Journal of Mathematical Analysis and Applications **262,** 733–748 (2001)
doi:10.1006/jmaa.2001.7615, available online at http://www.idealibrary.com on IDEAL®

On C. Neumann's Method for Second-Order Elliptic Systems in Domains with Non-smooth Boundaries

O. Steinbach and W. L. Wendland

Mathematisches Institut A, Universität Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany
E-mail: {steinbach,wendland}@mathematik.uni-stuttgart.de

Submitted by William F. Ames

Received April 17, 2001

DEDICATED TO GEORGE LEITMANN ON THE OCCASION OF HIS 75TH BIRTHDAY

In this paper we investigate the convergence of Carl Neumann's method for the solution of Dirichlet or Neumann boundary values for second-order elliptic problems in domains with non-smooth boundaries. We prove that $\frac{1}{2}I + K$, where K is the double-layer potential, is a contraction in $H^{1/2}(\Gamma)$ when an energy norm is used that is induced by the inverse of the single-layer potential.

Key Words: boundary integral equations; Neumann series.

1. INTRODUCTION

Carl Neumann's famous method in potential theory goes back to C. F. Gauss [10], who proposed to solve the Dirichlet problem for the Laplacian in a sufficiently smoothly bounded domain Ω by using a double-layer potential of the form

$$u(x) = (Wv)(x) := \int_\Gamma \left(\frac{\partial}{\partial n_y} U^*(x, y) \right) v(y) ds_y$$

0022-247X/01 $35.00

with a double-layer charge v to be determined as the solution of the boundary integral equation

$$v = \left(\frac{1}{2}I + K\right)v - g, \quad \text{where } (Kv)(x) = \int_{\Gamma\setminus\{x\}}\left(\frac{\partial}{\partial n_y}U^*(x, y)\right)v(y)\,ds_y.$$

$U^*(x, y)$ is the full-space fundamental solution of the Laplacian, and $\Gamma := \partial\Omega$. Carl Neumann showed in his papers [26, 27] that, for a convex domain Ω, the operator

$$\left(\tfrac{1}{2}I + K\right)$$

is on the space of continuous functions strictly contracting with respect to the oscillation which, in fact, is equivalent to

$$\|\tfrac{1}{2}I + K\| < 1, \tag{1.1}$$

with the operator norm generated by the norm

$$\|u\| := \sup_{x, y\in\Gamma}|u(x) - u(y)| + \alpha\sup_{x\in\Gamma}|u(x)|, \tag{1.2}$$

with an appropriate positive constant α, on the Banach space $C^0(\Gamma)$ of continuous functions on Γ (see [16]). In C. Neumann's proof, completed in 1877 (not recognized in [16, 35]), only specific piecewise plane boundaries were excluded. For smooth (including also non-convex) boundaries, J. Plemelj [29] showed for $(\frac{1}{2}I + K)$ that its spectral radius on $C^0(\Gamma)$ is less than 1, and this result was extended by J. Radon to two-dimensional domains having boundaries of bounded rotation [30]; J. Kral finally found [18] necessary and sufficient conditions on the boundary curve. On $L_2(\Gamma)$, Shelepov [37] obtained a similar result under more restrictive assumptions.

For three-dimensional domains and for smooth boundaries, e.g., a closed Lyapounov surface Γ, (1.1) and Plemelj's proof remain valid, as do C. Neumann's result and (1.1), (1.2) for convex Ω [16]. For non-smooth and non-convex boundaries and the Laplacian, one finds sufficient but rather restrictive conditions on the geometry of Γ for (1.1) to hold with respect to the essential norm associated with $C^0(\Gamma)$ (i.e., K is taken modulo compact perturbations) in the work by J. Kral [17] and by V. Maz'ya *et al.* in [1] (for the current state see Maz'ya's survey in [22]) and in [41]. J. Kral and D. Medkova showed [23] that the essential Fredholm radius defined by I. Gohberg and A. Marcus [11] is here equivalent to finding an appropriate weighted supremum norm on $C^0(\Gamma)$. For arbitrary polyhedral domains and the Laplacian, O. Hansen shows [13] how to construct the appropriate weight function. If $L_2(\Gamma)$ is used then again (1.1) is valid in the spectral norm, as has been shown by J. Elschner [8] for a restricted class of boundaries Γ.

Carl Neumann's iteration still converges in one of the Banach spaces on Γ if the corresponding spectral radius of the operator $(\frac{1}{2}I + K)$ is less than 1. This property not only holds for the double-layer potential of the Laplacian but also for the double-layer potential of the Lamé system governing linearized elasticity if Γ is sufficiently regular. For a Lyapunov boundary Γ and the space of Hölder continuous boundary functions these estimates for the spectrum of $(\frac{1}{2}I + K)$ for the Lamé system go back to Mikhlin [24] and have further been elaborated in [19, 28, 32]. For boundary functions in L_p, corresponding results are due to Shelepov [37] for two-dimensional problems and, for three-dimensional problems, are due to Maz'ya [21] and B. Dahlberg [6].

For a Lipschitz boundary Γ and the Laplacian, G. Verchota in [39] and, with the Lamé system, B. Dahlberg, C. Kenig, and G. Verchota in [7] showed that $\frac{1}{2}I - K$ has closed range and is invertible on $L_2(\Gamma)$. However, these results do not provide that the spectral radius is less than 1.

In this paper we employ coerciveness properties of the single layer and the hypersingular boundary integral operators which are still valid on a Lipschitz boundary Γ and which can be shown for the rather big class of second-order formally positive elliptic systems of partial differential equations [25, 34, 40], based on results by M. Costabel [3]. Then the classical Neumann boundary integral equations of the second kind need to be considered in the appropriate boundary trace spaces $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$, respectively. Since the Calderon projection holds even on a Lipschitz boundary Γ, we can show the strict contraction properties for $(\frac{1}{2}I \pm K)$ and for the adjoint operators on appropriate subspaces of these trace spaces by using appropriate norms.

Consequently, Carl Neumann's classical iteration method, originally invented for solving the Dirichlet and Neumann problems of the Laplacian, converges for an astonishingly large class of problems. The applications are numerous, e.g., in domain decomposition with boundary elements as in [15] and for preconditioning as in [38].

The paper is organized as follows. In Section 2 we introduce the boundary integral operators of the first and second kind and appropriate norms on the trace spaces on Γ. In Section 3 we formulate Carl Neumann's method and the main results of the paper concerning the strict contracting properties. In Section 4 we comment on the so-called direct approach for boundary integral equations and apply appropriate Neumann series. In Section 5 we present the proofs of the main results (which are formulated in Section 3).

The authors gratefully acknowledge the substantial support by the collaborative research centre "Sonderforschungsbereich 404, Mehrfeldprobleme" of the German Research Foundation Deutsche Forschungsgemeinschaft at the University of Stuttgart. The second author also expresses his sincere gratitude to the Mittag–Leffler Institute of the Royal Academy of Sciences, Sweden, where he carried out some part of this work.

2. BOUNDARY INTEGRAL OPERATORS

Let $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ be a bounded domain with Lipschitz continuous boundary $\Gamma = \partial\Omega$. We consider a second-order, self-adjoint, formally positive elliptic partial differential operator L [25, 34, 40] given by

$$Lu(x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} a_{ji}(x) \frac{\partial}{\partial x_i} u(x) + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} u(x) + c(x)u(x) \tag{2.1}$$

and the corresponding conormal derivative operator T_x defined as

$$T_x u(x) = \sum_{i,j=1}^{n} n_j(x) a_{ji}(x) \frac{\partial}{\partial x_i} u(x), \tag{2.2}$$

where $n(x)$ is the outer normal vector defined for almost every $x \in \Gamma$. Then there exists a fundamental solution $U^*(x, y)$ of L. The single-layer potential in the domain Ω and Ω^c, respectively, is defined by

$$(Vw)(x) = \int_\Gamma U^*(x, y) w(y)\, ds_y \qquad \text{for } x \in \Omega \cup \Omega^c. \tag{2.3}$$

For $x \in \Gamma$ we define the standard boundary integral operators, in particular, the single-layer potential,

$$(Vw)(x) = \lim_{x' \to x} \int_\Gamma U^*(x', y) w(y)\, ds_y \qquad \text{for almost every } x \in \Gamma, \tag{2.4}$$

where $x' \in C_x(\Omega)$ and where $C_x(\Omega) \subset \Omega$ denotes any cone with vertex $x \in \Gamma$ associated with the Lipschitz boundary Γ since it has the uniform cone property (see, e.g., [12, Theorem 1.2.2.2]). In (2.4) we also let $x' \in C_x(\Omega^c)$. We will also consider the double-layer potential,

$$(Wv)(x) := \int_\Gamma \left(T_y U^*(x, y)\right)^\top v(y)\, ds_y \qquad \text{for } x \in \Omega \cup \Omega^c.$$

For appropriate v, the corresponding double-layer boundary integral operator is defined in terms of its boundary values, namely by

$$(Ku)(x) := \lim_{x' \to x} (Wu)(x') + \tfrac{1}{2} u(x) \qquad \text{for } x \in \Gamma \text{ and } x' \in C_x(\Omega). \tag{2.5}$$

Correspondingly,

$$(Ku)(x) := \lim_{x'' \to x} (Wu)(x'') - \tfrac{1}{2}u(x) \qquad \text{if } x'' \in C_x(\Omega^c).$$

If Γ is locally Lyapunov in the vicinity of $x \in \Gamma$ then

$$(Ku)(x) = \int_{\Gamma\setminus\{x\}} \big(T_y U^*(x,y)\big)^\top u(y)\,ds_y, \tag{2.6}$$

where the integral is defined as a Cauchy principal value integral. (If the principal part of Γ is the Laplacian then the kernel function of the integral operator in (2.6) is only weakly singular.) If x is a vertex point of Γ and the tangential cone to Γ at x approximates Γ in the vicinity of x "well enough," then, for continuous u,

$$(Ku)(x) = \big[\sigma(x) - \tfrac{1}{2}\big]u(x) + \int_{\Gamma\setminus\{x\}} \big(T_y U^*(x,y)\big)^\top u(y)\,ds_y,$$

and $\sigma(x)$ is given explicitly (see, e.g., [41] for the Laplacian and [9] for the Lamé system). For a Lipschitz boundary, (2.6) holds for almost every $x \in \Gamma$ only. The operator adjoint to K with respect to the duality

$$\langle v, w\rangle_{L_2(\Gamma)} := \int_\Gamma v(y)\cdot w(y)\,ds_y,$$

on appropriate pairs of function spaces on Γ, is given in terms of the boundary values of the conormal derivative of the single-layer potential (2.3), namely by

$$(K'w)(x) := \lim_{x' \to x} \sum_{i,j=1}^{n} n_j(x) a_{ji}(x) \frac{\partial}{\partial x_i'} (Vw)(x') - \tfrac{1}{2}w(x), \tag{2.7}$$

where $x' \in C_x(\Omega)$. Again, if Γ is locally Lyapunov in the vicinity of $x \in \Gamma$, then

$$(K'w)(x) = \int_{\Gamma\setminus\{x\}} T_x U^*(x,y) w(y)\,ds_y, \tag{2.8}$$

and the integral is defined as a Cauchy principal value integral. For a Lipschitz boundary, (2.8) holds for almost every $x \in \Gamma$.

The hypersingular operator is defined by

$$(Du)(x) := -T_x(Wu)(x) = -\lim_{x' \to x} \sum_{i,j=1}^{n} n_j(x) a_{ji}(x) \frac{\partial}{\partial x_i'} (Wu)(x') \tag{2.9}$$

for $x \in \Gamma$, where $x' \in C_x(\Omega)$ or $x' \in C_x(\Omega^c)$. If Γ is locally Lyapunov in the vicinity of $x \in \Gamma$ and u is smooth enough, then

$$(Du)(x) = -\text{p.f.} \int_{\Gamma \setminus \{x\}} T_x\big(T_y U^*(x,y)\big)^\top u(y)\, ds_y \tag{2.10}$$

for $x \in \Gamma$, where now the integral is defined as Hadamard's finite part integral with respect to the Euclidean distance $|x - y|$ (see [36]).

All of these operators as defined above will be considered here on the Sobolev spaces $H^s(\Gamma)$ on Γ for $|s| \le 1$. The mapping properties of these boundary integral operators are well known (see, for example, [3, 5]), in particular, the boundary integral operators

$$\begin{aligned} V\colon H^{-1/2+s}(\Gamma) &\to H^{1/2+s}(\Gamma), \\ K\colon H^{1/2+s}(\Gamma) &\to H^{1/2+s}(\Gamma), \\ K'\colon H^{-1/2+s}(\Gamma) &\to H^{-1/2+s}(\Gamma), \\ D\colon H^{1/2+s}(\Gamma) &\to H^{-1/2+s}(\Gamma) \end{aligned}$$

are bounded for $|s| < \frac{1}{2}$ (see [3]). For the Laplacian and for the Lamé system, one still has continuity of K and K' even in $L_2(\Gamma)$ due to the results in [39] and [6]. Furthermore, we assume that the single-layer potential in (2.4) is $H^{-1/2}(\Gamma)$ elliptic, i.e., there exists a constant $c_1^V > 0$ such that

$$\langle Vw, w\rangle_{L_2(\Gamma)} \ge c_1^V \cdot \|w\|^2_{H^{-1/2}(\Gamma)} \qquad \text{for all } w \in H^{-1/2}(\Gamma). \tag{2.11}$$

Remark 2.1. Assumption (2.11) is known for a wide class of partial differential operators L (see [5]). However, in the case $n = 2$, appropriate scaling conditions are needed (see [14] for the Laplacian and [4] for the Bilaplacian; then Airy's stress function yields corresponding scaling in 2D elasticity).

We denote by

$$\mathscr{R} := \left\{ v \in H^{1/2}(\Gamma) : v = \tilde{v}_{|\Gamma}, \tilde{v} \in H^1(\Omega) : L\tilde{v} = 0 \text{ in } \Omega, T\tilde{v} = 0 \text{ on } \Gamma \right\} \tag{2.12}$$

the finite-dimensional solution space of the homogeneous Neumann boundary value problem. Note that

$$Dv = \left(\tfrac{1}{2}I + K\right)v = 0 \qquad \text{for all } v \in \mathscr{R}. \tag{2.13}$$

Let

$$\begin{aligned} H_0^{-1/2}(\Gamma) &:= \left\{ w \in H^{-1/2}(\Gamma) : \langle w, v\rangle_{L_2(\Gamma)} = 0 \text{ for all } v \in \mathscr{R} \right\}, \\ H_0^{1/2}(\Gamma) &:= \left\{ u \in H^{1/2}(\Gamma) : \langle u, v\rangle_{L_2(\Gamma)} = 0 \text{ for all } v \in \mathscr{R} \right\}. \end{aligned}$$

As for V, we assume that the hypersingular integral operator D in (2.9) is $H_0^{1/2}(\Gamma)$-elliptic, i.e., there exists a constant $c_1^D > 0$ such that

$$\langle Du, u\rangle_{L_2(\Gamma)} \geq c_1^D \cdot \|u\|^2_{H^{1/2}(\Gamma)} \qquad \text{for all } u \in H_0^{1/2}(\Gamma). \tag{2.14}$$

Since the single-layer potential V: $H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is positive definite and bounded, we may define on $H^{-1/2}(\Gamma)$ an equivalent Sobolev norm induced by V, i.e.,

$$\|w\|_V := \sqrt{\langle Vw, w\rangle_{L_2(\Gamma)}} \qquad \text{for all } w \in H^{-1/2}(\Gamma). \tag{2.15}$$

Correspondingly, we may equip $H^{1/2}(\Gamma)$ with an equivalent norm induced by the inverse of the single-layer potential V^{-1}:

$$\|u\|_{V^{-1}} := \sqrt{\langle V^{-1}u, u\rangle_{L_2(\Gamma)}} \qquad \text{for all } u \in H^{1/2}(\Gamma). \tag{2.16}$$

These norms will be used to investigate the convergence of the Neumann series to be introduced in the next section.

3. CARL NEUMANN'S CLASSICAL METHOD

Due to C. F. Gauss [10], for the solution of the Dirichlet boundary value problem

$$Lu(x) = 0 \quad \text{for } x \in \Omega, \qquad u(x) = g(x) \quad \text{for } x \in \Gamma, \tag{3.1}$$

C. Neumann investigated in [26, 27] the double-layer potential ansatz

$$u(\tilde{x}) = (Wv)(\tilde{x}) \qquad \text{for } \tilde{x} \in \Omega \tag{3.2}$$

with an unknown density $v \in H^{1/2}(\Gamma)$ for its solution. With the boundary values of the double-layer potential W, given by (2.5), we get from (3.2) the boundary integral equation of the second kind,

$$\tfrac{1}{2}v(x) - (Kv)(x) = -g(x) \qquad \text{for all } x \in \Gamma, \tag{3.3}$$

for the yet unknown dipole boundary charge v. To solve this equation, C. Neumann in [26, 27] showed for continuous g and L the Laplacian that the series

$$v(x) = -\sum_{l=0}^{\infty} \Big(\big(\tfrac{1}{2}I + K\big)^l g\Big)(x) \tag{3.4}$$

converges uniformly on Γ provided Γ is convex. (This was also the birth of the well-known Neumann series.) We consider now (3.4) on the trace space $H^{1/2}(\Gamma)$ instead and can show the convergence of (3.4) under very weak conditions on Γ and for a rather large class of equations due to the following main result of our paper.

THEOREM 3.1. *Let L be a formally positive elliptic partial differential operator of second order with a fundamental solution $U^*(\cdot,\cdot)$ generating all of the boundary integral operators* (2.4)–(2.9). *Let the operators V and D satisfy the coerciveness estimates* (2.11) *and* (2.14). *Then*,

$$\|(\tfrac{1}{2}I + K)u\|_{V^{-1}} \leq c_K \cdot \|u\|_{V^{-1}} \qquad \textit{for all } u \in H^{1/2}(\Gamma) \tag{3.5}$$

and

$$\|(\tfrac{1}{2} + K')w\|_V \leq c_K \cdot \|w\|_V \qquad \textit{for all } w \in H^{-1/2}(\Gamma) \tag{3.6}$$

with the positive constant

$$c_K := \tfrac{1}{2} + \sqrt{\tfrac{1}{4} - c_1^V c_1^D} < 1. \tag{3.7}$$

Here, c_1^V is the ellipticity constant of the single-layer potential V in (2.11) *and c_1^D is the ellipticity constant of the hypersingular integral operator D in* (2.14).

For the Neumann boundary value problem, i.e., to find the solution u of

$$Lu(x) = 0 \qquad \text{for } x \in \Omega, \qquad T_x u(x) = f(x) \qquad \text{for } x \in \Gamma, \tag{3.8}$$

with given $f \in H_0^{-1/2}(\Gamma)$, where u is unique only modulo $\mathscr{R}$, C. Neumann employed the single-layer potential ansatz

$$u(\tilde{x}) = (Vw)(\tilde{x}) \qquad \text{for } \tilde{x} \in \Omega. \tag{3.9}$$

Due to (2.7), here the boundary condition implies the boundary integral equation

$$\tfrac{1}{2}w(x) + (K'w)(x) = f(x) \qquad \text{for } x \in \Gamma, \tag{3.10}$$

now for the yet unknown boundary charge w. Since K' is the operator adjoint to K, J. Radon investigated in [30, Eq. (3.10)], for the Laplacian in the space of bounded variation C^*, the space dual to the continuous functions on Γ. Here, we now consider (3.10) in $H_0^{-1/2}(\Gamma)$, which is dual to $H_0^{1/2}(\Gamma)$ with respect to $\langle \cdot, \cdot \rangle_{L^2(\Gamma)}$. Now, we may find the solution again in the form of the Neumann series

$$w(x) = \sum_{l=0}^{\infty} \left((\tfrac{1}{2}I - K')^l f \right)(x) \tag{3.11}$$

in $H_0^{-1/2}(\Gamma)$.

LEMMA 3.1. *The operator* $\frac{1}{2}I - K'$ *maps* $H_0^{-1/2}(\Gamma)$ *into itself boundedly.*

Proof. Let $w \in H_0^{-1/2}(\Gamma) \subset H^{-1/2}(\Gamma)$. Then the image $z := (\frac{1}{2}I - K')w$ belongs to $H^{-1/2}(\Gamma)$ and satisfies

$$\langle z, v\rangle_{L_2(\Gamma)} = \langle w, v\rangle_{L_2(\Gamma)} - \langle z, \left(\tfrac{1}{2}I + K\right)v\rangle_{L_2(\Gamma)} = 0 \qquad \text{for all } v \in \mathscr{R}$$

due to $w \in H_0^{-1/2}(\Gamma)$ and because of (2.13). Hence, $z \in H_0^{-1/2}(\Gamma)$. ∎

The following theorem ensures the convergence of (3.11) in $H_0^{-1/2}(\Gamma)$.

THEOREM 3.2. *Let* L *and* $U^*(\cdot,\cdot)$ *be as in Theorem* 3.1. *Then,*

$$\|\left(\tfrac{1}{2}I - K'\right)w\|_V \le c_K \cdot \|w\|_V \qquad \textit{for all } w \in H_0^{-1/2}(\Gamma) \tag{3.12}$$

with the positive constant c_K *given by* (3.7) *and*

$$(1 - c_K) \cdot \|u\|_{V^{-1}} \le \|\left(\tfrac{1}{2}I - K\right)u\|_{V^{-1}} \le c_K \cdot \|u\|_{V^{-1}} \qquad \textit{for all } u \in H_0^{1/2}(\Gamma). \tag{3.13}$$

Hence, this theorem guarantees the convergence of the Neumann series (3.11) in $H_0^{-1/2}(\Gamma)$. The proof of Theorem 3.2 will be given in Section 5.

4. BOUNDARY INTEGRAL EQUATIONS OF THE DIRECT APPROACH

In the previous section we considered boundary integral equations resulting from the classical single- or double-layer potential ansatz, respectively. Now, we consider corresponding boundary integral equations of the second kind arising from the so-called direct approach based on Green's representation formula. In general, the solution of our second-order boundary value problems in the interior is given by

$$u(\tilde{x}) = \int_\Gamma U^*(\tilde{x}, y)t(y)\,ds_y - \int_\Gamma \left(T_y U^*(\tilde{x}, y)\right)^\top u(y)\,ds_y \qquad \text{for } \tilde{x} \in \Omega. \tag{4.1}$$

Here, $t(x) = T_x u(x)$ for $x \in \Gamma$; and $u\,|_\Gamma$ and t are the Cauchy data of L in Ω. When the Dirichlet boundary value problem (3.1) is considered, then the Neumann datum $t(x)$ on Γ is unknown. Hence, for finding t we may consider either a boundary integral equation of the first kind (see, e.g., [14]), or a boundary integral equation of the second kind, i.e.,

$$\left(\tfrac{1}{2}I - K'\right)t(x) = (Dg)(x) \qquad \text{for } x \in \Gamma. \tag{4.2}$$

Again, we may use a Neumann series to compute t, the solution of (4.2), i.e.,

$$t(x) = \sum_{l=0}^{\infty} \Big(\big(\tfrac{1}{2} I + K' \big)^l Dg \Big)(x) \qquad \text{for } x \in \Gamma. \tag{4.3}$$

The second assertion (3.6) of Theorem 3.1 now ensures the convergence of the Neumann series (4.3) in the space $H^{-1/2}(\Gamma)$.

In the case of the Neumann boundary value problem (3.8), the Dirichlet datum $u(x)$ on Γ is unknown. Hence, the solution of the boundary integral equation

$$\big(\tfrac{1}{2} I + K \big) u(x) = (Vg)(x) \qquad \text{for } x \in \Gamma \tag{4.4}$$

will provide us with the desired boundary values of u.

Note that $u \in H^{1/2}(\Gamma)$ is unique only modulo $\mathscr{R}$ since (4.1) implies that

$$\big(\tfrac{1}{2} I + K \big) u_0 = 0 \qquad \text{for every } u_0 \in \mathscr{R}. \tag{4.5}$$

We therefore define the projection operator $P_{\mathscr{R}}$: $H^{1/2}(\Gamma) \to H_0^{1/2}(\Gamma)$ by the mapping $u \mapsto P_{\mathscr{R}} u$ via

$$\langle P_{\mathscr{R}} u, v \rangle_{L_2(\Gamma)} = 0 \qquad \text{for all } v \in \mathscr{R}. \tag{4.6}$$

Instead of (4.4), now we consider the uniquely solvable boundary integral equation

$$\big[I - P_{\mathscr{R}} \big(\tfrac{1}{2} I - K \big) \big] u = P_{\mathscr{R}} Vg, \tag{4.7}$$

to find $u \in H_0^{1/2}(\Gamma)$. The Neumann series for the solution of (4.7) is given by

$$u(x) = \sum_{l=0}^{\infty} \big[P_{\mathscr{R}} \big(\tfrac{1}{2} I - K \big) \big]^l P_{\mathscr{R}} (Vg)(x) \qquad \text{for } x \in \Gamma. \tag{4.8}$$

Because of Assertion (3.13), this Neumann series converges in $H_0^{1/2}(\Gamma)$.

5. PROOFS

The proofs are based on the properties of the Calderon projection operator associated with L in Ω. We write both boundary integral equations as a system resulting from the direct approach in the form of the Calderon projection, i.e.,

$$\begin{pmatrix} u \\ T_x u \end{pmatrix} = \begin{pmatrix} \frac{1}{2} I - K & V \\ D & \frac{1}{2} I + K' \end{pmatrix} \begin{pmatrix} u \\ T_x u \end{pmatrix}. \tag{5.1}$$

From (5.1) we derive the Dirichlet–Neumann map

$$(Tu)(x) := (Su)(x) \qquad \text{for } x \in \Gamma \tag{5.2}$$

in the form of the Steklov–Poincaré operator S given by

$$S = V^{-1}\left(\tfrac{1}{2}I + K\right) = D + \left(\tfrac{1}{2}I + K'\right)V^{-1}\left(\tfrac{1}{2}I + K\right). \tag{5.3}$$

Now we are in a position to formulate some results based on the projection property of the Calderon projection (5.1) as well as on the mapping properties of all of the boundary integral operators in (2.4)–(2.10):

PROPOSITION 5.1. *The double-layer potential operator K as defined in* (2.5), (2.6) *can be symmetrized*; *in particular*,

$$KV = VK', \quad DK = K'D \qquad \text{and} \qquad VD = \tfrac{1}{4}I - K^2, \quad DV = \tfrac{1}{4}I - K'^2. \tag{5.4}$$

Proof. The relations in (5.4) are a direct consequence of the projection property of (5.1). In the special case of the Laplacian in a smoothly bounded two-dimensional domain, this result was already given by J. Plemelj [29] and for boundaries of bounded rotation by J. Radon [30]. For the Laplacian and boundaries with bounded cyclic variations, the first relation in (5.4) was shown by J. Kral (see [17]). ∎

PROPOSITION 5.2. *Let V: $H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ satisfy* (2.11). *Then there holds*

$$c_1^V \cdot \langle V^{-1}u, u\rangle_{L_2(\Gamma)} \le \|u\|^2_{H^{1/2}(\Gamma)} \qquad \textit{for all } u \in H^{1/2}(\Gamma) \tag{5.5}$$

with the same positive constant c_1^V as in (2.11).

Proof. For $w \in H^{-1/2}(\Gamma)$, $w \neq 0$, by duality,

$$\begin{aligned}\|Vw\|_{H^{1/2}(\Gamma)} &= \sup_{\tau \in H^{-1/2}(\Gamma)} \frac{|\langle Vw, \tau\rangle_{L_2^*(\Gamma)}|}{\|\tau\|_{H^{-1/2}(\Gamma)}} \ge \frac{|\langle Vw, w\rangle_{L_2(\Gamma)}|}{\|w\|_{H^{-1/2}(\Gamma)}} \\ &\ge c_1^V \cdot \|w\|_{H^{-1/2}(\Gamma)}.\end{aligned}$$

Since V: $H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is bijective, for every chosen $v \in H^{/1}(\Gamma)$ there exists a unique $w \in H^{-1/2}(\Gamma)$ such that $v = Vw$. Then,

$$\begin{aligned}\|V^{-1}u\|_{H^{-1/2}(\Gamma)} &= \sup_{0 \neq v \in H^{1/2}(\Gamma)} \frac{|\langle V^{-1}u, v\rangle_{L_2(\Gamma)}|}{\|v\|_{H^{1/2}(\Gamma)}} \\ &= \sup_{0 \neq Vw \in H^{1/2}(\Gamma)} \frac{|\langle u, w\rangle_{L_2(\Gamma)}|}{\|Vw\|_{H^{1/2}(\Gamma)}} \\ &\le \frac{1}{c_1^V} \cdot \|u\|_{H^{1/2}(\Gamma)}.\end{aligned}$$

Therefore,

$$\langle V^{-1}u, u\rangle_{L_2(\Gamma)} \leq \|V^{-1}u\|_{H^{-1/2}(\Gamma)}\|u\|_{H^{1/2}(\Gamma)} \leq \frac{1}{c_1^V} \cdot \|u\|^2_{H^{1/2}(\Gamma)}.$$

This is the proposed inequality (5.5). ∎

PROPOSITION 5.3. *The Steklov–Poincaré operator* S: $H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ *as defined in* (5.3) *is* $H_0^{1/2}(\Gamma)$*-elliptic*, *i.e.*,

$$\langle Su, u\rangle_{L_2(\Gamma)} \geq c_1^D \cdot \|u\|^2_{H^{1/2}(\Gamma)} \qquad \text{for all } u \in H_0^{1/2}(\Gamma), \tag{5.6}$$

where c_1^D *is the same positive constant as in* (2.14).

Proof. Since V^{-1}: $H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is positive definite, we find

$$\begin{aligned}\langle Su, u\rangle_{L_2(\Gamma)} &= \langle Du, u\rangle_{L_2(\Gamma)} + \langle V^{-1}\left(\tfrac{1}{2}I + K\right)u, \left(\tfrac{1}{2}I + K\right)u\rangle_{L_2(\Gamma)}\\ &\geq \langle Du, u\rangle_{L_2(\Gamma)}\end{aligned}$$

for all $u \in H^{1/2}(\Gamma)$. Then the assertion follows from (2.14). ∎

PROPOSITION 5.4. *Since the operator* V *is self-adjoint*, *there exists the self-adjoint square root* $V^{1/2}$ *satisfying* $V = V^{1/2}V^{1/2}$, $(V^{1/2})' = V^{1/2}$, *and* $V^{-1/2} := (V^{1/2})^{-1}$ *with* $V^{-1/2} = V^{1/2}V^{-1}$. *Moreover*,

$$\|V^{-1/2}u\|_{L_2(\Gamma)} = \|u\|_{V^{-1}}. \tag{5.7}$$

These properties are consequences of a well-known theorem in Hilbert spaces, (*see*, *e.g*., [33, *Theorem* 12.33]).

THEOREM 5.1. *For every* $u \in H_0^{1/2}(\Gamma)$, *there holds*

$$(1 - c_K) \cdot \|u\|_{V^{-1}} \leq \|\left(\tfrac{1}{2}I + K\right)u\|_{V^{-1}} \leq c_K \cdot \|u\|_{V^{-1}} \tag{5.8}$$

with

$$c_K := \tfrac{1}{2} + \sqrt{\tfrac{1}{4} - c_0} \qquad \text{and} \qquad c_0 := c_1^V c_1^D.$$

Proof. With relation (5.4), we obtain

$$\begin{aligned}\|\left(\tfrac{1}{2}I + K\right)u\|^2_{V^{-1}} &= \langle V^{-1}\left(\tfrac{1}{2}I + K\right)u, \left(\tfrac{1}{2}I + K\right)u\rangle_{L_2(\Gamma)}\\ &= \langle\left(\tfrac{1}{2}I + K'\right)V^{-1}\left(\tfrac{1}{2}I + K\right)u, u\rangle_{L_2(\Gamma)}\\ &= \langle Su, u\rangle_{L_2(\Gamma)} - \langle Du, u\rangle_{L_2(\Gamma)}\end{aligned} \tag{5.9}$$

due to the symmetric representation in (5.3). Using Proposition 5.4 and the Cauchy–Schwarz inequality, we get

$$\begin{aligned}\langle Su, u\rangle_{L_2(\Gamma)} &= \langle V^{-1/2}VSu, V^{-1/2}u\rangle_{L_2(\Gamma)}\\ &\le \|V^{-1/2}VSu\|_{L_2(\Gamma)}\|V^{-1/2}u\|_{L_2(\Gamma)}\\ &= \|VSu\|_{V^{-1}}\|u\|_{V^{-1}}\\ &= \|\big(\tfrac{1}{2}I + K\big)u\|_{V^{-1}}\|u\|_{V^{-1}}. \end{aligned}\tag{5.10}$$

due to the first identity in (5.3).

On the other hand, from (2.14) together with Proposition 5.2, we have

$$\begin{aligned}\langle Du, u\rangle_{L_2(\Gamma)} &\ge c_1^D \cdot \|u\|^2_{H^{1/2}(\Gamma)}\\ &\ge c_1^D c_1^V \cdot \langle V^{-1}u, u\rangle_{L_2(\Gamma)} = c_0 \cdot \|u\|^2_{V^{-1}}. \end{aligned}\tag{5.11}$$

Hence, combining (5.9), (5.10), and (5.11) gives

$$\|\big(\tfrac{1}{2}I + K\big)u\|^2_{V^{-1}} \le \|\big(\tfrac{1}{2}I + K\big)u\|_{V^{-1}}\|u\|_{V^{-1}} - c_0 \cdot \|u\|^2_{V^{-1}}.$$

Now, set $a := \|(\tfrac{1}{2}I + K)u\|_V > 0$ (since $u \in H_0^{1/2}(\Gamma)$; see also (2.13)) and $b := \|u\|_V > 0$ and require, as above, the inequality

$$a^2 \le a \cdot b - c_0 \cdot b^2,$$

which is equivalent to

$$\left(\frac{a}{b}\right)^2 - \frac{a}{b} + c_0 \le 0.$$

Then,

$$\tfrac{1}{2} - \sqrt{\tfrac{1}{4} - c_0} \le \tfrac{a}{b} \le \tfrac{1}{2} + \sqrt{\tfrac{1}{4} - c_0};$$

and the theorem is proved. ∎

Note that with (4.5), the upper estimate in (5.8) is (3.5). The estimate (3.6), however, follows from the following corollary, which will complete the proof of Theorem 3.1.

COROLLARY 5.1. *Let $u \in H_0^{1/2}(\Gamma)$. Then,*

$$(1 - c_K) \cdot \|u\|_{V^{-1}} \le \|\big(\tfrac{1}{2}I - K\big)u\|_{V^{-1}} \le c_K \cdot \|u\|_{V^{-1}}. \tag{5.12}$$

For $u \in \mathscr{R}$, there holds

$$\|\big(\tfrac{1}{2}I - K\big)u\|_{V^{-1}} = \|u\|_{V^{-1}}. \tag{5.13}$$

Proof. By the triangle inequality and by using (5.8), we get

$$\begin{aligned}\|u\|_{V^{-1}} &= \left\| \left[\left(\tfrac{1}{2} I - K \right) + \left(\tfrac{1}{2} I + K \right) \right] u \right\|_{V^{-1}} \\ &\le \| \left(\tfrac{1}{2} I - K \right) u \|_{V^{-1}} + \| \left(\tfrac{1}{2} I + K \right) u \|_{V^{-1}} \\ &\le \| \left(\tfrac{1}{2} I - K \right) u \|_{V^{-1}} + c_K \cdot \|u\|_{V^{-1}}\end{aligned}$$

and, hence, the lower inequality in (5.12). Using both representations of the Steklov–Poincaré operator in (5.3), and (5.8) and (5.11), as well as the definition of c_K, we get

$$\begin{aligned}&\left\| \left[\tfrac{1}{2} I - K \right] u \right\|_{V^{-1}}^2 \\ &\quad = \left\| \left[I - \left(\tfrac{1}{2} I + K \right) \right] u \right\|_{V^{-1}}^2 \\ &\quad = \|u\|_{V^{-1}}^2 + \| \left(\tfrac{1}{2} I + K \right) u \|_{V^{-1}}^2 - 2 \langle V^{-1} \left(\tfrac{1}{2} I + K \right) u, u \rangle_{L_2(\Gamma)} \\ &\quad = \|u\|_{V^{-1}}^2 + \| \left(\tfrac{1}{2} I + K \right) u \|_{V^{-1}}^2 - 2 \langle S u, u \rangle_{L_2(\Gamma)} \\ &\quad = \|u\|_{V^{-1}}^2 + \| \left(\tfrac{1}{2} I + K \right) u \|_{V^{-1}}^2 \\ &\qquad - 2 \left\langle \left[D + \left(\tfrac{1}{2} I + K' \right) V^{-1} \left(\tfrac{1}{2} I + K \right) \right] u, u \right\rangle_{L_2(\Gamma)} \\ &\quad = \|u\|_{V^{-1}}^2 - \| \left(\tfrac{1}{2} I + K \right) u \|_{V^{-1}}^2 - 2 \langle D u, u \rangle_{L_2(\Gamma)} \\ &\quad \le \left[1 - (1 - c_K)^2 - 2 c_0 \right] \cdot \|u\|_{V^{-1}}^2 = c_K^2 \cdot \|u\|_{V^{-1}}^2.\end{aligned}$$

This is the upper inequality in (5.12). The relation (5.13) follows immediately from (4.5). ∎

Note that (5.12) in Corollary 5.1 is just (3.13) of Theorem 3.2. Then, with (5.4) and (5.12), we have

$$\| \left(\tfrac{1}{2} I - K' \right) w \|_V^2 = \| \left(\tfrac{1}{2} I - K' \right) V w \|_{V^{-1}}^2 \le c_K^2 \| V w \|_{V^{-1}}^2 = c_K^2 \| w \|_V^2,$$

which is the proposed estimate (3.12), and Theorem 3.2 is proved completely.

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