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Uniformization by rectangular domains: A path from slits to squares

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ABSTRACT

Let $\Sigma(\Omega)$ be the class of functions $f(z) = z + \frac{a_1}{z} + \cdots$ univalent on a finitely connected domain Ω, $\infty \in \Omega \subset \overline{\mathbb{C}}$. By a classical result due to H. Grötzsch, the function f_0 maximizing $\Re a_1$ over the class $\Sigma(\Omega)$ maps Ω onto $\overline{\mathbb{C}}$ slit along horizontal segments. Recently, M. Bonk found a similar extremal problem, which maximizer $f_1 \in \Sigma(\Omega)$ maps Ω onto a domain on $\overline{\mathbb{C}}$, whose complementary components are squares. In this note, we discuss a parametric family of extremal problems on the class $\Sigma(\Omega)$ with maximizers f_m, $0 < m < 1$, mapping Ω onto domains on $\overline{\mathbb{C}}$, whose complementary components are rectangles with horizontal and vertical sides and with module m.

1. Introduction

Let $\Sigma(\Omega)$ denote the class of functions $f : \Omega \to \overline{\mathbb{C}}$ meromorphic and univalent in the domain $\Omega \subset \overline{\mathbb{C}}$ with $\infty \in \Omega$, which are normalized by condition

$$f(z) = z + \frac{a_1}{z} + \cdots \tag{1}$$

for z near ∞. This class includes several so-called "canonical mappings" that are mappings onto domains whose complementary components have specific, usually simple, geometry. The following two canonical mappings are classical. One of them, first studied by P. Koebe in 1931 [7], is a mapping $f_\varnothing$ from Ω onto a domain on $\overline{\mathbb{C}}$ whose complementary components are either closed geometric disks or single points. Koebe proved in [7] that $f_\varnothing$ exists and is unique in the class $\Sigma(\Omega)$ if Ω is finitely connected. Koebe's famous "*Kreisnormierungsproblem*" to prove or disprove that such a mapping $f_\varnothing$ exists for all infinitely connected

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domains remains open and challenging. A partial progress toward its solution was achieved by Z.H. He and O. Schramm [4] who proved that $f_\varnothing$ exists for domains Ω having countably many boundary components.

One more canonical mapping, which is more relevant to our study, is a mapping $f_\theta \in \Sigma(\Omega)$, $0 \le \theta < \pi$, from Ω onto the Riemann sphere $\overline{\mathbb{C}}$ slit along rectilinear segments in the direction θ; those are segments forming angle θ with the direction of the positive real axis. Its existence, for finitely connected domains Ω, was proved by D. Hilbert in 1909 [5] and then, in 1931, Hilbert's result was extended by G. Grötzsch [2] for the case of all infinitely connected domains; see Theorems 5.8 and 5.16 in [6]. It is clear that f_θ maps Ω onto $\overline{\mathbb{C}}$ slit along segments in the direction θ if and only if $f_0(z) = e^{-i\theta} f_\theta(e^{i\theta} z)$ maps Ω onto $\overline{\mathbb{C}}$ slit along horizontal segments. Thus, to study properties of f_θ, it would be enough to work with f_0. The following theorem summarizes important results on existence, uniqueness and certain extremal properties of functions in the class $\Sigma(\Omega)$ mapping Ω onto $\overline{\mathbb{C}}$ slit along horizontal segments.

Theorem 1 *(see Theorems 5.8 and 5.16 in [6]).*

(1) *There is a unique function $f_0 \in \Sigma(\Omega)$, which maximizes the functional*

$$S_0(f, \Omega) = 2\pi \Re\, a_1 \tag{2}$$

over all functions $f(z) = z + \frac{a_1}{z} + \cdots$ in the class $\Sigma(\Omega)$.

(2) *If Ω is finitely connected, then the function f_0 maximizing (2) is the only function in $\Sigma(\Omega)$ which maps Ω onto $\overline{\mathbb{C}}$ slit along horizontal segments.*

(3) *If Ω is infinitely connected, then f_0 maps Ω onto $\overline{\mathbb{C}}$ slit along horizontal segments and is the only function in $\Sigma(\Omega)$ possessing the following property:*
   - (∗) *If $L > 0$ is large enough and $Q_L(f_0) = f_0(\Omega) \cap \{w : |\Re w| < L, |\Im w| < L\}$, then the module of $Q_L(f_0)$ for the family of locally rectifiable curves in $Q_L(f_0)$ joining the pair of vertical sides of $Q_L(f_0)$ is one.*

For the definition and properties of the module of a family of curves we refer to Jenkins' book [6]. As an example, we want to mention that, if R is a rectangle with horizontal sides of length $a > 0$ and vertical sides of length $b > 0$, then the module of the family of curves $\gamma \in R$ joining its vertical sides, also known as the module of R, is $m = b/a$, see Theorem 2.3 in [6].

As part (3) of Theorem 1 suggests and, indeed, as it was shown by example by H. Grötzsch, f_0 is not, in general, the only function in $\Sigma(\Omega)$ mapping Ω onto $\overline{\mathbb{C}}$ slit along horizontal segments if Ω is infinitely connected. But f_0 is the only function in $\Sigma(\Omega)$ maximizing the functional in (2). The latter extremal property is a key ingredient in the proof of the existence of a function with required mapping properties.

In contrast with Theorem 1, it is not known whether or not a function $f_\varnothing$ mapping Ω onto a circular domain maximizes any reasonable functional. And this lack of information is a reason why Koebe's Kreisnormierungsproblem remains open.

In the literature, theorems which establish conformal equivalence of a given domain with some "canonical" domain, such as Theorem 1 above, are customary called *uniformization* theorems. Thus, Theorem 1 assures that every domain Ω with $\infty \in \Omega$ can be uniformized by a domain, which complementary components are horizontal slits. In the case of finitely connected domains, a very general uniformization theorem was established by A.N. Harrington [3]. We will need one particular case of Harrington's result (see Theorem and Corollary in [3]), which we state as the following proposition.

Proposition 1. *[3] Let $m > 0$ and let Ω be a finitely connected domain with nondegenerate boundary components such that $\infty \in \Omega$. Then there is a function $f_m \in \Sigma(\Omega)$, which maps Ω onto a domain on $\overline{\mathbb{C}}$ whose complementary components are rectangles with module m.*

In what follows, a domain $\Omega \subset \overline{\mathbb{C}}$, which boundary components are rectangles with horizontal and vertical sides, will be called a *rectangular domain*. Furthermore, if all boundary rectangles of Ω have module $m > 0$, then we say that Ω is a rectangular domain with module m. In particular, a *square domain* is a rectangular domain with module 1; i.e. Ω is a square domain when all its boundary components are squares.

Thus, when $m = 1$, the function f_1 defined in Proposition 1 maps Ω onto a square domain. We note that Harrington's proof in [3] rely on Sard's theorem on the existence of smooth homotopy paths and on Brower's fixed point theorem and does not use any extremal property of the function f_1; for details, see [3]. Thus, Harrington's method cannot be extended to the case of infinitely connected domains. In a recent paper [1], in a search for an extremal property of functions mapping Ω onto domains complementary to squares, M. Bonk succeeded to find a functional with an extremal function f_1. To state his main result, we first introduce necessary notations.

Let Ω be finitely connected domain with n nondegenerate boundary components such that $\infty \in \Omega$. For $f \in \Sigma(\Omega)$, let $K_j = K_j(f)$, $j = 1, \ldots, n$, be complementary components of $D = f(\Omega)$. Furthermore, let $A_j = A_j(f)$ denote the area of K_j and let $H_j = H_j(f)$ and $V_j = V_j(f)$ denote, respectively, the horizontal and the vertical variations of K_j defined as

$$H_j = \max_{w \in K_j} \Re w - \min_{w \in K_j} \Re w, \quad V_j = \max_{w \in K_j} \Im w - \min_{w \in K_j} \Im w. \tag{3}$$

With these notations, M. Bonk main result in [1] can be restated in the following form more convenient for our purposes.

Theorem 2 *(cf. Theorem 1.1 [1]). Let Ω be a finitely connected domain on $\bar{\mathbb{C}}$ with $\infty \in \Omega$ having n nondegenerate boundary components. There is a unique function $f_1 \in \Sigma(\Omega)$, which maximizes the functional*

$$S_1(f, \Omega) = 2\pi \Re a_1 + \sum_{j=1}^{n} (A_j - H_j^2) \tag{4}$$

over all functions $f(z) = z + \frac{a_1}{z} + \cdots$ in the class $\Sigma(\Omega)$.

The function f_1 maximizing (4) is the only function in $\Sigma(\Omega)$, which maps Ω onto a square domain.

In [1], instead of the maximization problem for $S_1(f, \Omega)$, M. Bonk considers the minimization problem for the functional

$$S_1^{\perp}(f, \Omega^{\perp}) = 2\pi \Re a_1 + \sum_{j=1}^{n} (V_j^2 - A_j) \tag{5}$$

over the class $\Sigma(\Omega^{\perp})$ of functions $f(z) = z + \frac{a_1}{z} + \cdots$ univalent in the domain $\Omega^{\perp} = \{z : -iz \in \Omega\}$. Switching from the domain Ω to the domain $\Omega^{\perp}$ and from a function $f \in \Sigma(\Omega)$ to a function $f^{\perp}(z) = if(-iz) \in \Sigma(\Omega^{\perp})$, we conclude that $S_1^{\perp}(f^{\perp}, \Omega^{\perp}) = -S_1(f, \Omega)$. The latter shows that the minimization problem for the functional (5) on the class $\Sigma(\Omega^{\perp})$ is equivalent to the maximization problem (4) on the class $\Sigma(\Omega)$. The only reason why we prefer the form (4) is because it allows us to include the function f_0 of Theorem 1 and the function f_1 of Theorem 2 into a family $\{f_m : 0 \le m \le 1\}$ of functions f_m continuously depending on the parameter m, $0 \le m \le 1$, where $f_m \in \Sigma(\Omega)$ maps Ω onto a rectangular domain with module m. This family of functions can be considered as a continuous path from a mapping onto a slit domain to a mapping onto a conformally equivalent square domain as it is reflected in the title of this paper. Moreover, as we will show in the next section, each of the functions f_m maximizes a modified version of the functional (4).

2. Extremal problem and uniformization by rectangular domains

Our main result in this paper is the following.

Theorem 3. *Let $m > 0$ be fixed and let Ω be a finitely connected domain on $\overline{\mathbb{C}}$ with $\infty \in \Omega$ having n nondegenerate boundary components. There is a unique function $f_m \in \Sigma(\Omega)$, which maximizes the functional*

$$S_m(f, \Omega) = 2\pi \Re\, a_1 + \sum_{j=1}^{n} (A_j - mH_j^2) \tag{6}$$

over all functions $f(z) = z + \frac{a_1}{z} + \cdots$ in the class $\Sigma(\Omega)$.

The function f_m maximizing (6) is the only function in $\Sigma(\Omega)$, which maps Ω onto a rectangular domain with module m.

The proof of this theorem consists of two major steps. First, using Harrington's result stated in Proposition 1, we show that it is enough to solve maximization problem of Theorem 3 for a particular case when Ω is already a rectangular domain with module m. Then, in the second part, which is rather technical, we use a special metric to find relation between three quantities: coefficient $a_1 = a_1(f)$, area of the complement $\overline{\mathbb{C}} \setminus f(\Omega)$, and horizontal variations $H_j = H_j(f)$. As it was mentioned by Jenkins (see Sections 5.3 – 5.5 in [6]), the background ideas of these proofs go back to Koebe, del Possel, and Grötzsch. More recently, similar approach was used by He and Schramm in [4] and by Bonk in [1]. In our presentation, we follow closely to the lines of the proof of Theorem 2.1 in [1].

Proof. Let Ω be a finitely connected domain as in Theorem 3. By Proposition 1, there is a function

$$\varphi(z) = z + \frac{b_1}{z} + \cdots \tag{7}$$

in the class $\Sigma(\Omega)$, which maps Ω onto a rectangular domain Ω_m with module m. Since φ is a conformal bijection between Ω and Ω_m it follows that f belongs to the class $\Sigma(\Omega)$ if and only if the function $g = f \circ \varphi^{-1}$ belongs to the class $\Sigma(\Omega_m)$. Using expansions (1) and (7), we find:

$$g(z) = z + \frac{a_1 - b_1}{z} + \cdots \tag{8}$$

Since $f(\Omega) = g(\Omega_m)$, taking into account (1) and (8), we obtain

$$S_m(f, \Omega) = S_m(g, \Omega_m) + 2\pi \Re\, b_1. \tag{9}$$

Thus, as equation (9) shows, a function $f_m \in S(\Omega)$ maximizes the functional $S_m(f, \Omega)$ if and only if the function $g_m = f_m \circ \varphi^{-1} \in \Sigma(\Omega_m)$ maximizes the functional $S_m(g, \Omega_m)$. Now, Theorem 3 will follow from Proposition 2 stated below. $\square$

Proposition 2. *Let $\Omega \subset \overline{\mathbb{C}}$ be a rectangular domain with module $m > 0$, which has $n \geq 1$ boundary components, such that $\infty \in \Omega$. Then, for $f \in \Sigma(\Omega)$,*

$$S_m(f, \Omega) = 2\pi \Re\, a_1 + \sum_{j=1}^{n} (A_j - mH_j^2) \leq 0 \tag{10}$$

with equality if and only if f is the identity on Ω.

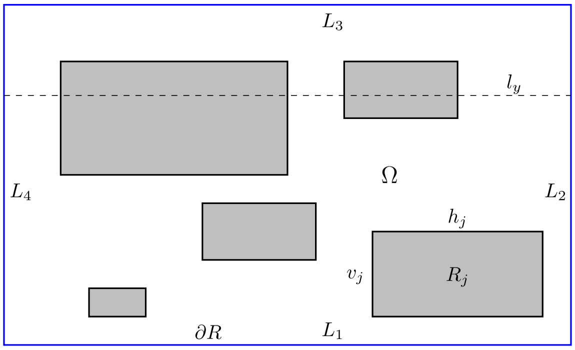

Fig. 1. Rectangular domain Ω and rectangle R.

Moreover, the function $f(z) = z$ is the only function in $\Sigma(\Omega)$ mapping Ω onto a rectangular domain with module m.

Proof. To prove (10), we will estimate certain area and line integrals. Let $R = \{z : |\Re z| \le r, |\Im z| \le l\}$ be a rectangle on $\mathbb{C}$ with large $r > 0$. This rectangle is shown in Fig. 1, which illustrates notations used in this proof. We choose $l = r^{\frac{2}{3}}$ so that

$$l/r \to 0 \quad \text{and} \quad r/l^2 \to 0 \quad \text{as} \quad r \to \infty. \tag{11}$$

The choice of relations (11) between r and l, which was suggested by M. Bonk in [1], is, in a certain sense, crucial for this proof.

We assume that r is large enough and that $\bar{\mathbb{C}} \setminus \Omega$ is contained in the interior of the rectangle R. Then, it is clear that $\partial R \subset \Omega$ and $J = f(\partial R)$ is a Jordan curve in $\mathbb{C}$. We will use line integral over J, assuming its positive orientation, to estimate the area $A = A(r)$ of the region bounded by J up to a term $o(1)$ as $r \to \infty$.

Applying the complex version of the Green's formula, we obtain

$$\begin{aligned} A &= \frac{1}{2i}\int\limits_J \bar{w}\,dw = \frac{1}{2i}\int\limits_{\partial R} \overline{f(z)} f'(z)\,dz \\ &= \frac{1}{2i}\int\limits_{\partial R} \left(\bar{z} + \frac{\bar{a_1}}{\bar{z}} + \ldots\right)\left(1 - \frac{a_1}{z^2} + \ldots\right) dz \\ &= \frac{1}{2i}\int\limits_{\partial R} \left(\bar{z} + \frac{\bar{a_1}}{\bar{z}} - \frac{a_1\bar{z}}{z^2} + O(|z|^{-2})\right) dz \\ &= \frac{1}{2i}\int\limits_{\partial R} \bar{z}\,dz + \frac{1}{2i}\int\limits_{\partial R}\left(\frac{\bar{a_1}z}{\bar{z}} - \frac{a_1\bar{z}}{z}\right)\frac{dz}{z} + o(1). \end{aligned}$$

The above calculation gives the following asymptotic expression for the area A:

$$A = 4rl + \int\limits_{\partial R} \Im\left(\frac{\bar{a_1}z}{\bar{z}}\right)\frac{dz}{z} + o(1). \tag{12}$$

To estimate the integral $\int_{\partial R} \Im\left(\frac{\bar{a_1}z}{\bar{z}}\right)\frac{dz}{z}$, we will integrate over horizontal sides and vertical sides of R separately. Let $L_1 = \{z = t - il : -r \le t \le r\}$, $L_2 = \{z = r + it : -l \le t \le l\}$, $L_3 = \{z = -t + il : -r \le t \le r\}$, and $L_4 = \{z = -r - it : -l \le t \le l\}$. We also put $a_1 = \alpha + i\beta$, $\alpha, \beta \in \mathbb{R}$.

Since the area A in the equation (12) is real and positive, we may ignore the imaginary part when estimating our integrals. Integrating over horizontal sides of R and simplifying, we get the following:

$$\begin{aligned}
\Re \int_{L_1 \cup L_3} \Im\left(\frac{\bar{a_1} z}{\bar{z}}\right) \frac{dz}{z} &= 2\Re \int_{-r}^{r} \Im\left(\frac{(\alpha - i\beta)(t - il)}{t + il}\right) \frac{dt}{t - il} \\
&= 2\int_{-r}^{r} \Im\left(\frac{(\alpha - i\beta)(t^2 - l^2 - 2ilt)}{t^2 + l^2}\right) \frac{tdt}{t^2 + l^2} \\
&= 2\int_{-r}^{r} \left(\frac{-2\alpha lt + \beta(l^2 - t^2)}{t^2 + l^2}\right) \frac{tdt}{t^2 + l^2} \\
&= -4\alpha l \int_{-r}^{r} \frac{t^2}{(t^2 + l^2)^2}\, dt - 2\beta \int_{-r}^{r} \frac{t(t^2 - l^2)}{(t^2 + l^2)^2}\, dt \\
&= -4\alpha \left(\arctan(r/l) - \frac{rl}{r^2 + l^2}\right) \\
&= -2\pi \Re\, a_1 + o(1).
\end{aligned}$$

Similarly, integration over the vertical sides leads to the following:

$$\begin{aligned}
\Re \int_{L_2 \cup L_4} \Im\left(\frac{\bar{a_1} z}{\bar{z}}\right) \frac{dz}{z} &= 2\Re \int_{-l}^{l} \Im\left(\frac{(\alpha - i\beta)(r + it)}{r - it}\right) \frac{idt}{r + it} \\
&= 2\int_{-l}^{l} \Im\left(\frac{(\alpha - i\beta)(r^2 - t^2 + 2irt)}{t^2 + r^2}\right) \frac{tdt}{t^2 + r^2} \\
&= 2\int_{-l}^{l} \frac{2\alpha rt + \beta(t^2 - r^2)}{(t^2 + r^2)^2}\, tdt \\
&= 4\alpha \left(\arctan(l/r) - \frac{rl}{r^2 + l^2}\right) = o(1).
\end{aligned}$$

Combining (12) with asymptotic expressions for the integrals above, we find the following asymptotic expression for the area A:

$$A = 4rl - 2\pi \Re\, a_1 + o(1), \quad \text{where } o(1) \to 0 \text{ as } r \to \infty. \tag{13}$$

One more formula involving the area A can be obtained by using a special metric $\rho(z) \geq 0$, which we introduce below. We will use the following notations. Let R_j, $j = 1, \ldots, n$, denote rectangles complementary to the domain Ω and let $K_j = f(R_j)$ and $A_j = \text{area}\,(K_j)$. Then, by h_j and v_j we denote the lengths of horizontal and vertical sides of R_j, respectfully. Also, we will use horizontal and vertical variations H_j and V_j defined by (3). With these notations, we consider the following metric:

$$\rho(z) = \begin{cases} |f'(z)| & \text{if} \quad z \in \Omega, \\ \frac{H_j}{h_j} & \text{if} \quad z \in R_j. \end{cases} \tag{14}$$

Integrating $\rho^2(z)$ over the rectangle R, we obtain:

$$\begin{aligned}\int\limits_R \rho^2(z)\,dxdy &= \int\limits_{R\cap\Omega} |f'(z)|^2\,dxdy + \sum_{j=1}^n \int\limits_{R_j} \left(\frac{H_j}{h_j}\right)^2\,dxdy \\ &= \text{Area}\,(f(R\cap\Omega)) + \sum_{j=1}^n H_j^2\frac{v_j}{h_j} \\ &= \text{Area}\,(f(R\cap\Omega)) + \sum_{j=1}^n A_j - \sum_{j=1}^n (A_j - mH_j^2) \\ &= A - \sum_{j=1}^n (A_j - mH_j^2).\end{aligned}$$

Combining this with formula (13), we obtain the following:

$$\begin{aligned}\int\limits_R \rho^2(z)\,dxdy &= 4rl - 2\pi\Re\,a_1 - \sum_{j=1}^n (A_j - mH_j^2) + o(1) \\ &= 4rl - S_m(f,\Omega) + o(1) \quad \text{as } r\to\infty.\end{aligned} \tag{15}$$

Next, we will estimate integrals of $\rho(z)$ over the line segments $l_y = \{t+iy : -r \le t \le r\}$, where $-l \le y \le l$. We have

$$\begin{aligned}\int\limits_{l_y} \rho(z)\,|dz| &= \int\limits_{l_y\cap\Omega} |f'(z)|\,|dz| + \sum_{j=1}^n \int\limits_{R_j\cap l_y\neq\emptyset} \frac{H_j}{h_j}\,|dz| \\ &= \int\limits_{l_y\cap\Omega} |f'(z)||dz| + \sum_{j=1}^n \int\limits_{R_j\cap l_y\neq\emptyset} \frac{H_j}{h_j}\quad dt \\ &= \int\limits_{l_y\cap\Omega} |f'(z)||dz| + \sum_{R_j\cap l_y\neq\emptyset} H_j \\ &\ge \Re\,(f(r+iy) - f(-r+iy)) = 2r + O(1/r).\end{aligned} \tag{16}$$

The latter equation implies that

$$\frac{1}{2r}\left(\int\limits_{l_y} \rho(z)\,dx\right)^2 \ge 2r + o(1). \tag{17}$$

Integrating (17) over the interval $-l \le y \le l$, we obtain:

$$\frac{1}{2r}\int\limits_{-l}^{l}\left(\int\limits_{l_y} \rho(z)\,dx\right)^2\,dy \ge \int\limits_{-l}^{l} 2r\,dy + o(1) = 4rl + o(1). \tag{18}$$

Applying Cauchy-Bunyakovsky inequality to the inner integral in (18), we get the following:

$$4rl + o(1) \le \frac{1}{2r}\int\limits_{-l}^{l}\left(\int\limits_{l_y} \rho^2(z)\,dx \cdot \int\limits_{-r}^{r} dx\right)\,dy \tag{19}$$

$$= \int_{-l}^{l} \left(\int_{l_y} \rho^2(z)\, dx \right) dy = \int_R \rho^2(z)\, dxdy.$$

Combining this with (15), we get the following inequality:

$$4rl + o(1) \le 4rl - S_m(f, \Omega) + o(1). \tag{20}$$

Taking the limit in (20) as $r \to \infty$, we conclude that $S_m(f, \Omega) \le 0$, as required.

Next, we discuss when equality occurs in (10). If $f(z) = z$, then $a_1 = 0$, $A_j = h_j v_j$, $mH_j^2 = \frac{v_j}{h_j}(h_j)^2 = h_j v_j$ and therefore $S_m(f, \Omega) = 0$ in this case.

Now, suppose that $S_m(f, \Omega) = 0$ for some $f \in \Sigma(\Omega)$. Then it follows from (15) that

$$\int_R \rho^2(z)\, dxdy = 4rl + o(1) \quad \text{as } r \to \infty, \tag{21}$$

where $\rho(z)$ is defined by (14).

Let us consider a modified metric $\tilde{\rho}(z) = \rho(z) + 1$. Using (16), we get the following:

$$\int_{l_y} \tilde{\rho}(z)\, |dz| = \int_{l_y} (\rho(z) + 1)\, dy = \int_{l_y} \rho(z)\, dy + 2r \ge 4r + O(1/r). \tag{22}$$

Using estimate (22), we replace $\rho(z)$ by $\tilde{\rho}(z)$ in inequalities (17) – (19) and then we find that $\tilde{\rho}(z)$ satisfies the following inequality:

$$\int_R \tilde{\rho}^2(z)\, dxdy \ge 16rl + o(1). \tag{23}$$

Next, we estimate the nonnegative integral $\int_R (1 - \rho(z))^2\, dxdy$. Using equations (21) and (23), we obtain

$$\begin{aligned} 0 \le \int_R (1 - \rho(z))^2\, dxdy &= \int_R (2 + 2\rho^2(z) - \tilde{\rho}^2(z))\, dxdy \\ &\le 8lr + 8lr - 16lr + o(1) = o(1). \end{aligned}$$

Letting $r \to \infty$, we have that $\int_{\mathbb{C}} (1 - \rho)^2(z)\, dxdy = 0$. So it is immediate that $\rho(z) = 1$ almost everywhere on $\mathbb{C}$. It also follows that $|f'(z)| = 1$ for all $z \in \Omega$. Hence $f'(z)$ is a constant and with the given normalization it follows that $f'(z) = 1$ and therefore $f(z)$ is the identity on Ω. This completes the proof of the first part of Proposition 2.

It remains to prove uniqueness statement of the second part of Proposition 2. To prove this suppose that there exists a function $f \in \Sigma(\Omega)$ mapping Ω onto another rectangular domain Ω' with module m. Then the inverse function f^{-1} is in the class $\Sigma(\Omega')$ and the relevant coefficients of $f(z)$ and $f^{-1}(z)$ are related by $a_1(f) = -a_1(f^{-1})$. Since, the domains Ω and Ω' play the symmetric roles there is no generality loss if we assume that $\Re\, a_1 \ge 0$, otherwise, we start with the function f^{-1} instead of $f(z)$. Since $f(\Omega) = \Omega'$ is a rectangular domain, we have $m = \frac{V_j}{H_j}$ for $j = 1, ..., n$. Therefore, we have $S_m(f, \Omega) = 2\pi \Re\, a_1 \ge 0$ and by plugging $m = \frac{V_j}{H_j}$ and $V_j H_j = A_j$ in the inequality 10 we have $S_m(f, \Omega) = 2\pi \Re\, a_1 \le 0$. Hence, we have $S_m(f, \Omega) = 0$. Thus, under our assumption, the function $f(z)$ maximizes the functional in (10). As we proved before, the latter happens if and only if $f(z)$ is the identity mapping. This completes the proof of Proposition 2. $\square$

M. Bonk's Theorem 2 for the square domains immediately follows from Theorem 3, when $m = 1$. In the other extreme case, when $m = 0$, we have the following.

Corollary 1. *Let Ω be as in Theorem 3 and let $f_0 \in \Sigma(\Omega)$ maps Ω onto a domain bounded by slits parallel to the real axis. If $f \in \Sigma(\Omega)$, then*

$$\Re\, a_1(f) + \frac{1}{2\pi}\sum_{j=1}^{n} A_j(f) \le \Re\, a_1(f_0). \tag{24}$$

Equality occurs in (24) if and only if $f(z) = f_0(z)$.

The inequality (24) strengthens the classical inequalities; see, for instance, Lemma 5.6 and the proof of Lemma 5.7 in [6].

3. Further results and questions

Here we add few observations how the maximal value of the functional $S_m(f,\Omega)$ and related extremal functions $f_m \in \Sigma(\Omega)$ depend on the parameter m. Let $S_m(\Omega)$ denote the maximal value of $S_m(f,\Omega)$ over the class $\Sigma(\Omega)$. Then

$$S_m(\Omega) = S_m(f_m,\Omega) = 2\pi\Re\, a_1(f_m) = \max_{f\in\Sigma(\Omega)} S_m(f,\Omega). \tag{25}$$

Proposition 3. *With our notations introduced above, the following holds.*

(1) *If $m_k \ge 0$ for $k = 0, 1, \ldots,$ and $m_k \to m_0$ as $k \to \infty$, then $f_{m_k}(z) \to f_{m_0}(z)$ uniformly on compact subsets of Ω.*
(2) *The quantities $a_1(f_m)$, $A_j(f_m)$ and $H_j(f_m)$, and therefore the functional $S_m(\Omega)$, are continuous functions of m, $m \ge 0$.*

Proof. This result follows from standard convergence theorems for sequences of normalized univalent functions and uniqueness part of Theorem 3. Indeed, if $m_k \to m_0$ as $k \to \infty$ but the sequence $f_{m_k}(z)$ does not converge to $f_{m_0}(z)$ uniformly on compact subsets of Ω, then there is a subsequence, we may assume that it is the sequence $f_{m_k}(z)$ itself, which converges uniformly on compact subsets of Ω to some function $\tilde{f} \in \Sigma(\Omega)$. We may also assume that the sequences of vertices of complementary components $K_j = K_j(f_{m_k})$ defined in Section 1, those are rectangles in our case, also converge to some points on $\mathbb{C}$. Then, one can easily see that the limit function $\tilde{f}(z)$ maps Ω onto a rectangular domain with module m. But, by the uniqueness part of Theorem 3, such a rectangular domain is unique. Since both $f_{m_0}(z)$ and $\tilde{f}(z)$ are normalized as in (1), we must have $\tilde{f}(z) = f_{m_0}(z)$ and therefore $f_{m_k}(z) \to f_{m_0}(z)$, proving part (1) of the proposition.

Now, since $f_{m_k}(z) \to f_{m_0}(z)$ as $m \to \infty$, it follows that $a_1(f_{m_k}) \to a_1(f_{m_0})$ and the sequences of vertices of boundary rectangles of the domains $f_{m_k}(\Omega)$ converge to appropriate vertices of the boundary components of the domain $f_{m_0}(\Omega)$. The latter immediately implies the continuity properties of the areas $A_j(f_m)$ and horizontal variations $H_j(f_m)$, which, in turn, implies the continuity property of the functional $S_m(\Omega)$. □

Theorem 4. *The functional $S_m(\Omega)$, defined by equation (25), is a strictly decreasing smooth function of m, $m \ge 0$, which derivative is given by*

$$\frac{d}{dm} S_m(\Omega) = -\sum_{j=1}^{n} H_j^2(f_m). \tag{26}$$

Proof. We proceed as in the proof of Theorem 5.1 in [9]. Same idea was used earlier in some proofs in the paper [8]. Since $f_m(z)$ is admissible for the maximization problem for the functional $S_{m+\triangle m}(f,\Omega)$, assuming $\triangle m > 0$, we have

$$\begin{aligned}\liminf_{\triangle m\to 0}\frac{S_{m+\triangle m}(\Omega)-S_m(\Omega)}{\triangle m} &\geq \liminf_{\triangle m\to 0}\frac{S_{m+\triangle m}(f_m,\Omega)-S_m(\Omega)}{\triangle m} \\ = \lim_{\triangle m\to 0}&\frac{2\pi a_1(f_m)+\sum_{j=1}^n(A_j(f_m)-(m+\triangle m)H_j^2(f_m))-S_m(f_m,\Omega)}{\triangle m} \\ = -\sum_{j=1}^n H_j^2(f_m).&\end{aligned} \tag{27}$$

On the other hand, $f_{\widetilde{m}}(z)$, with $\widetilde{m} = m + \triangle m$, is admissible for the maximization problem for $S_m(\Omega)$. Thus, assuming $\triangle m < 0$ and using continuity property of horizontal variations $H_j(f_m)$ established in Proposition 3, we have

$$\begin{aligned}\limsup_{\triangle m\to 0}\frac{S_{m+\triangle m}(\Omega)-S_m(\Omega)}{\triangle m} &\leq \limsup_{\triangle m\to 0}\frac{S_{m+\triangle m}(\Omega)-S_m(f_{\widetilde{m}},\Omega)}{\triangle m} \\ = \lim_{\triangle m\to 0}&\frac{2\pi a_1(f_{\widetilde{m}})+\sum_{j=1}^n(A_j(f_{\widetilde{m}})-(m+\triangle m)H_j^2(f_{\widetilde{m}}))-S_m(f_{\widetilde{m}},\Omega)}{\triangle m} \\ = -\sum_{j=1}^n H_j^2(f_m).&\end{aligned} \tag{28}$$

For $\triangle m < 0$, we obtain inequalities similar to (27) and (28). Combining all these inequalities, we obtain (26). By Proposition 3, for each $j = 1, \ldots, n$, the horizontal variation $H_j(f_m)$ is a continuous function of m. This, together with formula (26), implies that $S_m(\Omega)$ is a smooth strictly decreasing function on the interval $m \geq 0$. $\square$

Next, we provide an example illustrating results of Theorems 3 and 4 when all calculations can be performed explicitly in terms of special functions. Let $\Omega = \mathbb{D}^* = \{z \in \overline{\mathbb{C}} : |z| > 1\}$ and let $f_m \in \Sigma(\mathbb{D}^*)$ maps $\mathbb{D}^*$ onto a domain exterior to the rectangle $R(a,b) = \{z : |\Re z| \leq a, |\Im z| \leq b\}$ with module $m = b/a$. The function $f_m(z)$ can be represented as the composition $f_m(z) = g_k(J(z))$, where $J(z)$ is the Joukowski function defined as

$$J(z) = (1/2)(z + 1/z), \tag{29}$$

and the function $g_k(\zeta)$, that is a Schwarz-Christoffel integral depending on the parameter $k = k(m)$, $0 < k < 1$, which is given by the following formula:

$$g_k(\zeta) = 2k^2\int_0^{\zeta/k}\sqrt{\frac{1-t^2}{1-k^2t^2}}\,dt + 2ik^2\int_1^{1/k}\sqrt{\frac{t^2-1}{1-k^2t^2}}\,dt. \tag{30}$$

Using (29) and (30), we find the following expansion near $z = \infty$:

$$f_m(z) = g_k(J(z)) = z + \frac{2k^2-1}{z} + \cdots \tag{31}$$

Therefore,

$$S_m(\mathbb{D}^*) = 2\pi(2k^2 - 1). \tag{32}$$

Also, using (30), we find that

$$H_1(f_m) = 4k^2 \int_0^1 \sqrt{\frac{1-t^2}{1-k^2t^2}}\, dt, \quad V_1(f_m) = 4k^2 \int_1^{1/k} \sqrt{\frac{t^2-1}{1-k^2t^2}}\, dt. \tag{33}$$

From the latter equations, we find the following relation between m and k:

$$m = \frac{\int_1^{1/k} \sqrt{\frac{t^2-1}{1-k^2t^2}}\, dt}{\int_0^1 \sqrt{\frac{1-t^2}{1-k^2t^2}}\, dt}. \tag{34}$$

Finally, using (26) and (31)–(34), we conclude that

$$\frac{d}{dm} S_m(\mathbb{D}^*) = 8\pi k \frac{dk}{dm} = -H_1^2(k) = -16k^4 \left(\int_0^1 \sqrt{\frac{1-t^2}{1-k^2t^2}}\, dt \right)^2. \tag{35}$$

The integral in equation (35) can be expressed in terms of complete elliptic integrals as follows:

$$\int_0^1 \sqrt{\frac{1-t^2}{1-k^2t^2}}\, dt = \frac{\mathcal{E}(k) - (1-k^2)\mathcal{K}(k)}{k^2}, \tag{36}$$

where

$$\mathcal{K}(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \qquad \mathcal{E}(k) = \int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}}\, dt.$$

From equations (34)-(36), we derive the following formula for the derivative of the quotient of the integrals in the equation (34):

$$\frac{dm}{dk} = \frac{d}{dk}\left(\frac{\int_1^{1/k} \sqrt{\frac{t^2-1}{1-k^2t^2}}\, dt}{\int_0^1 \sqrt{\frac{1-t^2}{1-k^2t^2}}\, dt} \right) = -\frac{\pi}{2} \frac{k}{(\mathcal{E}(k) - (1-k^2)\mathcal{K}(k))^2}. \tag{37}$$

We were not able to locate differentiation formula (37) in the accessible literature but it might be known to experts in the area of elliptic functions.

We finish this paper with few questions. For the case $m = 1$, the first and the second of these questions were posed in an equivalent form by M. Bonk in [1].

(1) What function minimizes the functional $S_m(f, \Omega)$ defined by (6)?
(2) It would be interesting to see whether the functional (6) with $m > 0$ can be used to give an independent existence proof for a conformal map of a finitely connected domain Ω onto a rectangular domain with module m without resorting to the Harrington's result stated in Proposition 1.
(3) The previous problem is open even in the simplest special case, when $\Omega = \mathbb{D}^*$. Thus, we restate it as follows. Find a variational method to prove that a function $f_m \in \Sigma(\mathbb{D}^*)$ maximizing the functional $S_m(f, \mathbb{D}^*)$ maps $\mathbb{D}^*$ onto a rectangular domain $\overline{\mathbb{C}} \setminus R(a, b)$ defined above with $a = H_1(f_m)$, $b = V_1(f_m)$, where $H_1(f_m)$ and $V_1(f_m)$ are defined by formulas (33) and (34).

(4) Prove that the sum $\sum_{j=1}^{n} H_j^2(f_m)$ is a decreasing function for $m \geq 0$. This will imply that the functional $S_m(\Omega)$ is a convex function for $m \geq 0$.

(5) We suspect that even stronger result, than stated in (4) is true. Precisely, we conjecture that for each j, $1 \leq j \leq n$, the individual horizontal variation $H_j(f_m)$ is a decreasing function for $m \geq 0$. For simply connected domains Ω, i.e. when $n = 1$, the monotonicity of $H_1(f_m)$, and therefore the convexity property of $S_m(\Omega)$, easily follows from normalization of functions $f \in \Sigma(\Omega)$ and from a version of the Schwarz lemma applied to this class of functions.

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