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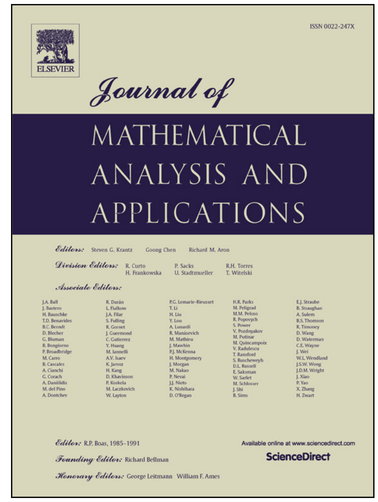

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The convective eigenvalues of the one–dimensional p–Laplacian as $p \to 1$

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Abstract

This paper studies the limit behavior as $p \to 1$ of the eigenvalue problem

$$\begin{cases} -(|u_x|^{p-2}u_x)_x - c|u_x|^{p-2}u_x = \lambda|u|^{p-2}u, & 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases}$$

We point out that explicit expressions for both the eigenvalues λ_n and associated eigenfunctions are not available (see [16]). In spite of this hindrance, we obtain the precise values of the limits $\lim_{p\to 1+} \lambda_n$. In addition, a complete description of the limit profiles of the eigenfunctions is accomplished. Moreover, the formal limit problem as $p \to 1$ is also addressed. The results extend known features for the special case $c = 0$ ([6], [28]).

Keywords: Eigenvalues and eigenfunctions, One-dimensional p-Laplacian operator, One-dimensional 1-Laplacian operator, Asymptotic behaviour.

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1. Introduction

This work deals with the nonlinear eigenvalue problem

$$\begin{cases} -(\varphi_p(u_x))_x - c\varphi_p(u_x) = \lambda\varphi_p(u), & 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{1}$$

where $p > 1$, $\varphi_p(\cdot)$ stands for the function $\varphi_p(t) = |t|^{p-2}t$ while $\mathcal{L}_p(u) := (\varphi_p(u_x))_x$ is the well–known p–Laplacian operator acting in a distributional sense on functions $u \in C_0^1(0,1)$ (see Section 2). There is no loss of generality in assuming that the constant c is positive. Otherwise, the transformation $x \to 1 - x$ restores the problem to this normalized situation.

The symmetric case $c = 0$ of (1), namely

$$\begin{cases} -(\varphi_p(u_x))_x = \lambda\varphi_p(u), & 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{2}$$

goes back at least to [11]. The set of eigenvalues to (2) consists of the sequence $\sigma_n = n^p t_1^p$, $n = 1, 2, \ldots$, where $t_1 = \frac{2\pi}{p\sin(\pi/p)}(p-1)^{1/p}$. They are simple and a normalized eigenfunction associated to σ_n is given by

$$\tilde{u}_n(x) = v(t), \qquad t = nt_1 x,$$

$v(t)$ being the odd and $2t_1$–periodic function such that

$$t = \left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_0^{v(t)} \frac{ds}{(1-s^p)^{1/p}} \qquad \& \qquad v(t_1 - t) = v(t)$$

for every $0 \le t \le \frac{t_1}{2}$ (see further details in [7], [18] and [30] for a generalization of the classical Sturm–Liouville theory). However, ascertaining the structure of the full Dirichlet spectrum of the p-Laplacian $-\Delta_p$ in a N–dimensional domain still remains a hard open question ([23]). In this N–dimensional setting, the radially symmetric case is the only one where all the eigenvalues are essentially known ([8], [30], [28]).

As far as problem (1) is concerned, a detailed knowledge of the full set of eigenvalues and associated eigenfunctions has not been attained until quite recently (see [16]). Observe that in the linear scenario $p = 2$, problem (1) is easily reduced to (2). Nevertheless, it has been shown in [16] that the eigenvalues to (1) form a sequence,

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots,$$

such that $\lambda_n \to \infty$ (more exactly $\lambda_n \sim \sigma_n$ as $n \to \infty$). Each λ_n is simple and a normalized eigenfunction can be expressed as

$$u_n(x,p) = v(\lambda_n^{\frac{1}{p}} x),$$

where $v(t)$ is a suitably chosen oscillatory solution to the equation

$$(\varphi_p(v_t))_t + \gamma\varphi_p(v_t) + \varphi_p(v) = 0, \tag{3}$$

with $\gamma = c\lambda_n^{-\frac{1}{p}}$. Specifically, $v(t)$ is required to satisfy

$$v(0) = 0 \qquad \text{and} \qquad \max_{[0,\infty)} v = \max_{[0,\infty)} |v| = 1. \tag{4}$$

The latter condition is equivalent to $v'(0) = v_0'$ with a proper choice of $v_0' > 0$. However, it can be checked that $v_0' \to \infty$ as $p \to 1$. Our objective being to study the limit behavior as $p \to 1$, the maximum condition turns out to be more convenient. See further details in Section 3 below.

It should be stressed that while the study of (2) only involves direct integration arguments, the corresponding analysis for its convective perturbation (1) is far from immediate. It requires upgrading to the p–Laplacian case well known devices from the qualitative ODE theory (see Lemmas 5 and 6 of Section 2). On the other hand, the situation for the N–dimensional version of (1) is even worse. In fact, the variational nature of the p–Laplacian is lost under the presence of a convective term. We refer to [17] for recent results on this considerably more delicate problem.

In this work our interest is focused on two goals. The first one is elucidating the existence of the limits

$$\bar{\lambda}_n := \lim_{p\to 1} \lambda_n(p), \tag{5}$$

together with a detailed description of the limit profiles,

$$\bar{u}_n(x) := \lim_{p\to 1} u_n(x,p) \tag{6}$$

of the associated eigenfunctions. The notation $(\lambda_n(p), u_n(\cdot,p))$ for the eigenpairs of (1) will be employed when necessary to emphasize their dependence on p.

The second one is addressing the limit eigenvalue problem of (1) as $p \to 1$. This allows us to give a meaning to the formal eigenpairs $(\bar{\lambda}_n, \bar{u}_n)$. Higher order eigenvalues to the 1–Laplacian have not been considered until quite recently (see [6], [25] and further references in [28]). Accordingly, an explicit description of the eigenvalues and associated eigenfunctions to the convective perturbation (11) of the one–dimensional 1–Laplacian is introduced in this work.

It is worth recalling that equations involving the p-Laplacian operator arise in the study of a variety of physical phenomena, in particular non-Newtonian fluids (see [10]), glaciology ([12]), also in reaction–diffusion systems and population dynamics. Regarding the latter subjects, the classical logistic equation under nonlinear diffusion was extended and thoroughly analyzed in the series of works [14], [13] and [15] (see also [20] for a previous discussion). The 1–Laplacian was first studied as a limit of p–Laplacian type equations and the interest in such limit process originated in an optimal design problem raised in the theory of torsion and related geometrical problems (see [21]). Since the 1–Laplacian describes isotropic diffusion within each level surface, with no diffusion across different level surfaces, this operator also constitutes a powerful tool in PDE based image processing (see the pioneering paper [27] and [4]). Although our objectives in the present work are mainly theoretical, we point out that our results could find applications in these areas.

In our previous work [28] the N–dimensional version of (2) is considered. The existence of the limits (5) and (6) is discussed when the λ_n are the Ljusternik–Schnirelman eigenvalues to $-\Delta_p$ ([23]). In addition, $\bar{u}_n$ is shown to define a "reference" eigenfunction to a suitable limit eigenvalue problem (see [24] for related results). For the special case of the radial eigenvalues, the delicate question of ascertaining the limit profiles of the eigenfunctions u_n is addressed in [28] by direct arguments in the spirit of Section 4 below. In contrast to this, more precise phase space methods allow us, in this work, to provide a detailed description of the limits $(\bar{\lambda}_n, \bar{u}_n)$ of the eigenpairs (λ_n, u_n) to (1) as $p \to 1$. It should be mentioned that similar results were obtained in [6] for the simpler problem (2). Therefore, the analysis of (1) constitutes an interesting intermediate case of study linking the one–dimensional and the radial problems. However, the extension to dimension N of the results of this work concerning (1) stands for the moment as a challenging problem.

Our main results on the limit behavior of problem (1) as $p \to 1$ are next stated. It is worthwhile recalling that every eigenfunction u_n to (2) vanishes exactly at $x = k/n$, $0 \le k \le n$. In addition, it exhibits a unique critical point $x = \xi_{k-1} \in \left(\frac{k-1}{n}, \frac{k}{n}\right)$, $1 \le k \le n$. These assertions are properly reviewed in the course of the proofs of Section 3. In what follows $C(U)$ will stand for the space of continuous functions in an open $U \subset \mathbb{R}$, endowed with the uniform convergence in compact sets.

Theorem 1. *Let $\lambda_n = \lambda_n(p,c)$ be the n–th eigenvalue to problem* (1) *and let $u_n = u_n(x,p,c)$ be the associated eigenfunction normalized so as*

$$\max_{x\in(0,1)} u_n(x) = 1.$$

Then the following properties hold.

a) The limits $\bar\lambda_n = \lim_{p\to1} \lambda_n(p,c)$ exist for every $n \in \mathbb{N}$. In fact,

$$\bar\lambda_n = c\frac{e^{\frac{c}{n}}+1}{e^{\frac{c}{n}}-1}.$$

In particular,

$$\bar\lambda_n \sim \bar\sigma_n,$$

both as $c \to 0+$ and as $n \to \infty$, where $\bar\sigma_n = 2n$ and $\bar\sigma_n = \lim_{p\to1}\sigma_n(p)$, $\sigma_n(p)$ being the n–th eigenvalue to problem (2).

b) The unique critical point ξ_{k-1} of u_n in the interval $\frac{k-1}{n} < x < \frac{k}{n}$ satisfies

$$\lim_{p\to1}\xi_{k-1} = \frac{k}{n} - \frac{1}{c}\log\left(\frac{e^{-\frac{c}{n}}+1}{2}\right). \qquad (7)$$

Moreover,

$$\bar\alpha_{k-1} := \lim_{p-1}\alpha_{k-1} = e^{-(k-1)\frac{c}{n}}, \qquad 1 \le k \le n, \qquad (8)$$

where $\alpha_{k-1} > 0$ is defined through $u_n(\xi_{k-1}) = (-1)^{k-1}\alpha_{k-1}$, $\alpha_0 = 1$.

c) For each n

$$\bar u_n(x) := \lim_{p\to1} u_n(x,p,c) = \sum_{k=1}^{n}(-1)^{k-1}\bar\alpha_{k-1}\chi(nx-(k-1)), \qquad (9)$$

where χ stands for the characteristic function of the unit open interval $I = (0,1)$. Moreover, convergence (9) *holds in $C(I \setminus \{\frac{1}{n},\dots,\frac{n-1}{n}\})$.*

d) The limit

$$\lim_{p\to1}\frac{du_n}{dx}(x,p,c) = 0, \qquad (10)$$

also holds true in $C(I \setminus \{\frac{1}{n},\dots,\frac{n-1}{n}\})$.

Remark 1. As shown in [28], the n–th eigenfunction $\tilde{u}_n(x,p)$ to problem (2), normalized so that $\max_{x\in(0,1)} \tilde{u}_n = 1$, satisfies

$$\lim_{p\to 1} \tilde{u}_n(x,p) = \sum_{k=1}^{n} (-1)^{k-1} \chi(nx - (k-1)).$$

This expression coincides with the limit of the right–hand side of (9) as $c \to 0+$.

Once the existence of the limits $\bar{\lambda}_n$ and

$$\bar{u}_n(x) = \sum_{k=1}^{n} (-1)^{k-1} \bar{\alpha}_{k-1} \chi(nx - (k-1))$$

is settled, their connection with the formal limit problem

$$\begin{cases} -\left(\dfrac{u_x}{|u_x|}\right)_x - c\dfrac{u_x}{|u_x|} = \lambda \dfrac{u}{|u|}, & 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{11}$$

should be addressed. As expected, we are going to show that each $(\bar{\lambda}_n, \bar{u}_n)$ defines an eigenpair of problem (11), in a certain sense to be introduced in Section 4. At this early stage, we point out the presence in (11) of the so–called 1–Laplacian operator $\left(\frac{u_x}{|u_x|}\right)_x$, a natural feature in this kind of degenerate limit problems (see [28]).

In the following statement solutions are understood in a distributional sense (see precise details in Section 4).

Theorem 2. *For every* $n \in \mathbb{N}$, $\bar{\lambda}_n = c\dfrac{e^{\frac{c}{n}} + 1}{e^{\frac{c}{n}} - 1}$ *is an eigenvalue to problem* (11) *having*

$$\bar{u}_n(x) = \sum_{k=1}^{n} (-1)^{k-1} e^{-(k-1)\frac{c}{n}} \chi(nx - (k-1))$$

as the associated eigenfunction normalized with $\max \bar{u}_n = 1$.

Moreover,

$$\bar{u}_n(x) = v(\bar{\lambda}_n x),$$

where $v = v(t)$ *is the unique* normalized *solution to equation*

$$\left(\frac{v_t}{|v_t|}\right)_t + \gamma\frac{v_t}{|v_t|} + \frac{v}{|v|} = 0, \qquad t \in (0,+\infty), \tag{12}$$

with $\gamma = c\bar{\lambda}_n^{-1}$*, which satisfies the additional condition*

$$\frac{d|v|}{dt} = -\gamma\left|\frac{dv}{dt}\right|. \tag{13}$$

Remark 2. Normalized solutions to (12) are properly defined in Section 4. They satisfy conditions analogous to those in (4) (see Remark 8). We point out that equation (12) could exhibit infinitely many normalized solutions. Therefore, energy condition (13) is crucial since it allows us to choose a normalized solution with uniqueness.

The paper is organized as follows. Section 2 studies the relevant initial value problems for equation (3). In addition, an analysis of its orbits is discussed. Section 3 addresses the eigenvalue problem (1) and contains the proof of Theorem 1. The asymptotic analysis of the auxiliary integrals involved in such a proof is also included there. Section 4 deals with the limit problem from the alternative approach provided by the 1–Laplacian. It contains the proof of Theorem 2.

2. Initial value problems

In this section the analysis of (1) is revisited. The tools to attain the existence of eigenvalues developed in [16] are introduced for the purposes of the present work. Since we are mainly concerned with the behavior of problem (1) as $p \to 1$ we are restricting ourselves in what follows to the case where $1 < p \le 2$.

By a weak eigenpair $(\lambda, u) \in \mathbb{R} \times W_0^{1,p}(I)$, $I = (0,1)$, it is understood that $u \neq 0$ solves (1) in a weak sense. That is,

$$\int_0^1 \{\varphi_p(u_x)\psi_x - c\varphi_p(u_x)\psi\} = \lambda\int_0^1 \varphi_p(u)\psi \tag{14}$$

for all $\psi \in W_0^{1,p}(I)$. Thus both $\varphi_p(u)$ and $\varphi_p(u_x)$ belongs to $L^{p'}(I)$, $p' = \frac{p}{p-1}$, while (14) entails that $\varphi_p(u_x) \in W^{1,p'}(I)$. By modifying both u and u_x in a null set we conclude that $u, \varphi_p(u_x) \in C^1(I) \cap C(\bar{I})$ (see [5]) whereas

$$(\varphi_p(u_x))_x + c\varphi_p(u_x) + \lambda\varphi_p(u) = 0 \tag{15}$$

is pointwise satisfied in I. In addition, u vanishes at $x = 0, 1$. Finally, observe that $\varphi_{p'}(\cdot) = \varphi_p(\cdot)^{-1}$, where $p' = p/(p-1)$, and hence $u_x = \varphi_{p'}(\varphi_p(u_x))$. As $1 < p \leq 2$, we achieve that $u \in C^2(I) \cap C(\overline{I})$. Summarizing all these remarks, we can state the following lemma.

Lemma 3. *Every weak eigenfunction $u \in W_0^{1,p}(I)$ associated to an eigenvalue λ to* (1)*, with $1 < p \leq 2$, can be regarded as a function $u \in C^2(I) \cap C(\overline{I})$ defining a classical solution to* (14) *such that $u(0) = u(1) = 0$. Moreover, the eigenvalues λ of* (1) *are necessarily positive.*

Proof. Equation (14) can be rewritten as

$$-(e^{cx}\varphi_p(u_x))_x = \lambda e^{cx}\varphi_p(u) \qquad x \in I,$$

whereby, after multiplication by u and an integration, it follows that

$$\lambda = \frac{\displaystyle\int_0^1 e^{cx}|u_x|^p}{\displaystyle\int_0^1 e^{cx}|u|^p} > 0.$$

□

Let $u \in W_0^{1,p}(I)$ be a solution to (1). Then the scale change $t = \lambda^{\frac{1}{p}} x$, $\lambda > 0$, leads to a classical solution $v(t) := u(\lambda^{-\frac{1}{p}} t)$ to equation

$$(\varphi_p(v_t))_t + \gamma\varphi_p(v_t) + \varphi_p(v) = 0,$$

with $\gamma = c\lambda^{-\frac{1}{p}}$. In addition, $v(0) = v(\lambda^{\frac{1}{p}}) = 0$. Therefore, a discussion of the initial value problem

$$\begin{cases} v' = \varphi_{p'}(w), & v(0) = v_0, \\ w' = -(\varphi_p(v) + \gamma w), & w(0) = w_0, \end{cases} \qquad ' = \frac{d}{dt}, \tag{16}$$

becomes necessary. Problem (16) is obtained from (3) by setting $w = \varphi_p(v')$. The prominent rôle of the function w will be shown when dealing with the limit problem as $p \to 1$. Also, it should be stressed that (16) does not fall into the scope of the standard existence and uniqueness theory when $p \neq 2$. In fact, as $1 < p < 2$, (16) fails to be smooth when $v = 0$, while the first equation in (16) loses its differentiability near $w = 0$ if $p > 2$. Figures 2 and 3 depict the orbit of (16) corresponding to $v_0 = 1$, $w_0 = 1$. The singular features of the equation as $p \to 1$ are suggested by these numerical simulations.

Lemma 4. *Suppose $1 < p \leq 2$. Then problem (16) possesses a unique classical solution $v \in C^2(0,\infty) \cap C[0,\infty)$ for every initial data $v_0, w_0 \in \mathbb{R}$. Moreover, $(v(t), v'(t)) \to (0,0)$ as $t \to \infty$. Furthermore, for every $b > 0$, $(v(\cdot), v'(\cdot))$ defines a continuous function of the parameters $(p,\gamma) \in (1,2) \times \mathbb{R}$ when it is regarded as taking values in $C^1[0,b] \times C^1[0,b]$.*

Proof. We refer to [16] for a proof of the existence and uniqueness assertions. The decay of solutions to $(0,0)$ follows from La Salle's invariance principle (see [22]) by employing the Lyapunov function

$$V(v,w) = \frac{1}{p'}|w|^{p'} + \frac{1}{p}|v|^p, \tag{17}$$

whose derivative along trajectories is given by

$$\frac{dV}{dt} = -\gamma |v'|^p. \tag{18}$$

The continuous dependence assertion is a well-known consequence of uniqueness of solutions to problem (16) ([19]). Actually, sharper results on the dependence of the solution v on initial data v_0, w_0 and parameter γ can be achieved (see [13]). $\square$

Remark 3. Lemma 4 also holds true if $p > 2$. However, solutions to (16) are not twice differentiable at the discrete set of points where u' vanishes. This fact is more clearly expressed through the asymptotic estimate

$$v(t) = -v_0 \varphi_{p'}(t - t_0) + o(|t - t_0|^{p'-1}), \qquad t \to t_0, \tag{19}$$

which is valid when $v'(t_0) = 0$.

Remark 4. We point out that if we let p go to 1 both in (17) and (18), we formally obtain $V(v,w) = |v|$ and $\frac{dV}{dt} = -\gamma|v'|$. This suggests the extra equation (13) in Theorem 2.

Suppose again that $u \in W_0^{1,p}(I)$ is a weak eigenfunction to (1). Since the equation (3) is invariant with respect to similarities $v \to \mu v$, $\mu \in \mathbb{R}$, it is clear from Lemma 4 that $u(x) = \mu \phi(\lambda^{\frac{1}{p}} x)$ for certain $\mu \neq 0$, where $v = \phi(t)$ is the unique solution to

$$\begin{cases} (\varphi_p(v_t))_t + \gamma \varphi_p(v_t) + \varphi_p(v) = 0, & t > 0, \\ v(0) = 0,\ v'(0) = 1, \end{cases} \tag{20}$$

for $\gamma = c\lambda^{-\frac{1}{p}}$. Since ϕ must vanish at $t = \lambda^{\frac{1}{p}} > 0$, then an analysis of the oscillatory properties of the solutions to (3) is required.

In spite of equation (3) being non linear, the ansatz $v = e^{zt}$ turns out to be helpful to shed some light on this. It leads to the equation

$$(p-1)|z|^p + \gamma\varphi_p(z) + 1 = 0, \tag{21}$$

which is, of course, the equivalent to the characteristic equation in the linear case. By setting,

$$h(z) = (p-1)|z|^p + \gamma\varphi_p(z) + 1, \tag{22}$$

it is easily seen that $h(z) > 0$ for all z if $0 \le \gamma < p$, while h vanishes exactly at $z_1 = -1$ for $\gamma = p$. Similarly, h exhibits exactly two negative zeros $z_2 < z_1 < 0$ when $\gamma > p$. The next two results show that similar consequences to those of the linear case can be extracted from this discussion on the roots of (21). A sketch of the proof of Lemmas 5 and 6 below is provided for future use in Section 3. We are restricting ourselves to the regime $1 < p \le 2$ (see [16] for the complementary range $p > 2$).

Lemma 5. *Assume that v is a nontrivial solution to (3) defined in $J = [0, \infty)$. Then v vanishes at most once in J provided $\gamma \ge p$. In particular, the solution ϕ to (20) becomes positive in $(0, \infty)$.*

Lemma 6. *Let v be a nontrivial solution of (3) defined in $J = [0, \infty)$ and suppose that $0 \le \gamma < p$. Then there exists $T = T(\gamma)$ so that*

i) v vanishes exactly at $t = t_0 + (n-1)T$ for a certain $t_0 \in J$ and all $n \in \mathbb{N}$. In particular ϕ vanishes at $t = nT$, $n \in \mathbb{N}$.

ii) Function $T(\gamma)$ is smooth and increasing in $0 \le \gamma < p$,

$$T(0) = \frac{2\pi}{p\sin\left(\frac{\pi}{p}\right)}(p-1)^{1/p}, \tag{23}$$

while $T(\gamma) \to \infty$ as $\gamma \to p-$.

iii) An integral expression for T is given by

$$T(\gamma) = 2(p-1)\int_0^\infty \frac{t^p + p - 1}{(t^p + p - 1)^2 - \gamma^2 t^2}\, dt. \tag{24}$$

Proof of Lemmas 5 and 6. Let v be a nontrivial solution of (3). It gives rise to the nontrivial solution $\mathbf{v} = (v_1, v_2) = (v, v')$ to

$$\begin{cases} v_1' = v_2 \\ v_2' = -\frac{1}{p-1}|v_2|^{2-p}\{\varphi_p(v_1) + \gamma\varphi_p(v_2)\}, \end{cases} \tag{25}$$

satisfying $\mathbf{v}(0) = (v(0), v'(0))$. It should be remarked that such solution $\mathbf{v}(t)$ attains at finite time t the values $(v_0, 0) \neq (0, 0)$, even though the latter are critical points to (25) (see below). In that case, it is implicit in the estimate (19) that a solution $\mathbf{v}$ to (16) can only assume such value $(v_0, 0)$ at a unique instant t. Indeed, although all solutions to (3) generate a solution of (25), not all solutions to the latter equation come from a solution to (3) (see [16] for further details).

Moreover, there exist C^1 functions ρ and θ defined in $[0, \infty)$ so that ([29])

$$v_1(t) = \rho(t)\cos\theta(t), \qquad v_2(t) = \rho(t)\sin\theta(t), \qquad t \geq 0. \tag{26}$$

Thus, ρ and θ solve the equations

$$\begin{cases} \rho' = \ \rho f_1(\theta), \\ \theta' = \ \ f_2(\theta), \end{cases} \tag{27}$$

where

$$f_1 = -\frac{1}{p-1}\frac{\sin^2\theta}{|\tan\theta|^p}\left\{\gamma|\tan\theta|^p + \tan\theta - (p-1)\varphi_p(\tan\theta)\right\},$$

$$f_2 = -\frac{1}{p-1}\frac{\sin^2\theta}{|\tan\theta|^p}h(\tan\theta),$$

and h is the function defined in (22). Observe that as a consequence of Lemma 3 we find that $\rho(t) \to 0$ as $t \to \infty$ for every nontrivial solution v to (3).

Let us suppose next that $0 \leq \gamma < p$. Taking into account that $f_2 \to -1$ as $\theta \to \frac{\pi}{2} + n\pi$, $n \in \mathbb{Z}$, it follows that $f_2 < 0$ for all $\theta \neq n\pi$. In addition,

$$f_2(\theta) \sim -\frac{1}{p-1}|\sin\theta|^{2-p} \qquad \text{as } \theta \to n\pi.$$

This means that the integral

$$\int_{n\pi}^{\theta} \frac{ds}{f_2(s)}$$

converges for every $n \in \mathbb{Z}$. Thus $\theta(t)$ reaches the values $\theta = n\pi$ at finite t and, in view of (19), $\theta(t)$ progressively crosses all these values in decreasing sense. Therefore, $v(t) = v_1(t)$ exhibits infinitely many zeros in $[0, \infty)$. To compute the distance between two consecutive zeros $t_1 < t_2$, we observe that $\theta(t_1) = n\pi + \frac{\pi}{2}$ and $\theta(t_2) = n\pi - \frac{\pi}{2}$. From the equation for θ in (27) we obtain

$$t_2 - t_1 = \int_{n\pi+\frac{\pi}{2}}^{n\pi-\frac{\pi}{2}} \frac{ds}{f_2(s)} = -\left(\int_{-\frac{\pi}{2}}^{0} + \int_{0}^{\frac{\pi}{2}}\right) \frac{ds}{f_2(s)},$$

and hence

$$t_2 - t_1 = (p-1)\int_0^\infty \frac{d\tau}{\tau^p + \gamma\tau + p - 1} + (p-1)\int_0^\infty \frac{d\tau}{\tau^p - \gamma\tau + p - 1}, \quad (28)$$

which is the desired expression (24) for T. The smoothness and increasing character of T are also implicit in (24).

As for the case $\gamma \geq p$, suppose that $v(t) \not\equiv 0$ solves (3) and is different from $e^{z_i t}$, where $h(z_i) = 0$, $i = 1, 2$. Assume without loss of generality that $v(t_0) = 0$ and $v'(t_0) > 0$. Then, $\theta(t_0) = \frac{\pi}{2}$ (mod π) whereas $\theta(t) \to \theta_1$ as $t \to \infty$ in a decreasing way. Here $-\frac{\pi}{2} < \theta_1 < 0$ with $\tan\theta_1 = z_1$, z_1 being the maximum negative root of h. This proves Lemma 5. $\square$

3. Eigenvalues for $p > 1$ and their limits

The following result is a shortened version of Theorem 4.1 in [16]. Details of the proof will be instrumental for the forthcoming issues in this paper.

Theorem 7. *The set of eigenvalues of* (1) *consists of a sequence*

$$\left(\frac{c}{p}\right)^p < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots,$$

so that $\lambda_n \sim \sigma_n$ as $n \to \infty$, where σ_n is the n–th eigenvalue to (2). *Moreover, every λ_n is simple while its associated eigenfunctions are scalar multiples of*

$$u(x) = \phi(\lambda_n^{\frac{1}{p}} x, \gamma_n),$$

where $\gamma_n = c\lambda_n^{-\frac{1}{p}}$ and $\phi(\cdot, \gamma)$ stands for the solution to (20). *In particular, every eigenfunction vanishes at the points $x = k/n$, $0 \leq k \leq n$.*

Proof. An eigenfunction u associated to a possible eigenvalue λ is necessarily of the form

$$u = \phi(\lambda^{\frac{1}{p}} x, \gamma), \qquad \gamma = c\lambda^{-\frac{1}{p}},$$

modulus a scalar factor. Thus

$$\lambda^{\frac{1}{p}} = nT(\gamma) \tag{29}$$

for some $n \in \mathbb{N}$. Hence we get $\gamma = \gamma_n$ as the unique solution to

$$c = n\gamma T(\gamma), \qquad 0 \leq \gamma < p. \tag{30}$$

We remark that existence and uniqueness for (30) is a consequence of the increasing character of the function $\gamma \mapsto \gamma T(\gamma)$ together with the limit $\lim_{\gamma \to p-} \gamma T(\gamma) = +\infty$. Notice also that the sequence $0 < \gamma_n < p$ is decreasing and $\gamma_n \to 0$. Then the eigenvalues to (1) are exactly

$$\lambda_n = \left(\frac{c}{\gamma_n}\right)^p,$$

and from the previous remarks we achieve

$$\lambda_n \sim n^p T(0)^p = \sigma_n \qquad \text{as } n \to \infty.$$

□

Remark 5. The normalized eigenfunction in Theorem 1 is obtained as $u_n = v_0'\phi$, where $v_0' > 0$ is chosen so as to satisfy

$$\max_{[0,\infty)} v_0'\phi(t) = 1,$$

with ϕ the solution to (20). Since function V in (17) decreases, it follows that

$$(p-1){v_0'}^p > 1,$$

and so $v_0' \to \infty$ when $p \to 1+$.

We proceed now to show the assertion in Theorem 1 regarding the limit of the λ_n's as $p \to 1$. In the next statement, the parameter p is incorporated as an extra argument in the expression of the relevant functions.

Lemma 8. *Let $T(\gamma, p)$, $0 \le \gamma < p$, the function defined in (24). Then*

$$\lim_{p \to 1+} T(\gamma, p) = \frac{1}{\gamma} \log\left(\frac{1+\gamma}{1-\gamma}\right), \quad \gamma \in (0,1), \tag{31}$$

where the value of the limit is equal to 2 if $\gamma = 0$.

Proof. Let $p > 1$ and $\gamma \in [0,1)$. Set

$$T(\gamma, p) = T(\gamma, p)^+ + T(\gamma, p)^-,$$

where

$$T(\gamma, p)^+ = (p-1) \int_0^{+\infty} \frac{1}{t^p + \gamma t + p - 1}\, dt \tag{32}$$

and

$$T(\gamma, p)^- = (p-1) \int_0^{+\infty} \frac{1}{t^p - \gamma t + p - 1}\, dt. \tag{33}$$

To achieve (31) it suffices to take $a = p - 1$ and $b = \pm\gamma$ in formula (34) proved in Lemma 9. □

Lemma 9. *Let $b > -1$. It holds that*

$$\lim_{a \to 0+} \int_0^{\infty} \frac{a}{t^{1+a} + bt + a}\, dt = \frac{\log(1+b)}{b}, \tag{34}$$

where the value of the limit is equal to 1 if $b = 0$.

Proof. We first check that the integrand appearing in (34) is positive and bounded for $t \ge 0$ and $a > 0$. It is obvious that

$$t^{1+a} + bt + a \ge t^{1+a} - t + a, \quad b > -1, \quad a > 0, \quad t \ge 0.$$

Set $g(t) = t^{1+a} - t + a$. Then $g'(t) = (1+a)t^a - 1$ and the function g attains its minimum at the point

$$t_a = \frac{1}{(1+a)^{1/a}} < 1.$$

Thus, we can write $t_a = 1/x_a$, with $x_a \in [1, +\infty)$. For each $x \in [1, +\infty)$ let us consider the function

$$h(a) = \frac{1}{x^{1+a}} - \frac{1}{x} + a, \quad a \in [0, \infty).$$

It is clear that $h(0) = 0$, whereas

$$h'(a) = 1 - \frac{\log x}{x^{1+a}} > 1 - \frac{\log x}{x} > 0, \quad a > 0,$$

which shows that $h(a) > 0$ for all $x \geq 1$ and $a > 0$. In particular, $g(t_a) > 0$, as we wanted to prove.

Next, we will prove the formula

$$\lim_{a \to 0+} \int_0^M \frac{a}{t^{1+a} + bt + a} \, dt = 0, \quad b > -1, \quad M > 0. \tag{35}$$

Let us fix an arbitrary $\delta \in (0, M)$ and $b > -1$. We have

$$\int_0^\delta \frac{a}{t^{1+a} + bt + a} \, dt \leq \frac{a\delta}{t_a^{1+a} - t_a + a}.$$

Note that

$$\lim_{a \to 0+} \frac{t_a^{1+a} - t_a + a}{a} = \frac{e-1}{e}, \tag{36}$$

since

$$\frac{t_a^{1+a} - t_a + a}{a} = 1 - \frac{t_a - t_a^{1+a}}{a} = 1 - t_a \frac{1 - t_a^a}{a} = 1 - \frac{t_a}{1+a}$$

and

$$\lim_{a \to 0+} \frac{t_a}{1+a} = \lim_{a \downarrow 0} t_a = \frac{1}{e}.$$

Then it follows from (36) that

$$\limsup_{a \to 0+} \int_0^\delta \frac{a}{t^{1+a} + bt + a} \, dt \leq \frac{e}{e-1} \, \delta. \tag{37}$$

Furthermore, the dominated convergence theorem implies

$$\lim_{a \to 0+} \int_\delta^M \frac{a}{t^{1+a} + bt + a} \, dt = 0,$$

which, together with (37), proves the estimate

$$\limsup_{a \to 0+} \int_0^M \frac{a}{t^{1+a} + bt + a} \, dt \leq \frac{e}{e-1} \, \delta. \tag{38}$$

Since $\delta > 0$ is arbitrary, we obtain (35) by taking limits in (38) as δ goes to 0.

Now we are ready to prove (34). Let us consider an arbitrary $M > 1$. Since the function $f(t) = t^{1+a} + bt$ is strictly positive on the interval $[1, +\infty)$, the integral

$$\int_M^{+\infty} \frac{1}{t^{1+a} + bt}\, dt, \quad b > -1, \quad a > 0,$$

is convergent. To solve it, we perform the change of variable $x = t^a$. Then

$$\int_M^{+\infty} \frac{1}{t^{1+a} + bt}\, dt = \frac{1}{a}\int_{M^a}^{+\infty} \frac{1}{x(x+b)}\, dx = \frac{1}{ab}\int_{M^a}^{+\infty} \left(\frac{1}{x} - \frac{1}{x+b}\right) dx$$

$$= -\frac{1}{ab}\left[\log\left(1 + \frac{b}{x}\right)\right]_{M^a}^{+\infty} = \frac{1}{ab}\log\left(1 + \frac{b}{M^a}\right).$$

Consequently,

$$\lim_{a\to 0+} \int_M^{+\infty} \frac{a}{t^{1+a} + bt}\, dt = \frac{\log(1+b)}{b}. \tag{39}$$

These calculations are valid provided that $b \neq 0$. In the case $b = 0$, it is very easy to see that

$$\lim_{a\to 0+} \int_M^{+\infty} \frac{a}{t^{1+a}}\, dt = 1. \tag{40}$$

In addition, it is clear that

$$0 \leq \int_M^{+\infty} \frac{a}{t^{1+a} + bt + a}\, dt \leq \int_M^{+\infty} \frac{a}{t^{1+a} + bt}\, dt$$

and, therefore, we have

$$0 \leq \int_M^{+\infty} \left(\frac{a}{t^{1+a} + bt} - \frac{a}{t^{1+a} + bt + a}\right) dt$$

$$= \int_M^{+\infty} \frac{a^2}{(t^{1+a} + bt + a)(t^{1+a} + bt)}\, dt \leq \int_M^{+\infty} \frac{a^2}{(t^{1+a} + bt)^2}\, dt.$$

At this point we need to further specify the constant M depending on the constant b. Fix $b > -1$ and choose $M > 1$ such that $t^{1+a} + bt > 1$ for all

$t \geq M$. For instance, it is enough to take $M > 1/(1+b)$, since $t^{1+a} + bt > (1+b)t$. Then

$$\int_M^{+\infty} \frac{a^2}{(t^{1+a}+bt)^2}\, dt < \int_M^{+\infty} \frac{a^2}{t^{1+a}+bt}\, dt$$

and, taking account of (39) and (40), we arrive at the expression

$$\lim_{a\to 0+} \int_M^{+\infty} \left(\frac{a}{t^{1+a}+bt} - \frac{a}{t^{1+a}+bt+a} \right) dt = 0.$$

Now, this last equality and formulas (35), (39) y (40) imply

$$\lim_{a\to 0+} \int_0^{\infty} \frac{a}{t^{1+a}+bt+a}\, dt = \lim_{a\to 0^+} \int_M^{+\infty} \frac{a}{t^{1+a}+bt}\, dt = \frac{\log(1+b)}{b},$$

which proves (34). □

Remark 6. Proof of Lemma 9 can be adapted to show the slightly more general relation

$$\lim_{(a,b)\to(0+,\bar{b})} \int_0^{\infty} \frac{a}{t^{1+a}+bt+a}\, dt = \frac{\log\left(1+\bar{b}\right)}{\bar{b}},$$

provided that $\bar{b} > -1$.

Proof of Theorem 1–a). As shown in Theorem 7,

$$\lambda_n(p,c) = \left(\frac{c}{\gamma_n(p,c)} \right)^p,$$

where $\gamma = \gamma_n(p,c)$, $0 < \gamma < p$, is the unique root of the equation (30). Therefore, $\gamma_n(p,c) \to \bar{\gamma}_n$ as $p \to 1$, where $\gamma = \bar{\gamma}_n \in (0,1)$ is the solution to the equation

$$c = n \log\left(\frac{1+\gamma}{1-\gamma} \right),$$

and so,

$$\bar{\gamma}_n = \frac{e^{\frac{c}{n}} - 1}{e^{\frac{c}{n}} + 1}. \tag{41}$$

□

Proof of Theorem 1–b). Set $v = v(t, \gamma, p)$ the solution to (3) such that $v(0) = 0$ together with $\sup_{(0,\infty)} v = 1$ (see Figure 1). Explicit reference to p in the expression for v will be omitted for short. As was pointed out (Remark 5), v is a scalar multiple of the solution ϕ to (20), while the normalized n–th eigenfunction associated to (1) is defined by

$$u_n(x) = v(\lambda_n^{\frac{1}{p}} x, \gamma), \qquad \gamma = \gamma_n, \tag{42}$$

γ_n being the root of (30). Function $v(t, \gamma)$ vanishes exactly at $t = nT(\gamma)$, $n \in \mathbb{N}$. Set

$$\alpha_{n-1} = \max_{t \in ((n-1)T, nT)} |v(t, \gamma)|.$$

Then, the decreasing character of the energy V (see Lemma 4) implies that α_n decreases. In particular,

$$\alpha_0 = \max_{t \in (0,T)} v(t, \gamma) = 1.$$

In order to study the behavior of v in the interval $((n-1)T, nT)$ a first fact to be observed is that the relation

$$v(t, \gamma) = (-1)^{n-1} \alpha_{n-1} v(t - (n-1)T, \gamma), \tag{43}$$

holds for $(n-1)T \leq t \leq nT$. In fact, it follows from the analysis in Lemma 6 that the extremum value $(-1)^{n-1}\alpha_{n-1}$ is achieved at

$$t = t_{n-1} := (n-1)T + T^+,$$

T and T^+ being given by (24) and (32) respectively. Accordingly, the shape of v in $((n-1)T, nT)$ is controlled by the amplitude α_{n-1} and the restriction of $v(t, \gamma)$ to the initial interval $(0, T)$.

Let us analyze α_{k-1} as $p \to 1$. We are also keeping track of the dependence of α_{k-1} on the parameter γ.

By employing the expression (26) for v and v', we notice that $\theta(t_k) = -k\pi$ together with $\rho(t_k) = \alpha_k$. In addition, the orbital equation associated to (27) is

$$\frac{d\rho}{d\theta} = \rho \frac{f_1(\theta)}{f_2(\theta)}.$$

Thus,

$$\frac{\alpha_k}{\alpha_{k-1}} = \exp\left\{ - \int_0^{\pi} \frac{f_1(\theta)}{f_2(\theta)} \right\} = \exp\left\{ - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{f_1(\theta)}{f_2(\theta)} \right\},$$

wherein the fact that $\frac{f_1(\theta)}{f_2(\theta)}$ is π–periodic has been employed.

Define
$$J = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{f_1(\theta)}{f_2(\theta)}\, d\theta.$$
Setting $\tau = \tan\theta$ we find
$$\int_0^{\frac{\pi}{2}} \frac{f_1(\theta)}{f_2(\theta)}\, d\theta = \int_0^\infty \frac{1}{1+\tau^2}\frac{g(\tau)}{h(\tau)}\, d\tau,$$
where h is defined in (22) while
$$g(\tau) = \gamma|\tau|^p + \tau - (p-1)\varphi_p(\tau).$$
Similarly,
$$\int_{-\frac{\pi}{2}}^{0} \frac{f_1(\theta)}{f_2(\theta)}\, d\theta = \int_0^\infty \frac{1}{1+\tau^2}\frac{g(-\tau)}{h(-\tau)}\, d\tau.$$
It can be seen that
$$\frac{1}{1+\tau^2}\left(\frac{g(\tau)}{h(\tau)} + \frac{g(-\tau)}{h(-\tau)}\right) = 2\gamma(p-1)\frac{\tau^{2(p-1)}}{((p-1)\tau^p+1)^2 - \gamma^2\tau^{2(p-1)}}.$$
Thus,
$$J = 2\gamma(p-1)\int_0^\infty \frac{\tau^{2(p-1)}}{((p-1)\tau^p+1)^2 - \gamma^2\tau^{2(p-1)}}\, d\tau.$$
At this point it is worth considering J as a function of both p and γ, $0 \le \gamma < 1$. Then, by proceeding as in Lemma 9 and putting $a = p-1$, we next show that
$$\lim_{(a,\gamma)\to(0+,\bar\gamma)} J = -\log\left(\frac{1-\bar\gamma}{1+\bar\gamma}\right).$$
In fact,
$$\begin{aligned} J &= 2\gamma a\int_0^\infty \frac{\tau^{2a}}{(a\tau^{a+1}+1)^2 - \gamma^2\tau^{2a}}\, d\tau = 2\gamma\int_0^\infty \frac{s^{2a}}{(s^{a+1}+a^a)^2 - \gamma^2 s^{2a}}\, ds \\ &= 2\gamma\int_0^\infty \frac{ds}{(s+a^a s^{-a})^2 - \gamma^2} = 2\gamma\left(\int_0^{\bar\gamma+2\varepsilon} + \int_{\bar\gamma+2\varepsilon}^\infty\right)\frac{ds}{(s+a^a s^{-a})^2 - \gamma^2}, \end{aligned}$$
with a small $\varepsilon > 0$. Assume that $\gamma \le \bar\gamma + \varepsilon$. The latter integrand is estimated as
$$\frac{2\gamma}{(s+a^a s^{-a})^2 - \gamma^2} \le \frac{2(\bar\gamma+\varepsilon)}{s^2 - (\bar\gamma+\varepsilon)^2},$$

while the former integrand can be estimated in the form

$$\frac{2\gamma}{(s+a^a s^{-a})^2-\gamma^2} \le \frac{2(\bar\gamma+\varepsilon)}{(s+a^a(\bar\gamma+2\varepsilon)^{-a})^2-(\bar\gamma+\varepsilon)^2}.$$

Indeed, taking into account that $\bar\gamma < 1$, a small ε can be chosen such that

$$a^a(\bar\gamma+2\varepsilon)^{-a} \ge \bar\gamma+2\varepsilon$$

when a is small enough.

Hence, dominated convergence enables us to conclude that

$$\lim_{(a,\gamma)\to(0+,\bar\gamma)} J = 2\bar\gamma\int_0^\infty \frac{ds}{(s+1)^2-\bar\gamma^2} = 2\bar\gamma\int_1^\infty \frac{ds}{s^2-\bar\gamma^2} = -\log\left(\frac{1-\bar\gamma}{1+\bar\gamma}\right).$$

Thus,

$$\lim_{(a,\gamma)\to(0+,\bar\gamma)} \frac{\alpha_k}{\alpha_{k-1}} = \frac{1-\bar\gamma}{1+\bar\gamma}. \tag{44}$$

Since $\alpha_0 = 1$ this implies that

$$\lim_{(a,\gamma)\to(0+,\bar\gamma)} \alpha_k = \left(\frac{1-\bar\gamma}{1+\bar\gamma}\right)^k.$$

Going back to the expression (42) for the normalized eigenfunction u_n, we observe that in the interval $\left(\frac{k}{n},\frac{k+1}{n}\right)$ it takes its extremum with value $(-1)^k\alpha_k$ at the point

$$\xi_k = \frac{k}{n} + \frac{T^+}{nT},$$

where both T and T^+ are evaluated at $\gamma=\gamma_n$. Thus, in view of (41), we obtain

$$\lim_{p\to 1}\alpha_k = \left(\frac{1-\bar\gamma_n}{1+\bar\gamma_n}\right)^k = e^{-c\frac{k}{n}},$$

together with

$$\lim_{p\to 1+}\xi_k = \frac{k}{n} + \frac{1}{c}\log(1+\bar\gamma_n).$$

This proves both (7) and (8).

Next, we are going to show c). As before, χ stands for the characteristic function of the interval $(0,1)$. First observe that, in view of (43), the convergence

$$u_n(x) \to (-1)^{k-1}\bar\alpha_{k-1}\chi\left(nx-(k-1)\right) \quad \text{as} \quad p\to 1$$

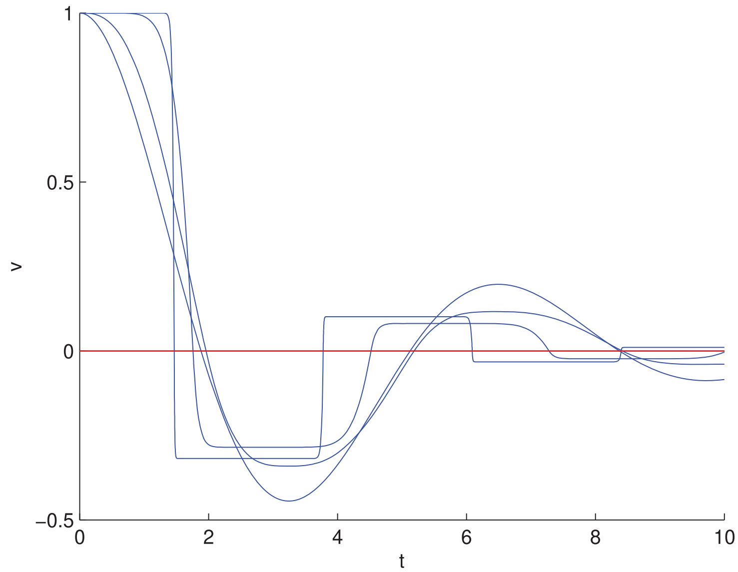

Figure 1: Profile of $v(t) := v(t + T^+, \gamma)$, $t \geq 0$. Notice that v solves (3) with initial conditions $v(0) = 1$, $v'(0) = 0$. Parameter γ is fixed to $\gamma = 0.5$ while p is successively chosen $p = 2$, $p = 1.5$, $p = 1.1$ and $p = 1.01$. The smaller p is the steeper v becomes. Solution v vanishes at $t = T^-(\gamma, p)$, its limit value as $p \to 1$ being, according to (34), $\overline{T}^-(0.5) = 1.39$. In the figure, the first computed zero of v for $p = 1.01$ is $T^- = 1.46$.

in $C\left(\frac{k-1}{n}, \frac{k}{n}\right)$, amounts to prove that

$$v(t, \gamma_n) \to \chi\left(\overline{T}^{-1} t\right) \quad \text{as} \quad p \to 1,$$

in $C(0, \overline{T})$ (see Figures 1, 2, 3), where the value $\overline{T}$ is defined by

$$\overline{T} = \frac{c}{n\bar{\gamma}_n}\,.$$

Setting $v(t) = v(t, \gamma_n)$ for short we begin by studying its limit profile in $(0, \overline{T}^+)$, where

$$\overline{T}^+ = \frac{1}{\bar{\gamma}_n} \log(1 + \bar{\gamma}_n).$$

Accordingly, we fix $0 < \eta < 1$ and define t_η as the unique value $0 < t < T^+(\gamma_n)$ where $v = 1 - \eta$. From the equation for θ in (27) an expression for t_η is

$$t_\eta = (p-1) \int_0^{\cot\theta(t_\eta)} \frac{dt}{t^p + \gamma_n t + (p-1)}.$$

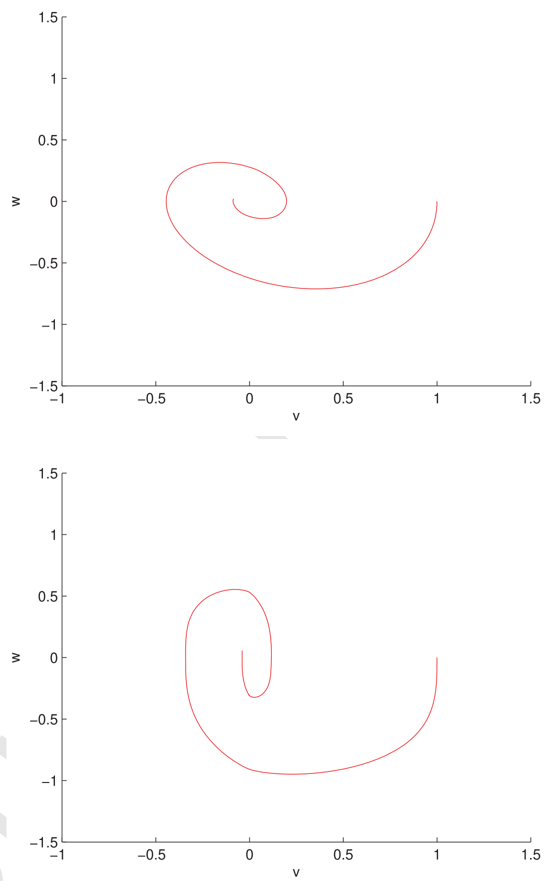

Figure 2: The orbit corresponding to the solution $(v, w) = (v(t), \varphi_p(v'(t)))$ to (16) with $(v_0, w_0) = (1, 0)$. The parameter γ is taken to be 0.5 while $p = 2$ and $p = 1.5$, respectively.

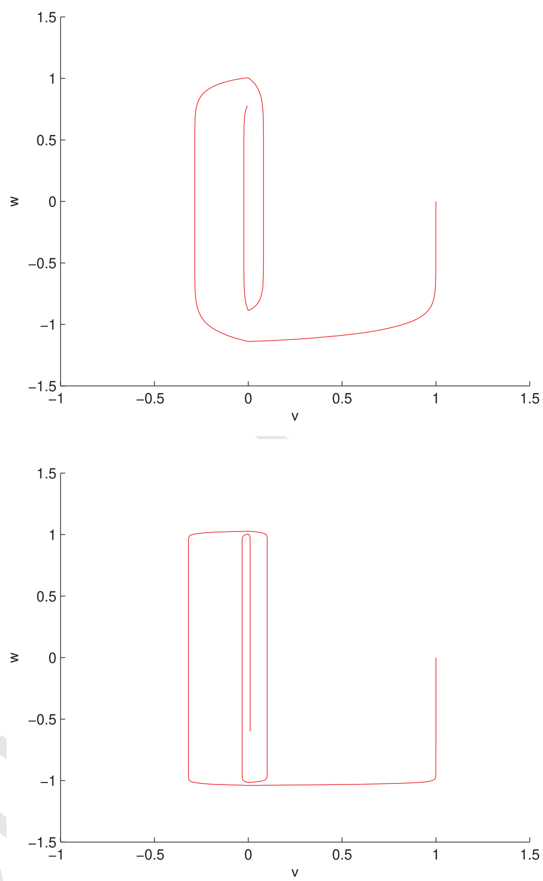

Figure 3: The orbit in Figure 2 is drawn now for the values $p = 1.1$ and $p = 1.01$. The closer to 1 the value of p is, the more "squared" the shape of the orbit becomes.

Equivalently,

$$t_\eta = (p-1)^{\frac{1}{p}} \int_0^{\frac{\cot\theta(t_\eta)}{(p-1)^{\frac{1}{p}}}} \frac{d\tau}{\tau^p + \gamma_n (p-1)^{-\frac{1}{p'}}\tau + 1}. \tag{45}$$

Since v is increasing in $[0, T^+(\gamma_n)]$ and the energy $V(v, v') = \frac{p-1}{p}|v'|^p + \frac{1}{p}|v|^p$ decreases, the inequality

$$(p-1)v'(t_\eta)^p + (1-\eta)^p > 1$$

holds. Now, it follows from

$$\cot\theta(t_\eta) = \frac{1-\eta}{v'(t_\eta)},$$

that

$$\frac{\cot\theta(t_\eta)}{(p-1)^{\frac{1}{p}}} < \frac{1-\eta}{(1-(1-\eta)^p)^{\frac{1}{p}}},$$

and so,

$$\frac{\cot\theta(t_\eta)}{(p-1)^{\frac{1}{p}}} = O(1) \quad \text{as} \quad p \to 1.$$

This in turn says that

$$\int_0^{\frac{\cot\theta(t_\eta)}{(p-1)^{\frac{1}{p}}}} \frac{d\tau}{\tau^p + \gamma_n (p-1)^{-\frac{1}{p'}}\tau + 1} \le \int_0^{\frac{\cot\theta(t_\eta)}{(p-1)^{\frac{1}{p}}}} \frac{d\tau}{\tau^p + 1} = O(1) \quad \text{as} \quad p \to 1.$$

The relation (45) then implies that $t_\eta \to 0$ as $p \to 1$. This entails that

$$v(t, \gamma_n) \to 1 \quad \text{as} \quad p \to 1,$$

in $C(0, \overline{T}^+)$. We now show that

$$v(t_0, \gamma_n) \to 1 \quad \text{as} \quad p \to 1 \tag{46}$$

at every fixed $\overline{T}^+ < t_0 < \overline{T}$. In fact, direct integration yields

$$v(t_0) = 1 - \int_{T^+(\gamma_n)}^{t_0} \psi(t)^{\frac{1}{p-1}} \, dt, \tag{47}$$

where

$$\psi(t) = \int_{T^+}^{t} e^{\gamma_n(s-t)} v(s)^{p-1}\, ds.$$

Notice that

$$\psi(t) \le \frac{1}{\gamma_n}(1 - e^{-\gamma_n(t_0 - T^+(\gamma_n))}) \qquad \text{for} \qquad T^+(\gamma_n) \le t \le t_0.$$

Since

$$\lim_{p\to 1+} \frac{1}{\gamma_n}(1 - e^{-\gamma_n(t_0 - T^+(\gamma_n))}) = \frac{1}{\bar{\gamma}_n}(1 - e^{-\bar{\gamma}_n(t_0 - \overline{T}^+)})$$

and $t_0 - \overline{T}^+ < \overline{T}^-$, where

$$\overline{T}^- = -\frac{1}{\overline{\gamma}_n}\log(1 - \overline{\gamma}_n),$$

we find that

$$\frac{1}{\gamma_n}(1 - e^{-\gamma_n(t_0 - T^+(\gamma_n))}) \le k < 1$$

for p close enough to 1. This estimate combined with (47) yields (46).

Finally, for arbitrary $0 < a < \overline{T}^+ < b < \overline{T}$ we observe that

$$1 \ge \min_{[a,b]} v(\cdot, \gamma_n) \ge \min\{v(a,\gamma_n), v(b,\gamma_n)\} \to 1 \quad \text{as} \quad p \to 1. \tag{48}$$

We thus achieve the desired conclusion.

As for d) observe that

$$w(t,p) := \varphi_p(v'(t,p)) \to \frac{1}{\bar{\gamma}_n}(e^{\bar{\gamma}_n(\overline{T}^+ - t)} - 1) \quad \text{as} \quad p \to 1, \tag{49}$$

pointwise in any fixed interval $[a,b] \subset (0,\overline{T})$. By appealing both to equation (3) and the behavior of $v(\cdot,p)$, it follows that the family $\{w(\cdot,p)\}_p$ is equicontinuous and uniformly bounded in $[a,b]$. Hence, limit (49) is uniform. Therefore the limit

$$v'(t,p) \to 0 \qquad \text{as } p \to 1, \tag{50}$$

is actually uniform in $[a,b]$. This fact is transferred to the remaining intervals $((k-1)\overline{T}, k\overline{T})$ by means of (43).

Thus we have completed the proof of Theorem 1. $\square$

4. Eigenvalues and eigenfunctions of the limit problem

This section is concerned with the limit problem in (1) when $p \to 1$. It goes without saying that the analysis is more delicate than in the case $p > 1$. Due to the presence of the 1–Laplacian operator $\left(\frac{u_x}{|u_x|}\right)_x$, the natural framework to study problem (11) is the space $BV(0,1)$ of functions of bounded variation in the interval $(0,1)$. A function $u \in L^1(0,1)$ belongs to $BV(0,1)$ provided that its distributional derivative $u_x \in \mathcal{D}'(0,1)$ is a signed Radon measure with finite total variation. In that case, $|u_x|$ will designate the total variation measure of the distributional derivative of u. A first feature is the validity of the embedding $BV(0,1) \subset L^\infty(0,1)$. Additionally, functions that are equal almost everywhere are identified. However, it is remarkable that in any equivalence class there always exist distinguished elements, the so–called good representatives. A good representative u satisfies

$$\sup\left\{\sum_{i=1}^{n-1} |u(x_{i+1}) - u(x_i)|\right\} < \infty,$$

where $0 < x_1 < \cdots < x_n < 1$ varies in the partitions of $(0,1)$ with $n \geq 2$ elements. Regarding $u_x \in \mathcal{D}'(0,1)$ as a measure, its set of atoms is defined as $A = \{x : u_x(\{x\}) \neq 0\}$ (for instance $A = \{\frac{1}{2}\}$ for Dirac's $\delta(x - \frac{1}{2})$). It can be shown that every good representative u is continuous in $(0,1)\backslash A$ and has just jump discontinuities at every point of A. In what follows, we are only dealing with good representatives of functions in $BV(0,1)$. We refer to [1, Section 3.2] for further information on this class of functions.

The next definition involves a concept of solution for the 1–Laplacian that goes back to [2, 3, 9]. For the specific case of the N–dimensional Dirichlet eigenvalue problem, it has been recently analyzed in [28] (see also [6] for the one–dimensional setting). We continue to denote $I = (0,1)$.

Definition 10. *We say that* (λ, u) *is a* weak *eigenpair to problem* (11)

$$\begin{cases} -\left(\dfrac{u_x}{|u_x|}\right)_x - c\dfrac{u_x}{|u_x|} = \lambda\dfrac{u}{|u|}, & 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases}$$

if $u \in BV(I)$ *and there exist functions* $\mathbf{z} \in W^{1,\infty}(I)$ *and* $\beta \in L^\infty(I)$ *such that*

1) $-\mathbf{z}_x - c\mathbf{z} = \lambda\beta$ *in the sense of* $\mathcal{D}'(0,1)$*,*

2) $\|\mathbf{z}\|_\infty \le 1$ *and* $(\mathbf{z}, u_x) = |u_x|$ *as measures (recall* $|u_x|$ *is the total variation of* u_x*),*

3) $\|\beta\|_\infty \le 1$ *and* $\beta u = |u|$ *holds a.e. in* I*,*

4) boundary conditions are understood in the following sense:

$$z(0)u(0+) = |u(0+)|, \qquad -z(1)u(1-) = |u(1-)|,$$

where $u(0+)$ *and* $u(1-)$ *stand for the corresponding side limits.*

Remarks 7.

1) Functions $\mathbf{z}$ and β play the rôle of the fractions $\frac{u_x}{|u_x|}$ and $\frac{u}{|u|}$, respectively. Moreover, these functions provide a meaning to these quotients even when u_x or u vanish.

2) Condition $\mathbf{z} \in W^{1,\infty}(I)$ is specific to the one–dimensional case. It is just a consequence of $\mathbf{z}, \beta \in L^\infty(I)$ and that $\mathbf{z}$ solves the equation in 1). Notice that this is consistent with the fact $\varphi_p(u_x) \in W^{1,p'}(I)$ when $p > 1$ (see Section 2). In addition, it follows from $\mathbf{z} \in W^{1,\infty}(I)$, that $\mathbf{z}$ is a Lipschitz–continuous function in $\overline{I}$ (see [5]). In particular, $\mathbf{z}$ admits a derivative $\mathbf{z}_x$ a. e. in I with $\mathbf{z}_x \in L^\infty(I)$.

3) Distribution $(\mathbf{z}, u_x)$ in 2) is defined through the expression

$$\langle (\mathbf{z}, u_x), \varphi \rangle = -\int_0^1 u(\mathbf{z}\varphi)_x \,,$$

for $\varphi \in \mathcal{D}(0,1)$. Since $\mathbf{z}$ is a continuous function and u_x is a Radon measure then $(\mathbf{z}, u_x)$ also constitutes a Radon measure. In addition, it can be shown that the following integration by parts formula holds true:

$$\int_0^1 u(x)\mathbf{z}_x(x)\,dx + \int_0^1 (\mathbf{z}, u_x) = \mathbf{z}(1)u(1-) - \mathbf{z}(0)u(0+)\,. \tag{51}$$

4) In general, the boundary condition does not hold in the sense of traces when the 1–Laplacian operator is involved. The last condition of Definition 10 is a weak version of the usual Dirichlet condition. As pointed out above, functions $u \in BV(I)$ always admit finite side limits $u(x\pm)$ at every $0 < x < 1$ as well as $u(0+)$ and $u(1-)$. Therefore, the weak boundary condition 4) makes sense.

Regarding the terminology "weak eigenvalue", it should be mentioned that a variational definition of eigenpair to (11) can be given, by straightforward extension of the one in [6] for the case $c = 0$. From this point of view "weak" eigenvalue becomes a necessary condition to be a "variational" eigenvalue. See Remark 9 below.

We next show that the limits $(\bar{\lambda}_n, \bar{u}_n)$ introduced in Theorem 1 define weak eigenpairs to problem (11).

Proposition 11. *For every $n \in \mathbb{N}$, $\bar{\lambda}_n = c\dfrac{e^{\frac{c}{n}}+1}{e^{\frac{c}{n}}-1}$ is a weak eigenvalue to problem* (11) *with*

$$\bar{u}_n(x) = \sum_{k=1}^{n} (-1)^{k-1} e^{-(k-1)\frac{c}{n}} \chi(nx-(k-1))$$

as associated normalized eigenfunction. Here χ denotes the characteristic function of the unit open interval $(0,1)$.

Proof. It is enough to check that the functions

$$\mathbf{z}(x) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{c} \left[(\bar{\lambda}_n + c) e^{-\frac{c}{n}(nx-(k-1))} - \bar{\lambda}_n \right] \chi(nx-(k-1))$$

and

$$\beta(x) = \sum_{k=1}^{n} (-1)^{k-1} \chi(nx-(k-1))$$

satisfy the requirements of Definition 10. $\square$

Let us now introduce the notion of a normalized solution to equation (12). In the next definition $BV_{loc}(0,+\infty)$ denotes the space

$$BV_{loc}(0,+\infty) = \bigcap_{b>0} BV(0,b).$$

Definition 12. *It is said that $v \in BV_{loc}(0,+\infty)$ is a* normalized *solution to equation* (12) *if there exist $w \in W^{1,\infty}(0,+\infty)$ and $\beta \in L^\infty(0,+\infty)$ such that*

1) $w_t + \gamma w + \beta = 0$ holds in $\mathcal{D}'(0,\infty)$,
2) $w(0) = 1$, $\|w\|_\infty \le 1$ and $(w, v_t) = |v_t|$ as measures,
3) $\|\beta\|_\infty \le 1$ and $\beta v = |v|$ a.e. in $(0,\infty)$,

4) $v(0+) = 1$.

Remark 8. Under condition (13), normalized solutions satisfy analogous features as those in (4). In fact, (13) says that $|v|$ is nonincreasing which, together with 4), implies that $\max_{[0,+\infty)} v = 1$. In addition, the equality $w(0)v(0+) = |v(0+)|$ follows from 2) and 4) and so the initial condition $v(0) = 0$ is fulfilled in the sense of Definition 10.

Existence of a normalized solution to (12) verifying (13) is now shown. We remark that the condition on the maximum of v below is equivalent to $\max_{[0,\infty)} v = 1$ together with $v'(0) > 0$. It means that v reaches its maximum before the first zero.

Proposition 13. *Fix* $0 \le \bar{\gamma} < 1$ *and for* $1 < p \le 2$, $0 \le \gamma < 1$, *let* $v_p \in C^2(0,\infty) \cap C[0,\infty)$ *be the solution to*

$$\begin{cases} (\varphi_p(v_t))_t + \gamma\varphi_p(v_t) + \varphi_p(v) = 0, & t > 0, \\ v(0) = 0, \qquad \max_{[0,\infty)} v = \max_{[0,\infty)} |v| = 1. \end{cases} \tag{52}$$

Then

$$v_p \to v \quad as \quad (p,\gamma) \to (1,\bar{\gamma})$$

in $L^1(0,b)$ *for all* $b > 0$, *where* v *is a normalized solution to equation*

$$\left(\frac{v_t}{|v_t|}\right)_t + \bar{\gamma}\frac{v_t}{|v_t|} + \frac{v}{|v|} = 0,$$

satisfying condition (13).

Proof. Let $T(\gamma,p)$ be the value defined in (24) (see Lemma 6) and $\overline{T}$ its limit as (p,γ) goes to $(1,\bar{\gamma})$. According to Lemma 8, we have

$$\overline{T} = \begin{cases} \dfrac{1}{\bar{\gamma}}\log\left(\dfrac{1+\bar{\gamma}}{1-\bar{\gamma}}\right) & \text{if } \bar{\gamma} \in (0,1)\,; \\ 2 & \text{if } \bar{\gamma} = 0\,. \end{cases}$$

First of all, notice that as a consequence of (48) the convergence

$$v_p(t) \to v(t) := 1, \qquad (p,\gamma) \to (1,\bar{\gamma})$$

holds in $C(0,\overline{T})$. Due to (43) and (44) it further implies that

$$v_p(t) \to v(t) := (-1)^k \alpha^k, \qquad \alpha := \frac{1-\bar{\gamma}}{1+\bar{\gamma}}, \tag{53}$$

in $C(k\overline{T},(k+1)\overline{T})$. This defines v with the exception of points $t = k\overline{T}$. Moreover, by taking side limits we arrive at

$$|v(k\overline{T}+)| - |v(k\overline{T}-)| = \frac{-2\bar{\gamma}}{1+\bar{\gamma}}\alpha^{k-1} = -\bar{\gamma}\left|v(k\overline{T}+) - v(k\overline{T}-)\right|. \tag{54}$$

Since v is constant in $(0,\infty)\setminus\{k\overline{T} : k = 1,2,\dots\}$, then its derivative vanishes except at the jumps. Hence, $v \in BV_{loc}(0,+\infty)$ and its derivative in the sense of $\mathcal{D}'(0,\infty)$ satisfies the equation (13). In addition, condition 3) holds under the choice of the function β defined as

$$\beta(t) = (-1)^k, \qquad t \in (k\overline{T},(k+1)\overline{T}). \tag{55}$$

Moreover, we have

$$\varphi_p(v_p'(t)) = -\int_{kT(\gamma,p)+T(\gamma,p)^+}^{t} e^{-\gamma(t-s)}\varphi_p(v_p)\, ds$$

and, as a consequence of Lemma 9, $\lim_{(p,\gamma)\to(1,\bar{\gamma})} T(\gamma,p)^+ = \overline{T}^+$. So, it follows from (53) that

$$\varphi_p(v_p'(t)) \to w(t) := \frac{(-1)^k}{\bar{\gamma}}(e^{-\bar{\gamma}(t-k\overline{T}-\overline{T}^+)} - 1) \tag{56}$$

as $(p,\gamma) \to (1,\bar{\gamma})$ in $C(k\overline{T},(k+1)\overline{T})$ for all k. By taking into account (Remark 6)

$$\overline{T}^{\pm} = \pm\frac{\log(1\pm\bar{\gamma})}{\bar{\gamma}},$$

it is found that

$$w(k\overline{T}+) = (-1)^k, \qquad w((k+1)\overline{T}-) = (-1)^{k+1}. \tag{57}$$

Hence w can be extended as a Lipschitz continuous function to the whole of $[0,\infty)$, conditions $w(0) = 1$ and $\|w\|_\infty \leq 1$ being clearly fulfilled.

Now observe that w, as defined in $[k\overline{T},(k+1)\overline{T}]$ through (56) and (57), just coincides with the solution of the initial value problem

$$\begin{cases} w_t + \bar{\gamma} w + \beta = 0, & k\overline{T} \le t \le (k+1)\overline{T}, \\ w(k\overline{T}) = (-1)^k, \end{cases}$$

for all k. Hence w and β satisfy 1) in Definition 12.

It only remains to check the equality $(w, v_t) = |v_t|$ as measures. The derivative of v is

$$v_t = \sum_{k=1}^{\infty} (v(k\overline{T}+) - v(k\overline{T}-)\delta(t - k\overline{T}),$$

where $\delta(t - k\overline{T})$ stands for the Dirac's δ shifted to $t = k\overline{T}$. Thus,

$$\begin{aligned} (w, v_t) = \sum_{k=1}^{\infty} & w(k\overline{T})(v(k\overline{T}+) - v(k\overline{T}-))\delta(t - k\overline{T}) \\ &= \sum_{k=1}^{\infty} (-1)^k (v(k\overline{T}+) - v(k\overline{T}-))\delta(t - k\overline{T}) \\ &\qquad = \sum_{k=1}^{\infty} \left| v(k\overline{T}+) - v(k\overline{T}-) \right| \delta(t - k\overline{T}) = |v_t| \end{aligned}$$

as desired. □

Notice that, in full concordance with Lemma 6 ii), $\overline{T} = \overline{T}(\gamma)$ is smooth and increasing in $0 \le \gamma < 1$, $\overline{T}(0) = 2$ and $\overline{T}(\gamma) \to \infty$ as $\gamma \to 1-$.

Regarding uniqueness, we now show that equation (12) possesses at most a normalized solution which satisfies condition (13). Furthermore, the proof furnishes the explicit form of this solution.

Theorem 14. *Assume $\gamma \ge 0$. Then there exists at most a normalized solution to equation* (12) *which satisfies condition* (13).

Proof. Let $v \in BV_{loc}(0, \infty)$, $w \in W^{1,\infty}(0, +\infty)$ and $\beta \in L^\infty(0, +\infty)$ be the functions involved in Definition 12. We first analyze the case $0 < \gamma < 1$, delaying $\gamma \ge 1$ until the end of the proof.

Consider the open set $\{|w(t)| < 1\}$ and let (a, b) be one of its components. We assert that v keeps constant in (a, b). In fact, choose $J := [a_1, b_1] \subset (a, b)$. Then

$$|v_t|_J = (w, v_t) \le \|w\|_{\infty,J}|v_t|_J,$$

where $|v_t|_J := \sup\langle v_t, \varphi\rangle$, functions $\varphi \in C(J)$ with supp $\varphi \subset J$ and $\sup_J |\varphi| \le 1$. So, v is constant in (a, b). Recall that we are using a good representative for v.

We next suppose $a > 0$ together with $v = c$ a constant, $c \neq 0$, in the mentioned component (a, b). We claim that: 1) $b = a + \overline{T}$, 2) sign $w(a) =$ sign c while $w(a)w(b) = -1$. Fix $t_0 \in (a, b)$ with $w(t_0) = w_0$ (notice that $|w_0| < 1$). By employing the equation for w in 1) we obtain

$$w(t) = \left(w_0 + \frac{\text{sign } c}{\gamma}\right) e^{-\gamma(t-t_0)} - \frac{\text{sign } c}{\gamma}.$$

Hence w decreases if $c > 0$ or increases when $c < 0$. Since $|w| = 1$ at $t = a$ and $t = b$, taking $c > 0$ we observe that

$$a = t_0 + \frac{1}{\gamma}\log\left(\frac{\text{sign } c + w_0\gamma}{\text{sign } c + \gamma}\right) \qquad \& \qquad b = t_0 + \frac{1}{\gamma}\log\left(\frac{\text{sign } c + w_0\gamma}{\text{sign } c - \gamma}\right).$$

In this case $w(a) = 1$, $w(b) = -1$ while $b - a = \overline{T}$. When $c < 0$ conclusions remain the same after interchanging the rôles of a and b. This proves the claim.

We now prove that, as a consequence of (13), function v cannot jump from $v(t_1-) \neq 0$ to $v(t_1+) = 0$ at any $t_1 > 0$. In fact, it follows from this relation and the distributional derivatives of both $|v|$ and v at $t = t_1$ that

$$-|v(t_1-)| = -\gamma|v(t_1-)|.$$

This is not possible since $0 < \gamma < 1$.

Now let $(a, a + \overline{T})$ be a component of $\{|w| < 1\}$ with $v = c$ a constant, $c \neq 0$. Then we assert that $(a + \overline{T}, a + 2\overline{T})$ is a further component where $v = c'$ is a constant so that $cc' < 0$. Moreover, it holds that

$$|c'| = \frac{1-\gamma}{1+\gamma}|c|. \tag{58}$$

Indeed by putting $b = a + \overline{T}$ and since $v(b+) \neq 0$ then sign $v =$ sign $v(b+)$ in a small interval $(b, b + \delta)$. Suppose $c > 0$ (the argument is symmetric if

$c < 0$) and so $w(b) = -1$. In this interval $(b, b+\delta)$, it holds that

$$w(t) = \left(-1 + \frac{\text{sign } v(b+)}{\gamma}\right) e^{-\gamma(t-b)} - \frac{\text{sign } v(b+)}{\gamma},$$

and so it must be $v(b+) < 0$. Otherwise, w would become less than -1 which is not possible. Accordingly, $-1 < w < 1$ in $(b, b+\delta)$ (by reducing δ if necessary). Thus $(b, b+\delta)$ falls into a further component (b, b_1) where, as shown above, $b_1 = b + \overline{T}$ and $v = c'$ is a constant that exhibits the opposite sign to c. To show the jump relation (58) we make use of (13) and the (distributional) derivatives of $|v|$ and v at $t = b$ to obtain

$$|c'| - |c| = |v(b+)| - |v(b-)| = -\gamma|v(b-) - v(b+)| = -\gamma(|c| + |c'|),$$

which provides the relation (58).

We finally show that v is uniquely determined from the "initial data" $w(0) = v(0+) = 1$. By gathering together all the preliminary features of w and v we first obtain that $v = 1$ in $(0, \overline{T})$ with $w(\overline{T}) = -1$. Then v jumps to

$$v = -\alpha, \qquad \alpha = \frac{1-\gamma}{1+\gamma},$$

in the interval $(\overline{T}, 2\overline{T})$. Proceeding by induction we see that

$$v = (-1)^k \alpha^k \qquad \text{in } \ (k\overline{T}, (k+1)\overline{T}).$$

Therefore, function v is completely defined in $(0, \infty)$ and the discussion of case $0 < \gamma < 1$ is over.

As for the range $\gamma \geq 1$, notice that equation 1) provides the expression

$$w = \left(1 + \frac{1}{\gamma}\right) e^{-\gamma t} - \frac{1}{\gamma},$$

which is valid for all $t > 0$. This means that $v = 1$ is the unique normalized solution in this case. $\square$

Corollary 15. *Let v be the normalized solution to equation* (12) *which satisfies condition* (13)*. Then,*

$$\lim_{(p,\gamma') \to (1,\gamma)} v_p = v \ \text{ in } \ L^1(0, b) \ \text{ for all } \ b > 0, \tag{59}$$

where v_p stands for the solution to (52) *with γ replaced by γ'.*

In our final result we directly construct a sequence of weak eigenvalues to (11) by an alternative approach. Namely, the eigenpairs are computed by means of the normalized solution of Theorem 14 and the scaling argument of Theorem 7. We would like to remark that the proof of Theorem 14 does not involve any limit process as $p \to 1$. However, we are just obtaining the same eigenpairs $(\bar{\lambda}_n, \bar{u}_n)$ as in Proposition 11. Of course, this should be expected if one takes (59) into account.

Proposition 16. *Let $v = v(t,\gamma)$ the normalized solution to equation* (12) *which satisfies* (13)*. Then*

$$\lambda = \bar{\lambda}_n, \quad \textit{where} \quad \bar{\lambda}_n = c\frac{e^{\frac{c}{n}}+1}{e^{\frac{c}{n}}-1},$$

is a weak eigenvalue to (24) *and*

$$\hat{u}_n = v(\lambda_n x, \gamma), \qquad \lambda_n \gamma = c,$$

is its normalized associated eigenfunction. Moreover

$$\hat{u}_n = \bar{u}_n,$$

where $\bar{u}_n$ is given by (9)*.*

Proof. Let $v(t,\gamma)$ and $w(t)$ be the functions obtained in Theorem 14, and let $\beta(t)$ be the function in (55). By setting

$$u(x) = v(\lambda x, \gamma), \qquad z(x) = w(\lambda x),$$

conditions 1), 2), 3) together with $z(0+)u(0+) = |u(0+)|$ in Definition 10 hold. To this purpose $\beta(\lambda x)$ plays the rôle of β. To get an eigenpair only condition

$$-z(1-)u(1-) = |u(-1)|$$

remains to be checked. It is just equivalent to

$$-w(\lambda-)v(\lambda-) = |v(\lambda-)|.$$

Since $v(\lambda-) \neq 0$ for all $\lambda > 0$, the latter equality holds only if

$$\lambda = n\overline{T}(\gamma)$$

for a certain $n \geq 1$. It amounts to $c = n\gamma\overline{T}(\gamma)$, which is equivalent to

$$\gamma = \frac{e^{\frac{c}{n}} - 1}{e^{\frac{c}{n}} + 1}.$$

This shows that $(\bar{\lambda}_n, \hat{u}_n)$ is a weak eigenvalue to (11) being $\hat{u}_n = v(\bar{\lambda}_n x, \gamma)$. That $\hat{u}_n = \bar{u}_n$ follows by simple checking. □

Remarks 9.

1) When $\gamma = 0$, condition (13) becomes $\dfrac{d|v|}{dt} = 0$ and so $|v|$ must be constant (although v could change sign). This fact agrees with the results of [6].

2) Variational eigenvalues to (24) with $c = 0$ have been introduced in [6]. A main result in this work characterizes the variational eigenvalues as the limit as $p \to 1$ of the corresponding eigenvalues to (2). It can be shown by the same arguments as in [6] that the eigenpairs $(\bar{\lambda}_n, \bar{u}_n)$ introduced in Theorem 1 constitute the variational eigenpairs to (11). Details are delayed to a future work.

3) Property of being a weak eigenvalue (Definition 10) is only a necessary condition to be a variational eigenvalue. More importantly, a whole interval of weak (and so non variational) eigenvalues to (11) can be constructed (see [6] for the case $c = 0$ and a further N–dimensional example in [26]). Such an immoderate amount of weak eigenvalues clearly reveals that they do not capture, by themselves, the "proper" spectrum of (11).

4) It is stressed that condition (13) has been crucial in the present work to discriminate the proper eigenvalues $\bar{\lambda}_n$ from the spurious ones. Right eigenvalues to (11) are just those obtained as the limit as $p \to 1$ of the eigenvalues to (1). As was mentioned in Section 1, all these features have been recently studied in [28] for the radially symmetric 1–Laplacian.

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