{aligned}
&\int_{t_0}^{t} \mathrm{e}^{\hat{\Lambda}\tau}\Big(\int_\Omega |\Delta v(x,\tau)|^{\frac{\hat{p}+r-\beta}{r-\beta}}dx\Big)d\tau \\
&\le L_3\Big\{\|w\|_{L^\infty((0,T_{\max});L^q(\Omega))}^{\frac{(\hat{p}+r-\beta)(1-\tilde{\iota}_3)}{r-\beta}}\Big(\int_{t_0}^{t}\mathrm{e}^{\hat{\Lambda}\tau}\|u(\cdot,\tau)\|_{L^{\hat{p}+r-\beta}(\Omega)}^{\frac{(\hat{p}+r-\beta)\tilde{\iota}_3}{r-\beta}}d\tau \\
&\quad + \mathrm{e}^{\hat{\Lambda}t_0}\|w(\cdot,t_0)\|_{\hat{p}+r-\beta,\frac{(\hat{p}+r-\beta)\tilde{\iota}_3}{r-\beta}}^{\frac{(\hat{p}+r-\beta)\tilde{\iota}_3}{r-\beta}}\Big) + \mathrm{e}^{\hat{\Lambda}t_0}\|v(\cdot,t_0)\|_{\frac{\hat{p}+r-\beta}{r-\beta},\frac{\hat{p}+r-\beta}{r-\beta}}^{\frac{\hat{p}+r-\beta}{r-\beta}}\Big\} \\
&\le L_3\Big\{M_\mu^{\frac{(\hat{p}+r-\beta)(1-r+\beta)}{r-\beta}}\Big(\int_{t_0}^{t}\mathrm{e}^{\hat{\Lambda}\tau}\int_\Omega (u+1)^{\hat{p}+r-\beta}(x,\tau)dxd\tau \\
&\quad + \mathrm{e}^{\hat{\Lambda}t_0}\|w(\cdot,t_0)\|_{\hat{p}+r-\beta,\hat{p}+r-\beta}^{\hat{p}+r-\beta}\Big) + \mathrm{e}^{\hat{\Lambda}t_0}\|v(\cdot,t_0)\|_{\frac{\hat{p}+r-\beta}{r-\beta},\frac{\hat{p}+r-\beta}{r-\beta}}^{\frac{\hat{p}+r-\beta}{r-\beta}}\Big\} \qquad (4.16)
\end{aligned}$$

for all $t \in (t_0, T_{\max})$, where

$$\hat{\Lambda} = \frac{1}{2}\min\Big\{\frac{(\hat{p}+r-\beta)a_1}{r-\beta}, (\hat{p}+r-\beta)a_2\Big\},$$

and the second inequality is due to (4.13). By the expression of M_μ, we can take $\mu_* > 0$ large enough such that when $\mu > \mu_*$,

$$b_0^{\frac{\hat{p}+r-\beta}{r-\beta}}(2^{-r-1}\mu)^{-\frac{\hat{p}}{r-\beta}}L_3 M_\mu^{\frac{(\hat{p}+r-\beta)(1-r+\beta)}{r-\beta}} \le 2^{-r-1}\mu,$$

and hence (4.16) turns into

$$\begin{aligned}
\int_{t_0}^{t}\mathrm{e}^{\hat{\Lambda}\tau}\hat{h}(\tau)d\tau &\le b_0^{\frac{\hat{p}+r-\beta}{r-\beta}}(2^{-r-1}\mu)^{-\frac{\hat{p}}{r-\beta}} \\
&\quad \times L_3\Big\{M_\mu^{\frac{(\hat{p}+r-\beta)(1-r+\beta)}{r-\beta}}\mathrm{e}^{\hat{\Lambda}t_0}\|w(\cdot,t_0)\|_{\hat{p}+r-\beta,\hat{p}+r-\beta}^{\hat{p}+r-\beta} + \mathrm{e}^{\hat{\Lambda}t_0}\|v(\cdot,t_0)\|_{\frac{\hat{p}+r-\beta}{r-\beta},\frac{\hat{p}+r-\beta}{r-\beta}}^{\frac{\hat{p}+r-\beta}{r-\beta}}\Big\} \\
&=: C_2
\end{aligned}$$

for all $t \in (t_0, T_{\max})$, which together with (4.15) implies $u \in L^\infty((t_0, T_{\max}); L^{\hat{p}-\beta+1}(\Omega))$ thanks to Lemma 2.4 (ii). So, $u \in L^\infty((0, T_{\max}); L^{\hat{p}-\beta+1}(\Omega))$ by the continuity of u on $\bar{\Omega} \times [0, t_0]$.

Next, with $\mu > \mu_*$, we further assert the L^p-boundedness of u for any $p > 1$, and thereby achieve the L^∞-boundedness of u. It is known from the choice of $\hat{p}$ that

$$\frac{1}{\hat{p}-\beta+1} - \frac{2(r-\beta)}{n(1-r+\beta)} < \frac{2}{n},$$

which allows us to pick

$$\hat{q} > \frac{n(1-r+\beta)}{2(r-\beta)}, \quad \text{or equivalently,} \ \ r > \beta + \frac{n}{n+2\hat{q}}$$

such that

$$\frac{1}{\hat{p}-\beta+1} - \frac{1}{\hat{q}} < \frac{2}{n}.$$

Since $u \in L^\infty((0, T_{\max}); L^{\hat{p}-\beta+1}(\Omega))$ under $\mu > \mu_*$, we have $w \in L^\infty((0, T_{\max}); L^{\hat{q}}(\Omega))$ by Proposition 2.1 (i), and furthermore $u \in L^\infty((0, T_{\max}); L^p(\Omega))$ for any $p > 1$ according to Lemma 4.4. The proof is complete. □

4.4. Self-diffusion and logistic source both balancing cross-diffusion. Proof of Theorem 4

Finally, we focus on the proof of Theorem 4 concerning the global boundedness of solutions for the case that the self-diffusion and the logistic source both balance the cross-diffusion.

Proof of Theorem 4. Since $\alpha = 0$, $\beta = 1$ and $r = 3/2$, (4.1) with $\bar{p} = 3/2$ takes the form

$$\begin{aligned} \frac{d}{dt}\int_\Omega (u+1)^{\frac{3}{2}} \le& -\frac{4}{3}a_0 \int_\Omega |\nabla(u+1)^{\frac{3}{4}}|^2 + \frac{1}{2}b_0 \int_\Omega (u+1)^{\frac{3}{2}}|\Delta v| \\ &+ \frac{3}{2}(b+\mu)\int_\Omega (u+1)^{\frac{1}{2}} - \frac{3\sqrt{2}}{4}\mu \int_\Omega (u+1)^2 \end{aligned} \tag{4.17}$$

for all $t \in (0, T_{\max})$. By the Hölder inequality, the Gagliardo-Nirenberg inequality in dimension $n = 4$ and the Young inequality, we derive

$$\begin{aligned} &\frac{1}{2}b_0 \int_\Omega (u+1)^{\frac{3}{2}}|\Delta v| \\ &\le \frac{1}{2}b_0 \|(u+1)^{\frac{3}{2}}\|_{L^{\frac{6}{5}}(\Omega)} \|\Delta v\|_{L^6(\Omega)} \\ &= \frac{1}{2}b_0 \|(u+1)^{\frac{3}{4}}\|^2_{L^{\frac{12}{5}}(\Omega)} \|\Delta v\|_{L^6(\Omega)} \\ &\le \frac{1}{2}b_0 C_1 \Big(\|\nabla(u+1)^{\frac{3}{4}}\|^{\frac{2}{3}}_{L^2(\Omega)} \|(u+1)^{\frac{3}{4}}\|^{\frac{4}{3}}_{L^2(\Omega)} + \|(u+1)^{\frac{3}{4}}\|^2_{L^2(\Omega)} \Big) \|\Delta v\|_{L^6(\Omega)} \\ &\le \frac{4}{3}a_0 \|\nabla(u+1)^{\frac{3}{4}}\|^2_{L^2(\Omega)} + \frac{\sqrt{2}}{12} a_0^{-\frac{1}{2}} (b_0 C_1)^{\frac{3}{2}} \|(u+1)^{\frac{3}{4}}\|^2_{L^2(\Omega)} \|\Delta v\|^{\frac{3}{2}}_{L^6(\Omega)} \\ &\quad + \frac{1}{2}b_0 C_1 \|(u+1)^{\frac{3}{4}}\|^2_{L^2(\Omega)} \Big(\frac{2}{3}\|\Delta v\|^{\frac{3}{2}}_{L^6(\Omega)} + \frac{1}{3} \Big) \end{aligned}$$

for all $t \in (0, T_{\max})$ with $C_1 = C_1(\Omega) > 0$. Inserting it into (4.17), we gain

$$\frac{d}{dt}\int_\Omega (u+1)^{\frac{3}{2}} \le C_2\|\Delta v\|^{\frac{3}{2}}_{L^6(\Omega)}\int_\Omega (u+1)^{\frac{3}{2}} + \frac{1}{6}b_0C_1\int_\Omega (u+1)^{\frac{3}{2}} + \frac{3}{2}(b+\mu)\int_\Omega (u+1)^{\frac{1}{2}} - \frac{3\sqrt{2}}{4}\mu\int_\Omega (u+1)^2$$

for all $t\in(0,T_{\max})$, where $C_2=(\sqrt{2}/12)a_0^{-\frac{1}{2}}(b_0C_1)^{\frac{3}{2}}+(1/3)b_0C_1$. Because

$$\frac{1}{6}b_0C_1(s+1)^{\frac{3}{2}}+\frac{3}{2}(b+\mu)(s+1)^{\frac{1}{2}}-\frac{3\sqrt{2}}{4}\mu(s+1)^2\le C_3 \quad \text{for } s\ge 0$$

with $C_3>0$ relying on b_0, b, μ and C_1, we further have

$$\frac{d}{dt}\int_\Omega (u+1)^{\frac{3}{2}} \le C_2\|\Delta v\|^{\frac{3}{2}}_{L^6(\Omega)}\int_\Omega (u+1)^{\frac{3}{2}} + C_3|\Omega| \quad \text{for all } t\in(0,T_{\max}).$$

Note that it follows from Lemma 3.2 that

$$\int_{t_0+(t-1-t_0)_+}^{t}\int_\Omega (u+1)^{\frac{3}{2}}(x,\tau)\,dxd\tau \le \int_{(t-1)_+}^{t}\int_\Omega (u+1)^{\frac{3}{2}}(x,\tau)\,dxd\tau \le \sqrt{2}(M_2+|\Omega|) \quad \text{for all } t\in(t_0,T_{\max}),$$

and Lemma 3.5 gives

$$\int_{t_0+(t-1-t_0)_+}^{t} \|\Delta v(\cdot,\tau)\|^{\frac{3}{2}}_{L^6(\Omega)}d\tau \le C_4 \quad \text{for all } t\in(t_0,T_{\max})$$

with $C_4>0$ depending on a_i,b_i $(i=1,2)$, Ω, M_2, $\|w(\cdot,t_0)\|_{3/2,3/2}$ and $\|v(\cdot,t_0)\|_{6,3/2}$. Thus we can know by Lemma 2.5 that $u\in L^\infty((t_0,T_{\max});L^{\frac{3}{2}}(\Omega))$, and so $u\in L^\infty((0,T_{\max});L^{\frac{3}{2}}(\Omega))$. Moreover, applying Proposition 2.1 (i), we get $w\in L^\infty((0,T_{\max});L^q(\Omega))$ with $q\in[1,6)$. Fix $q\in(2,6)$. Then

$$\alpha+\beta<1+\frac{2}{n}-\frac{n}{n+2q}$$

with $\alpha=0$, $\beta=1$ and $n=4$. By Lemma 4.3, $u\in L^\infty((0,T_{\max});L^p(\Omega))$ for all $p>1$. Hence invoking Lemma 4.1 yields the global boundedness of solutions. □

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