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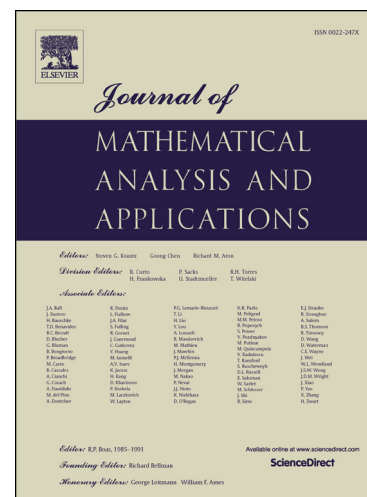
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# Finite time blowup of solutions to semilinear wave equation in an exterior domain

Motohiro Sobajima<sup>\*†</sup> and Kyouhei Wakasa<sup>‡</sup>

**Abstract.** We consider the initial-boundary value problem of semilinear wave equation with nonlinearity  $|u|^p$  in exterior domain in  $\mathbb{R}^N$  ( $N \geq 3$ ). Especially, the lifespan of blowup solutions with small initial data are studied. The result gives upper bounds of lifespan which is essentially the same as the Cauchy problem in  $\mathbb{R}^N$ . At least in the case  $N = 4$ , their estimates are sharp in view of the work by Zha–Zhou [22]. The idea of the proof is to use special solutions to linear wave equation with Dirichlet boundary condition which are constructed via an argument based on Wakasa–Yordanov [16].

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*Key words and phrases:* Semilinear wave equations, blowup, upper bound of lifespan, exterior domain

## 1 Introduction

In this paper we consider the semilinear wave equations in an exterior domain in  $\mathbb{R}^N$  ( $N \geq 3$ ):

$$\begin{cases} \partial_t^2 u(x, t) - \Delta u(x, t) = |u(x, t)|^p, & (x, t) \in \Omega \times (0, T), \\ u(x, t) = 0 & (x, t) \in \partial\Omega \times (0, T), \\ u(x, 0) = \varepsilon f(x) & x \in \Omega, \\ \partial_t u(x, 0) = \varepsilon g(x) & x \in \Omega, \end{cases} \quad (1.1)$$

where  $\partial_t = \partial/\partial t$ ,  $\Delta = \sum_{j=1}^N \partial^2/\partial x_j^2$ ,  $1 < p \leq \frac{N}{N-2}$ ,  $T > 0$  and  $\Omega \subset \mathbb{R}^N$  satisfies that  $\mathbb{R}^N \setminus \Omega$  is bounded and  $\partial\Omega$  is a smooth boundary. The pair  $(f, g)$  is given (the shape of) initial data and the parameter  $\varepsilon > 0$  describes the size (smallness) of the initial data. The interest of the present paper is to study the profile of solutions to (1.1) with sufficiently small initial data. Here the pair  $(f, g)$  satisfies

$$(f, g) \in C_0^\infty(\Omega), \quad \text{supp}(f, g) \subset \overline{B(0, r_0)}, \quad (1.2)$$

where  $C_0^\infty(K)$  is the set of all smooth functions supported in  $K$  and  $B(0, r) = \{x \in \mathbb{R}^N; |x| < r\}$ .

This kind of study of global existence and blowup of solutions to (1.1) has been discussed since the pioneering work of John [6] when  $\Omega = \mathbb{R}^N$  with  $N = 3$ :

$$\begin{cases} \partial_t^2 u(x, t) - \Delta u(x, t) = |u(x, t)|^p, & (x, t) \in \mathbb{R}^N \times (0, T), \\ u(x, 0) = \varepsilon f(x) & x \in \mathbb{R}^N, \\ \partial_t u(x, 0) = \varepsilon g(x) & x \in \mathbb{R}^N. \end{cases} \quad (1.3)$$

It is shown in [6] that the critical exponent of (1.3) is determined as  $p = 1 + \sqrt{2}$ , that is,

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- if  $1 < p < 1 + \sqrt{2}$ , then the solution of (1.3) blows up in finite time for “positive” initial data;
- if  $p > 1 + \sqrt{2}$ , then there exists a global solution with small initial data.

After that, Strauss [14] conjectured that the critical exponent of (1.3) for general dimension  $N$  is given by

$$p_S(N) = \sup\{p > 1 ; \gamma(N, p) > 0\}, \quad \gamma(N, p) = 2 + (N + 1)p - (N - 1)p^2.$$

Now  $p_S(N)$  is called the Strauss exponent. Including blowup phenomena in the critical situation, this conjecture was solved until the works Yordanov–Zhang [17] and Zhou [19]. The further study for blowup solutions can be found in the literature. Especially, the behavior of lifespan (maximal existence time) of blowup solutions with small initial data is intensively discussed (see e.g., Lindblad [12], Takamura–Wakasa [15], Zhou–Han [21], Ikeda–Sobajima–Wakasa [5] and the references therein). Here the definition of lifespan is given as follows:

$$T_\varepsilon := T(\varepsilon f, \varepsilon g) = \sup\{T > 0 ; \text{there exists a unique weak solution } u \text{ of (1.1) in } (0, T)\}.$$

The precise behavior of lifespan of small solutions is given by

$$T_\varepsilon \approx \begin{cases} C\varepsilon^{-\frac{2p(p-1)}{\gamma(N,p)}} & \text{if } 1 < p < p_S(N), \\ \exp(C\varepsilon^{-p(p-1)}) & \text{if } p = p_S(N) \end{cases}$$

when  $\varepsilon > 0$  is sufficiently small.

Of course, there are many investigations dealing with the exterior problem (1.1) of semilinear wave equations. The significant difference to the initial value problem is the effect of reflection at the boundary and the lack of symmetry such as scale-invariance, rotation-invariance and so on. For the existence of global-in-time solutions to (1.1) has been discussed in Du–Metcalfe–Sogge–Zhou [2] and Hidano–Metcalfe–Smith–Sogge–Zhou [3] when  $p_S(N) < p < \frac{N+3}{N-1}$  and  $N = 3, 4$ . The sharp lower bounds for lifespan of solutions are shown in Yu [18]  $2 < p < p_S(3)$  with  $N = 3$ ; Zhou–Han [20] proved sharp upper bounds in the case  $1 < p < p_S(N)$  and  $N \geq 3$ . For the critical case  $p = p_S(N)$ , Zha–Zhou [22] discussed the case  $N = 4$  and  $p = p_S(4) = 2$  and proved the lower bound  $T_\varepsilon \geq \exp(C\varepsilon^{-2})$  which seems sharp from the lifespan estimate for the Cauchy problem. The upper bounds for the critical cases are shown in Lai–Zhou [8] for  $N = 3$  and Lai–Zhou [9] for  $N \geq 5$ . We should point out that in the two dimensional case there are some blowup results for small initial data (see Li–Wang [11] and Lai–Zhou [10]), however, precise estimates for lifespan are not treated so far.

As far as the author’s knowledge, sharp upper bound of lifespan for two and four dimensional cases are unknown. Moreover, the proofs of the blowup in previous works (including studies of (1.3)) depend on the positivity of initial data, especially in the higher dimensional case  $N \geq 4$ . In contrast, such a restriction in the whole space case is recently removed by using a framework of test function methods in Ikeda–Sobajima–Wakasa [5].

The purpose of the present paper is to prove blowup of solutions to (1.1) with sharp upper bound of lifespan when  $\Omega_0 = \mathbb{R}^N \setminus \overline{B(0, 1)}$  without positivity assumption in the pointwise sense as in [5].

To state the result of the present paper, we give the definition of (weak) solutions to (1.1).

**Definition 1.1.** Let  $f, g \in C_c^\infty(\Omega)$  with  $\text{supp}(f, g) \subset \overline{B(0, r_0)}$  and  $p > 1$ . The function

$$u \in C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega)) \cap L^p(0, T; L^p(\Omega))$$

is called a weak solution of (1.1) in  $(0, T)$  if  $u(0) = \varepsilon f$ ,  $\partial_t u(0) = \varepsilon g$  and for every  $\Psi \in C_0^\infty(\overline{\Omega} \times [0, T])$  satisfying  $\Psi \equiv 0$  on  $\partial\Omega \times [0, T]$ ,

$$\begin{aligned} & \varepsilon \int_{\Omega} g(x) \Psi(x, 0) dx + \int_0^T \int_{\Omega} |u(x, t)|^p \Psi(x, t) dx dt \\ &= \int_0^T \int_{\Omega} \left( -\partial_t u(x, t) \partial_t \Psi(x, t) + \nabla u(x, t) \cdot \nabla \Psi(x, t) \right) dx dt. \end{aligned}$$

The following is the main result of the present paper.

**Theorem 1.1.** Let  $N \geq 3$ ,  $\Omega = \Omega_0 (= \mathbb{R}^N \setminus \overline{B(0, 1)})$  and  $U(x) = 1 - |x|^{2-N}$ . Let the pair  $(f, g)$  satisfy (1.2) with

$$\int_{\Omega_0} g(x) U(x) dx > 0. \quad (1.4)$$

If  $1 < p \leq p_S(N)$ , then  $T_\varepsilon < \infty$  for every  $\varepsilon > 0$  with the following upper bounds: there exist positive constants  $\varepsilon_0 > 0$  and  $C > 0$  (independent of  $\varepsilon$ ) such that for every  $\varepsilon \in (0, \varepsilon_0]$ ,

$$T_\varepsilon \leq \begin{cases} C\varepsilon^{-\frac{2p(p-1)}{\gamma(N,p)}} & \text{if } 1 < p < p_S(N), \\ \exp(C\varepsilon^{-p(p-1)}) & \text{if } p = p_S(N). \end{cases}$$

In the proof of Theorem 1.1, the positive harmonic function  $U$  satisfying boundary condition

$$\begin{cases} \Delta U = 0 & \text{in } \Omega_0, \\ U = 0 & \text{on } \partial\Omega_0, \\ U > 0 & \text{in } \Omega_0 \end{cases}$$

plays an important role as in the previous works concerning upper bounds for lifespan for exterior problem. However, the strategy of the proof in the present paper is quite different from those. Our technique is based on the test function method for wave equations developed in Ikeda–Sobajima–Wakasa [5]. This argument requires the special solutions of corresponding linear wave equation having slowly decaying property. To construct this kind of solution, we used the construction by Wakasa–Yordanov [16].

*Remark 1.1.* Comparing the previous results for upper bounds for lifespan, we do not assume positivity of initial data in the pointwise sense. Moreover, our assumption (1.4) means the quantity  $\int_{\Omega} \partial_t u U dx$  is always positive

$$\frac{d}{dt} \int_{\Omega} \partial_t u U dx = \int_{\Omega} (\Delta u + |u|^p) U dx \geq 0.$$

This may be meaningful, in fact, in the whole space case the condition  $\int_{\mathbb{R}^N} g dx \neq 0$  is sometimes imposed to see the precise behavior of lifespan with respect to  $\varepsilon \ll 1$ . This poses that  $\int_{\mathbb{R}^N} g U_{\mathbb{R}^N} dx$  with the positive harmonic function  $U_{\mathbb{R}^N} = 1$  is crucial for the lifespan estimates.

*Remark 1.2.* In Lai–Zhou [9], to find the lifespan estimates they essentially assumed that the support of initial data is far away from boundary. Theorem 1.1 allows us to consider the initial data which have the support close to the boundary.

*Remark 1.3.* Our technique is also applicable to the problem with nonlinearity  $|\partial_t u|^p$  and the one of their weakly coupled system.

The present paper is organized as follows. In Section 2, we construct special solutions of corresponding linear wave equation having slowly decaying property by separation of variables and the argument in Wakasa–Yordanov [16]. We also give their fundamental profiles in Section 2. Section 3 is devoted to prove Theorem 1.1 by using test function method based on the one in Ikeda–Sobajima–Wakasa [5].

## 2 Preliminaries

First we consider a class of special solutions to the linear wave equation with Dirichlet boundary condition

$$\begin{cases} \partial_t^2 \Phi(x, t) - \Delta \Phi(x, t) = 0, & (x, t) \in \Omega_0 \times (0, T), \\ \Phi(x, t) = 0, & (x, t) \in \partial\Omega_0 \times (0, T). \end{cases} \quad (2.1)$$

The aim of this section is to construct a positive solution of the linear wave equation in the space-time domain

$$\mathcal{Q}_1 = \{(x, t) \in \Omega_0 \times (0, t) ; |x| < t\} \quad (2.2)$$

having polynomial decay of arbitrary order.

### 2.1 Solutions of wave equation by separation of variables

To begin with, we consider solutions by separation of variables of the form

$$\Phi(x, t) = e^{-\lambda t} \varphi_\lambda(x), \quad (x, t) \in \Omega_0 \times (0, \infty)$$

which has an exponential decay. Then by (2.1) this is equivalent to the following elliptic equation related to the eigenvalue problem of Laplace operator with Dirichlet boundary condition:

$$\begin{cases} \lambda^2 \varphi_\lambda(x) - \Delta \varphi_\lambda(x) = 0, & x \in \Omega_0, \\ \varphi_\lambda(x) = 0, & x \in \partial\Omega_0, \\ \varphi_\lambda(x) > 0, & x \in \Omega_0, \end{cases} \quad (2.3)$$

where  $\lambda \geq 0$  is a parameter. Here we will construct a family  $\{\varphi_\lambda\}_{\lambda>0}$  which is continuous with respect to  $\lambda$  in a suitable sense. If  $\lambda = 0$ , then  $\varphi_0$  is nothing but a positive harmonic function on  $\Omega_0$  satisfying the Dirichlet boundary condition, and therefore, we first fix

$$\varphi_0(x) = U(x) = 1 - |x|^{2-N}, \quad x \in \Omega_0.$$

Then by using modified Bessel functions  $I_\nu$  and  $K_\nu$ , we define the family of functions  $\{\varphi_\lambda\}_{\lambda>0}$  as follows:

**Definition 2.1.** For  $N \geq 3$  and  $\lambda > 0$ , define

$$\varphi_\lambda(x) := \psi_1(\lambda|x|) - \frac{I_\nu(\lambda)}{K_\nu(\lambda)} \psi_2(\lambda|x|), \quad x \in \Omega_0,$$

where  $\nu = \frac{N-2}{2} > 0$  and

$$\psi_1(z) = 2^\nu \Gamma(\nu+1) z^{-\nu} I_\nu(z), \quad \psi_2(z) = 2^\nu \Gamma(\nu+1) z^{-\nu} K_\nu(z)$$

(for the detailed information about modified Bessel functions, see e.g., Beals–Wong [1]).

*Remark 2.1.* The function  $\psi_1(|x|)$  can be represented by

$$\psi_1(|x|) = \frac{1}{|S^{N-1}|} \int_{S^{N-1}} e^{x \cdot \omega} d\omega$$

which has been introduced in Yordanov–Zhang [17] and used many times in the previous papers listed in Section 1.

To analyse the behavior of  $\varphi_\lambda$ , we use the precise behavior of  $I_\nu$  and  $K_\nu$  listed in the following lemma. The precise behavior of  $I_\nu$  and  $K_\nu$  can be found in Olver [13, Sections 2.10.1, and 7.8.2–7.8.4].

**Lemma 2.1.** *Let  $\mu > 0$ . Then  $I_\mu$  and  $K_\mu$  are smooth positive functions satisfying*

$$z^2 y''(z) + z y'(z) = (z^2 + \mu^2) y(z), \quad z > 0 \quad (2.4)$$

*with the following properties*

$$\lim_{z \downarrow 0} (z^{-\mu} I_\mu(z)) = \frac{1}{2^\mu \Gamma(\mu+1)}, \quad \lim_{z \downarrow 0} (z^\mu K_\mu(z)) = 2^{\mu-1} \Gamma(\mu), \quad (2.5)$$

$$\lim_{z \rightarrow \infty} \left( \frac{z^{\frac{1}{2}}}{e^z} I_\mu(z) \right) = \frac{1}{\sqrt{2\pi}}, \quad \lim_{z \rightarrow \infty} (z^{\frac{1}{2}} e^z K_\mu(z)) = \sqrt{\frac{\pi}{2}}, \quad (2.6)$$

$$\frac{d}{dz} (z^{-\mu} I_\mu(z)) = z^{-\mu} I_{\mu+1}(z), \quad \frac{d}{dz} (z^{-\mu} K_\mu(z)) = -z^{-\mu} K_{\mu+1}(z). \quad (2.7)$$

Using the profile of  $I_\nu$  and  $K_\nu$  stated in Lemma 2.1, we have

**Lemma 2.2.** *The family  $\{\varphi_\lambda\}_{\lambda>0}$  has the following properties.*

- (i) *for every  $\lambda > 0$ ,  $\varphi_\lambda$  satisfies (2.3).*
- (ii) *the map  $(x, \lambda) \in \Omega_0 \times (0, \infty) \mapsto \varphi_\lambda(x)$  is continuous.*
- (iii) *for every  $x \in \Omega_0$ , one has*

$$\lim_{\lambda \downarrow 0} \varphi_\lambda(x) = U(x).$$

- (iv) *for every  $\lambda > 0$ ,*

$$\varphi_\lambda(x) \geq U(x) \psi_1(\lambda|x|), \quad x \in \Omega_0.$$

- (v) *there exists a constant  $C_\nu > 0$  such that for every  $\lambda \in (0, 1]$ ,*

$$\varphi_\lambda(x) \leq C_\nu U(x) \psi_1(\lambda|x|), \quad x \in \Omega_0.$$

*Proof.* The assertion **(ii)** is obvious by the construction of  $\psi_1$  and  $\psi_2$ . **(i)** is also verified because the pair  $(\psi_1, \psi_2)$  is the fundamental system of the following ordinary differential equation:

$$\psi''(r) + \frac{N-1}{r}\psi'(r) = \psi(r), \quad r > 0$$

which is equivalent (via the change of functions  $v(z) = z^{\frac{N-2}{2}}\psi(z)$ ) to the modified Bessel equation (2.4) with the parameter  $\mu = \nu = \frac{N-2}{2}$ . By using (2.7) with  $\varphi_\lambda \equiv 0$  on  $\partial\Omega_0$ ,  $\varphi_\lambda$  satisfies (2.3). For **(iii)**, noting that

$$\varphi_\lambda(x) = 2^\nu \Gamma(\nu+1)(\lambda r)^{-\nu} I_\nu(\lambda r) \left( 1 - \frac{\lambda^{-\nu} I_\nu(\lambda)}{(\lambda r)^{-\nu} I_\nu(\lambda r)} \cdot \frac{(\lambda r)^\nu K_\nu(\lambda r)}{\lambda^\nu K_\nu(\lambda)} \cdot r^{2-N} \right)$$

with the notation  $r = |x|$ , we have  $\lim_{\lambda \downarrow 0} \varphi_\lambda(x) = 1 - r^{2-N} = U(x)$ . For **(iv)**, we define

$$\tilde{\varphi}_\lambda(x) := U(x)\psi_1(\lambda|x|) - \varphi_\lambda(x), \quad x \in \overline{\Omega}_0$$

with arbitrary fixed  $\lambda > 0$ . Note that by (i) and (2.7), we have

$$\begin{aligned} \Delta \tilde{\varphi}_\lambda(x) &= (\Delta U)\psi_1(\lambda|x|) + 2\nabla U \cdot \nabla(\psi_1(\lambda|x|)) + U\Delta(\psi_1(\lambda|x|)) - \Delta\varphi_\lambda(x) \\ &= 2\nabla U \cdot \nabla(\psi_1(\lambda|x|)) + \lambda^2 U\psi_1(\lambda|x|) - \lambda^2 \varphi_\lambda(x) \\ &\geq \lambda^2 \tilde{\varphi}_\lambda(x). \end{aligned}$$

Moreover, in view of (2.6), there exists a positive constant  $z_0$  such that for every  $z \geq z_0$ ,

$$I_\nu(z) \geq \frac{1}{2\sqrt{2\pi}} z^{-\frac{1}{2}} e^z, \quad K_\nu(z) \leq 2\sqrt{\frac{\pi}{2}} z^{-\frac{1}{2}} e^{-z},$$

that is, for  $x \in \Omega_0$  satisfying  $|x| \geq \frac{z_0}{\lambda}$ ,

$$\frac{K_\nu(\lambda|x|)}{I_\nu(\lambda|x|)} \leq 4\pi e^{-2\lambda|x|}.$$

Hence we see that for  $x \in \Omega_0$  satisfying  $|x| \geq \frac{z_0}{\lambda}$ ,

$$\begin{aligned} \tilde{\varphi}_\lambda(x) &= (U(x) - 1)\psi_1(\lambda|x|) + \frac{I_\nu(\lambda)}{K_\nu(\lambda)}\psi_2(\lambda|x|) \\ &= \psi_1(\lambda|x|) \left( -|x|^{-2\nu} + \frac{I_\nu(\lambda)}{K_\nu(\lambda)} \frac{K_\nu(\lambda|x|)}{I_\nu(\lambda|x|)} \right) \\ &\leq \psi_1(\lambda|x|) \left( -|x|^{-2\nu} + \frac{4\pi I_\nu(\lambda)}{K_\nu(\lambda)} e^{-2\lambda|x|} \right). \end{aligned}$$

Taking  $R_\lambda \geq \frac{z_0}{\lambda}$  large enough, we deduce  $\tilde{\varphi}_\lambda(x) \leq 0$  in  $\mathbb{R}^N \setminus B(0, R_\lambda)$ . Therefore we have

$$\begin{cases} \lambda^2 \tilde{\varphi}_\lambda(x) - \Delta \tilde{\varphi}_\lambda(x) \leq 0, & x \in \Omega_0, \\ \tilde{\varphi}_\lambda(x) = 0, & x \in \partial\Omega_0, \\ \tilde{\varphi}_\lambda(x) \leq 0, & \text{for } x \in \mathbb{R}^N \setminus B(0, R_\lambda). \end{cases}$$

The maximum principle implies  $\tilde{\varphi}_\lambda \leq 0$  on  $\Omega_0$  and hence  $\varphi_\lambda(x) \geq U(x)\psi_1(\lambda|x|)$  ( $x \in \Omega_0$ ) is verified. Finally we prove (v). We put

$$\begin{aligned} c_{1,\nu} &= \inf_{z \in (0,2)} \left( z^{-\nu} I_\nu(z) \right) \leq \sup_{z \in (0,2)} \left( z^{-\nu} I_\nu(z) \right) = C_{1,\nu}, \\ c_{2,\nu} &= \inf_{z \in (0,2)} \left( z^\nu K_\nu(z) \right) \leq \sup_{z \in (0,2)} \left( z^\nu K_\nu(z) \right) = C_{2,\nu}, \end{aligned}$$

which are all finite by (2.5). By (2.7) we see that for every  $x \in \Omega_0$  and  $\lambda > 0$ ,

$$\begin{aligned} \frac{\partial \varphi_\lambda}{\partial r}(x) &= \left( \lambda \psi_1'(\lambda r) - \frac{I_\nu(\lambda)}{K_\nu(\lambda)} \lambda \psi_2'(\lambda r) \right) \\ &= 2^\nu \Gamma(\nu+1) \left( \lambda (\lambda r)^{-\nu} I_{\nu+1}(\lambda r) + \frac{I_\nu(\lambda)}{K_\nu(\lambda)} \lambda (\lambda r)^{-\nu} K_{\nu+1}(\lambda r) \right). \end{aligned}$$

If  $|x| \leq 2$  and  $\lambda \in (0, 1]$ , then

$$\begin{aligned} \frac{\partial \varphi_\lambda}{\partial r}(x) &\leq 2^\nu \Gamma(\nu+1) \left( C_{1,\nu+1} \lambda^2 r + \frac{C_{1,\nu} C_{2,\nu+1}}{c_{2,\nu}} r^{-2\nu-1} \right) \\ &\leq 2^\nu \Gamma(\nu+1) \left( 4^{\nu+1} C_{1,\nu+1} + \frac{C_{1,\nu} C_{2,\nu+1}}{c_{2,\nu}} \right) r^{-2\nu-1}. \end{aligned}$$

Combining the above estimate with Dirichlet boundary condition implies that for  $x \in \Omega_0 \cap \overline{B(0, 2)}$ ,

$$\varphi_\lambda(x) \leq 2^{\nu-1} \Gamma(\nu) \left( 4^{\nu+1} C_{1,\nu+1} + \frac{C_{1,\nu} C_{2,\nu+1}}{c_{2,\nu}} \right) U(x).$$

If  $|x| \geq 2$ , by the definition of  $\varphi_\lambda$  and the monotonicity of  $U(x)$  (with respect to  $r = |x|$ ), we see

$$\varphi_\lambda(x) \leq \psi_1(\lambda|x|) = [U(x)]^{-1} U(x) \psi_1(\lambda|x|) \leq (1 - 2^{2-N})^{-1} U(x) \psi_1(\lambda|x|).$$

We obtain the desired upper bound for  $\varphi_\lambda$ .  $\square$

## 2.2 Slowly decaying solutions of wave equation

Next we construct a family of solutions having polynomial decay of arbitrary order (with respect to the variable  $t$ ).

Before the construction of solutions to the problem with Dirichlet boundary condition (2.1), we consider the following formula describing the connection between the modified Bessel function  $I_\nu(z)$  and the Gauss hypergeometric function  $F(\cdot, \cdot, \cdot; z)$  in the “light cone”

$$\mathcal{Q}_0 = \{(x, t) \in \mathbb{R}^N \times (0, \infty) ; |x| < t\}.$$

**Lemma 2.3.** *Let  $\beta > 0$ . If  $(x, t) \in \mathcal{Q}_0$ , then*

$$\frac{1}{\Gamma(\beta)} \int_0^\infty e^{-\lambda t} \psi_1(\lambda|x|) \lambda^{\beta-1} d\lambda = t^{-\beta} F\left(\frac{\beta}{2}, \frac{\beta+1}{2}, \frac{N}{2}; \frac{|x|^2}{t^2}\right),$$

where  $F(a, b, c; z)$  is the Gauss hypergeometric function defined as

$$F(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad |z| < 1$$

with the Pochhammer symbol  $(d)_0 = 1$  and  $(d)_n = \prod_{k=1}^n (d+k-1)$  for  $n \in \mathbb{N}$ .



*Proof.* By the asymptotic profile of  $I_\nu$ , we have  $\psi_1(x) \leq e^{|x|}$  and then

$$e^{-\lambda t} \psi_1(\lambda|x|) \lambda^{\beta-1} \leq e^{-\lambda(t-|x|)} \lambda^{\beta-1}.$$

This implies that if  $|x| < t$ , then the function

$$v(x, t) = \int_0^\infty e^{-\lambda t} \psi_1(\lambda|x|) \lambda^{\beta-1} d\lambda$$

is well-defined. By similar argument, we also have  $v \in C^2(\mathcal{Q}_0)$ . Observe that for every  $\lambda > 0$ ,  $v_\lambda = e^{-\lambda t} \psi_1(\lambda|x|)$  satisfies  $\partial_t^2 v_\lambda - \Delta v_\lambda = 0$  in  $\mathcal{Q}_0$ . Therefore  $v$  also satisfies  $\partial_t^2 v - \Delta v = 0$  on  $\mathcal{Q}_0$ . Moreover, we see from the change of variables  $\mu = \lambda s$  that

$$\begin{aligned} s^\beta v(sx, st) &= s^\beta \int_0^\infty e^{-\lambda st} \psi_1(\lambda|sx|) \lambda^{\beta-1} d\lambda \\ &= \int_0^\infty e^{-\mu t} \psi_1(\mu|x|) \mu^{\beta-1} d\mu \\ &= v(x, t). \end{aligned}$$

Noting that  $v(0, t) = \Gamma(\beta)t^{-\beta}$ , by [4, Lemma 2.1] (with  $\mu' = 0$  in their notation) we obtain the desired equality.  $\square$

Now we introduce the family of solutions to (2.1), which plays a crucial role in the present paper.

**Definition 2.2.** For  $\beta > 0$ ,

$$\Phi_\beta(x, t) = \frac{1}{\Gamma(\beta)} \int_0^1 e^{-\lambda t} \varphi_\lambda(x) \lambda^{\beta-1} d\lambda, \quad (x, t) \in \mathcal{Q},$$

where  $\mathcal{Q}$  is as in (2.2). Note that  $\Phi_\beta$  is well-defined by virtue of Lemma 2.2 (v) and Lemma 2.3.

The following lemma is mainly used in the proof of main result in this paper.

**Lemma 2.4.** *The functions  $\{\Phi_\beta\}_{\beta>0}$  satisfy the following properties:*

- (i) *for every  $\beta > 0$ ,  $\Phi_\beta$  satisfies (2.1) in  $\mathcal{Q}$ .*
- (ii) *for every  $\beta > 0$ ,  $\Phi_\beta$  satisfies  $\partial_t \Phi_\beta = -\beta \Phi_{\beta+1}$  in  $\mathcal{Q}$ .*
- (iii) *(Upper bound) the following inequality holds with the same constant  $C_\nu$  as in Lemma 2.2:*

$$\Phi_\beta(x, t) \leq C_\nu U(x) t^{-\beta} F\left(\frac{\beta}{2}, \frac{\beta+1}{2}, \frac{N}{2}, \frac{|x|^2}{t^2}\right), \quad (x, t) \in \mathcal{Q}.$$

- (iv) *(Lower bound) there exists a constant  $C'_\nu > 0$  such that for every  $(x, t) \in \mathcal{Q}$  with  $t \geq 1$ ,*

$$\Phi_\beta(x, t) \geq C'_\nu U(x) t^{-\beta}.$$

- (v) *(Large time behavior) For every  $x \in \Omega_0$ ,*

$$\lim_{t_0 \rightarrow \infty} \left( t_0^\beta \Phi_\beta(x, t_0) \right) = U(x).$$

*Proof.* (i) Since for every  $\lambda > 0$ ,  $e^{-\lambda t}\varphi_\lambda(x)$  satisfies (2.1),  $\Phi_\beta$  is also the solution of the same problem. (ii) By direct computation, we have for every  $(x, t) \in \mathcal{Q}$ ,

$$\begin{aligned}\partial_t \Phi_\beta(x, t) &= \frac{1}{\Gamma(\beta)} \frac{\partial}{\partial t} \left( \int_0^1 e^{-\lambda t} \varphi_\lambda(x) \lambda^{\beta-1} d\lambda \right) \\ &= -\frac{\beta}{\Gamma(\beta+1)} \int_0^1 e^{-\lambda t} \varphi_\lambda(x) \lambda^{(\beta+1)-1} d\lambda \\ &= -\beta \Phi_{\beta+1}(x, t).\end{aligned}$$

(iii) Using Lemma 2.2 (v), we deduce that for every  $(x, t) \in \mathcal{Q}$ ,

$$\begin{aligned}\Phi_\beta(x, t) &\leq \frac{C_\nu}{\Gamma(\beta)} \int_0^1 e^{-\lambda t} \left( U(x) \psi_1(\lambda|x|) \right) \lambda^{\beta-1} d\lambda \\ &\leq \frac{C_\nu U(x)}{\Gamma(\beta)} \int_0^\infty e^{-\lambda t} \psi_1(\lambda|x|) \lambda^{\beta-1} d\lambda \\ &= C_\nu U(x) t^{-\beta} F\left(\frac{\beta}{2}, \frac{\beta+1}{2}, \frac{N}{2}, \frac{|x|^2}{t^2}\right).\end{aligned}$$

(iv) Employing Lemma 2.2 (iv) with  $\psi_1(z) \geq 1$  implies that for every  $(x, t) \in \mathcal{Q}$  with  $t \geq 1$ ,

$$\begin{aligned}\Phi_\beta(x, t) &\geq \frac{1}{\Gamma(\beta)} \int_0^1 e^{-\lambda t} U(x) \lambda^{\beta-1} d\lambda \\ &= \frac{U(x)}{\Gamma(\beta)} t^{-\beta} \int_0^t e^{-\mu} \mu^{\beta-1} d\mu \\ &\geq \left( \frac{1}{\Gamma(\beta)} \int_0^1 e^{-\mu} \mu^{\beta-1} d\mu \right) U(x) t^{-\beta}.\end{aligned}$$

(v) For  $t_0 > \max\{1, 2|x|\}$ , employing change of variables  $\mu = \lambda t_0$  gives

$$\begin{aligned}t_0^\beta \Phi_\beta(x, t_0) &= \frac{t_0^\beta}{\Gamma(\beta)} \int_0^1 e^{-\lambda t_0} \varphi_\lambda(x) \lambda^{\beta-1} d\lambda \\ &= \frac{1}{\Gamma(\beta)} \int_0^{t_0} e^{-\mu} \varphi_{\mu t_0^{-1}}(x) \mu^{\beta-1} d\mu \\ &= \frac{1}{\Gamma(\beta)} \int_0^\infty \chi_{(0, t_0)}(\mu) e^{-\mu} \varphi_{\mu t_0^{-1}}(x) \mu^{\beta-1} d\mu,\end{aligned}$$

where  $\chi_I$  denotes the indicator function on the interval  $I$ . Noting that  $\varphi_{\mu t_0^{-1}}(x) \rightarrow U(x)$  as  $t_0 \rightarrow \infty$  (Lemma 2.2 (iii)) and the consequence of Lemma 2.2 (iv) as

$$\chi_{(0, t_0)}(\mu) e^{-\mu} \varphi_{\mu t_0^{-1}}(x) \leq e^{-\mu} U(x) \psi_1(\mu t_0^{-1} x) \leq U(x) e^{-\mu + \frac{\mu|x|}{t_0}} \leq U(x) e^{-\frac{\mu}{2}},$$

from the dominated convergence theorem we obtain the desired convergence.  $\square$

### 3 Proof of blowup with lifespan estimates

Here we give a proof of blowup phenomena with the sharp upper bound of lifespan estimates via the similar strategy as in [5]. The proof of Theorem 1.1 for weak solutions are almost the

same as the one for the case where the solution is smooth enough. Therefore we prove only the case of smooth solutions (for the general situation see [5]). The difference from the case  $\Omega = \mathbb{R}^N$  is to use the special solutions satisfying Dirichlet boundary condition. For simplicity, we use  $\Omega = \Omega_0$  in this section.

*Proof of Theorem 1.1.* First we observe that for smooth function  $\Psi$  on  $\text{supp } u$  satisfying Dirichlet boundary condition on  $\partial\Omega$ , we see by multiplying  $\Psi$  to the equation in (1.1) and by using integration by parts that

$$\int_{\Omega} |u|^p \Psi \, dx = \frac{d}{dt} \int_{\Omega} (\partial_t u \Psi - u \partial_t \Psi) \, dx + \int_{\Omega} u (\partial_t^2 \Psi - \Delta \Psi) \, dx. \quad (3.1)$$

We frequently use a cut-off function  $\eta \in C^\infty([0, \infty); [0, 1])$  satisfying  $\eta(s) = 1$  for  $s \in [0, \frac{1}{2}]$  and  $\eta(s) = 0$  for  $s \in [1, \infty)$  with  $\eta'(s) \leq 0$ . Also we assume without loss of generality that  $T_\varepsilon > 1$  (otherwise the solution blows up until  $t = 1$ ).

**(Subcritical case  $1 < p < p_S(N)$ )** Taking  $\Psi = \eta_R(t)^{2p'} U(x)$  with  $\eta_R(t) = \eta(t/R)$  in (3.1), we have

$$\begin{aligned} \int_{\Omega} |u|^p \eta_R^{2p'} U \, dx &= \frac{d}{dt} \int_{\Omega} (\partial_t u \eta_R^{2p'} U - u \partial_t [\eta_R^{2p'} U]) \, dx \\ &\quad + \int_{\Omega} u (\partial_t^2 [\eta_R^{2p'} U] - \eta_R^{2p'} \Delta U) \, dx \\ &\leq \frac{d}{dt} \int_{\Omega} (\partial_t u \eta_R^{2p'} U - u \partial_t [\eta_R^{2p'} U]) \, dx \\ &\quad + CR^{-2} \int_{\Omega} |u| [\eta_R^{2p'}]^{\frac{1}{p}} U \, dx. \end{aligned}$$

Taking  $R \in (1, T_\varepsilon)$  and integrating it over  $[0, T_\varepsilon]$ , we deduce

$$\begin{aligned} \varepsilon \int_{\Omega} gU \, dx + \int_0^{T_\varepsilon} \int_{\Omega} |u|^p \eta_R^{2p'} U \, dx \, dt \\ \leq CR^{-2} \int_0^{T_\varepsilon} \int_{\Omega} [|u|^p \eta_R^{2p'}]^{\frac{1}{p}} U \, dx \, dt \\ \leq \frac{C^{p'}}{p'} R^{-2p'} \int_0^R \int_{\Omega(t)} U \, dx \, dt + \frac{1}{p} \int_0^{T_\varepsilon} \int_{\Omega} |u|^p \eta_R^{2p'} U \, dx \, dt, \end{aligned}$$

where  $\Omega(t) = \Omega \cap B(0, R_0 + t)$ . Noting that  $U(x) \leq 1$  and  $|\Omega(t)| \leq C(R_0 + t)^N \leq C'R^N$ , we have

$$p' \varepsilon \int_{\Omega} gU \, dx + \int_0^{T_\varepsilon} \int_{\Omega} |u|^p \eta_R^{2p'} U \, dx \, dt \leq CR^{N-1-\frac{2}{p-1}}. \quad (3.2)$$

Next we note by (1.4) that

$$\int_{\Omega} g(x) (t_0^\beta \Phi_\beta(x, t_0)) \, dx + \beta \int_{\Omega} f(x) (t_0^\beta \Phi_{\beta+1}(x, t_0)) \, dx \rightarrow \int_{\Omega} gU \, dx > 0$$

as  $t_0 \rightarrow \infty$ . Therefore there exists  $t_\beta > R_0$  such that

$$I_\beta = \int_{\Omega} g(x) \tilde{\Phi}_\beta(x, 0) \, dx + \frac{\beta}{t_\beta} \int_{\Omega} f(x) \tilde{\Phi}_{\beta+1}(x, 0) \, dx \geq \frac{1}{2} \int_{\Omega} gU \, dx$$

with  $\tilde{\Phi}_\beta(x, t) = t_\beta^\beta \Phi_\beta(x, t_\beta + t)$ . Now we take  $\Psi(x, t) = \eta_R^{2p'} \tilde{\Phi}_\beta$  in (3.1). Then

$$\begin{aligned} & \int_{\Omega} |u|^p \eta_R^{2p'} \tilde{\Phi}_\beta(t) dx \\ &= \frac{d}{dt} \int_{\Omega} \left( \partial_t u \eta_R^{2p'} \tilde{\Phi}_\beta(t) + \frac{\beta}{t_\beta} u \eta_R^{2p'} \tilde{\Phi}_{\beta+1}(t) - 2p' u \eta_R^{2p'-1} \eta'_R \tilde{\Phi}_\beta(t) \right) dx \\ &+ 2p' \int_{\Omega} u [\eta_R^*]^{\frac{2p'}{p}} \left( [(2p' - 1)(\eta'_R)^2 + \eta''_R \eta_R] \tilde{\Phi}_\beta(t) + \frac{2(N-1)}{t_\beta} \eta'_R \eta_R \tilde{\Phi}_{\beta+1}(t) \right) dx. \end{aligned}$$

Here we have introduced  $\eta_R^*(t) = \eta^*(t/R)$  with  $\eta^*(s) = \chi_{[\frac{1}{2}, \infty)} \eta(s)$  ( $\chi_I$  is the indicator function on the interval  $I$ ). Integrating it over  $[0, T_\varepsilon]$ , we see

$$I_\beta + \int_0^{T_\varepsilon} \int_{\Omega} |u|^p \eta_R^{2p'} \tilde{\Phi}_\beta(t) dx dt \leq C \int_0^{T_\varepsilon} \int_{\Omega} [|u|^p (\eta_R^*)^{2p'}]^{\frac{1}{p}} \left( \frac{\tilde{\Phi}_\beta}{R^2} + \frac{\tilde{\Phi}_{\beta+1}}{t_\beta R} \right) dx dt. \quad (3.3)$$

Putting  $\beta = N - 1$  and using Lemma 2.4 (iii) with the formula  $F(a, b, a; z) = F(b, a, a; z) = (1 - z)^{-b}$ , we have

$$\frac{\tilde{\Phi}_{N-1}}{R^2} + \frac{\tilde{\Phi}_N}{t_\beta R} \leq C R^{-2-\beta} \left( 1 - \frac{|x|^2}{t_\beta + t} \right)^{-\frac{N+1}{2}} U(x), \quad x \in \text{supp } u(t), \quad t \in (R/2, R).$$

Computing the second integral on the right hand side of the above inequality with Hölder's inequality, we deduce

$$\delta \left( \frac{\varepsilon}{2} \int_{\Omega} gU dx \right)^p R^{N - \frac{N-1}{2}p} \leq \int_0^T \int_{\Omega} |u|^p (\eta_R^*)^{2p'} U dx dt. \quad (3.4)$$

Combining (3.4) with (3.2), we obtain

$$\delta \left( \frac{\varepsilon}{2} \int_{\Omega} gU dx \right)^p \leq C R^{-\frac{\gamma(N,p)}{2(p-1)}}$$

which implies the desired upper bound for lifespan of  $u$  when  $1 < p < p_S(N)$ .

**(Critical case  $p = p_S(N)$ )** In view of (3.4) together with Lemma 2.4 (iv), we have

$$\tilde{\delta} \left( \frac{\varepsilon}{2} \int_{\Omega} gU dx \right)^p \leq \int_0^{T_\varepsilon} \int_{\Omega} |u|^p (\eta_R^*)^{2p'} \tilde{\Phi}_{\beta_p} dx dt, \quad (3.5)$$

where  $\beta_p = \frac{N-1}{2} - \frac{1}{p} = N - \frac{N-1}{2}p > 0$  by the condition  $p = p_S(N)$ . Take  $\beta = \beta_p$  in (3.3). Then

$$\begin{aligned} & \frac{\varepsilon}{2} \int_{\Omega} gU dx + \int_0^{T_\varepsilon} \int_{\Omega} |u|^p \eta_R^{2p'} \tilde{\Phi}_\beta dx dt \\ & \leq C \left( \int_{\frac{R}{2}}^R \int_{\Omega(t)} \left( \frac{1}{R^2} + \frac{\tilde{\Phi}_{\beta+1}}{t_\beta R} \right)^{p'} \tilde{\Phi}_\beta dx dt \right)^{\frac{1}{p'}} \left( \int_0^{T_\varepsilon} \int_{\Omega} |u|^p (\eta_R^*)^{2p'} \tilde{\Phi}_\beta dx dt \right)^{\frac{1}{p}}. \end{aligned}$$

Applying Lemma 2.4 (iii) and (iv), we obtain

$$\int_0^{T_\varepsilon} \int_{\Omega} |u|^p \eta_R^{2p'} \tilde{\Phi}_\beta dx dt \leq C (\log R)^{\frac{1}{p'}} \left( \int_0^{T_\varepsilon} \int_{\Omega} |u|^p (\eta_R^*)^{2p'} \tilde{\Phi}_\beta dx dt \right)^{\frac{1}{p}}. \quad (3.6)$$

By introducing the function

$$Y(R) = \int_0^R \left( \int_0^{T_\varepsilon} \int_\Omega |u|^p (\eta_\rho^*)^{2p'} \tilde{\Phi}_\beta dx dt \right) \rho^{-1} d\rho$$

(as in [5, Lemma 3.9]), the inequalities (3.5) and (3.6) can be translated into

$$\begin{cases} \tilde{\delta} \left( \frac{\varepsilon}{2} \int_\Omega gU dx \right)^p \leq RY'(R), \\ Y(R)^p \leq (\log R)^{p-1} RY'(R) \end{cases}$$

for every  $R \in (1, T_\varepsilon)$ . Employing [5, Lemma 3.10], we obtain

$$T_\varepsilon \leq \exp \left( C \left( \frac{\varepsilon}{2} \int_\Omega gU dx \right)^{-p(p-1)} \right).$$

This gives the desired upper bound of the lifespan of  $u$  in the critical case  $p = p_S(N)$ . The proof is complete.  $\square$

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