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Schrödinger means in higher dimensions

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ABSTRACT

Maximal estimates for Schrödinger means and convergence almost everywhere of sequences of Schrödinger means are studied.

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1. Introduction

For $f \in L^2(\mathbb{R}^n), n \geq 1$ and $a > 0$ we set

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} \mathrm{e}^{-i\xi\cdot x} f(x)\, dx, \xi \in \mathbb{R}^n, \tag{1}$$

and

$$S_t f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \mathrm{e}^{i\xi\cdot x} \mathrm{e}^{it|\xi|^a} \hat{f}(\xi)\, d\xi, \quad x \in \mathbb{R}^n, t \geq 0.$$

We introduce Sobolev spaces $H_s = H_s(\mathbb{R}^n)$ by setting

$$H_s = \{f \in \mathscr{S}'; \|f\|_{H_s} < \infty\}, s \in \mathbb{R},$$

where

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$$\|f\|_{H_s} = \left(\int_{\mathbb{R}^n} (1 + [\xi[^2)^s |\hat{f}(\xi)|^2 \, d\xi \right)^{1/2}$$

and $\mathscr{S} = \mathscr{S}(\mathbb{R}^n)$ denotes the Schwartz class.
In the case $n = 2, a = 2$ Du, Guth and Li [2] proved that

$$\lim_{t \to 0} S_t f(x) = f(x) \text{ almost everywhere,}$$

if $s > \frac{1}{3}$ and $f \in H_s(\mathbb{R}^2)$.
In the case $n \geq 3$ Du and Zhang [3] proved that the above convergence holds also for $s > n/(2(n+1))$ and $f \in H_s(\mathbb{R}^n)$.
We let the sequence $\{t_m\}_1^\infty$ have the properties that

$$1 > t_1 > t_2 > t_3 > \cdots > 0 \text{ and } \lim_{m \to \infty} t_m = 0. \tag{2}$$

We shall study the problem of deciding for which sequences $\{t_m\}_1^\infty$ and functions f one has

$$\lim_{m \to \infty} S_{t_m} f(x) = f(x) \text{ almost everywhere.} \tag{3}$$

For $r > 0$ we say that $\{t_m\}_1^\infty \in l^r$ if $\sum_1 t_m^r < \infty$, and that $\{t_m\}_1^\infty \in l^{r,\infty}$ if $\#\{m; t_m > b\} \lesssim b^{-r}$ for $b > 0$.

In Sjölin and Strömberg [4] we proved that if $a > 0, n \geq 1, 0 < s < a/2$, and $\{t_m\}_1^\infty \in l^r$ for some $r < 2s/(a-2s)$, then (3) holds if $f \in H_s(\mathbb{R}^n)$.
In the case $n = 1$ Dimou and Seeger [1] have proved that if $a > 0, a \neq 1, 0 < s < a/4$ and $\{t_m\}_1^\infty \in l^{r\infty}$, where $r = 2s/(a-4s)$, then (3) holds if $f \in H_s(\mathbb{R}^)$. They also proved that here r cannot be replaced by r_1 if $r_1 > r$.
In Section 3 of this paper we shall improve the result from [4] mentioned above in the case $n \geq 2$. We shall prove that (3) holds for $f \in H_s(\mathbb{R}^n)$ if $a > 0, a \neq 1, n \geq 2, 0 < s < a/2$, and $\{t_m\}_1^\infty \in l^{r,\infty}$ where $r = 2s/(a-2s)$. In the proof of this result we shall use the following theorem.

Theorem 1. *Assume $a > 0, n \geq 1, \lambda \geq 1$, and let the interval $J \subset [0,1]$. Assume also that $f \in L^2(\mathbb{R}^n)$ and supp $\hat{f} \subset B(0;\lambda) = \{\xi \in \mathbb{R}^n; |\xi| \leq \lambda\}$. Then one has*

$$\| \sup_{t \in J} |S_t f| \|_2 \leq (1 + C|J|^{1/2} \lambda^{a/2}) \|f\|_2.$$

Let the ball B be a subset of R^n and let the interval $J \subset [0,1]$. Set $E = B \times J$. In Section 2 we shall also study maximal functions of the type

$$S_E^* f(x) = \sup_{(y,t) \in E} |S_t f(x+y)|, \; x \in \mathbb{R}^n, f \in \mathscr{S}(\mathbb{R}^n).$$

In Section 4 we study relations between maximal estimates in one variable and maximal estimates in dimension $n \geq 2$.
In Section 5 finally we give a counter-example which shows that in the case $n \geq 2, 0 < s < a/4, r = 2s/(a-4s), \{t_m\}_1^\infty \in l^{r,\infty}, (t_n - t_{n+1})_1^\infty$ decreasing, there is no estimate

$$\| \sup_m |S_{t_m} f| \|_2 \lesssim \|f\|_{H_s}$$

for all radial functions $f \in \mathscr{S}(\mathbb{R}^n)$.

We write $A \lesssim B$ if there is a positive constant C such that $A \leq CB$, and we write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$.

2. Maximal estimates

The following theorem was proved in Sjölin and Strömberg [4]. We shall here give a new proof which is simpler than the proof in [4].

Theorem A *(Sjölin-Strömberg). Assume $a > 0, n \geq 1, \lambda \geq 1$, and J interval in $\mathbb{R}$. Assume also that $f \in L^2(\mathbb{R}^n)$ and supp $\hat{f} \subset B(0;\lambda)$.*
Then one has

$$\| \sup_{t\in J} |S_t f| \|_2 \leq (1 + C|J|\lambda^a)\|f\|_2.$$

Proof. We can write

$$S_t f(x) = c \int \mathrm{e}^{i\xi\cdot x} \mathrm{e}^{it|\xi|^a} \hat{f}(\xi)\, d\xi$$

and $J = [t_0, t_0 + r]$ where $r = |J|$.
We have

$$\mathrm{e}^{it|\xi|^a} = \Delta + \mathrm{e}^{it_0|\xi|^a}$$

where $\Delta = \mathrm{e}^{it|\xi|^a} - \mathrm{e}^{it_0|\xi|^a}$. It follows that

$$\Delta = i|\xi|^a \int_{t_0}^{t} \mathrm{e}^{i|\xi|^a s}\, ds,$$

and

$$\begin{aligned} S_t f(x) &= c \int_{\mathbb{R}^n} \int_{t_0}^{t} \mathrm{e}^{i\xi\cdot x} i|\xi|^a \mathrm{e}^{is|\xi|^a} \hat{f}(\xi)\, d\xi\, ds + c \int_{\mathbb{R}^n} \mathrm{e}^{i\xi\cdot x} \mathrm{e}^{it_0|\xi|^a} \hat{f}(\xi)\, d\xi \\ &= S_1(x,t) + S_2(x,t). \end{aligned}$$

Hence

$$|S_1(x,t)| \lesssim \int_{t_0}^{t} \left| \int_{\mathbb{R}^n} \mathrm{e}^{i\xi\cdot x} i|\xi|^a \mathrm{e}^{is|\xi|^a} \hat{f}(\xi)\, d\xi \right| ds$$

and

$$\sup_{t\in J} |S_1(x,t)| \lesssim \int_{t_0}^{t_0+r} \left| \int_{\mathbb{R}^n} \mathrm{e}^{i\xi\cdot x} i|\xi|^a \mathrm{e}^{is|\xi|^a} \hat{f}(\xi)\, d\xi \right| ds.$$

Using Minkovski's inequality and Plancherel's theorem we obtain

$$\left(\int \sup_{t\in J}|S_1(x,t)|^2\,dx\right)^{1/2} \lesssim \int_{t_0}^{t_0+r} \||\xi|^a \hat{f}(\xi)\|_2\,ds$$

$$\leq r\lambda^a \left(\int_{B(0;\lambda)} |\hat{f}(\xi)|^2|\right)^{1/2} \lesssim r\lambda^a \|f\|_2.$$

Also

$$\left(\int \sup_{t\in J}|S_2(x,t)|^2\,dx\right)^{1/2} = \left(\int |S_2(x,t_0)|^2\,dx\right)^{1/2} = \|f\|_2,$$

and we obtain

$$\|\sup_{t\in J}|S_t f|\|_2 \leq (1+Cr\lambda^a)\|f\|_2,$$

which proves the theorem. □

In the following theorems let J denote an interval. In the one-dimensional case Dimou and Seeger obtained the following result.

Theorem B *(Dimou-Seeger). Assume $a>0, a\neq 1, n=1, \lambda\geq 1$, and $J\subset[0,1]$. Assume also that $f\in L^2(\mathbb{R})$ and supp $\hat{f}\subset\{\xi;\lambda/2\leq|\xi|\leq\lambda\}$.*
Then one has

$$\|\sup_{t\in J}|S_t f|\|_2 \lesssim \left(1+|J|^{1/4}\lambda^{a/4}\right)\|f\|_2.$$

We have the following result.

Theorem 1. *Assume $a>0, n\geq 1, \lambda\geq 1$, and let the interval $J\subset[0,1]$. Assume also that $f\in L^2(\mathbb{R}^n)$ and supp $\hat{f}\subset B(0;\lambda)=\{\xi\in\mathbb{R}^n;|\xi|\leq\lambda\}$. Then one has*

$$\|\sup_{t\in J}|S_t f|\|_2 \leq (1+C|J|^{1/2}\lambda^{a/2})\|f\|_2.$$

Proof. First we study the case $|J|\lambda^a\leq 1$. Then $(1+C|J|\lambda^a))\leq(1+C|J||^{1/2}\lambda^{a/2})$. Applying Theorem A we get

$$\|\sup_{t\in J}|S_t f|\|_2 \leq (1+C|J|\lambda^a))\|f\|_2 \leq (1+C|J||^{1/2}\lambda^{a/2})\|f\|_2.$$

It remains to study the case $|J|\lambda^a>1$.
Cover J with intervals $J_i, i=1,2,\dots,N$, of length λ^{-a}. We may take $N\leq|J|\lambda^a+1$. Using Theorem A we obtain

$$\|\sup_{t\in J}|S_t f|\|_2^2 \leq \sum_{i=1}^N \|\sup_{t\in J_i}|S_t|f\|_2^2 \lesssim \sum_{i=1}^N (1+|J_i|\lambda^a)\,\|f\|_2^2 \lesssim N\|f\|_2^2 \leq (|J|\lambda^a+1)\,\|f\|_2^2$$

and the inequality in the theorem follows. □

We have the following extension of Theorem B.

Theorem 2. *Assume $a > 0, a \neq 1, \lambda \geq 1$ and $J \subset [0,1]$. Assume also that $f \in L^2(\mathbb{R})$ with supp $\hat{f} \subset B(0,\lambda)$. Then one has*

$$\| \sup_{t\in J} |S_t f| \|_2 \lesssim C \left(|J|^{1/4} \lambda^{a/4} + 1 \right) \|f\|_2.$$

Proof. First we study the case $|J|\lambda^a \leq 1$. From Theorem A we obtain

$$\| \sup_{t\in J} |S_t f| \|_2 \lesssim \|f\|_2,$$

which proves the theorem in this case.
We then consider the case $|J|\lambda^a > 1$, i.e. $\lambda > |J|^{-1/a}$.
We choose k such that $2^{-k-1}\lambda < |J|^{-1/a} \leq 2^{-k}\lambda$ and then write $f = \sum_{j=0}^{k} f_j + g$ where supp $\hat{f}_j \subset \{\xi; 2^{-j-1}\lambda \leq |\xi| \leq 2^{-j}\lambda\}$ for $j = 0, 1, \ldots, k$, and supp $\hat{g} \subset B(0; 2^{-k-1}\lambda)$.
From Theorem A we conclude that

$$\| \sup_{t\in J} |S_t g| \|_2 \lesssim \left(1 + |J||J|^{-1}\right) \|g\|_2 \lesssim \|f\|_2,$$

and it follows from Theorem B that

$$\| \sup_{t\in J} |S_t f_j| \|_2 \lesssim \left(1 + |J|^{1/4} 2^{-ja/4} \lambda^{a/4}\right) \|f\|_2 \lesssim |J|^{1/4} 2^{-ja/4} \lambda^{a/4} \|f\|_2,$$

since $|J|2^{-ja}\lambda^a \geq 1$. Hence we have

$$\sum_{j=0}^{k} \| \sup_{t\in J} |S_t f_j| \|_2 \lesssim |J|^{1/4} \lambda^{a/4} \|f\|_2,$$

and the theorem follows from the above estimates. □

The method of Dimou and Seeger to prove Theorem B can be extended to all dimensions n an gives the following result.

Lemma 1. *Assume $n \geq 1, a > 0$ and $\lambda \geq 1$. Let the interval $J \subset [0,1]$, let $f \in L^2(\mathbb{R}^n)$ with supp $\hat{f} \subset \{\lambda/2 \leq |\xi| \leq \lambda\}$.*
Then one has

$$\| \sup_{t\in J} |S_t f| \|_2 \leq C \left(|J|^{n/4} \lambda^{na/4} + 1 \right) \|f\|_2 \text{ when } a \neq 1,$$

and

$$\| \sup_{t\in J} |S_t f| \|_2 \leq C \left(|J|^{(n+1)/4} \lambda^{(n+1)/4} + 1 \right) \|f\|_2 \text{ when } a = 1.$$

We shall then study a more general problem. Let E denote a bounded set in $\mathbb{R}^{n+1}$. For $f \in \mathscr{S}(\mathbb{R}^n)$ we introduce the maximal function

$$S_E^* f(x) = \sup_{(y,t)\in E} |S_t f(x+y)|, \quad x \in \mathbb{R}^n.$$

The method used to prove Lemma 1 can also be used to prove the following result.

Lemma 2. *Assume $a > 0, n \geq 1$ and $\lambda \geq 1$. Let the interval $J \subset [0,1]$. Let B be a ball in $\mathbb{R}^n$ with radius r, let $E = B \times J = \{(x,t); x \in B, t \in J\}$ and let $f \in L^2(\mathbb{R}^n)$ with supp $\hat{f} \subset \{\lambda/2 \leq |\xi| \leq \lambda\}$.*
Then one has

$$\|S_E^* f\|_2 \leq C\left(|J|^{n/4}\lambda^{na/4} + r^{n/2}\lambda n/2 + 1\right)\|f\|_2 \ \text{ when } a \neq 1,$$

and

$$\|S_E^* f\|_2^2 \leq C\left(|J|^{(n+1)/4}\lambda^{(n+1)/4} + r^{n/2}\lambda^{n/2} + 1\right)\|f\|_2 \ \text{ when } a = 1.$$

We observe that Lemma 1 is a special case of Lemma 2 by taking $B = B(0,\epsilon)$ with $\epsilon > 0$ small enough.

Proof of Lemma 2. We let χ denote a smooth non-negative function on $\mathbb{R}$ supported on $[1/3, 4/3]$ and identically 1 on $[1/2, 1]$. We also use the same notation for the radial function on χ $\mathbb{R}^n$ with $\chi(\xi) = \chi(|\xi|)$. We set

$$S_\lambda f(x,t) = \int_{R^n} \mathrm{e}^{i(x\cdot\xi + t|\xi|^a)}\hat{f}(\xi)\chi(\xi/\lambda)\,d\xi \quad x \in \mathbb{R}^n, t \in \mathbb{R}.$$

We introduce measurable functions $t : \mathbb{R}^n \to J$ and $b : \mathbb{R}^n \to B$ and then have

$$\begin{aligned} S_\lambda f(x + b(x), t(x)) &= \int_{R^n} \mathrm{e}^{i(x+b(x))\cdot\xi + t(x)|\xi|^a)}\hat{f}(\xi)\chi(\xi/\lambda)\,d\xi \\ &= [\eta = \xi/\lambda] = \int_{\mathbb{R}^n} \mathrm{e}^{i(\lambda(x+b(x))\cdot\eta + t(x)\lambda^a|\eta|^a)}\hat{f}(\lambda\eta)\chi(\eta)\,d\eta\,\lambda^n \\ = \lambda^n T_\lambda(\hat{f}(\lambda\cdot))(x) \end{aligned}$$

where

$$T_\lambda g(x) = \int \mathrm{e}^{i(\lambda(x+b(x))\cdot\xi + t(x)\lambda^a|\xi|^a)}g(\xi)\chi(\xi)\,d\xi.$$

We have $\|\hat{f}(\lambda\cdot)\|_2 = c\lambda^{-n/2}\|f\|_2$ and $T_\lambda T_\lambda^*$ has kernel

$$K_\lambda(x,y) = \int_{\mathbb{R}^n} \mathrm{e}^{i[\lambda(x-y+b(x)-b(y))\cdot\xi + \lambda^a(t(x)-t(y))|\xi|^a]}\chi(\xi)^2\,d\xi$$

We will majorize the kernel K_λ by a convolution kernel G_λ, that is $|K(x,y)| \lesssim G(x-y)$. One then has the L^2-operator norm $\|T_\lambda T_\lambda^*\| \lesssim \|G\|_1$.
First we have the trivial estimate

$$|K_\lambda(x,y)| \lesssim 1$$

holds for all x and y. We shall use this estimate when $\lambda|x-y| \leq C_0 + 2\lambda d$ where $d = 2r$.
For $\lambda|x-y| > C_0 + 2\lambda d$ we have

$$|x-y|\left(1 - \frac{d}{|x-y|}\right) = |x-y| - d < |x-y+b(x)-b(y)| < |x-y| + d = |x-y|\left(1 + \frac{d}{|x-y|}\right),$$

and

$$(1/2)|x-y| < |x-y+b(x)-b(y)| < (3/2)|x-y|. \tag{4}$$

Introducing polar coordinates we have

$$K_\lambda(x,y) = \int_0^\infty \mathrm{e}^{i\lambda^a(t(x)-t(y))r^a}\chi(r)^2 \left(\int_{S^{n-1}} \mathrm{e}^{i\lambda r(x-y+b(x)-b(y))\cdot\xi'}\, d\sigma(\xi') \right) r^{n-1}\, dr.$$

We observe that inner integral is $\hat{\sigma}(\lambda|x-y+b(x)-b(y)|r)$. According to Stein ([6], p. 347) one has

$$\hat{\sigma}(\xi) = c|\xi|^{1-n/2} J_{(n-2)/2}(|\xi|).$$

We take C_0 large so that

$$\begin{aligned} J_{(n-2)/2}(r) =& a_0 \frac{\mathrm{e}^{ir}}{r^{1/2}} + a_1 \frac{\mathrm{e}^{ir}}{r^{3/2}} + \cdots + a_N \frac{\mathrm{e}^{ir}}{r^{N+1/2}} + \\ & b_0 \frac{\mathrm{e}^{-ir}}{r^{1/2}} + b_1 \frac{\mathrm{e}^{-ir}}{r^{3/2}} + \cdots + b_N \frac{\mathrm{e}^{-ir}}{r^{N+1/2}} + R(r) \text{ for } r \geq C_0, \end{aligned}$$

where $|R(r)| \lesssim \frac{1}{r^{N+3/2}}$ for $r \geq C_0$ (see [6] p. 338).
From this we get

$$\begin{aligned} K_\lambda(x,y) = \int_0^\infty \mathrm{e}^{i\lambda^a(t(x)-t(y))r^a}\chi(r)^2\, r^{n-1} \Bigg(& a_0 \frac{\mathrm{e}^{i\lambda|x-y+b(x)-b(y)|r}}{(\lambda|x-y+b(x)-b(y)|r)^{\frac{n}{2}-\frac{1}{2}}} + \\ & \cdots + b_N \frac{\mathrm{e}^{-i\lambda|x-y+b(x)-b(y)|r}}{(\lambda|x-y+b(x)-b(y)|r)^{N+\frac{n}{2}-\frac{1}{2}}} + R_1(\lambda|x-y+b(x)-b(y)|r) \Bigg)\, dr \end{aligned} \tag{5}$$

where $R_1(r) = r^{1-n/2}R(r)$. It follows from (4) that

$$R_1(\lambda|x-y|-b(x)-b(y)r) \lesssim \frac{1}{(\lambda|x-y+b(x)-b(y)|)^{N+n/2+1/2}} \leq \frac{1}{(\lambda|x-y|/2)^{N+n/2+1/2}}.$$

The remainder term contributes with a remainder part of the kernel

$$K_{\lambda,\mathrm{rem}}(x,y) \lesssim (\lambda|x-y|)^{-N-n/2-1/2}.$$

Set $\Phi_\lambda(r) = \lambda^a(t(x)-t(y))r^a + \lambda|x-y+b(x)-b(y)|r$
The main term in ((5)) gives the following contribution to $K_\lambda(x-y)$:

$$K_{\lambda,0}(x,y) = \frac{a_0}{(\lambda|x-y+b(x)-b(y)|)^{n/2-1/2}} \cdot \int_0^\infty \mathrm{e}^{i\Phi_\lambda(r)}\chi(r)^2 r^{n-1} r^{\frac{1}{2}-\frac{n}{2}}\, dr.$$

We consider two cases:
Case 1: $|x-y| >> \lambda^{a-1}|t(x)-t(y)|$ gives $\Psi'_\lambda \gtrsim \lambda|x-y|$ and integrations by parts gives $|K_{\lambda,0}(x,y)| \lesssim (\lambda|x-y|)^{-N}$ for any large N.
Case 2: $|x-y| \lesssim \lambda^{a-1}|t(x)-t(y)|$. In this case we have

$$\Phi''_\lambda = \lambda^a\,(t(x)-t(y))\,a(a-1)r^{a-2},$$

and in the case $a \neq 1$ van der Corput gives

$$|K_{\lambda,0}(x,y)| \lesssim \lambda^{-a/2}|t(x)-t(y)|^{-1/2}\,(\lambda|x-y|)^{\frac{1}{2}-\frac{n}{2}} \lesssim (\lambda|x-y|)^{-n/2}\,.$$

In Case 2a with $a=1$ we have the trivial estimate estimate

$$|K_{\lambda,0}(x,y)| \lesssim (\lambda|x-y|)^{(1-n)/2}\,,$$

and the other terms in (5) can be estimated in the same way.
We note that Case 2 is contained in the set $|x-y| \lesssim \lambda^{a-1}|J|$.

The other terms in (2) can be estimated in the same way.

To summarize the estimates we see that $|K_\lambda(x,y)| \lesssim G_\lambda(x-y)$ where
when $a \neq 1$:

$$G_\lambda(x) = \chi_{\{|x|<C_0\lambda^{-1}+2d\}}(x) + \chi_{\{|x|\geq\lambda^{-1}\}}\lambda^{-N}|x|^{-N} + \chi_{\{|x|\leq C\lambda^{a-1}|J|\}}\lambda^{-n/2}|x|^{-n/2},$$

and when $a=1$:

$$G_\lambda(x) = \chi_{\{|x|<C_0\lambda^{-1}+2d\}}(x) + \chi_{\{|x|\geq\lambda^{-1}\}}\lambda^{-N}|x|^{-N} + \chi_{\{|x|\leq C|J|\}}\lambda^{(1-n)/2}|x|^{(1-n)/2}.$$

In the case when $a \neq 1$ we have

$$\|G\|_1 \lesssim (\lambda^{-1}+d)^n + \lambda^{-n} + \int\limits_{|x|\leq C\lambda^{a-1}|J|} \lambda^{-n/2}|x|^{-n/2}\,dx,$$

and the above integral is majorized by

$$\lambda^{-n/2}\int\limits_0^{C\lambda^{a-1}|J|} r^{n/2-1}\,dr \lesssim \lambda^{-n/2}\lambda^{(a-1)n/2}|J|^{n/2} = \lambda^{\frac{n}{2}(a-2)}|J|^{n/2}.$$

Hence

$$\|G\|_1 \lesssim \lambda^{-n} + d^n + \lambda^{\frac{n}{2}(a-2)}|J|^{n/2}$$

in the case $a \neq 1$. In the case $a=1$ we get

$$\|G\|_1 \lesssim \lambda^{-n} + d^n + \lambda^{(1-n)/2}\int\limits_0^{C|J|} r^{1/2+n/2}\,dr \lesssim \lambda^{-n} + d^n + \lambda^{(1-n)/2}|J|^{(n+1)/2}.$$

In the case $a \neq 1$ we obtain

$$\|T_\lambda T_\lambda^*\| \lesssim \lambda^{-n} + d^n + \lambda^{\frac{n}{2}(a-2)}|J|^{n/2},$$

and

$$\|T_\lambda\| \lesssim \lambda^{-n/2} + d^{n/2} + \lambda^{\frac{n}{4}(a-2)}|J|^{n/4}.$$

From this we get

$$\begin{aligned}\|S_\lambda f(x+b(x),t(x))\|_2 &\le \lambda^n \|T_\lambda[\hat{f}(\lambda\cdot)]\|_2 \le \lambda^n \|T_\lambda\| \cdot \|\hat{f}(\lambda\cdot)\|_2 \\ &\lesssim \lambda^n \left(\lambda^{-n/2} + d^{n/2} + \lambda^{\frac{n}{4}(a-2)}|J|^{n/4}\right) \lambda^{-n/2}\|f\|_2 \\ &= \left(1 + d^{n/2}\lambda^{n/2} + \lambda^{na/4}|J|^{n/4}\right)\|f\|_2.\end{aligned}$$

Finally in the case $a = 1$ we obtain

$$\begin{aligned}\|T_\lambda T_\lambda^*\| &\lesssim \lambda^{-n} + d^n + \lambda^{(1-n/)2}|J|^{(n+1)/2}, \\ \|T_\lambda\| &\lesssim \lambda^{-n/2} + d^{n/2} + \lambda^{(1-n)/4}|J|^{(n+1)/4},\end{aligned}$$

and

$$\begin{aligned}\|S_\lambda f(x+b(x),t(x))\|_2 &\le \lambda^n \|T_\lambda[\hat{f}(\lambda\cdot)]\|_2 \le \lambda^n \|T_\lambda\| \cdot \|\hat{f}(\lambda\cdot)\|_2 \\ &\lesssim \lambda^n \left(\lambda^{-n/2} + d^{n/2} + \lambda^{(1-n)/4)}|J|^{(n+1)/4}\right) \lambda^{-n/2}\|f\|_2 \\ &= \left(1 + d^{n/2}\lambda^{n/2} + \lambda^{(n+1)/4}|J|^{(n+1)/4}\right)\|f\|_2.\end{aligned}$$

This completes the proof of Lemma 2. □

We shall then extend Lemma 2.

Lemma 3. *In Lemma 1 and Lemma 2 above the condition supp* $\hat{f} \subset \{\xi; \lambda/2 \le |\xi| \le \lambda\}$ *can be replaced by the weaker condition supp* $\hat{f} \subset \{\xi; |\xi| \le \lambda\}$.

In the proof of Lemma 3 we shall use a result in Sjölin and Strömberg [5]. Let $y_0 \in \mathbb{R}^n, t_0 \in \mathbb{R}^n, 0 < r \le 1$, and let $f \in L^2(\mathbb{R}^n)$ with supp $\hat{f} \subset B(0;\lambda)$ and $\lambda \ge 1$. Set

$$F = \{(y,t); y_{y_0,j} \le y_j \le y_{y_0,j} + r \text{ for } 1 \le j \le n, \text{ and } t_0 \le t \le t_0 + r^a\}.$$

It is proved that

$$\|S_F^* f\|_2 \lesssim (1 + r^a\lambda^a)(1 + r\lambda)^n \|f\|_2,$$

and the method in [5] can be used to prove that

$$\|S_E^* f\|_2 \lesssim (1 + |J|\lambda^a)(1 + r\lambda)^n \tag{6}$$

if $E = B \times J$ with B and J as in Lemma 2.
The method to prove this is a generalisation of the method we used to prove Theorem A.

Proof of Lemma 3. Let N be the smallest integer such that $|J|2^{-aN}\lambda^a + r2^{-N}\lambda < 2$. We write $f = \sum_0^N f_j$ where supp $\hat{f}_j \subset \{2^{-j-1}\lambda \le |\xi| \le 2^{-j}\lambda\}$ for $0 \le j < N$ and supp $\hat{f}_N \subset B(0; 2^{-N}\lambda)$. It follows from (6) that

$$\|S_E^* f_N\|_2 \lesssim (1 + |J|2^{-aN}\lambda^a)(1 + r2^{-N}\lambda)^n \|f_N\|_2 \lesssim \|f\|_2.$$

Also

$$\|S_E^* f\|_2 \le \sum_{j=0}^N \|S_E^* f_j\|_2,$$

and according to Lemma 2 we have for $a \neq 1$

$$\|S_E^* f_j|\|_2 \leq C(2^{-jan/4}|J|^{n/4}\lambda^{na/4} + r^{n/2}\lambda^{n/2}2^{-jn/2})\|f\|_2$$

for $0 \leq j < N$. It follows that

$$\left\| S_E^* \left(\sum_0^{N-1} f_j \right) \right\|_2 \lesssim (|J|^{n/4}\lambda^{na/4} + r^{n/2}\lambda^{n/2})\|f\|_2,$$

and we obtain

$$\|S_E^* f\|_2 \lesssim (|J|^{n/4}\lambda^{na/4} + r^{n/2}\lambda^{n/2} + 1)\|f\|_2$$

for $a \neq 1$.
The same proof works also for $a = 1$ and this completes the proof of Lemma 3. □

We shall then prove the following theorem.

Theorem 3. *Assume $a > 0, n \geq 1$ and $\lambda \geq 1$. Let the interval $J \subset [0,1]$, let B be a ball in $\mathbb{R}^n$ with radius r and set $E = B \times J$. Let f be an function in $L^2(\mathbb{R}^n)$ with supp $\hat{f} \subset B(0;\lambda)$.*
Then the following holds
when $n = 1$ and $a \neq 1$:

$$\|S_E^* f\|_2 \leq \left(|J|^{1/4}\lambda^{a/2} + r^{1/2}\lambda^{1/2} + 1 \right) \|f\|_2,$$

when $n = 1$ and $a = 1$:

$$\|S_E^* f\|_2 \leq \left(|J|^{1/2}\lambda^{a/2} + r^{1/2}\lambda^{1/2} + 1 \right) \|f\|_2,$$

when $n \geq 2$ and $a \neq 1$:

$$\|S_E^* f\|_2 \leq \left(|J|^{1/2}\lambda^{a/2} + r\lambda + 1 \right) (r\lambda + 1)^{(n-2)/2} \|f\|_2,$$

when $n \geq 2$ and $a = 1$:

$$\|S_E^* f\|_2 \leq \left(|J|^{1/2}\lambda^{1/2} + r^{n/(n+1)}\lambda^{n/(n+1)} + 1 \right) (r^{n/(n+1)}\lambda^{n/(n+1)} + 1)^{(n-1)/2} \|f\|_2.$$

Proof. The cases with $n = 1$ in Theorem 3 follow directly from Lemma 3.
In the cases with $n \geq 2$ we shall use an argument similar to the proof of Theorem 1 by covering the Interval J with intervals J_i of equal length.
In the case $a \neq 1$ we have by Lemma 3 the estimate

$$\|S_E^* f\|_2^2 \leq C \left(|J|^{n/2}\lambda^{na/2} + r^n\lambda^n + 1 \right) \|f\|_2^2.$$

We have

$$|J|^{n/2}\lambda^{na/2} + r^n\lambda^n + 1 \sim (|J|\lambda^a)^{n/2} + \left(r^2\lambda^2 + 1\right)^{n/2}.$$

Cover J with intervals $J_i, i = 1, 2, \ldots, N$, of intervals of length $|J_i|$ such that $|J_i|\lambda^a = r^2\lambda^2 + 1$ with $N \leq |J|/|J_i| + 1 = |J|\lambda^a \left(r^2\lambda^2 + 1\right)^{-1} + 1$. Set $E_i = B \times J_i$, then we have

$$\|S^*_{E_i} f\|_2^2 \lesssim \left((|J_i|\lambda^a)^{n/2} + \left(r^2\lambda^2+1\right)^{n/2} \right) \|f\|_2^2 = 2\left(r^2\lambda^2+1\right)^{n/2} \|f\|_2^2,$$

and

$$\begin{aligned}
\|S^*_E f\|_2^2 &\le \sum_{i=1}^N \|S^*_{E_i} f\|_2^2 \lesssim N\left(r^2\lambda^2+1\right)^{n/2} \|f\|_2^2 \\
&\le \left(|J|\lambda^a \left(r^2\lambda^2+1\right)^{-1} + 1 \right) \left(r^2\lambda^2+1\right)^{n/2} \|\|f\|_2^2 \\
&= \left(|J|\lambda^a + r^2\lambda^2 + 1 \right) \left(r^2\lambda^2+1\ \right)^{(n-2)/2} \|f\|_2^2 \\
&\le \left(|J|^{1/2}\lambda^{a/2} + r\lambda + 1 \right)^2 \left(r^{(n-2)/2}\lambda^{(n-2)/2} + 1 \right)^2 \|f\|_2^2,
\end{aligned}$$

which gives the desired estimate in this case.
In the case $a = 1$ we have by Lemma 3 the estimate

$$\|S^*_E f\|_2^2 \le C\left(|J|^{(n+1)/2}\lambda^{(n+1)a/2} + r^n\lambda^n + 1 \right) \|f\|_2^2.$$

We have

$$|J|^{(n+1)/2}\lambda^{(n+1)a/2} + r^n\lambda^n + 1 \sim (|J|\lambda^a)^{(n+1)/2} + \left(r^{2n/(n+1)}\lambda^{2n/(n+1)} + 1 \right)^{(n+1)/2}.$$

Cover J with intervals $J_i, i = 1, 2, \dots, N$, of intervals of length $|J_i|$ such that $|J_i|\lambda^a = r^{2n/(n+1)}\lambda^{2n/(n+1)} + 1$ and $N \le |J|/|J_i| + 1 = |J|\lambda^a \left(r^{2n/(n+1)}\lambda^{2n/(n+1)} + 1 \right)^{-1} + 1$. Set $E_i = B \times J_i$, then we have

$$\begin{aligned}
\|S^*_{E_i} f\|_2^2 &\lesssim \left((|J_i|\lambda^a)^{(n+1)/2} + \left((r^{2n/(n+1)}\lambda^{2n/(n+1)} + 1 \right)^{(n+1)/2} \right) \|f\|_2^2 \\
&= 2\left(r^{2n/(n+1)}\lambda^{2n/(n+1)} + 1 \right)^{(n+1)/2} \|f\|_2^2,
\end{aligned}$$

and

$$\begin{aligned}
\|S^*_E f\|_2^2 &\le \sum_{i=1}^N \|S^*_{E_i} f\|_2^2 \lesssim N\left((r^{2n/(n+1)}\lambda^{2n/(n+1)} + 1 \right)^{(n+1)/2} \|f\|_2^2 \\
&\le \left(|J|\lambda^a \left(r^{2n/(n+1)}\lambda^{2n/(n+1)} + 1 \right)^{-1} + 1 \right) \left(r^{2n/(n+1)}\lambda^{2n/(n+1)} + 1 \right)^{(n+1)/2} \|\|f\|_2^2 \\
&= \left(|J|\lambda^a + r^{2n/(n+1)}\lambda^{2n/(n+1)} + 1 \right) \left(r^{2n/(n+1)}\lambda^{2n/(n+1)} + 1\ \right)^{(n-1)/2} \|f\|_2^2 \\
&\le \left(|J|^{1/2}\lambda^{a/2} + r^{n/(n+1)}\lambda^{n/(n+1)} + 1 \right)^2 \left(r^{n/(n+1)}\lambda^{n/(n+1)} + 1 \right)^{n-1} \|f\|_2^2,
\end{aligned}$$

which gives the desired estimate in this case. □

3. A convergence result

We shall here prove a convergence result for function in $H_s(\mathbb{R}^n), n \ge 2$, and begin with two lemmas.

Lemma 4. *Assume $a > 0, a \ne 1, n \ge 2, \lambda \ge 1$ and $0 < s < a/2$. Also let $(t_m)_1^\infty \in l^{r,\infty}$ where $r = 2s/(a-2s)$ Then one has*

$$\|\sup_m |S_{t_m} f|\|_2 \lesssim \lambda^s \|f\|_2,$$

if $f \in L^2(\mathbb{R}^n)$ *and supp* $\hat{f} \subset \{\xi; \lambda/2 \le |\xi| \le \lambda\}$.

Proof. Let $0 < b < 1$. We have

$$\|\sup_m |S_{t_m} f|\|_2 \le \|\sup_{t_m \le b} |S_{t_m} f|\|_2 + \|\sup_{t_m > b} |S_{t_m} f|\|_2 = T_1 + T_2.$$

Theorem 1 gives the estimate

$$T_1 \lesssim \left(b^{1/2}\lambda^{a/2} + 1\right) \|f\|_2.$$

We also have $\#\{m; t_m > b\} \lesssim b^{-r}$ and it follows that

$$T_2 \lesssim b^{-r/2} \|f\|_2.$$

We then choose b such that

$$b^{1/2}\lambda^{a/2} = b^{-r/2}.$$

One gets

$$b^{1/2+r/2} = \lambda^{-a/2},$$

and

$$b = \lambda^{-a/(1+r)}) < 1.$$

Hence $b^{-r/2} \ge 1$ and

$$T_1 \lesssim b^{1/2+r/2} \|f\|_2 = b^{-r/2} \|f\|_2.$$

We have shown that

$$\|\sup_m |S_{t_m} f|\|_2 \lesssim b^{-r/2} \|f\|_2,$$

where

$$b^{-r/2} = \lambda^{ra/2(1+r)} = \lambda^s.$$

This completes the proof of the lemma. □

We shall then improve Lemma 4 by proving the following lemma

Lemma 5. *Let* $a, n, s, r,$ *and* $(t_m)_1^\infty$ *have the same properties as in Lemma 4, and assume that* $f \in H_s(\mathbb{R}^n)$. *Then*

$$\|\sup_m S_{t_m} f|\|_2 \lesssim \|f\|_{H_s} \text{ for } f \in \mathscr{S}(\mathbb{R}^n).$$

Before proving Lemma 5 we remark that the following theorem follows from Lemma 5 (see proof of Corollary 4 in Sjölin and Strömberg [4])

Theorem 4. *Let a, n, s, r, and $(t_m)_1^\infty$ have the same properties as in Lemma 4 and assume that $f \in H_s(\mathbb{R}^n)$. Then*

$$\lim_{m\to\infty} S_{t_m} f(x) = f(x) \text{ for almost every } x.$$

Proof of Lemma 5. It follows from the theorem on monotone convergence that instead of estimating $\sup_m |S_{t_m} f|$ it is sufficient to estimate $\sup_{m\le M} |S_{t_m} f|$ for large integers M (as long as the estimates do not dependend on M). We can find a measurable function $t(x)$ such that

$$\sup_{m\le M} |S_{t_m} f| = |S_{t(x)} f(x)|,$$

and $t(x)$ takes only finitely many values. We then define intervals $I_k = (2^{-k-1}, 2^{-k}], k = 0, 1, \dots$, and sets $F_k = \{x \in \mathbb{R}^n, t(x) \in I_k\}$. The Sets F_k are disjoint and $\mathbb{R}^n = \bigcup k \ge 0 F_k$. We let X_k denote the characteristic function of F_k and then have $\sum_{k\ge 0} \chi_k = 1$ and

$$S_{t(x)} f(x) = \sum_{k\ge 0} X_k(x) S_{t(x)} f(x). \tag{7}$$

We write a function $f = \sum_{j\ge 0} f_j$ by splitting its Fourier transform $\hat{f} = \sum_{j\ge 0} \hat{f}_j$,
where $\hat{f}_j$ i supported in Ω_j where $\Omega_0 = \{|\xi| \le 1\}$ and $\Omega_j = \{2^{-j-1}|\xi| \le 2^j\}$ for $j > 0$.
We shall then split the sum (7) into three parts. For $j \ge 0$ set

$$k(j) = (a - 2s)j = j2s/r, \qquad b(j) = 2^{-k(j)},$$
$$b_1(j) = 2^{-k(j)-\epsilon_1 j} \quad \text{and} \quad b_2(j) = 2^{-k(j)+\epsilon_2 j},$$

where $\epsilon_1 = 2\epsilon$ and $\epsilon_2 = 2\epsilon/r$ and ϵ is a small positive number.

We have

$$S_{t(x)} f(x) = \sum_{k\ge 0} \sum_{j\ge 0} X_k(x) S^a_{t(x)} f j(x) = S^1_{t(x)} f(x) + S^2_{t(x)} f(x) + S^3_{t(x)} f(x),$$

where

$$S^1_{t(x)} f(x) = \sum_{j\ge 0} \sum_{\substack{k\ge 0,\\ k\ge k(j)+\epsilon_1 j}} X_k(x) S_{t(x)} f_j(x),$$

$$S^2_{t(x)} f(x) = \sum_{j\ge 0} \sum_{\substack{k\ge 0\\ k\le k(j)-\epsilon_2 j}} X_k(x) S_{t(x)} f_j(x),$$

$$S^3_{t(x)} f(x) = \sum_{j\ge 0} \sum_{\substack{k\ge 0\\ k(j)-\epsilon_2 j<k<k(j)+\epsilon_1 j}} X_k(x) S_{t(x)} f_j(x).$$

Invoking Theorem 1 we obtain

$$|S^1_{t(x)}f(x)| \le \sum_{j\ge 0} \sum_{\substack{k\ge 0\\ k\ge k(j)+\epsilon_1 j}} X_k(x)|S_{t(x)}f_j(x)| \le \sum_{j\ge 0} \sup_{t_m\le b_1(j)} |S_{t_m}f_j(x)|,$$

and

$$\|S^1_{t(x)}f\|_2 \le \sum_{j\ge 0} \| \sup_{t_m\le b_1(j)} |S_{t_m}f_j|\|_2 \lesssim \sum_{j\ge 0}(1+b_1(j)^{1/2}2^{aj/2})\|f_j\|_2.$$

We have

$$b_1(j)^{1/2}2^{aj/2} = 2^{-\frac{1}{2}k(j)-\frac{1}{2}\epsilon_1 j}2^{aj/2} = 2^{-\frac{1}{2}(a-2s+\epsilon_1-a)j} = 2^{j(s-\epsilon)},$$

and it follows that

$$\|S^1_{t(x)}f\|_2 \lesssim \sum_{j\ge 0} 2^{j(s-\epsilon)}\|f_j\|_2 \le \|f\|_{H_s}.$$

We shall then estimate $S^2_{t(x)}f(x)$. One has

$$|S^2_{t(x)}f(x)| \le |\sum_{j\ge 0} \sum_{\substack{k\ge 0\\ k\le k(j)-\epsilon_2 j}} X_k(x)|S_{t(x)}f_j(x)| \le \sum_{j\ge 0} \sup_{t_m\ge b_2(j)} |S_{t_m}f_j(x)|,$$

and we obtain

$$\| \sup_{t_m\ge b_2(j)} |S_{t_m}f_j(x)|\|_2^2 \lesssim \sum_{t_m>b_2(j)/2} \|f_j\|_2^2 = \#\{m; t_m > b_2(j)/2\}\|f_j\|_2^2 \lesssim b_2(j)^{-r}\|f_j\|_2^2.$$

We also have

$$b_2(j)^{-r} = 2^{rk(j)-\epsilon_2 rj},$$

and

$$rk(j) - \epsilon_2 rj = 2j(s-\epsilon).$$

It follows that

$$\|S^2_{t(x)}f\|_2 \lesssim \sum_{j\ge 0} 2^{j(s-\epsilon)}\|f_j\|_2 \le \|f\|_{H_s}.$$

It remains to study $S^3_{t(x)}f(x)$. We let $[k(j)]$ denote the integral part of $k(j)$, and setting $l = k - [k(j)]$ we obtain

$$|S^3_{t(x)}f(x)| \le \sum_{j\ge 0} \sum_{\substack{k\ge 0\\ k(j)-\epsilon_2 j<k<k(j)+\epsilon_1 j}} X_k(x)|S_{t(x)}f_j(x)|$$

$$= \sum_{l=-\infty}^{\infty} \sum_{j>\max\{(l-1)/\epsilon_1, -l/\epsilon_2\}} X_{[k(j)]+l}(x)|S_{t(x)}f_j(x)|.$$

Using the fact that $X_k = X_k^2$ and applying Cauchy-Schwarz inequality one obtains

$$\left(\sum_{j>\max\{(l-1)/\epsilon_1,-l/\epsilon_2\}} X_{[k(j)]+l}(x)|S_{t(x)}f_j(x)|\right)^2$$
$$\lesssim \left(\sum_{j>\max\{(l-1)/\epsilon_1,-l/\epsilon_2\}} X^2_{[k(j)]+l}(x)\right)\left(\sum_{j>\max\{(l-1)/\epsilon_1,-l/\epsilon_2\}} X_{[k(j)]+l}(x)|S_{t(x)}f_j(x)|^2\right).$$

The first sum on the second line is majorized by

$$C_0 \max_k \#\{j;[k(j)]=k\} \lesssim 1,$$

and it follows that

$$\|\sum_{j>\max\{(l-1)/\epsilon_1,-l/\epsilon_2\}} X_{k(j)+l}(x)|S_{t(x)}f_j(x)|\|_2^2 \lesssim \sum_{j>\max\{(l-1)/\epsilon_1,-l/\epsilon_2\}} \int X_{[k(j)]+l}|S_{t(x)}f_j||^2\,dx$$
$$\lesssim \sum_{j>\max\{(l-1)/\epsilon_1,-l/\epsilon_2\}} \int \sup_{t_m\in I_{[k(j)]+l}} |S_t(x)f_j(x)|^2\,dx.$$

Invoking Minkovski's inequality we then obtain

$$\|S^3_{t(x)}f\|_2 \lesssim= \sum_{l=-\infty}^{\infty}\left(\sum_{j>\max\{(l-1)/\epsilon_1,-l/\epsilon_2\}} \|\sup_{t_m\in I_{[k(j)]+l}} |S_{t_m}f_j|\|_2^2\right)^{1/2}.$$

Furthermore for $l\geq 0$ we have by Theorem 1.

$$\|\sup_{t_m\in I_{[k(j)]+l}} |S_{t_m}f_j|\|_2^2 \lesssim (1+2^{[-k(j)]-l}2^{aj})\|f_j\|_2^2 \lesssim (1+2^{-k(j)-l+aj})\|f_j\|_2^2$$
$$\lesssim (1+2^{2sj}2^{-l})\|f_j\|_2^2 \lesssim 2^{-l}2^{2sj}\|f_j\|_2^2.$$

For $l\leq 0$ we have

$$\|\sup_{t_m\in I_{[k(j)]+l}} |S_{t_m}f_j|\|_2^2 \lesssim (\#\{m;t_m\in I_{[k(j)]+l}\})\|f_j\|_2^2 \lesssim 2^{-r(-[k(j)]-l)}\|f_j\|_2^2 \lesssim 2^{rl}2^{2sj}\|f_j\|_2^2.$$

We conclude that

$$\|S^3_{t(x)}f\|_2 \lesssim \sum_{l=-\infty}^{0}\left(\sum_{j>-l/\epsilon_2} 2^{rl}2^{2sj}\|f_j\|_2^2\right)^{1/2} + \sum_{l=1}^{\infty}\left(\sum_{j>(l-1)/\epsilon_1} 2^{-l}2^{sj}\|f_j\|_2^2\right)^{1/2}$$
$$\leq \left(\sum_{l=-\infty}^{0} 2^{rl/2}+\sum_{l=1}^{\infty}2^{-l/2}\right)\left(\sum_{j>0}2^{2sj}\|f_j\|_2^2\right)^{1/2} \lesssim \|f\|_{H_s}.$$

This completes the proof of Lemma 5. □

4. Relations between maximal estimates in one variable and maximal estimates in dimension $n\geq 2$

Next we shall consider the Schrödinger equation on radial or symmetric functions on $\mathbb{R}^n$ and will see how it can be reduced to a one-dimensional problem

Remark. In this paper we have the Fourier transform $\hat{f}$ of a function on $\mathbb{R}^n$ defined by (1), and then yields $\|\hat{f}\|_{L(\mathbb{R}^n)} = (2\pi)^{n/2}\|f\|_{L(\mathbb{R}^n)}$. We set

$$\alpha_n = (2\pi)^{n/2}, \text{ for } n \geq 1$$

in this section.

Let $S_t^{(k)}$ denote the k-dimensional Schrödinger operator (with a given $a > 0$ in its definition) and let

$$S_E^{*(k)} f(x) = \sup_{t\in E} |S_t^{(k)} f(x)|$$

where f is a function on $\mathbb{R}^k$ and the supremum is taken over a set $E \subset [0,1]$.

Remark. We may in this section replace the Fourier multiplier functions $\{e^{it|\xi|^s}\}_t$ with any family of radial Fourier multiplier functions $\{\hat{k}_t(|\xi|)\}_t$ satisfying $|\hat{k}_t(|\xi|)| \leq 1$.

We shall prove the following theorem.

Theorem 5. *Let $s \geq 0$, let $n \geq 2$ and let E be a given subset of the interval $[0,1]$-*
If

$$\|S_E^{*(1)} f\|_2 \leq C\|f\|_{H_s}$$

for all functions f in $\mathscr{S}(\mathbb{R})$,
then

$$\|S_E^{*(n)} f\|_2 \leq C_{n,k}\|f\|_{H_s}$$

for all functions f in $\mathscr{S}(\mathbb{R}^n)$ of the form $f(x) = f_0(x)P(x)$, where $f_0 \in \mathscr{S}(\mathbb{R}^n)$ and is radial and P is a solid spherical harmonic on $\mathbb{R}^n$ of degree $k \geq 0$.

Theorem 5 will follow with some approximation arguments from the following theorems.

Theorem 6. *Let f_1 be a function $L^2(\mathbb{R})$, supported on $\mathbb{R}_+$, not identically zero; and equal to a function in $\mathscr{S}(\mathbb{R})$ on $[1,\infty)$. Let P be a solid spherical harmonic on $\mathbb{R}^n$ of degree k and normalised so that $\|P\|_{L^2(\mathbf{S}^{n-1})} = 1$.*
Let f_P be the symmetric function on $\mathbb{R}^n$ defined by its Fourier transform

$$\hat{f}_P(\xi) = P(\xi')f_1(|\xi|)|\xi|^{1/2-n/2} \text{ for } \xi = \xi'|\xi| \in \mathbb{R}^n.$$

Let $\check{f}_1$ be the inverse Fourier transform of f_1 on $\mathbb{R}$. Note that $\alpha_1\|\check{f}\|_{L^2(\mathbb{R})} = \|f_1\|_{L^2(\mathbb{R}_+)}$. Assume that E is subset of the interval $[0,1]$. Then there is a constant $C_{n,k}$ dependent only on n,k but independent of $a > 0, E$ and f_1 such that

$$\alpha_n\|S_E^{*(n)} f_P\|_{L^2(\mathbb{R}^n)} \leq \alpha_1\sqrt{2}\|S_E^{*(1)}\check{f}_1\|_{L^2(\mathbb{R})} + C_{n,k}\|f_1\|_{L^2(\mathbb{R}_+)}. \tag{8}$$

Remark. Most of the results in this section can also be formulated with norms on spheres, where the $L^2(\mathbb{R}^n)$ is replaced by the norms $L^2(\mathbf{S}^{n-1}(r))$ for $r > 0$.
For instance (8) may be replaced by

$$\alpha_n \|S_E^{*(n)} f_P\|_{L^2(\mathbf{S}^{n-1}(r))} \le \alpha_1 \left(|S_E^{*(1)} \check{f}_1(r)|^2 + |S_E^{*(1)} \check{f}_1(-r)|^2 \right)^{1/2} + T f_1(r), \text{ for } r > 0,$$

where $\|Tf_1\|_{L^2(\mathbb{R}+)} \le C_{n,k} \|f_1\|_{L^2(\mathbb{R}+)}$.

With two solid spherical harmonic the following estimates hold.

Theorem 7. *Let the functions f_1, P, f_P and the set E be as in Theorem 6 and let Q be a solid spherical harmonic on $\mathbb{R}^{n_1}$ of degree k_1 and normalised so that $\|Q\|_{L^2(\mathbf{S}^{n_1-1})} = 1$ and define f_Q by its Fourier transform*

$$\hat{f}_Q(\xi) = Q(\xi') f_1(|\xi|) |\xi|^{1/2-n_1/2} \text{ for } \xi = \xi'|\xi| \in \mathbb{R}^{n_1}.$$

Set $v = k + n/2 - 1$ and $v_1 = k_1 + n_1/2 - 1$.
If $v = v_1$ then

$$\alpha_n \|S_E^{*(n)} f_P\|_{L^2(\mathbb{R}^n)} = \alpha_{n_1} \|S_E^{*(n_1)} f_Q\|_{L^2(\mathbb{R}^{n_1})}.$$

It $2v_1 = 2v \mod (4)$ then

$$\left| \alpha_n \|S_E^{*(n)} f_P\|_{L^2(\mathbb{R}^n)} - \alpha_{n_1} \|S_E^{*(n_1)} f_Q\|_{L^2(\mathbb{R}^{n_1})} \right| \le (C_{n,k} + C_{n_1,k_1}) \|f_1\|_{L^2(\mathbb{R}+)}.$$

Remark. In the special case $n = 1$ we have $k = 0, 1$ and use a special definition of f_P. When $k = 0$ the even function $f_e = f_P$ is defined by

$$\hat{f}_e(\xi) = \begin{cases} \frac{1}{\sqrt{2}} f_1(\xi) \text{ for } \xi \ge 0, \\ \frac{1}{\sqrt{2}} f_1(-\xi) \text{ for } \xi < 0, \end{cases}$$

and when $k = 1$ the odd function $f_o = f_P$ is defined by

$$\hat{f}_o(\xi) = \begin{cases} \frac{1}{\sqrt{2}} f_1(\xi) \text{ for } \xi \ge 0, \\ -\frac{1}{\sqrt{2}} f_1(-\xi) \text{ for } \xi < 0. \end{cases}$$

Estimates in the opposite direction of (8) in Theorem 6 are somewhat more complicated.

We will state the key estimate in this section. First let us define the complex unit vectors

$$\gamma(v) = \mathrm{e}^{-i[\frac{\pi v}{2} + \frac{\pi}{4}]}.$$

We have the following

Proposition 1. *Let f be a function in $L^2(\mathbb{R})$ whose Fourier transform $\hat{f}$ is equal to a function in $\mathscr{S}(\mathbb{R})$ on $(-\infty, -1]$ and on $[1, \infty)$*
Let $n \ge 1, k \ge 0, v = n/2 + k - 1$ and $\gamma(v)$ as above. Assume that $\hat{f}$ satisfies the symmetry

$$\gamma(v) \hat{f}(-r) = \overline{\gamma}(v) \hat{f}(r), \text{ for all } r > 0.$$

Let P be a solid spherical harmonic on $\mathbb{R}^n$ of degree k normalised so that $\|P\|_{L^2(\mathbf{S}^{n-1})} = 1$. Let f_P be the symmetric function on $\mathbb{R}^n$ defined by its Fourier transform

$$\hat{f}_P(\xi) = P(\xi') \hat{f}(|\xi|) |\xi|^{1/2-n/2} \text{ for } \xi = \xi'|\xi| \in \mathbb{R}^n.$$

Let E be any subset of the interval $[0,])$ containing 0. Then there is a constant C_v dependent only on v but independent of $a > 0, E$ and f such that

$$\alpha_n \|S_E^{*(n)} f_P\|_{L^2(\mathbb{R}^n)} = \alpha_1 \left(\int_0^\infty |S_E^{*(1)} \ f(x)|^2 \, dx \right)^{1/2} + R(f,v)$$

where the remainder term

$$|R(f,v)| \le C_v \|f\|_{L^2(\mathbb{R})}.$$

The remainder term $R(f,v)$ depend on the parameter $a > 0$ in the definition of S_t and the set E.

Proposition 1 follows directly from Proposition 2 with Corollary 3 and Proposition 3 which are stated and proved at the end of this section.

Using Proposition 1 it is easy to prove Theorems 6 and 7

Proofs of Theorem 6 and Theorem 7. When $2v_1 = 2v \mod (4)$ then $\gamma(v_1) = \pm\gamma v$. Define the function f by its Fourier transform

$$c\hat{f}(\xi) = \begin{cases} f_1(\xi), \xi > 0, \\ (\overline{\gamma}(v))^2 f_1(-\xi) = (\overline{\gamma}(v_1))^2 f_1(-\xi), \xi < 0. \end{cases} \tag{9}$$

By Proposition 1

$$\alpha_n \|S_E^{*(n)} f_P\|_{L^2(\mathbb{R}^n)} = \alpha_1 \left(\int_0^\infty |S_E^{*(1)} \ f(x)|^2 \, dx \right)^{1/2} + R(f,v),$$

$$\alpha_n \|S_E^{*(n_1)} f_P\|_{L^2(\mathbb{R}^{n_1})} = \alpha_1 \left(\int_0^\infty |S_E^{*(1)} \ f(x)|^2 \, dx \right)^{1/2} + R(f,v_1).$$

We conclude that

$$\begin{aligned} \left| \alpha_n \|S_E^{*(n)} f_P\|_{L^2(\mathbb{R}^n)} - \alpha_n \|S_E^{*(n_1)} f_P\|_{L^2(\mathbb{R}^{n_1})} \right| &= |R(f,v) - R(f,v_1)| \\ &\le |R(f,v)| + |R(f,v_1)| \le (C_v + C_{v_1}) \|f_1\|_{L^2(\mathbb{R}+)}. \end{aligned}$$

In the special case $v = v_1$ we get

$$\alpha_n \|S_E^{*(n)} f_P\|_{L^2(\mathbb{R}^n)} = \alpha_n \|S_E^{*(n_1)} f_P\|_{L^2(\mathbb{R}^{n_1})}.$$

This completes the proof of Theorem 7.

For the proof of Theorem 6 we let f be defined by (9). We will use following property of the Fourier transform on $L^2(\mathbb{R}^n)$ Let the operator $g \to g_-$ is defined by $g_-(x) = g(-x)$. This operator commutes with the Fourier transform and also with the inverse Fourier transform. We have

$$(g_-)\hat{} = (\hat{g})_- = \alpha_n \check{g} \text{ and } (g_-)\check{} = (\check{g})_- = (\alpha_n)^{-1} \hat{g}.$$

From this we get

$$|S_t^{(1)} f(x)|^2 = |S_t^{(1)} \check{f}_1(x) + \overline{\gamma}^2 S_t^{(1)} \check{f}_1(-x)|^2 + \leq 2(|S_t^{(1)} \check{f}_1(x)|^2 + |S_t^{(1)} \check{f}(-x)|^2) \text{ for } x > 0$$

and hence

$$|S_E^{*(1)} f(x)|^2 \leq 2(|S_E^{*(1)} \check{f}_1(x)|^2 + |S_E^{*(1)} \check{f}_1(-x)|^2) \text{ for } x > 0.$$

Integrating over the positive interval we get

$$\int_0^\infty |S_E^{*(1)} \ f(x)|^2 \, dx \leq 2\|S_E^{*(1)} \check{f}_1\|^2_{L^2(\mathbb{R})}.$$

From this and Proposition 1 we get the desired estimate in Theorem 6. □

Now we consider estimates in the opposite direction of the estimate in Theorem 6. First a lemma which follows directly from Proposition 1.

Lemma 6. *Let f be a function in $L^2(\mathbb{R})$ equal to a function in $\mathscr{S}(\mathbb{R})$ on $(-\infty, -1]$ and on $[1, \infty)$. Let $\hat{f}$ be the Fourier transform of f. Let f_P be the symmetric function on $\mathbb{R}^n$ defined by its Fourier transform*

$$\hat{f}_P(\xi) = P(\xi')\hat{f}(|\xi|)|\xi|^{1/2-n/2} \text{ for } \xi = \xi'|\xi| \in \mathbb{R}^n.$$

If $\hat{f}$ satisfy the symmetry $\gamma(v)\hat{f}(-r) = \overline{\gamma}(v)\hat{f}(r)$ for all $r > 0$, then

$$\alpha_1 \left(\int_0^\infty |S_E^{*(1)} f(x)|^2 \, dx \right)^{1/2} \leq \alpha_n \|S_E^{*(n)} f_P\|_{L^2(\mathbb{R}^n)} + C_{n,k}\|f\|^2_{L^2(\mathbb{R})}.$$

Proof of Lemma 6. From Proposition 1 we get

$$\alpha_1 \left(\int_0^\infty |S_E^{*(1)} \ f(x)|^2 \, dx \right)^{1/2} = \alpha_n \|S_E^{*(n)} f_P\|_{L^2(\mathbb{R}^n)} - R(f, v)$$

with the remainder term satisfying $|-R(f,v)| \leq C_v\|f\|_{L^2(R_+)}$. This completes the proof of the lemma. □

Next we will see how we can combine several estimates like those in Lemma 7. We have

Lemma 7. *Let f be a function in $L^2(\mathbb{R})$ equal to a function in $\mathscr{S}(\mathbb{R})$ on $(-\infty, -1]$ and on $[1, \infty)$. Let $\hat{f}$ be the Fourier transform of f. Let $n \geq 1, k \geq 0$ and $n_1 \geq 1, k_1 \geq 0$ and let P be a solid spherical harmonic on $\mathbb{R}^n$ of degree k and Q be a solid spherical harmonic on $\mathbb{R}^{n_1}$ of degree k_1 and normalised so that $\|P\|_{L^2(\mathbf{S}^{n-1})} = \|Q\|_{L^2(\mathbf{S}^{n_1-1})} = 1$.*
Let $\boldsymbol{f}_P$ be the symmetric function on $\mathbb{R}^n$ defined by its Fourier transform

$$\hat{\boldsymbol{f}}_P(\xi) = P(\xi') \left(-\overline{\gamma}(v_1)\hat{f}(|\xi|) + \gamma(v_1)\hat{f}(-|\xi|) \right) |\xi|^{1/2-n/2} \text{ for } \xi = \xi'|\xi| \in \mathbb{R}^n,$$

and let $\boldsymbol{f}_Q$ be the symmetric function on $\mathbb{R}^{n_1}$ defined by its Fourier transform

$$\hat{\boldsymbol{f}}_Q(\xi) = Q(\xi') \left(\overline{\gamma(v)}\hat{f}(|\xi|) - \gamma(v)\hat{f}(-|\xi|) \right) |\xi|^{1/2-n_1/2} \text{ for } \xi = \xi'|\xi| \in \mathbb{R}^{n_1},$$

where $v = k + n/2 - 1$ *and* $v_1 = k_1 + n_1/2 - 1$ *and assume that* $2v_1 \neq 2v \mod (4)$, *i.e.* $\gamma(v_1) \neq \gamma(v)$
Let E *be any subset of the interval* $[0,1)$. *Then there is a constant* $C_{n,k}$ *and a constant* C_{n_1,k_1} *not dependent of* $a > 0, E$ *and* f *such that*

$$c\alpha_1 \left(\int_0^\infty |S_E^{*(1)} f(x)|^2 \, dx \right)^{1/2} \leq \alpha_n \|S_E^{*(n)} \boldsymbol{f}_P\|_{L^2(\mathbb{R}^n)} + \alpha_{n_1} \|S_E^{*(n_1)} \boldsymbol{f}_Q\|_{L^2(\mathbb{R}^{n_1})} + (C_{n,k} + C_{n_1,k_1}) \|f\|_{L^2(\mathbb{R})},$$

where $c = 2|\sin \frac{\pi(v-v_1)}{2}| \in \{2, \sqrt{2}\}$.

As a special case of Lemma 6 we have

Corollary 1. *Assume* f *satisfy the symmetry*

$$\hat{f}(\xi) = \begin{cases} c_1 g(\xi) & \text{for } \xi > 0, \\ c_2 g(-\xi) & \text{for } \xi < 0, \end{cases} \quad \text{with } (c_1, c_2) \in \mathbb{C}^2, (c_1, c_2) \neq 0,$$

and set $c = 2|\sin \frac{\pi(v-v_1)}{2}| \in \{2, \sqrt{2}\}$, $c_1' = c_1 \overline{\gamma(v_1)} - c_2 \gamma(v_1)$ *and* $c_2' = c_1 \overline{\gamma(v)} - c_2 \gamma(v)$.
Then

$$c\alpha_1 \left(\int_0^\infty |S_E^{*(1)} f(x)|^2 \, dx \right)^{1/2} \leq |c_1'| \alpha_n \|S_E^{*(n)} f_P\|_{L^2(\mathbb{R}^n)} + |c_2'| \alpha_{n_1} \|S_E^{*(n_1)} f_Q\|_{L^2(\mathbb{R}^{n_1})} + (C_{n,k} + C_{n_1,k_1}) \|f\|_{L^2(\mathbb{R})},$$

where

$$\hat{f}_P(\xi) = P(\xi') g(|\xi|) |\xi|^{1/2 - n/2} \text{ for } \xi = \xi' |\xi| \in \mathbb{R}^n,$$

and

$$\hat{f}_Q(\xi) = Q(\xi') g(|\xi|) |\xi|^{1/2 - n/2} \text{ for } \xi = \xi' |x| \in \mathbb{R}^{n_1}.$$

We are mostly interested in the cases when (c_1, c_2) is $(1,1)$, $(1,-1)$, $(1,0)$ or $(0,1)$ which corresponds to even or odd functions or functions with Fourier transform supported on the positive or negative half-axis.

Proof of Corollary 1. The functions $\boldsymbol{f}_P$ and $\boldsymbol{f}_Q$ defined in the Lemma 7 have Fourier transforms

$$\hat{\boldsymbol{f}}_P(\xi) = c_1' P(\xi') g(|\xi|) |\xi|^{1/2 - n/2} \text{ for } \xi = \xi' |\xi| \in \mathbb{R}^n,$$

and

$$\hat{\boldsymbol{f}}_Q(\xi) = c_2' Q(\xi') g(|\xi|) |\xi|^{1/2 - n/2} \text{ for } \xi = \xi' |x| \in \mathbb{R}^{n_1},$$

for $c_1' = c_1 \overline{\gamma(v_1)} - c_2 \gamma(v_1)$ and $c_2' = c_1 \overline{\gamma(v)} - c_2 \gamma(v)$. The desired estimates follow directly from Lemma 7. □

Proof of Lemma 7. We will essentially only do elementary calculation in $\mathbb{C}^2$, and use the triangle inequality for L^2 norms. Some details are left to the reader.
Set

$$\boldsymbol{f}_{P,1}(r) = -\overline{\gamma}(v_1)\hat{f}(r) + \gamma(v_1)\hat{f}(-r) \text{ for } r > 0,$$

and

$$\boldsymbol{f}_{Q,1}(r) = \overline{\gamma}(v)\hat{f}(r) - \gamma(v)\hat{f}(-r) \text{ for } r > 0.$$

Then

$$\|\sup_{t\in E} \tilde{S}_t \boldsymbol{f}_{P,1}\|_{L^2(\mathbb{R}_+)} = \alpha_n \|S_E^{*(n)} \boldsymbol{f}_P\|_{L^2(\mathbb{R}^n)},$$

and

$$\|\sup_{t\in E} \tilde{S}_t \boldsymbol{f}_{Q,1}\|_{L^2(\mathbb{R}_+)} = \alpha_{n_1} \|S_E^{*(n_1)} \boldsymbol{f}_Q\|_{L^2(\mathbb{R}^{n_1})},$$

where $\tilde{S}_t$ is defined as in (10). Let the functions $f_{P,2}$ and $f_{Q,2}$ on $\mathbb{R}$ be defined by

$$\boldsymbol{f}_{P,2}(r) = \begin{cases} \gamma(v) f_{P,1}(r) & \text{for } r \geq 0, \\ \overline{\gamma}(v) f_{P,1}(-r) & \text{for } r < 0, \end{cases}$$

and

$$\boldsymbol{f}_{Q,2}(r) = \begin{cases} \gamma(v_1) f_{Q,1}(r) & \text{for } r \geq 0, \\ \overline{\gamma}(v_1) f_{Q,1}(-r) & \text{for } r < 0. \end{cases}$$

We have

$$\begin{aligned} \boldsymbol{f}_{P,2} + \boldsymbol{f}_{Q,2} &= \begin{cases} (-\overline{\gamma}(v_1)\gamma(v) + \overline{\gamma}(v)\gamma(v_1))f(r) + (\gamma(v_1)\gamma(v) - \gamma(v)\gamma(v_1))f(-r) \text{ for } r \geq 0, \\ (-\overline{\gamma}(v_1)\overline{\gamma}(v) + \overline{\gamma}(v)\overline{\gamma}(v_1))f(r) + (\gamma(v_1)\overline{\gamma}(v) - \gamma(v)\overline{\gamma}(v_1))f(-r) \text{ for } r < 0, \end{cases} \\ &= (\gamma(v_1)\overline{\gamma}(v) - \gamma(v)\overline{\gamma}(v_1))\hat{f}(r) \text{ for } r \in \mathbb{R}. \end{aligned}$$

By the assumption $2v_1 \neq 2v \mod (4)$, we have

$$\begin{aligned} |\gamma(v_1)\overline{\gamma}(v) - \gamma(v)\overline{\gamma}(v_1)| &= |\mathrm{e}^{-i[\frac{\pi v_1}{2}+\frac{\pi}{4}]}\mathrm{e}^{i[\frac{\pi v}{2}+\frac{\pi}{4}]} - \mathrm{e}^{-i[\frac{\pi v}{2}+\frac{\pi}{4}]}\mathrm{e}^{i[\frac{\pi v_1}{2}+\frac{\pi}{4}]}| \\ &= |\mathrm{e}^{-i\frac{\pi}{2}(v_1-v)} - \mathrm{e}^{i\frac{\pi}{2}(v_1-v)}| = |2\sin(\pi(v_1-v)/2)| = c \in \{2, \sqrt{2}\}. \end{aligned}$$

Lemma 7 now follows from Lemma 6 and Proposition 3. □

Next we observe that $S_t^{(1)} f(-x) = S_t^{(1)} f_-(x)$, where $\hat{f}_-(\xi) = \hat{f}(-\xi)$. Thus if f is an even or an odd function on $L^2(\mathbb{R}^n)$ then $S_E^{*(1)} f$ is even and we obtain

$$\|S_E^{*(1)} f\|_{L^2(\mathbb{R})} = \sqrt{2} \left(\int_0^\infty |S_E^{*(1)} f(x)|^2 \, dx \right)^{1/2}.$$

We also observe that any function f on $\mathbb{R}$ can be written as a sum of an even and an odd function $f = f_e + f_o$. Then also the Fourier transforms $\hat{f}_e$ and $\hat{f}_o$ are even respective odd and we have $\hat{f} = \hat{f}_e + \hat{f}_o$ and that $\|f\|_2^2 = \|f_e\|_2^2 + \|f_o\|_2^2$.
Using Corollary 1 of Lemma 7 we get the following estimates in different cases.

Theorem 8. *Let f be a function in $L^2(\mathbb{R})$ equal to a function in $\mathscr{S}(\mathbb{R})$ on $(-\infty, -1]$ and on $[1, \infty)$. Decompose f into even and odd functions $f = f_e + f_o$ with Fourier transforms $\hat{f}_e$ and $\hat{f}_o$ Let $n \geq 1, k \geq 0$ and $n_1 \geq 1, k_1 \geq 0$ and let P be a solid spherical harmonic on $\mathbb{R}^n$ of degree k and Q be a solid spherical harmonic on $\mathbb{R}^{n_1}$ of degree k_1 and normalised so that $\|P\|_{L^2(\mathbf{S}^{n-1})} = \|Q\|_{L^2(\mathbf{S}^{n_1-1})} = 1$. Define the symmetric functions*

$$\begin{aligned}
\hat{f}_{e,P}(\xi) &= \hat{f}_e(|\xi|)P(\xi')|\xi|^{1/2-n/2} \text{ for } \xi = \xi'|\xi| \in \mathbb{R}^n,\\
\hat{f}_{e,Q}(\xi) &= \hat{f}_e(|\xi|)Q(\xi')|\xi|^{1/2-n_1/2} \text{ for } \xi = \xi'|\xi| \in \mathbb{R}^{n_1},\\
\hat{f}_{o,P}(\xi) &= \hat{f}_o(|\xi|)P(\xi')|\xi|^{1/2-n/2} \text{ for } \xi = \xi'|\xi| \in \mathbb{R}^n,\\
\hat{f}_{o,Q}(\xi) &= \hat{f}_o(|\xi|)Q(\xi')|\xi|^{1/2-n_1/2} \text{ for } \xi = \xi'|\xi| \in \mathbb{R}^{n_1}.
\end{aligned}$$

Let $\gamma(v) = e^{-i[\frac{\pi v}{2} + \frac{\pi}{4}]}$, $v = k + n/2 - 1$ and $v_1 = k_1 + n_1/2 - 1$ and assume that $2v_1 \neq 2v \mod (4)$. Then we have in different cases.

Case 1. Assume $\overline{\gamma}(v)/\gamma(v) = 1$ and $\overline{\gamma}(v_1)/\gamma(v_1) = -1$.
Then

$$\begin{aligned}
\alpha_1\|S_E^{*(1)}f_e\|_{L^2(\mathbb{R})} &\leq \alpha_n\sqrt{2}\|S_E^{*(n)}f_{e,P}\|_{L^2(\mathbb{R}^n)} + C'_{n,k}\|f_e\|,\\
\alpha_1\|S_E^{*(1)}f_o\|_{L^2(\mathbb{R})} &\leq \alpha_{n_1}\sqrt{2}\|S_E^{*(n_1)}f_{o,Q}\|_{L^2(\mathbb{R}^{n_1})} + C'_{n_1,k_1}\|f_o\|_2,
\end{aligned}$$

and

$$\begin{aligned}
\alpha_1\|S_E^{*(1)}f\|_{L^2(\mathbb{R})} \leq \alpha_n\sqrt{2}\|S_E^{*(n)}f_{e,P}\|_{L^2(\mathbb{R}^n)} &+ \alpha_{n_1}\sqrt{2}\|S_E^{*(n_1)}f_{o,Q}\|_{L^2(\mathbb{R}^{n_1})}\\
&+ (C'_{n,k} + C'_{n_1,k_1})\|f\|_{L^2(\mathbb{R})}.
\end{aligned}$$

Case 2. Assume that $\overline{\gamma}(v)/\gamma(v) = i$ and $\overline{\gamma}(v_1)/\gamma(v_1) = -i$.
Then

$$\begin{aligned}
\alpha_1\|S_E^{*(1)}f_e\|_{L^2(\mathbb{R})} \leq \alpha_n\|S_E^{*(n)}f_{e,P}\|_{L^2(\mathbb{R}^n)} &+ \alpha_{n_1}\|S_E^{*(n_1)}f_{e,Q}\|_{L^2(\mathbb{R}^{n_1})}\\
&+ (C'_{n,k} + C'_{n_1,k_1})\|f_e\|_{L^2(\mathbb{R})},\\
\alpha_1\|S_E^{*(1)}f_o\|_{L^2(\mathbb{R})} \leq \alpha_n\|S_E^{*(n)}f_{o,P}\|_{L^2(\mathbb{R}^n)} &+ \alpha_{n_1}\|S_E^{*(n_1)}f_{o,Q}\|_{L^2(\mathbb{R}^{n_1})}\\
&+ (C'_{n,k} + C'_{n_1,k_1})\|f_o\|_{L^2(\mathbb{R})},
\end{aligned}$$

and

$$\begin{aligned}
\alpha_1\|S_E^{*(1)}f\|_{L^2(\mathbb{R})} \leq \alpha_n\|S_E^{*(n)}f_{e,P}\|_{L^2(\mathbb{R}^n)} &+ \alpha_{n_1}\|S_E^{*(n_1)}f_{e,Q}\|_{L^2(\mathbb{R}^{n_1})}\\
&+ \alpha_n\|S_E^{*(n)}f_{o,P}\|_{L^2(\mathbb{R}^n)} + \alpha_{n_1}\|S_E^{*(n_1)}f_{o,Q}\|_{L^2(\mathbb{R}^{n_1})}\\
&+ (C'_{n,k} + C'_{n_1,k_1})\|f\|_{L^2(\mathbb{R})}.
\end{aligned}$$

Case 3: Assume $\overline{\gamma}(v)/\gamma(v) = 1$ and $\overline{\gamma}(v_1)/\gamma(v_1) = i$.
Then

$$\|S_E^{*(1)} f_e\|_{L^2(\mathbb{R})} \le \alpha_n \sqrt{2} \|S_E^{*(n)} f_{e,P}\|_{L^2(\mathbb{R}^n)} + C_{n,k} \|f_e\|_{L^2(\mathbb{R})},$$
$$\|S_E^{*(1)} f_o\|_{L^2(\mathbb{R})} \le \alpha_n \sqrt{2} \|S_E^{*(n)} f_{o,P}\|_{L^2(\mathbb{R}^n)} + 2\alpha_{n_1} \|S_E^{*(n_1)} f_{o,Q}\|_{L^2(\mathbb{R}^{n_1})} + (C'_{n,k} + C'_{n_1,k_1}) \|f_o\|_{L^2(\mathbb{R})},$$

and

$$\|S_E^{*(1)} f\|_{L^2(\mathbb{R})} \le \alpha_n \sqrt{2} \|S_E^{*(n)} f_{e,P}\|_{L^2(\mathbb{R}^n)} + \alpha_n \sqrt{2} \|S_E^{*(n)} f_{o,P}\|_{L^2(\mathbb{R}^n)} + 2\alpha_{n_1} \|S_E^{*(n_1)} f_{o,Q}\|_{L^2(\mathbb{R}^{n_1})} + (C'_{n,k} + C'_{n_1,k_1}) \|f\|_{L^2(\mathbb{R})}.$$

We have similar statements with $\overline{\gamma}(v)/\gamma(v) = -1$ and / or $\overline{\gamma}(v_1)/\gamma(v_1) = -i$.

It remains to prove Proposition 1. For this we will use Proposition 2 and Proposition 3 below.
First we define the operator $\tilde{S}_t$ for $t > 0$ by

$$\tilde{S}_t g(r) = \int_0^\infty J_v(rs)(rs)^{1/2} g(s) \mathrm{e}^{its^a}\, ds \tag{10}$$

for $g \in L^2(\mathbb{R}_+)$. Here J_m is the Bessel function of order m with integer or half-integer $m > -1$. Then we have

Proposition 2. *Let f_1 be a function in $L^2(\mathbb{R}_+)$. Let P be a solid spherical harmonic on $\mathbb{R}^n$ of degree k normalised so that $\|P\|_{L^2(\mathbf{S}^{n-1})} = 1$. Let $\hat{f}$ be the Fourier transform of f and Let f_P be the symmetric function on $\mathbb{R}^n$ defined by its Fourier transform*

$$\hat{f}_P(\xi) = f_1(|\xi|) P(\xi') |\xi|^{1/2-n/2} \text{ for } \xi = \xi'|\xi| \in \mathbb{R}^n.$$

Let $\tilde{S}_t$ defined as in (10) with $v = n/2 + k - 1$. Then

$$S_t^{(n)} f_P(x) = c_{k,n} \alpha_n^{-1} |x|^{1/2-n/2} \tilde{S}_t f_1(|x|) P(-x') \text{ for } x = x'|x| \in \mathbb{R}^n,$$

with $|c_{k,n}| = 1$.

We get the corollaries.

Corollary 2. *$\tilde{S}_t$ has norm 1 on $L^2(\mathbb{R}^n)$.*

Corollary 3. *We have*

$$S_E^{*(n)} f_P(x) r = \alpha_n^{-1} r^{1/2-n/2} \tilde{S}_E^* f_1(r) |P(-x')| \text{ for } x = rx' \in \mathbb{R}^n$$

and

$$\alpha_n \|S_E^{*(n)} f_P\|_{L^2(\mathbb{R}^n)} = \|\tilde{S}_E^* f_1\|_{L^2(\mathbb{R}_+)}.$$

Proof of Corollary 2. We have

$$\|\tilde{S}_t f_1\|_{L^2(\mathbb{R}_+)} = \alpha_n \|S_t^{(n)} f_P\|_{L^2(\mathbb{R}^n)} = \|e^{it|\xi|^a} \hat{f}_P\|_{L^2(\mathbb{R}^n)} = \|\hat{f}_P\|_{L^2(\mathbb{R}^n)} = \|f_1\|_{L^2(\mathbb{R}_+)}. \quad \square$$

Proof of Corollary 3. First part follows directly, as we may take out the factor $\alpha_n^{-1} r^{1/2-n/2} |P(-x')|$, from the supremum on the left hand side. Second part is obtained by integration over $\mathbb{R}^n$. $\square$

We will use the following estimate for the operator $\tilde{S}_t$

Proposition 3. *Let $\tilde{S}_t$ be defined as in (10) with $v = n/2 + k - 1$ the and let f be a function in $L^2(R)$ Let f_1 be a function in $L^2(\mathbb{R}+)$ and let f be the function in $L^2(\mathbb{R})$ with Fourier transform*

$$\hat{f}(\xi) = \begin{cases} \gamma(v) f_1(\xi) \text{ for } \xi > 0, \\ \overline{\gamma}(v) f_1(-\xi) \text{ for } \xi < 0. \end{cases}$$

Then

$$\tilde{S}_t f_1(x) = \alpha_1 S_t^{(1)} f(x) + R_{t,v} f_1(x)$$

where the remainder terms satisfy

$$\left(\int_0^\infty \sup_{t \in E} |R_{t,v} f_1(x)|^2 \, dx \right)^{1/2} \leq C_v \|f_1\|_{L^2(\mathbb{R}_+)}.$$

It remains to prove the propositions in the section. Proposition 1 follows directly from Proposition 2 with Corollary 3 and Proposition 3

Proof of Proposition 1. Let $f_1(\xi) = \overline{\gamma(v)} \hat{f}(\xi)$ for $\xi > 0$. We have

$$\alpha^n \|S_E^{*(n)} f_P\|_{L^2(\mathbb{R}^n)} = \|\tilde{S}_E^* f_1\|_{L^2(\mathbb{R}_+)} \leq \alpha_1 \|S_E^{*(1)} f\|_{L^2(\mathbb{R}_+)} + \||R_{t,v} f_1\|_{L^2(\mathbb{R}_+)}.$$

Set $R(f, v) = \int_0^\infty \sup_{t \in E} |R_{t,v} f_1(x)|^2 \, dx$. Then

$$R(f, v) = \left(\int_0^\infty \sup_{t \in E} |R_{t,v} f_1(x)|^2 \, dx \right)^{1/2} \leq C_v \|f_1\|_{L^2(\mathbb{R}_+)} = \frac{\alpha_1}{\sqrt{(2)}} C_v \|f\|_{L^2(\mathbb{R})},$$

and we obtain the desired estimate. This completes the proof of Proposition 1. $\square$

Proof of Proposition 2. Assume $f = f_0 P$ as in the above theorem. It follows from Stein and Weiss [7], p. 158 that (when $\hat{f}(x) = F_0(x) P(x)$ where

$$F_0(x) = c_{n,k} \alpha_n r^{1-n/2-k} \int_0^\infty f_0(s) J_{n/2+k-1}(rs) s^{n/2+k} \, ds$$

for $r = |x|$. Here J_m denotes Bessel functions of order m, and $c_{n,k}$ is a constant with $|c_{n,k}| = 1$. (The one-dimensional cases follow elementary as

$J_{-1/2}(r)) = \sqrt{2}\cos(r)/\sqrt{\pi r}$ and $J_{1/2}(r)) = \sqrt{2}\sin(r)/\sqrt{\pi r}$.)
Also

$$S_t f(x) = \alpha_n^{-1} c_{n,k} r^{1-n/2-k} \left(\int_0^\infty J_{n/2+k-1}(rs) F_0(s) \mathrm{e}^{its^a} s^{n/2+k}\, ds \right) P(-x),$$

where $r = |x| > 0$. We set

$$f_1(|\xi|) = F_0(|\xi|)|\xi|^{-1/2+n/2+k},$$

then $\|f_1\|_{L^2(\mathbb{R}+)} = \|\hat{f}_p\|_{L^2(\mathbb{R}^n)}$.
We obtain

$$\begin{aligned} S_t f_P(x) &= \alpha_n^{-1} c_{n,k} r^{1/2-n/2} \left(\int_0^\infty J_{n/2+k-1}(rs) f_1(s) (rs)^{1/2} \mathrm{e}^{its^a}\, ds \right) P(-x/r|) \\ &= \alpha_n^{-1} c_{n,k} r^{1/2-n/2} \tilde{S}_t f_1(r) P(-x/r), \end{aligned}$$

where the operator $\tilde{S}_t$ is defined by (10).
This completes the proof of Proposition 2. □

For proof of Proposition 3 we need the following

Lemma 8. *Let $K(r)$ be a non-negative function on $(0,\infty)$ satisfying*

$$\int_0^\infty \frac{K(r)}{\sqrt{r}}\, dr = A < \infty,$$

and let

$$Tf(s) = \int_0^\infty K(rs) f(r)\, dr$$

for $f \in L^2(0,\infty)$.
Then we have

$$\|Tf\|_{L^2(\mathbb{R}_+)} \le A \|f\|_{L^2(\mathbb{R}_+)}.$$

Proof of Proposition 3. We have the Bessel function

$$J_v(r) = \sqrt{\frac{2}{\pi r}} \cos(r - \pi v/2 - \pi/4) + \mathscr{O}(r^{-3/2}) \text{ as } r \to \infty.$$

See Stein and Weiss [7]. p. 158. It follows that

$$r^{1/2} J_v(r) = \gamma_v \mathrm{e}^{ir} + \overline{\gamma}_v \mathrm{e}^{-ir} + K_v(r)$$

for $r > 0$, where

$$\gamma_v = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-i(\pi v/2+\pi/4)} = \alpha_1 \gamma(v) \frac{1}{2\pi},$$

and

$$|K_v(r)| \le C_v \frac{1}{1+r}.$$

We get

$$\begin{aligned}
\tilde{S}_t f_1(r) &= \gamma(v)\alpha_1 \frac{1}{2\pi} \int_0^\infty \mathrm{e}^{irs} \mathrm{e}^{its^a} f_1(s)\, ds + \overline{\gamma}(v)\alpha_1 \frac{1}{2\pi} \int_0^\infty \mathrm{e}^{-irs} \mathrm{e}^{its^a} f_1(s)\, ds \\
&\quad + \int_0^\infty K_v(rs) \mathrm{e}^{its^a} f_1(s)\, ds \\
&= \alpha_1 \frac{1}{2\pi} \int_0^\infty \mathrm{e}^{irs} \mathrm{e}^{it|s|^a} \gamma(v) f_1(s)\, ds + \alpha_1 \frac{1}{2\pi} \int_{-\infty}^0 \mathrm{e}^{irs} \mathrm{e}^{it|s|^a} \overline{\gamma}(v) f_1(-s)\, ds + R_{t,v} f_1(r) \\
&= \alpha_1 \frac{1}{2\pi} \int_{-\infty}^\infty \mathrm{e}^{irs} \mathrm{e}^{it|s|^a} \hat{f}(s)\, ds = \alpha_1 S_t^{(1)} f(r) + R_{v,t} f_1(r)
\end{aligned}$$

for $r > 0$.
Let $\tilde{R}_v$ be the sublinear operator on $L^2[0,\infty)$ defined by

$$\tilde{R}_v g(r) = \int_0^\infty |K_v(rs)|\, |g(s)|\, ds.$$

Then

$$\sup_{t\ge 0} |R_{v,t} f_1(r)| \le \tilde{R}_v f_1(r).$$

Since $\int_0^\infty |K_v(r)| r^{-1/2}\, dr \le C_v \int_0^\infty r^{-1/2}(1+r)^{-1/2}\, dr < \infty$ we obtain by Lemma 8

$$\left(\int_0^\infty \sup_{t>0} |R_{v,t} \hat{f}_1(r)|^2 \right)^{1/2} \le \left(\int_0^\infty (\tilde{R}_v \hat{f}_1(r))^2\, dr \right)^{1/2} \le C_v \|f_1\|_2,$$

which completes the proof of the Proposition 3. □

In this section it remains only to prove Lemma 8.

Proof of Lemma 8. We have

$$Tf(s) = \int_0^\infty K(u) f(\frac{u}{s}) \frac{1}{s}\, du, s > 0,$$

and we set

$$h(s) = f\left(\frac{1}{s}\right)\frac{1}{s} \qquad \text{and} \qquad h_r(s) = h\left(\frac{s}{r}\right)\frac{1}{\sqrt{r}} \qquad \text{for } r > 0 \text{ and } s > 0.$$

It then follows that

$$f\left(\frac{r}{s}\right)\frac{1}{s} = f\left(\frac{1}{s/r}\right)\frac{1}{s/r}\frac{1}{r} = h\left(\frac{s}{r}\right)\frac{1}{r} = h_r(s)\frac{1}{\sqrt{r}}.$$

We have

$$Tf(s) = \int_0^\infty K(r)h_r(s)\frac{1}{\sqrt{r}}\,dr,$$

and an application of Minkovski's inequality gives

$$\|Tf\|_2 \le \int_0^\infty K(r)\frac{1}{\sqrt{r}}\,\|h_r\|_2\,dr.$$

We observe that

$$\|h_r\|_{L^2(\mathbb{R}_+)} = \|h\|_2 = \|f\|_{L^2(\mathbb{R}_+)},$$

and get

$$\|Tf\|_{L^2(\mathbb{R}_+)} \le \int_0^\infty K(r)\frac{1}{\sqrt{r}}\,dr\,\|f\|_{L^2(\mathbb{R}_+)} = A\|f\|_{L^2(\mathbb{R}_+)}.$$

This completes the proof of Lemma 8. □

5. A counter-example

We shall give a counter-example in dimension $n \ge 2$.

Theorem 9. *Assume that $(t_k)_1^\infty$ is decreasing and $(t_k - t_{k+1})_1^\infty$ is decreasing and $\lim_{k\to\infty} t_k = 0$. Assume $a > 0, a \ne 1$, and $0 < s < a/4$, and $n \ge 2$. Set $r(s) = 2s/(a-4s)$ and assume that $(t_k)_1^\infty \notin l^{r(s),\infty}$. Then there is no estimate*

$$\|\sup_k |S_{t_k} f|\|_2 \lesssim \|f\|_{H_s}$$

for all radial function $f \in \mathscr{S}(\mathbb{R}^n)$.

Proof. In the case $n = 1$ this theorem is proved in Dimou and Seeger [1] (with the exception that their counter-example is not radial) and we shall modify their proof. Assuming that $(t_k)_1^\infty \notin l^{r(s),\infty}$ Dimou and Seeger first construct sequences $(b_j)_1^\infty$ and $(M_j)_1^\infty$ of positive numbers such that $\lim_{j\to\infty} b_j = 0$ and $\lim_{j\to\infty} M_j = \infty$. Taking $\epsilon < 10^{-1}(a+2)^{-1}$ they then set

$$\lambda_j = M_j^{\frac{2}{a}} b_j^{-\frac{1}{a-4s}} \text{ and } \rho_j = \epsilon b_j^{-1/2}\lambda_j^{1-a/2} = \epsilon M_j^{\frac{2-a}{a}} b_j^{-\frac{1-2s}{a-4s}}$$

for $j = 1, 2, 3, \dots$. We shall consider these numbers for j large and observe that $\rho_j/\lambda_j = \epsilon M_j^{-1} b^{\frac{2s}{a-4s}} \le \epsilon$. In [1] the function

$$\Phi_{\lambda,\rho}(\xi, x, t) = x(\rho\xi - \lambda) + t(\lambda - \rho\xi)^a, |\xi| \le 1/2, x \in \mathbb{R}, t > 0,$$

is studied and it is proved that for $x \in I_j = [0, a\lambda_j^{a-1} b_j/2]$ there exists $t_{k(x,j)}$ such that

$$\max_{|\xi|\le 1/2} \left| e^{i\Phi_{\lambda_j,\xi_j}(\xi,x,t_{k(x,j)})} - 1 \right| \le 1/2. \tag{11}$$

We shall use the inequality (11) in our proof and shall also use that

$$\rho_j \lambda_j^{a-1} b_j \to \infty \text{ as } j \to \infty. \tag{12}$$

To prove (12) observe that

$$\begin{aligned} \rho_j \lambda_j^{a-1} b_j &= \epsilon M_j^{\frac{2-a}{a}} b_j^{-\frac{1-2s}{a-4s}} M_j^{\frac{2(a-1)}{a}} b_j^{-\frac{a-1}{a-4s}} b_j \\ &= \epsilon M_j^{\frac{2-a+2a-2}{a}} b_j^{\frac{-1+2s-a+1+a-4s}{a-4s}} = \epsilon M_j b_j^{-\frac{2s}{a-4s}} \end{aligned}$$

which implies (12).
Then set $J_j = [a\lambda_j^{a-1} b_j/4, a\lambda_j^{a-1} b_j/2]$ and let C_1 be a large constant. It follows from (12) that $\lambda_j \lambda_j^{a-1} b_j \to \infty$ as $j \to \infty$ and hence

$$2C_1 \le a\lambda_j \lambda_j^{a-1} b_j/4$$

and

$$\frac{2C_1}{\lambda_j} \le a\lambda_j^{a-1} b_j/4$$

for large j. We conclude that

$$|x| \in J_j \text{ implies } \lambda_j |x| \ge 2C_1. \tag{13}$$

Now let σ denote the surface measure on the unit sphere in $\mathbb{R}^n$. We have

$$\hat{\sigma}(y) = c_1 \frac{e^{i|y|}}{|y|^{n/2-1/2}} + c_2 \frac{e^{-i|y|}}{|y|^{n/2-1/2}} + R(y), \tag{14}$$

where

$$|R(y)| \lesssim \frac{1}{|y|^{n/2+1/2}} \le \frac{\delta}{|y|^{n/2-1/2}} \text{ for } |y| \ge C_1 \tag{15}$$

and δ is small. (See Stein [6], p. 347).
Then assume that $g \in C_0^\infty(\mathbb{R})$, supp $g \subset [-1/2, 1/2]$, $g \ge 0$, $\int g\, dx = 1$, and g even. We define a function $f \in \mathscr{S}(\mathbb{R}^n)$ by setting

$$\hat{f}(\xi) = \frac{1}{\rho} g\left(\frac{|\xi| - \lambda}{\rho} \right) \text{ for } \xi \in \mathbb{R}^n.$$

Here $\lambda = \lambda_j$, $\rho = \rho_j$ and $f = f_j$.
It is easy to see that $\hat{f}(\xi) \neq 0$ implies $\lambda - \rho/2 \leq |\xi| \leq \lambda + \rho/2$. We also have

$$\int |\hat{f}(\xi)|^2 \lambda^{2s}\, d\xi \lesssim \int_{\lambda-\rho/2}^{\lambda+\rho/2} \rho^{-2}\lambda^{2s} r^{n-1}\, dr \lesssim \rho^{-1}\lambda^{2s+n-1},$$

and

$$\|f\|_{H_s} \lesssim \rho^{-1/2}\lambda^{s+n/2-1/2}.$$

Using polar coordinates we have

$$\begin{aligned} S_t f(x) &= c\int \mathrm{e}^{i\xi\cdot x}\mathrm{e}^{it|\xi|^a}\hat{f}(\xi)\, d\xi = c\int \mathrm{e}^{i\xi\cdot x}\mathrm{e}^{it|\xi|^a}\frac{1}{\rho}\, g\left(\frac{|\xi|-\lambda}{\rho}\right) d\xi \\ &= c\int_0^\infty \frac{1}{\rho}\, g\left(\frac{r-\lambda}{\rho}\right)\mathrm{e}^{itr^a}\left(\int_{S^{n-1}} \mathrm{e}^{ir\xi'\cdot x}\, d\sigma(\xi')\right) r^{n-1} dr \\ &= c\int_0^\infty \frac{1}{\rho}\, g\left(\frac{r-\lambda}{\rho}\right)\mathrm{e}^{itr^a}\hat{\sigma}(rx) r^{n-1} dr \end{aligned}$$

where by (14)

$$\hat{\sigma}(rx) = c_1\frac{\mathrm{e}^{ir|x|}}{(r|x|)^{n/2-1/2}} + c_2\frac{\mathrm{e}^{-ir|x|}}{(r|x|)^{n/2-1/2}} + R(rx).$$

We assume $|x| \in J_j$ and (13) gives $\lambda|x| \geq 2C_1$ and $r|x| \geq C_1$ in the above integral. Hence by (15)

$$|R(rx)| \leq \delta(r|x|)^{-n/2+1/2}.$$

It follows that

$$\begin{aligned} S_t f(x) &= c_1 c\int_0^\infty \frac{1}{\rho}\, g\left(\frac{r-\lambda}{\rho}\right)\mathrm{e}^{itr^a}\frac{\mathrm{e}^{ir|x|}}{(r|x|)^{n/2-1/2}} r^{n-1}\, dr \\ &\quad + c_2 c\int_0^\infty \frac{1}{\rho}\, g\left(\frac{r-\lambda}{\rho}\right)\mathrm{e}^{itr^a}\frac{\mathrm{e}^{-ir|x|}}{(r|x|)^{n/2-1/2}} r^{n-1}\, dr + c\int_0^\infty \frac{1}{\rho}\, g\left(\frac{r-\lambda}{\rho}\right)\mathrm{e}^{itr^a} R(rx) r^{n-1}\, dr. \end{aligned}$$

Setting $\xi = (r-\lambda)/\rho$ so that $r = \lambda + \rho\xi$ we obtain

$$S_t f(x) = c_1 S_t^1 f(x) + c_2 S_t^2 f(x) + S_t^3 f(x)$$

where

$$\begin{aligned} S_t^1 f(x) &= \left(c\int g(\xi)\mathrm{e}^{i[t(\lambda+\rho\xi)^a + |x|(\lambda+\rho\xi)]}(\lambda+\rho\xi)^{n/2-1/2}\, d\xi\right)|x|^{1/2-n/2}, \\ S_t^2 f(x) &= \left(c\int g(\xi)\mathrm{e}^{i[t(\lambda+\rho\xi)^a - |x|(\lambda+\rho\xi)]}(\lambda+\rho\xi)^{n/2-1/2}\, d\xi\right)|x|^{1/2-n/2}, \end{aligned}$$

and

$$|S_t^3 f(x)| \lesssim \delta \int (\lambda+\rho\xi)^{n/2-1/2} g(\xi)\, d\xi\, |x|^{1/2-n/2} \le C\delta\lambda^{n/2-1/2}|x|^{1/2-n/2}$$

In $S_t^1 f(x)$ and $S_t^2 f(x)$ we can replace ξ by $-\xi$. We use that g is even and get the phase functions

$$\Phi_1(\xi) = |x|(\lambda-\rho\xi)] + t(\lambda-\rho\xi)^a,$$

and

$$\Phi_2(\xi) = -|x|(\lambda-\rho\xi)] + t(\lambda-\rho\xi)^a,$$

and replace $(\lambda+\rho\xi)^{n/2+1/2}$ by $(\lambda-\rho\xi)^{n/2-1/2}$.
We have

$$\Phi_2(\xi) = |x|(\rho\xi-\lambda)] + t(\lambda-\rho\xi)^a = \Phi_{\lambda,\rho}(\xi,|x|,t) = \Phi(\xi),$$

and we also have

$$S_t^2 f(x) = \left(\int \mathrm{e}^{i\Phi(\xi)}\Lambda(\xi) g(\xi) d\xi\right) |x|^{1/2-n/2}$$

where $\Lambda(\xi) = c(\lambda-\rho\xi)^{n/2-1/2}$. Choosing $t = t_{k(|x|),j}$ we obtain

$$|x|^{n/2-1/2}|S^2_{t_{k(|x|),j}} f(x)| \ge \int g\,\Lambda\, d\xi - \int |\mathrm{e}^{i\Phi}-1|\, g\,\Lambda\, d\xi$$
$$\ge \int g\,\Lambda\, d\xi - \max_{|\xi|\le 1/2} |\mathrm{e}^{i\Phi(\xi)}-1| \int g\,\Lambda\, d\xi \ge \frac{1}{2}\int g\,\Lambda\, d\xi$$

for $|x| \in J_j$ since $\max\limits_{|\xi|\le 1/2} |\mathrm{e}^{i\Phi(\xi)}-1| \le \frac{1}{2}$ according to inequality (11).
It follows that

$$\sup_k |S^2_{t_k} f(x)| \ge \frac{1}{2}\int g\,\Lambda\, d\xi\, |x|^{1/2-n/2} \ge c\lambda^{n/2-1/2}|x|^{1/2-n/2}$$

for $|x| \in J_j$. It remains to study $S_t^1 f(x)$. We set $h = g\Lambda$ an then have $h \lesssim \lambda^{n/2-1/2}$ and $|h'| \lesssim (\lambda+\rho)\lambda^{n/2-3/2} \le 2\lambda^{n/2-1/2}$. Integrating by parts we obtain

$$|x|^{n/2-1/2} S_t^1 f(x) = \int \mathrm{e}^{i\Phi_1} h\, d\xi = \int \mathrm{e}^{i\Phi_1} i\Phi_1' \frac{1}{i\Phi_1'} h\, d\xi$$
$$= -\int \mathrm{e}^{i\Phi_1} \frac{1}{i}\left(\frac{1}{\Phi_1'}h' - \frac{\Phi_1''}{(\Phi')^2}h\right) d\xi.$$

We have

$$\Phi_1' = -|x|\rho - \rho a t(\lambda-\rho\xi)^{a-1}$$

and

$$\Phi_1'' = ta(a-1)\rho^2(\lambda-\rho\xi)^{a-2}$$

and it follows that $|\Phi_1'| \geq \rho[x]$ and $|\Phi_1'| \geq \rho a t(\lambda - \rho\xi)^{a-1}$. We have

$$\frac{1}{|\Phi_1'|} \leq \frac{1}{\rho|x|},$$

and

$$\frac{|\Phi_1''|}{|\Phi'|^2} = \frac{1}{|\Phi_1'|}\frac{|\Phi_1''|}{|\Phi_1'|} \leq \frac{1}{\rho|x|}\frac{a|a-1|t\rho^2(\lambda-\rho\xi)^{a-2}}{at(\lambda-\rho\xi)^{a-1}} \lesssim \frac{1}{\rho|x|}\frac{\rho}{\lambda-\rho\xi} \lesssim \frac{1}{\rho|x|}.$$

It follows that

$$|x|^{n/2-1/2}|S_t^1 f(x)| \lesssim \frac{1}{\rho|x|}\int (|h| + |h'|)\, d\xi \lesssim \frac{1}{\rho|x|}\lambda^{n/2-1/2},$$

and if $|x| \in J_j$ we get

$$|S_t^1 f(x)| \lesssim |x|^{1/2-n/2}\frac{1}{\rho_j \lambda_j^{a-1} b_j}\lambda^{n/2-1/2} \lesssim \delta\lambda^{n/2-1/2}|x|^{1/2-n/2},$$

where we have used (12). Hence we have

$$S^* f(x) = \sup_k |S_{t_k} f(x)| \geq c\lambda^{n/2-1/2}|x|^{1/2-n/2}$$

for $|x| \in J_j$.
The theorem will follow if we show that with $f = f_j$ we have

$$\frac{\|S^* f_j\|_2}{\|f_j\|_{H_s}} \to \infty \text{ as } j \to \infty.$$

We have

$$\int_{R^n} |S^* f(x)|^2\, dx \geq \int_{|x|\in J_j} |S^* f(x)|^2|\, dx \gtrsim \int_{|x|\in J_j} \lambda^{n-1}|x|^{1-n}\, dx \geq |I_j|\lambda^{n-1},$$

and

$$\|f\|_{H_s}^2 \lesssim \rho^{-1}\lambda^{2s+n-1}.$$

With $f = f_j$ we get

$$\left(\frac{\|S^* f_j\|_2}{\|f_j\|_{H_s}}\right)^2 \gtrsim \frac{\lambda^{n-1}|I_j|}{\rho_j^{-1}\lambda_j^{2s+n-1}} = \rho_j \lambda_j^{-2s}|I_j|.$$

We have $|I_j| = a\lambda_j^{a-1}b_j/2$ and obtain

$$\begin{aligned}\left(\frac{\|S^* f_j\|_2}{\|f_j\|_{H_s}}\right)^2 &\gtrsim \rho_j\lambda_j^{-2s}\lambda_j^{a-1}b_j = \rho_j\lambda_j^{a-1-2s}b_j \\ &= \epsilon M_j^{\frac{2-a}{a}} b_j^{-\frac{1-2s}{a-4s}}\left(M_j^{\frac{2}{a}} b_j^{-\frac{1}{a-4s}}\right)^{a-1-2s} b_j \\ &= \epsilon M_j^{\frac{a-4s}{a}} b_j^{\frac{-1+2s-a+1+2s+a-4s}{a-4}} = \epsilon M_j^{\frac{a-4s}{a}}.\end{aligned}$$

Since $a - 4s > 0$ and $M_j \to \infty$ as $j \to \infty$ we conclude that

$$\frac{\|S^* f_j\|_2}{\|f_j\|_{H_s}} \to \infty \text{ as } j \to \infty.$$

This completes the proof of theorem. □

Now let $a > 0, a \neq 1, 0 < s < a/4$ and $r = 2s/(a - 4s)$. Also let $(t_m)_1^\infty$ satisfy (2) and let $(t_m - t_{m+1})_1^\infty$ be decreasing.
It is proved in Dimou and Seeger [1] that in the case $n = 1$ one has

$$\|\sup |S_{t_m} f|\|_2 \lesssim \|f\|_{H_s},\ f \in \mathscr{S}(\mathbb{R})$$

if $(t_m)_1^\infty \in l^{r,\infty}$.
It then follows from Theorems 5 and 9 that in the case $n \geq 2$ one has

$$\|\sup_m |S_{t_m} f|\|_2 \lesssim \|f\|_{H_s}$$

for all radial functions f in $\mathscr{S}(\mathbb{R}^n)$, if and only if $(t_m)_1^\infty \in l^{r,\infty}$.

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