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Extremal solutions and Green's functions of higher order periodic boundary value problems in time scales ☆

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Abstract

In this paper we develop the monotone method in the presence of lower and upper solutions for the problem

$$u^{\Delta^n}(t) + \sum_{j=1}^{n-1} M_j u^{\Delta^j}(t) = f\big(t, u(t)\big), \quad t \in [a, b],$$

$$u^{\Delta^i}(a) = u^{\Delta^i}\big(\sigma(b)\big), \quad i = 0, \ldots, n-1.$$

Here $f:[a,b] \times \mathbb{R} \to \mathbb{R}$ is such that $f(\cdot, x)$ is rd-continuous in I for every $x \in \mathbb{R}$ and $f(t, \cdot)$ is continuous in $\mathbb{R}$ uniformly at $t \in I$, $M_j \in \mathbb{R}$ are given constants and $[a,b] = \mathbb{T}^{\kappa^n}$ for an arbitrary bounded time scale $\mathbb{T}$. We obtain sufficient conditions in f to guarantee the existence and approximation of solutions lying between a pair of ordered lower and upper solutions α and β. To this end, given $M > 0$, we study some maximum principles related with operators

$$T_n^{\pm}[M]u(t) \equiv u^{\Delta^n}(t) + \sum_{j=1}^{n-1} M_j u^{\Delta^j}(t) \pm M u(t),$$

in the space of periodic functions.

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1. Introduction

The theory of dynamic equations has appeared recently in the Ph.D. thesis of Hilger [12]. This new theory unifies difference and differential equations and gives an abstract formulation that allows to the researchers to use the same notation in both fields. Recently, many papers devoted to the study of this kind of problems have been done, in the monograph of Bohner and Peterson [2] one can find a lot of fundamental tools to work with this type of equations.

In this paper we will study the existence of solutions of nonlinear periodic boundary value problems in time scales. To this end, we study the sign of related Green's functions and use the classical concept of lower and upper solutions for differential equations [4] and that has been used in recent years for difference [1,7,8,15] and dynamic equations [3,10,13]. Our results are in the same direction that many other recent papers in time scales, in which comparison results are given, mainly for second order equations; see, for instance, the papers of Erbe and Peterson [9] and Topal [11]. Here the results obtained in [5] for differential equations and in [7] for difference ones are unified.

In the next section we introduce the problem that we will consider along the paper. Fundamental concepts as lower and upper solutions, extremal solutions and inverse positive and inverse negative linear operators are exposed. Section 3 is devoted to give the expression of the Green's function and some fundamental properties. In Section 4 we develop the method of lower and upper solutions coupled with the monotone iterative techniques to derive existence of extremal solutions of the nonlinear considered problem. Finally, in Section 5 we obtain some estimations for nth order operators, which are optimal for first and, in some sense, for second order operators.

2. Preliminaries and hypothesis

In this paper we study the following periodic boundary value problem for $n \geqslant 1$ order dynamic equation:

$$u^{\Delta^n}(t) + \sum_{j=1}^{n-1} M_j u^{\Delta^j}(t) = f\bigl(t, u(t)\bigr) \quad \text{for all } t \in I = [a,b], \tag{2.1}$$

$$u^{\Delta^i}(a) = u^{\Delta^i}\bigl(\sigma(b)\bigr), \quad i = 0, 1, \ldots, n-1, \tag{2.2}$$

with $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ such that $f(\cdot, x) \in C_{\mathrm{rd}}(I) \equiv \{g : I \to \mathbb{R},\ g \text{ is rd-continuous in } I\}$ for every $x \in \mathbb{R}$, and $f(t, \cdot)$ is continuous in $\mathbb{R}$ uniformly at $t \in I$; $M_j \in \mathbb{R}$ given constants for $j \in \{1, \ldots, n-1\}$, $[a,b] = \mathbb{T}^{\kappa^n}$ and $\mathbb{T}$ an arbitrary bounded time scale.

A solution of problem (2.1)–(2.2) will be a function $u:\mathbb{T}\to\mathbb{R}$, such that

$$u\in C^n_{\mathrm{rd}}(I)=\left\{u:\mathbb{T}\to\mathbb{R},\ u^{\Delta^j}\in C(\mathbb{T}^{\kappa^j}),\ j=0,\dots,n-1,\ u^{\Delta^n}\in C_{\mathrm{rd}}(I)\right\},$$

and that satisfies both equalities.

For a pair of continuous functions $v\leqslant w$ in $\mathbb{T}$, we denote

$$[v,w]=\left\{x\in C(\mathbb{T}),\ v(t)\leqslant x(t)\leqslant w(t)\text{ for all } t\in\mathbb{T}\right\}.$$

We say that $x\in V$ is the maximal solution of (2.1)–(2.2) in the function's set V if any other solution of such problem $y\in V$ is such that $y\leqslant x$ in $\mathbb{T}$. We refer to a minimal solution if the reversed inequalities hold. We denote both functions as extremal solutions in V.

To deduce existence of extremal solutions for problem (2.1)–(2.2) we introduce the concept of lower and upper solutions.

Definition 2.1. Let $\alpha:\mathbb{T}\to\mathbb{R}$, $\alpha\in C^n_{\mathrm{rd}}(I)$. We say that α is a lower solution of problem (2.1)–(2.2) if α satisfies

$$\alpha^{\Delta^n}(t)+\sum_{j=1}^{n-1}M_j\alpha^{\Delta^j}(t)\leqslant f\big(t,\alpha(t)\big)\quad\text{for all } t\in I,$$
$$\alpha^{\Delta^i}(a)=\alpha^{\Delta^i}\big(\sigma(b)\big),\quad i=0,\dots,n-1.$$

Definition 2.2. Let $\beta:\mathbb{T}\to\mathbb{R}$, $\beta\in C^n_{\mathrm{rd}}(I)$. We say that β is an upper solution of problem (2.1)–(2.2) if β satisfies

$$\beta^{\Delta^n}(t)+\sum_{j=1}^{n-1}M_j\beta^{\Delta^j}(t)\geqslant f\big(t,\beta(t)\big)\quad\text{for all } t\in I,$$
$$\beta^{\Delta^i}(a)=\beta^{\Delta^i}\big(\sigma(b)\big),\quad i=0,\dots,n-1.$$

To deduce existence results via monotone iterative techniques, we assume that function f satisfies one of the following one sided Lipschitz conditions for a suitable constant $M>0$, depending on the circumstances:

(H_1) $\forall t\in I$, $\forall u,v\in[\alpha(t),\beta(t)]$: $u\leqslant v\Rightarrow f(t,u)+Mu\leqslant f(t,v)+Mv$.
(H_2) $\forall t\in I$, $\forall u,v\in[\beta(t),\alpha(t)]$: $u\leqslant v\Rightarrow f(t,u)-Mu\geqslant f(t,v)-Mv$.

As we will see, we translate the study of the existence of extremal solutions lying between α and β to find real parameters for which the Green's function related with an appropriate nth order linear operator has fixed sign. Before doing this, we define the concept of inverse positive and inverse negative operator.

Definition 2.3. Let $M_i\in\mathbb{R}$, $i=0,\dots,n-1$, be given constants. Let $Tu(t)=u^{\Delta^n}(t)+\sum_{i=0}^{n-1}M_iu^{\Delta^i}(t)$ be regressive in I, i.e., $1+\sum_{i=1}^{n}(-\mu(t))^iM_{n-i}\neq 0$ for all $t\in I$, defined in the set

$$W_n=\left\{u\in C^n_{\mathrm{rd}}(I),\ u^{\Delta^i}(a)=u^{\Delta^i}\big(\sigma(b)\big),\ i=0,\dots,n-1\right\}.$$

Suppose that there exists T^{-1} in W_n. We say that T is

(1) Inverse positive in W_n if: $u \in W_n$ and $Tu \geqslant 0$ in I implies $u \geqslant 0$ in $\mathbb{T}$.
(2) Inverse negative in W_n if: $u \in W_n$ and $Tu \geqslant 0$ in I implies $u \leqslant 0$ in $\mathbb{T}$.

Remark 2.1. In this paper we define the concept of lower and upper solutions assuming the equalities in all the derivatives of α and β in the boundary of $\mathbb{T}$. This was the case in [7] for nth order discrete problems. However, in [8] for difference equations and in [5] for differential ones, some adequate inequalities in the $(n-1)$th order derivatives are allowed.

One can see that the abstract formulation of the monotone iterative methods is the same in both situations, it is enough to study the inverse positive and inverse negative character of the linear operator T in the set

$$\begin{aligned}\Omega_n = \big\{&u \in C^n_{\mathrm{rd}}(I),\ u^{\Delta^i}(a) = u^{\Delta^i}\big(\sigma(b)\big),\ i = 0, \dots, n-2,\\ &u^{\Delta^{n-1}}(a) \geqslant u^{\Delta^{n-1}}\big(\sigma(b)\big)\big\},\end{aligned}$$

instead of W_n.

However, on the contrary to differential equations [5], where the values of M_i for which operator T is inverse positive or inverse negative in W_n are the same that in Ω_n, in difference equations the values are worse in this new situation (see [8]).

Since the aim of this paper is to explain the validity of the monotone iterative technique for dynamic equations, we have preferred this definition to do the paper more readable.

3. Expression of the Green's function

In this section we present a formula to obtain the expression of the Green's function associated to T^{-1}, where T is a general nth order linear operator invertible in W_n.

Theorem 3.1. *Let $M_i \in \mathbb{R}$, $i = 0, \dots, n-1$, be given constants, and let $Tu(t) = u^{\Delta^n}(t) + \sum_{i=0}^{n-1} M_i u^{\Delta^i}(t)$, $t \in I$, such that $1 + \sum_{i=1}^{n} (-\mu(t))^i M_{n-i} \neq 0$ for all $t \in I$. Then, if there is T^{-1} in W_n, the associated Green's function $G : \mathbb{T} \times I \to \mathbb{R}$ is given by the following expression*:

$$G(t,s) = \begin{cases} u(t,s) + v(t,s) & \text{if } \sigma(s) \leqslant t,\\ u(t,s) & \text{if } t < \sigma(s),\end{cases} \tag{3.1}$$

where, for every $s \in [a,b]$ fixed, $v(\cdot\,,s)$ is the unique solution of problem

$$\begin{cases} Tx_s(t) = 0, \quad t \in [\sigma(s), b],\\ x_s^{\Delta^i}(\sigma(s)) = 0, \quad i = 0, \dots, n-2,\\ x_s^{\Delta^{n-1}}(\sigma(s)) = 1,\end{cases} \tag{3.2}$$

and for every $s \in [a,b]$ fixed, $u(\cdot\,,s)$ is given as the unique solution of problem

$$\begin{cases} Ty_s(t) = 0, \quad t \in [a,b],\\ y_s^{\Delta^i}(a) - y_s^{\Delta^i}(\sigma(b)) = x_s^{\Delta^i}(\sigma(b)), \quad i = 0, \dots, n-1.\end{cases} \tag{3.3}$$

Proof. First, we note that function G is well defined, i.e., problems (3.2) and (3.3) are uniquely solvable. Since $1+\sum_{i=1}^{n}(-\mu(t))^i M_{n-i} \neq 0$ for all $t \in I$ we have, see Corollary 5.90 and Theorem 5.91 in [2], that the initial value problem (3.2) has a unique solution.

To verify that the periodic boundary value problem (3.3) has a unique solutions, we have that for any $h \in C_{\mathrm{rd}}(I)$ and $\lambda_i \in \mathbb{R}$, $i = 0, \dots, n-1$, $w \in C_{\mathrm{rd}}^n(I)$ is a solution of the periodic boundary problem

$$(\mathrm{P}_\lambda) \quad \begin{cases} w^{\Delta^n}(t) + \sum_{i=0}^{n-1} M_i w^{\Delta^i}(t) = h(t), & t \in I, \\ w^{\Delta^i}(a) - w^{\Delta^i}(\sigma(b)) = \lambda_i, & i = 0, \dots, n-1, \end{cases}$$

if and only if $W(t) = (w(t), w^{\Delta}(t), \dots, w^{\Delta^{n-1}}(t))^T$, is a solution of the matrix equation

$$W^{\Delta}(t) = AW(t) + H(t), \quad t \in [a,b], \qquad W(a) - W\bigl(\sigma(b)\bigr) = \lambda, \tag{3.4}$$

where $H(t) = (0, \dots, 0, h(t))^T$, $\lambda = (\lambda_0, \dots, \lambda_{n-1})^T$ and

$$A \equiv \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -M_0 & -M_1 & -M_2 & \dots & -M_{n-1} \end{pmatrix}. \tag{3.5}$$

Since operator T is regressive if and only if matrix A is regressive (see Definition 5.89 in [2]) we know that every solution of (3.4) is given by the expression

$$W(t) = e_A(t,a)W(0) + \int_a^t e_A\bigl(t, \sigma(\tau)\bigr) H(\tau)\, \Delta\tau,$$

with $W(0)$ satisfying

$$\bigl(I_n - e_A\bigl(\sigma(b), a\bigr)\bigr) W(0) = \int_a^{\sigma(b)} e_A\bigl(\sigma(b), \sigma(\tau)\bigr) H(\tau)\, \Delta\tau + \lambda,$$

where by I_n we denote the $n \times n$ identity matrix. Thus, T is invertible in W_n if and only if matrix $I_n - e_A(\sigma(b), a)$ is invertible. As consequence, under the assumptions of the enunciate, we have that problem (3.3) has a unique solution.

Now, let $z : \mathbb{T} \to \mathbb{R}$ be defined as

$$\begin{aligned} z(t) &= \int_a^{\sigma(b)} G(t,s) h(s)\, \Delta s \\ &= \int_a^t \bigl(u(t,s) + v(t,s)\bigr) h(s)\, \Delta s + \int_t^{\sigma(b)} u(t,s) h(s)\, \Delta s, \quad t \in \mathbb{T}. \end{aligned}$$

As consequence,

$$z^{\Delta}(t) = \int_a^t \big(u^{\Delta}(t,s) + v^{\Delta}(t,s)\big)h(s)\,\Delta s$$
$$+ \int_t^{\sigma(b)} u^{\Delta}(t,s)h(s)\,\Delta s + v\big(\sigma(t),t\big)h(t), \quad t \in \mathbb{T}^{\kappa}.$$

If $n = 1$ then $v(\sigma(t),t) = 1$ and we have finished. Otherwise, $v(\sigma(t),t) = 0$ and we can differentiate z^{Δ} once again. By recurrence, we have that for $i \in \{1,\dots,n-1\}$, the following equality holds:

$$z^{\Delta^i}(t) = \int_a^t \big(u^{\Delta^i}(t,s) + v^{\Delta^i}(t,s)\big)h(s)\,\Delta s + \int_t^{\sigma(b)} u^{\Delta^i}(t,s)h(s)\,\Delta s, \quad t \in \mathbb{T}^{\kappa^i}.$$

Finally,

$$z^{\Delta^n}(t) = \int_a^t \big(u^{\Delta^n}(t,s) + v^{\Delta^n}(t,s)\big)h(s)\,\Delta s + \int_t^{\sigma(b)} u^{\Delta^n}(t,s)h(s)\,\Delta s + h(t), \quad t \in I.$$

Now, it is clear, from the linearity of the integral and the definitions of u and v, that $Tz(t) = h(t)$ for all $t \in I$.

On the other hand, from the conditions imposed to u and v, we deduce

$$z^{\Delta^i}\big(\sigma(b)\big) = \int_a^{\sigma(b)} \big(u^{\Delta^i}\big(\sigma(b),s\big) + v^{\Delta^i}\big(\sigma(b),s\big)\big)h(s)\,\Delta s$$
$$= \int_a^{\sigma(b)} u^{\Delta^i}(a,s)h(s)\,\Delta s = z^{\Delta^i}(a), \quad i = 0,\dots,n-1.$$

In consequence, z is the unique solution of problem (P_0). □

Now, we prove the following properties of the Green's function.

Lemma 3.1. *Assume that* $1 + \sum_{i=1}^{n} (-\mu(t))^i M_{n-i} \neq 0$ *for all* $t \in I$. *Let* $G : \mathbb{T} \times I \to \mathbb{R}$ *be the Green's function of operator* T^{-1} *defined in* (3.1). *Then the following conditions are satisfied*:

(1) *There exists* $K > 0$ *such that* $|G(t,s)| \leqslant K$ *for all* $(t,s) \in \mathbb{T} \times I$.
(2) *If* $n = 1$, *for every* $t \in \mathbb{T}$, *function* $G(t,\cdot)$ *is rd-continuous at* $s \neq t$.
(3) *If* $n > 1$, *for every* $t \in \mathbb{T}$, *function* $G(t,\cdot)$ *is rd-continuous in* I.
(4) *If* b *is left dense and* $b = \sigma(b)$ *then* $G(t,\cdot)$ *is continuous at* b *for all* $t \in \mathbb{T}$.

Proof. As we have seen in the proof of Theorem 3.1, we know that the Green's function G related with operator T^{-1} in W_n is given as the $1 \times n$ term of the matrix Green's function:

$$F(t,s) = \begin{cases} e_A(t,\sigma(s)) + e_A(t,a)(I - e_A(\sigma(b),a))^{-1} e_A(\sigma(b),\sigma(s)), \\ \quad \sigma(s) \leqslant t, \\ e_A(t,a)(I - e_A(\sigma(b),a))^{-1} e_A(\sigma(b),\sigma(s)), \quad t < \sigma(s), \end{cases} \tag{3.6}$$

where A is given in (3.5).

Using, this expression, from the continuity of the exponential matrix function in both variables, see Theorems 5.18 and 5.23 in [2], we conclude that function G is bounded in the compact set $\mathbb{T} \times I$.

Due to the fact that $e_A(t,t) = I_n$, from Eq. (3.6) we have that when $t = \sigma(s) = s$, the matrix function F is not continuous in the diagonal terms. In consequence, from the continuity of the exponential matrix again, we have that for any $t_0 \in \mathbb{T}$ and $s_0 \neq t_0$ fixed, when $s \to s_0$ and $\sigma(s) \to \sigma(s_0)$ then $F(t_0,s) \to F(t_0,s_0)$. Since $G(t,s)$ $(\equiv F_{1,n}(t,s))$ belongs to the diagonal only when $n = 1$, we deduce assertions (2)–(4) of the lemma. □

Lemma 3.2. *Operator T is inverse positive on W_n if and only if the associated Green's function G given in* (3.1) *is nonnegative in $\mathbb{T} \times I$.*

Proof. Assume that operator T is inverse positive on W_n. Let $u \in W_n$ be such that $Tu \geqslant 0$ on I. In consequence there is a rd-continuous function $h : I \to [0,+\infty)$ satisfying

$$Tu(t) = h(t) \geqslant 0, \quad t \in I, \qquad u^{\Delta^i}(a) = u^{\Delta^i}\big(\sigma(b)\big), \quad i = 0,\dots,n-1, \tag{3.7}$$

and such that

$$0 \leqslant u(t) = T^{-1}h(t) \equiv \int_a^{\sigma(b)} G(t,s)h(s)\,\Delta s \quad \text{for all } t \in \mathbb{T}. \tag{3.8}$$

Obviously, from the uniqueness of the solutions of problem (3.7), the relationship between $h \in C_{\mathrm{rd}}(I)$ and $u \in W_n$ satisfying such properties is one-to-one.

Suppose that there is $(t_0,s_0) \in \mathbb{T} \times I$ such that $G(t_0,s_0) < 0$. We have four possibilities.

(1) If s_0 is an isolated point. Define $h(s_0) = 1$ and $h(t) = 0$ otherwise, then the unique solution of (3.7) satisfies

$$u(t_0) = \int_{s_0}^{\sigma(s_0)} G(t_0,s)h(s)\,\Delta s = \mu(s_0)G(t_0,s_0)h(s_0) < 0,$$

which contradicts (3.8).

(2) If s_0 is left dense and right scattered. Given $\epsilon \in (0,\mu(s_0))$, let $h \equiv 0$ outside of $(s_0 - \epsilon, s_0]$ and, in other case, define

$$h(s) = \exp\left(1 - \frac{1}{1 - [(s-s_0)/\epsilon]^2}\right). \tag{3.9}$$

It is clear that $h \in C(I)$. Moreover, using part (1) of Lemma 3.1, the unique solution u of (3.7) satisfies

$$u(t_0) = \int_{s_0-\epsilon}^{\sigma(s_0)} G(t_0, s)h(s)\,\Delta s \leqslant K\epsilon + \mu(s_0)G(t_0, s_0)h(s_0),$$

which is strictly negative for ϵ small enough and contradicts (3.8) again.

(3) If $s_0 < b$ is right dense, from the rd-continuity of $G(t_0, \cdot)$ at s_0 proved in Lemma 3.1 when $n > 1$ or $n = 1$ and $t_0 \neq s_0$, and using expression (3.6) if $n = 1$ and $t_0 = s_0$, we know that there is $\epsilon > 0$ small enough such that $G(t_0, s) < 0$ for $s \in [s_0, s_0 + \epsilon) \subset [a, b]$. Defining h as in (3.9) in the interval $[s_0, s_0 + \epsilon)$ by replacing in this case s_0 by $s_0 + \epsilon/2$ and ϵ by $\epsilon/2$ and as the trivial function otherwise, we get a contradiction in a similar way.

(4) If $s_0 = b = \sigma(b)$ (the case $b < \sigma(b)$ has been considered in (1) and (2)) and it is left scattered, since $[a, b] = \mathbb{T}^{\kappa^n}$ we deduce that $b = \sigma^n(b)$ and, as consequence, $\mathbb{T}^{\kappa} = [a, \rho(b)] \subsetneq [a, b]$, which contradicts the assumptions. If it is left dense, from the left continuity of $G(t_0, \cdot)$ at b proved in part (4) of Lemma 3.1, we attain a contradiction as in the previous case.

The other implication is trivial. □

In the same way, one can verify that the following result holds.

Lemma 3.3. *Operator T is inverse negative on W_n if and only if the associated Green's function G given in* (3.1) *is nonpositive in $\mathbb{T} \times I$.*

4. Lower and upper solutions

In this section, we prove the existence of extremal solutions lying between a pair of ordered lower and upper solutions. This existence result is derived from the comparison principles exposed in the previous section. As we will see, such existence results are, in some sense, optimal. First, given $M_j \in \mathbb{R}$, $j = 1, \ldots, n-1$, for every $M > 0$, we define the following operators in the set W_n:

$$T_n^{\pm}[M]u(t) = u^{\Delta^n}(t) + \sum_{j=1}^{n-1} M_j u^{\Delta^j}(t) \pm Mu(t), \quad t \in I. \tag{4.1}$$

Theorem 4.1. *Suppose that there exist $\alpha \leqslant \beta$ lower and upper solutions of problem* (2.1)–(2.2) *and that f is such that $f(\cdot, x)$ is rd-continuous in I for every $x \in \mathbb{R}$ and $f(t, \cdot)$ is continuous in $\mathbb{R}$ uniformly at $t \in I$, and satisfies condition* (H$_1$) *for some $M > 0$ such that operator $T_n^+[M]$ is inverse positive on W_n. Then there exist two monotone sequences in $\mathbb{T}$, $\{\alpha_m\}$ and $\{\beta_m\}$ with $\alpha_0 = \alpha$ and $\beta_0 = \beta$, which converge uniformly to the extremal solutions of problem* (2.1)–(2.2) *in $[\alpha, \beta]$.*

Proof. Fix $M > 0$ such that operator $T_n^+[M]$ is inverse positive on W_n and f satisfies condition (H_1). For each $\eta \in [\alpha, \beta]$, consider the following linear problem:

$$(\mathrm{P}_\eta) \qquad \begin{cases} T_n^+[M]u(t) = f(t, \eta(t)) + M\eta(t) \equiv h_\eta(t), & t \in I, \\ u^{\Delta^i}(a) = u^{\Delta^i}(\sigma(b)), & i = 0, \dots, n-1. \end{cases}$$

Clearly, $h_\eta \in C_{\mathrm{rd}}(I)$. Thus, since operator $T_n^+[M]$ is inverse positive on W_n, we know that (P_η) admits a unique solution u_η for each η given. Such solution is given by the expression

$$u_\eta(t) = \left(T_n^+[M]\right)^{-1} h_\eta(t) \equiv \int_a^{\sigma(b)} G_M^+(t,s) h_\eta(s)\, \Delta s, \quad t \in \mathbb{T},$$

with G_M^+ the associated Green's function of operator $T_n^+[M]$.

From the definition of α and condition (H_1), we know that $u_\eta - \alpha \in W_n$ and $T_n^+[M](u_\eta - \alpha) \geqslant 0$ on I. Now, from the fact that operator $T_n^+[M]$ is inverse positive on W_n, we deduce that $u_\eta \geqslant \alpha$ on $\mathbb{T}$.

On the other hand, let u_i, $i = 1, 2$, be the unique solutions of problem (P_{η_i}) with $\eta_1 \leqslant \eta_2$ on I. Clearly, $u_2 - u_1 \in W_n$ and $T_n^+[M](u_2 - u_1) \geqslant 0$ on I. The inverse positive character of operator $T_n^+[M]$ in W_n says us that $u_2 \geqslant u_1$ on $\mathbb{T}$.

From these two properties, defining $\alpha_0 = \alpha$, $\beta_0 = \beta$, $\alpha_m = (T_n^+[M])^{-1} h_{\alpha_{m-1}}$ and $\beta_m = (T_n^+[M])^{-1} h_{\beta_{m-1}}$, we construct two monotone sequences which, as a consequence of the Ascoli–Arzelá's theorem, converge uniformly to a continuous functions in $\mathbb{T}$. Using the integral representation of both sequences, we verify that such limits are solutions of problem (2.1)–(2.2). Obviously $\phi(t) = \lim_{m\to\infty} \alpha_m(t) \leqslant \lim_{m\to\infty} \beta_m(t) = \Phi(t)$ for all $t \in \mathbb{T}$ and belong to the sector $[\alpha, \beta]$.

Now, using that every solution u of problem (2.1)–(2.2) satisfies $u = (T_n^+[M])^{-1} h_u$ together with the fact that by condition (H_1) if $u_1 \leqslant u_2$ in I then $h_{u_1} \leqslant h_{u_2}$ in I, and that $G_M^+ \geqslant 0$ in $\mathbb{T} \times I$, we conclude that if $u \in [\alpha, \beta]$ then $u \in [\phi, \Phi]$, that is, ϕ and Φ are the extremal solutions of problem (2.1)–(2.2) in $[\alpha, \beta]$. □

In an analogous way, we can prove the following result.

Theorem 4.2. *Suppose that there exist $\alpha \geqslant \beta$ lower and upper solutions of problem* (2.1)–(2.2) *and f is such that $f(\cdot, x)$ is rd-continuous in I for every $x \in \mathbb{R}$ and $f(t, \cdot)$ is continuous in $\mathbb{R}$ uniformly at $t \in I$, and satisfies condition* (H_2) *for some $M > 0$ such that operator $T_n^-[M]$ is inverse negative on W_n. Then there exist two monotone sequences in $\mathbb{T}$, $\{\alpha_m\}$ and $\{\beta_m\}$ with $\alpha_0 = \alpha$ and $\beta_0 = \beta$, which converge uniformly to the extremal solutions of problem* (2.1)–(2.2) *in $[\beta, \alpha]$.*

Remark 4.1. Note that $T_n^+[M]$ cannot be inverse negative in W_n for any value of $M > 0$. It is enough to consider problem

$$T_n^+[M]u(t) = 1, \quad u \in W_n,$$

for which, $u \equiv 1/M$ is a positive solution.

Analogous assertion holds for $T_n^-[M]$.

Since conditions (H_1) and (H_2) can be true only for values of $M > 0$ very small, one can think that the sufficient conditions imposed in the two previous results are very restrictive, however, they are optimal in some sense, as we can see in the next result.

Theorem 4.3. *The assertions proved in Theorems* 4.1 *and* 4.2 *are optimal in the sense that*

(1) *For all $M > 0$, for which $T_n^+[M]$ is regressive in I and there is $(T_n^+[M])^{-1}$ on W_n, but $T_n^+[M]$ is not inverse positive on W_n, there are functions f, α and β satisfying the assumptions of Theorem* 4.1 *and for which problem* (2.1)–(2.2) *has no solution in* $[\alpha, \beta]$.
(2) *For all $M > 0$, for which $T_n^-[M]$ is regressive in I and there exists $(T_n^-[M])^{-1}$ on W_n, but $T_n^-[M]$ is not inverse negative on W_n, there are functions f, α and β satisfying the assumptions of Theorem* 4.2 *and for which problem* (2.1)–(2.2) *has no solution in* $[\beta, \alpha]$.

Proof. We only proof the first assertion, the second one is analogous.

Let $M > 0$ satisfying the conditions exposed in the enunciate. Fix $s_0 \in I$ and $\epsilon > 0$, define $f : (t, x) \in I \times \mathbb{R} \to f(t, x) = -Mx + Mh(t) \in \mathbb{R}$ with h given in the proof of Lemma 3.2 depending on the circumstances of the choice of s_0. Clearly, f satisfies condition (H_1) for such $M > 0$ and, since $h(t) \in [0, 1]$ for all $t \in I$, $\alpha \equiv 0$ and $\beta \equiv 1$ are a pair of lower and upper solutions for problem (2.1)–(2.2) such that $\alpha \leqslant \beta$. However, as it is proved in Lemma 3.2, for ϵ small enough the unique solution of such problem takes a negative value in at least one point and, in consequence, this problem has no solution lying between the lower and the upper solution. □

5. Estimates for comparison principles

As it is stated in the two previous sections, the existence of extremal solutions of the nonlinear nth order problem (2.1)–(2.2) lying between a pair of lower and upper solutions is equivalent to the fact that function f satisfies condition (H_1) or (H_2) for some adequate values of $M > 0$. To look for such values is not an easy problem, in [5] and [7,8] some estimates are obtained for nth order when differential ($\mu \equiv 0$) and difference ($\mu \equiv 1$) problems are considered.

To do the study for arbitrary time scales, we obtain the following result, where the structure of the M values set for which operators $T_n^{\pm}[M]$ satisfy comparison principles in W_n is given. Such result is a continuation of Theorem 3.1 in [6] for differential equations and Theorem 4.1 in [7] for difference ones. Here we unify both results. The result is the following

Theorem 5.1. *The following assertions hold*:

(1) *Suppose that there exists $\bar{M} > 0$ such that $T_n^+[\bar{M}]$ is regressive in I but not inverse positive on W_n. Then $T_n^+[M]$ is not inverse positive on W_n for all $M \geqslant \bar{M}$.*

(2) *Suppose that there exists $\bar{M} > 0$ such that $T_n^-[\bar{M}]$ is regressive in I but not inverse negative on W_n. Then $T_n^-[M]$ is not inverse negative on W_n for all $M \geqslant \bar{M}$.*

Proof. First, we suppose that $\bar{M}$ is a positive constant such that there exists $(T_n^+[\bar{M}])^{-1}$ on W_n. Define $f:(t,x) \in I \times \mathbb{R} \to f(t,x) = -\bar{M}x + \bar{M}h(t) \in \mathbb{R}$, with h given in the proof of Lemma 3.2. Clearly, f satisfies condition (H_1) for all $M \geqslant \bar{M}$ and, since $h(t) \in [0,1]$ for all $t \in I$, $\alpha \equiv 0$ and $\beta \equiv 1$ are a pair of lower and upper solutions for problem (2.1)–(2.2) such that $\alpha \leqslant \beta$. However, as it is proved in Lemma 3.2, the unique solution of such problem takes a negative value in at least one point and, in consequence, this problem has no solution lying between the lower and the upper solution. Thus, if there exists $M > \bar{M}$ for which $T_n^+[M]$ is inverse positive on W_n, we conclude, applying Theorem 4.1, that this problem has a solution between α and β, which is not true. Therefore the operator $T_n^+[M]$ is not inverse positive on W_n for all $M \geqslant \bar{M}$.

Now, we suppose the other case, that is, $T_n^+[M]$ is inverse positive on W_n for all $M \in (0, \bar{M})$ with $\bar{M} > 0$ such that there is no $(T_n^+[\bar{M}])^{-1}$ on W_n, i.e., problem (P_0) with $M_0 = \bar{M}$ does not have a unique solution. As we have seen in the proof of Theorem 3.1, if w is a solution of that problem then $W(t) = (w(t), w^{\Delta}(t), \ldots, w^{\Delta^{n-1}}(t))^T$ is a solution of the matrix equation

$$W^{\Delta}(t) = AW(t) + H(t), \quad t \in [a,b], \qquad W(a) = W\big(\sigma(b)\big),$$

where $H(t) = (0, \ldots, 0, h(t))^T$ and A is given in (3.5). Since matrix A is regressive, we have that the initial value problem

$$W^{\Delta}(t) = AW(t) + H(t), \quad t \in [a,b], \qquad W(a) = y_0,$$

has a unique solution given by

$$W(t) = e_A(t,a)y_0 + \int_a^t e_A\big(t, \sigma(\tau)\big)H(\tau)\,\Delta\tau.$$

On the other hand, due to the fact that the periodic problem has not a unique solution, we have that $\det(I_n - e_A(\sigma(b), a)) = 0$.

Denote by $I_n - e_A(\sigma(b), a) \equiv (b_{i,j})_{i,j \in \{1,\ldots,n\}}$; we have two possibilities:

(1) There exists $i_0 \in \{2, \ldots, n\}$ such that $b_{i_0,j} = \sum_{k=1}^{i_0-1} \lambda_k b_{k,j}$, $j = 1, \ldots, n$, with some $\lambda_k \neq 0$; or
(2) There exists $i_0 \in \{1, \ldots, n\}$ such that $b_{i_0,j} = 0$, $j = 1, \ldots, n$.

In the first case, let $C(s) = e_A(\sigma(b), s) \equiv (c_{i,j}(s))_{i,j \in \{1,\ldots,n\}}$. Thus, a necessary condition to assure the existence of solution of the periodic problem (P_0) is given by

$$\int_a^{\sigma(b)} \left(c_{i_0,n}\big(\sigma(s)\big) - \sum_{k=1}^{i_0-1} \lambda_k c_{k,n}\big(\sigma(s)\big) \right) h(s)\,\Delta s = 0. \tag{5.1}$$

Now, using Theorem 5.23 and Corollary 5.26 in [2], we arrive at

$$C^{\Delta}(s) = -AC\big(\sigma(s)\big), \quad s \in [a,b], \; C\big(\sigma(b)\big) = I_n. \tag{5.2}$$

If $c_{i_0,n}(s) = \sum_{k=1}^{i_0-1} \lambda_k c_{k,n}(s)$ for all $s \in [a,b]$, then $d(s) = (c_{1,n}(s), \dots, c_{i_0-1,n}(s))^T$ satisfies the equation $d^{\Delta}(s) = -Bd(\sigma(s))$, $d(\sigma(b)) = 0$, where

$$B \equiv \left(\begin{array}{c|c} 0 & I_{i_0-2} \\ \hline \lambda_1 & \lambda_2 \dots \lambda_{i_0-1} \end{array}\right),$$

and I_{i_0-2} is the $i_0 - 2 \times i_0 - 2$ identity matrix. In consequence, from Theorem 5.27 in [2], we have that $d(s) = e_{\ominus B}(s, \sigma(b))\, d(\sigma(b)) = 0$. Thus, using (5.2), we conclude by recurrence that $c_{i,n}(s) = 0$ in I, for all $i \in \{1, \dots, n\}$, which contradicts the fact that $C(\sigma(b)) = I_n$.

If the second situation holds, then a necessary condition to assure that problem (P_0) with $M_0 = \bar{M}$, is solvable is that

$$\int_a^{\sigma(b)} c_{i_0,n}(s) h(s)\, \Delta s = 0.$$

Now, if $c_{i_0,n}(s) = 0$ in I, as in the previous case we arrive at $c_{i,n}(s) = 0$ in I for all $i = i_0 + 1, \dots, n$. In particular $C_{n,n}(\sigma(b)) = 0$ which contradicts the fact that $e_A(\sigma(b), \sigma(b)) = I_n$.

In consequence, there exists $h \in C_{\mathrm{rd}}(I)$, $h \in [-\bar{M}, \bar{M}]$ such that (5.1) is not true and, therefore, problem (P_0) with $M_0 = \bar{M}$ has no solution for such h.

However, taking $\alpha = -1$ and $\beta = 1$, if there exists $M > \bar{M}$ such that $T_n^+[M]$ is inverse positive on W_n, we are in the hypothesis of Theorem 4.1, which is not possible. Then $T_n^+[M]$ is not inverse positive on W_n for all $M \geqslant \bar{M}$.

For operator $T_n^-[M]$ the same arguments hold. □

Now we study, in a first step first and second order operators.

5.1. First and second order operators

In this section, we give the optimal estimates on the values of $M > 0$ for which operators $T_1^+[M]$ and $T_1^-[M]$ are inverse positive and inverse negative, respectively, in W_1. First we study operator $T_1^-[M]$, to do this, we construct the Green's function G_M^- of such operator. Using the characterization given in Theorem 3.1 and since $1 + \mu(t)M \neq 0$ for all $t \in I$, it is not difficult to verify that $G_M^-(t,s)$ is given by the expression

$$G_M^-(t,s) = \begin{cases} \frac{e_M(t,\sigma(s))}{1 - e_M(\sigma(b),a)}, & \sigma(s) \leqslant t, \\ \frac{e_M(t,\sigma(s))}{e_M(a,\sigma(b)) - 1}, & t < \sigma(s). \end{cases}$$

In consequence, from the fact that $1 + \mu(t)M > 0$ for all $M > 0$ and $t \in I$, we conclude that $G_M^-(t,s) \leqslant 0$ for all $(t,s) \in \mathbb{T} \times I$. Now, from Lemma 3.3 we arrive at the following result.

Lemma 5.1. *Operator $T_1^-[M]$ is inverse negative in W_1 for all $M > 0$.*

If we refer to operator $T_1^+[M]$, such that $1 - \mu(t)M \neq 0$ for all $t \in I$, we have that its Green's function G_M^+ satisfies

$$G_M^+(t,s) = G_{-M}^-(t,s) = \begin{cases} \frac{e_{-M}(t,\sigma(s))}{1-e_{-M}(\sigma(b),a)}, & \sigma(s) \leqslant t, \\ \frac{e_{-M}(t,\sigma(s))}{e_{-M}(a,\sigma(b))-1}, & t < \sigma(s). \end{cases}$$

From the definition of the exponential function e_{-M} [2] we know that there is $(T_1^+[M])^{-1}$ in W_1 if and only if $M \neq 1/\mu(t)$ for all $t \in I$.

In this case, due to the fact that $e_{-M}(t,s)$ is decreasing with respect to t, we conclude that function G_M^+ is nonpositive on $\mathbb{T} \times I$ whenever $0 < M < 1/\mu(t)$ for all $t \in I$.

Now, if there is some $t_0 \in I$ such that $M > 1/\mu(t_0)$, choosing s_0 such that $\sigma(s_0) = t_0$ we have that

$$G_M^+(t_0,s_0)G_M^+\bigl(\sigma(t_0),s_0\bigr) = \frac{1 - M\mu(t_0)}{(1 - e_{-M}(\sigma(b),a))^2} < 0.$$

In consequence, we have the following result.

Lemma 5.2. *Operator $T_1^+[M]$ is inverse positive in W_1 if and only if $0 < M < 1/\mu(t)$ for all $t \in I$.*

Remark 5.1. As we have seen, the optimal estimates given in this section for operator $T_1^+[M]$ are valid for any arbitrary time scale. For it, we can give, in some particular situations, the optimal values of $M > 0$ for which operator $T_1^+[M]$ is inverse positive in W_1:

(1) $\mathbb{T} = \mathbb{R}$ and every $M > 0$.
(2) $\mathbb{T} = \mathbb{Z}$ and $M \in (0,1)$.
(3) $\mathbb{T} = h\mathbb{Z}$ and $M \in (0,1/h)$.
(4) $\mathbb{T} = \mathcal{C}$ the ternary Cantor set, and $M \in (0,3)$.
(5) $\mathbb{T} = q^{\mathbb{N}}$, $q > 1$, and $M \in (0,1/((q-1)b))$.
(6) $\mathbb{T} = \mathbb{N}_0^2$ and $M \in (0,1/(2\sqrt{b}+1))$.

Now, given $A \in \mathbb{R}$ and $B > 0$, we study of the following second order operator defined in the set W_2:

$$T_2[A,B]u(t) = u^{\Delta\Delta}(t) - 2Au^{\Delta}(t) + (A^2 + B^2)u(t). \tag{5.3}$$

In this situation we have that $1 + 2A\mu(t) + (A^2 + B^2)(\mu(t))^2 \neq 0$ for all $t \in I$, that is, operator $T_2[A,B]$ is regressive in I for all $A \in \mathbb{R}$ and $B > 0$. Thus, using the formula given in Theorem 3.1, we conclude that the Green's function $G_{A,B}$ associated with this operator follows the expression

$$G_{A,B}(t,s) = \begin{cases} u_{A,B}(t,s) + v_{A,B}(t,s), & \sigma(s) \leqslant t, \\ u_{A,B}(t,s), & t < \sigma(s), \end{cases}$$

where

$$v_{A,B}(t,s) = \frac{e_A(t,\sigma(s))}{B}\sin_p\bigl(t,\sigma(s)\bigr)$$

and

$$\begin{aligned} u_{A,B}(t,s) = {} & \frac{e_A(\sigma(b),a)e_A(t,\sigma(s))}{LB} \\ & \times \big[e_A\big(\sigma(b),a\big)e_{\mu p^2}\big(\sigma(b),a\big)e_{\mu p^2}\big(t,\sigma(s)\big)\sin_p\big(\sigma(s),t\big) \\ & + \big(\sin_p\big(\sigma(b),\sigma(s)\big)\cos_p(t,a) + \cos_p\big(\sigma(b),\sigma(s)\big)\sin_p(t,a)\big)\big]. \end{aligned}$$

Here

$$p(t) = \frac{B}{1+\mu(t)A}$$

and

$$L = \big(1 - \cos_p\big(\sigma(b),a\big)e_A\big(\sigma(b),a\big)\big)^2 + \big(\sin_p\big(\sigma(b),a\big)e_A\big(\sigma(b),a\big)\big)^2.$$

Of course, this expression has sense and in consequence there is $(T_2[A,B])^{-1}$ in W_2, if and only if e_A, $\sin_p$ and $\cos_p$ are well defined and moreover $L \neq 0$. One can verify that it is not true if and only if one of the following situations holds:

(1) There exists some $t \in I$ such that $1 + \mu(t)A = 0$.
(2) $\mu \equiv 0$, $A = 0$ and $B = 2k\pi/(b-a)$ for some $k \in \mathbb{N}_0$.
(3) $-B < A < 0$, $\mu \equiv -2A/(A^2+B^2)$ and

$$\frac{B\mu}{1+\mu A} = \tan\left(\frac{2k\pi\mu}{\sigma(b)-a}\right) \quad \text{for some } k \in \mathbb{N}_0.$$

As we have seen in Lemma 3.2, the values of $A \in \mathbb{R}$ and $B > 0$ for which operator $T_2[A,B]$ is inverse positive on W_2, are the same parameters for which $G_{A,B} \geqslant 0$ in $\mathbb{T} \times I$.

To this end, define

$$F_{A,B}(s) = \begin{cases} \frac{\arctan(B\mu(s)/(1+\mu(s)A))}{\mu(s)}, & \mu(s) > 0, \\ B, & \mu(s) = 0, \end{cases} \tag{5.4}$$

where, by $\arctan x$ we denote the angle $\theta \in [0,\pi)$ such that $\tan\theta = x$. It is not difficult to verify that function $F_{A,B}$ is continuous in I and also respect to the parameters A and B.

Note that [2] in this situation

$$\sin_p(r,l) = \exp\left(\int_l^r \frac{\log\sqrt{1+p^2(\tau)\mu(\tau)}}{\mu(\tau)}\,\Delta\tau\right)\sin\left(\int_l^r F_{A,B}(\tau)\,\Delta\tau\right)$$

and

$$\cos_p(r,l) = \exp\left(\int_l^r \frac{\log\sqrt{1+p^2(\tau)\mu(\tau)}}{\mu(\tau)}\,\Delta\tau\right)\cos\left(\int_l^r F_{A,B}(\tau)\,\Delta\tau\right).$$

As consequence, we have that

$$\sin_p\bigl(\sigma(b),\sigma(s)\bigr)\cos_p(t,a)+\cos_p\bigl(\sigma(b),\sigma(s)\bigr)\sin_p(t,a)$$
$$=\exp\Biggl(\int_a^t \frac{\log\sqrt{1+p^2(\tau)\mu(\tau)}}{\mu(\tau)}\,\Delta\tau+\int_{\sigma(s)}^{\sigma(b)}\frac{\log\sqrt{1+p^2(\tau)\mu(\tau)}}{\mu(\tau)}\,\Delta\tau\Biggr)$$
$$\times\sin\Biggl(\int_a^t F_{A,B}(\tau)\,\Delta\tau+\int_{\sigma(s)}^{\sigma(b)}F_{A,B}(\tau)\,\Delta\tau\Biggr).$$

Thus, when $t<\sigma(s)$, since $F_{A,B}\geqslant 0$ in $[a,\sigma(b)]$, we have that

$$\int_a^t F_{A,B}(\tau)\,\Delta\tau+\int_{\sigma(s)}^{\sigma(b)}F_{A,B}(\tau)\,\Delta\tau\leqslant\int_a^{\sigma(b)}F_{A,B}(\tau)\,\Delta\tau.$$

In consequence, if $1+\mu(t)A>0$ for all $t\in I$ we have that $e_A(r,l)>0$ in $\mathbb{T}\times I$.

Thus, we can assure that when $t<\sigma(s)$ and $1+\mu(t)A>0$ for all $t\in I$ then $u_{A,B}(t,s)\geqslant 0$ whenever

$$\int_a^{\sigma(b)}F_{A,B}(\tau)\,\Delta\tau\in[0,\pi].$$

When $t\geqslant\sigma(s)$ we have that

$$LB\bigl(u_{A,B}(t,s)+v_{A,B}(t,s)\bigr)$$
$$=-e_A\bigl(t,\sigma(s)\bigr)e_{\mu p^2}\bigl(t,\sigma(s)\bigr)\sin_p\bigl(\sigma(s),t\bigr)$$
$$+e_A\bigl(t,\sigma(s)\bigr)e_{\mu p^2}\bigl(t,\sigma(s)\bigr)e_A\bigl(\sigma(b),a\bigr)$$
$$\times\bigl(\sin_p\bigl(\sigma(s),t\bigr)\cos_p\bigl(\sigma(b),a\bigr)+\cos_p\bigl(\sigma(s),t\bigr)\sin_p\bigl(\sigma(b),a\bigr)\bigr).$$

One can verify, see [2], that

$$\sin_p\bigl(\sigma(s),t\bigr)\cos_p\bigl(\sigma(b),a\bigr)+\cos_p\bigl(\sigma(s),t\bigr)\sin_p\bigl(\sigma(b),a\bigr)\bigr)$$
$$=\exp\Biggl(\int_t^{\sigma(s)}\frac{\log\sqrt{1+p^2(\tau)\mu(\tau)}}{\mu(\tau)}\,\Delta\tau+\int_a^{\sigma(b)}\frac{\log\sqrt{1+p^2(\tau)\mu(\tau)}}{\mu(\tau)}\,\Delta\tau\Biggr)$$
$$\times\sin\Biggl(\int_a^{\sigma(b)}F_{A,B}(\tau)\,\Delta\tau+\int_t^{\sigma(s)}F_{A,B}(\tau)\,\Delta\tau\Biggr).$$

In this case, since $\sigma(s)\leqslant t$ and $F_{A,B}\geqslant 0$ in I we know that

$$\int_a^{\sigma(b)}F_{A,B}(\tau)\,\Delta\tau+\int_t^{\sigma(s)}F_{A,B}(\tau)\,\Delta\tau\leqslant\int_a^{\sigma(b)}F_{A,B}(\tau)\,\Delta\tau.$$

In consequence, as in the previous case we conclude that when $t\geqslant\sigma(s)$, $1+\mu(t)A>0$ for all $t\in I$ and $\int_a^{\sigma(b)}F_{A,B}(\tau)\,\Delta\tau\in[0,\pi]$, then $u_{A,B}(t,s)+v_{A,B}(t,s)\geqslant 0$.

Thus, operator $T_2[A, B]$ is inverse positive in W_2 when

$$1 + \mu(t)A > 0 \quad \text{for all } t \in I \text{ and } \int_a^{\sigma(b)} F_{A,B}(s)\,\Delta s \leqslant \pi.$$

Now, assume that $1 + \mu(t)A > 0$ for all $t \in I$ and $\int_a^{\sigma(b)} F_{A,B}(s)\,\Delta s \in (\pi, 2\pi)$; then for $\sigma(s) = t$ we have $G_{A,B}(t, s) = e_A(\sigma(b), a) \sin_p(\sigma(b), a)/(LB) < 0$.

Suppose now that $1 + \mu(t)A > 0$ for all $t \in I$ and $\int_a^{\sigma(b)} F_{A,B}(s)\,\Delta s > 2\pi$. Due to the fact that $\mathcal{F} : B \in [0, +\infty) \to \int_a^{\sigma(b)} F_{A,B}(s)\,\Delta s \in [0, +\infty)$ is a continuous function such that $\mathcal{F}(0) = 0$, we have that there is $0 < B_1 < B$ satisfying $\int_a^{\sigma(b)} F_{A,B_1}(s)\,\Delta s \in (\pi, 2\pi)$ and, in consequence, operator $T_2[A, B_1]$ is not inverse positive on W_2. Now, from Theorem 5.1, we conclude that $T_2[A, B]$ cannot be inverse positive on W_2.

As consequence of all this results we arrive at the following one.

Lemma 5.3. *Let $A \in \mathbb{R}$ such that $1 + \mu(t)A > 0$ for all $t \in I$. Then operator $T_2[A, B]$ is inverse positive in W_2 if and only if*

$$\int_a^{\sigma(b)} F_{A,B}(s)\,\Delta s \in [0, \pi].$$

Remark 5.2. Note that when μ is a constant, function $F_{A,B}$ is also a constant, which is independent of A when $\mu \equiv 0$. Thus, assuming that $1 + \mu A > 0$ (which is always true when $\mu \equiv 0$), we have that operator $T_2[A, B]$ is inverse positive on W_2 if and only if

(1) $\mu \equiv 0$ and $B \leqslant \pi/(b - a)$.
(2) $\mu > 0$ and $0 < B/(1 + \mu A) \leqslant \tan(\pi\mu/(\sigma(b) - a))$.

The first estimate, for differential equations, has been obtained in [14], the second one, for difference equations ($\mu \equiv 1$), has been given in [7].

5.2. Higher order equations

In this section we obtain some estimates in the values of $M > 0$ for which operators

$$L_n^{\pm}[M]u(t) = u^{\Delta^n}(t) \pm Mu(t) \tag{5.5}$$

are inverse positive or inverse negative on W_n.

These estimates will be used to deduce extremal solutions of the nth order problem

$$u^{\Delta^n}(t) = f\big(t, u(t)\big) \quad \text{for all } t \in I = [a, b],$$
$$u^{\Delta^i}(a) = u^{\Delta^i}\big(\sigma(b)\big), \quad i = 0, \ldots, n-1,$$

as a particular case of Theorems 4.1 and 4.2 for problem (2.1)–(2.2).

First, we enunciate the following result which gives us a property of the composition of two inverse positive operators. The proof follows from the fact that the composition of two regressive operators is a regressive operator too and from similar arguments to the ones used in Lemma 2.3 in [5] and in Lemma 2.1 in [7].

Lemma 5.4. *Let S_l and S_m be two lth and mth order linear operators, inverse positive in W_l and inverse positive (inverse negative) on W_m, respectively. Then the composition operator $S_l \circ S_m$ is inverse positive (inverse negative) on W_{l+m}.*

Now, we are in a position to prove the following result in which estimates for operator $L_n^{\pm}[M]$ are given.

Lemma 5.5. *If $m \in (0, \underline{m}^n) \cap (0, \bar{m}^n]$ then operator $L_n^+[M]$ is inverse positive on W_n. Here $\bar{m} = +\infty$ if $P(m) < \pi$ for all $m > 0$ and $P^{-1}(\pi)$ otherwise, where $P : m \in [0, +\infty) \to P(m) = \int_a^{\sigma(b)} F_m(s)\, \Delta s \in [0, +\infty)$, and*

(1) *If $n = 4p$, $p \in \{1, 2, \ldots\}$, $\underline{m} = 1/\max_{t \in I}\{\mu(t)\cos(\pi/n)\}$ and*

$$F_m(s) = \begin{cases} \dfrac{\arctan\left(\frac{m \sin(\pi(n+2)/(2n))\mu(s)}{1+\mu(s) m \cos(\pi(n-1)/n)}\right)}{\mu(s)} & \text{if } \mu(s) > 0, \\ m \sin(\pi(n+2)/(2n)) & \text{if } \mu(s) = 0. \end{cases}$$

(2) *If $n = 2 + 4p$, $p \in \{1, 2, \ldots\}$, $\underline{m} = 1/\max_{t \in I}\{\mu(t)\cos(\pi/n)\}$ and*

$$F_m(s) = \begin{cases} \dfrac{\arctan\left(\frac{m\mu(s)}{1+\mu(s) m \cos(\pi(n-1)/n)}\right)}{\mu(s)} & \text{if } \mu(s) > 0, \\ m & \text{if } \mu(s) = 0. \end{cases}$$

(3) *If n is odd, $\underline{m} = 1/\max_{t \in I}\{\mu(t)\}$ and*

$$F_m(s) = \begin{cases} \dfrac{\arctan\left(\frac{m \sin(\pi(n+1)/(2n))\mu(s)}{1+\mu(s) m \cos(\pi(n-2)/n)}\right)}{\mu(s)} & \text{if } \mu(s) > 0, \\ m \sin(\pi(n+1)/(2n)) & \text{if } \mu(s) = 0. \end{cases}$$

Proof. Let $n = 4p$, for some $p \in \{1, 2, \ldots\}$, and denote by $m > 0$ such that $m^n = M$. In this case, it is clear that

$$L_n^+[M] \equiv N_0 \circ N_1 \circ \cdots \circ N_{(n-2)/2},$$

where $N_l u(t) = u^{\Delta^2}(t) - 2A_l u^{\Delta}(t) + (A_l^2 + B_l^2)u(t)$, $k \in I$, and

$$A_l = m \cos\left(\frac{2l+1}{n}\pi\right) \quad \text{and} \quad B_l = m \sin\left(\frac{2l+1}{n}\pi\right).$$

It is easy to verify that $B_l \leqslant B_{n/4} = m\sin(\pi(n+2)/(2n))$ and $A_l \geqslant A_{(n-2)/2} = m \times \cos(\pi(n-1)/n)$. Then, if $1 + \mu(t)A_{(n-2)/2} > 0$ and $\int_a^{\sigma(b)} F_{A_{(n-2)/2}, B_{n/4}}(s)\, \Delta s \leqslant \pi$, as a consequence of Lemmas 5.3 and 5.4 and the fact that $F_{A,B}(s)$ is increasing in B and decreasing in A, we conclude that operator $L_n^+[M]$ is inverse positive on W_n.

Clearly, function F_m is strictly increasing in m, and, as consequence, function $P(m) = \int_a^{\sigma(b)} F_m(s)\, \Delta s$ is strictly increasing too. Thus we have that $P(m) \leqslant \pi$ if and only if $m \in (0, \bar{m}]$, $\bar{m}$ given in the enunciate.

If $n = 2 + 4p$ for some $p \in \{1, 2, \ldots\}$ (case $p = 0$ is proved in Lemma 5.3), the same decomposition of operator $L_n^+[M]$ holds.

In this situation one can verify that $A_l \geqslant A_{(n-2)/2}$ and $B_l \leqslant B_{(n-2)/4} = m$ for all $l \in \{0, 1, \ldots, (n-2)/2\}$. Thus $F_{A_l, B_l} \leqslant F_{A_{(n-2)/2}, B_{(n-2)/4}}$. Reasoning as in the previous case we conclude this part.

If n is odd, we have that

$$L_n^+[M] \equiv N_0 \circ N_1 \circ \cdots \circ N_{(n-3)/2} \circ T_1^+[m].$$

In this case $B_l \leqslant m \sin(\pi(n+1)/(2n))$ which is equal to $B_{(n-1)/4}$ when $n = 4p + 1$ and $B_{(n-3)/4}$ when $n = 4p - 1$. Moreover $A_l \geqslant A_{(n-3)/2}$ for all $l \in \{0, 1, \ldots, (n-3)/2\}$.

Thus, if $m < 1/\mu(t)$ for all $t \in I$ and $\int_a^{\sigma(b)} F_{A_{(n-3)/2}, B_{l_0}}(s)\, \Delta s \leqslant \pi$ we have that operator $L_n^+[M]$ is inverse positive on W_n. Here by l_0 we denote $(n-1)/4$ if $n = 4p + 1$ and $(n-3)/4$ when $n = 4p - 1$. $\square$

In the same way one can prove the following result for operator $L_n^-[M]$.

Lemma 5.6. *If $m \in (0, \underline{m}^n) \cap (0, \bar{m}^n]$ then operator $L_n^-[M]$ is inverse negative on W_n with $\bar{m}$ given in the enunciate of Lemma* 5.5, *and*

(1) *If $n = 4p$, $p \in \{1, 2, \ldots\}$, $\underline{m} = 1/\max_{t \in I}\{\mu(t)\}$ and*

$$F_m(s) = \begin{cases} \dfrac{\arctan\left(\frac{m\mu(s)}{1+\mu(s)m\cos(\pi(n-2)/n)}\right)}{\mu(s)} & \text{if } \mu(s) > 0, \\ m & \text{if } \mu(s) = 0. \end{cases}$$

(2) *If $n = 2 + 4p$, $p \in \{1, 2, \ldots\}$, $\underline{m} = 1/\max_{t \in I}\{\mu(t)\}$ and*

$$F_m(s) = \begin{cases} \dfrac{\arctan\left(\frac{m\sin(\pi(n+2)/(2n))\mu(s)}{1+\mu(s)m\cos(\pi(n-2)/n)}\right)}{\mu(s)} & \text{if } \mu(s) > 0, \\ m\sin(\pi(n+2)/(2n)) & \text{if } \mu(s) = 0. \end{cases}$$

(3) *If n is odd, $\underline{m} = 1/\max_{t \in I}\{\mu(t)\cos(\pi/n)\}$ and*

$$F_m(s) = \begin{cases} \dfrac{\arctan\left(\frac{m\sin(\pi(n+1)/(2n))\mu(s)}{1+\mu(s)m\cos(\pi(n-1)/n)}\right)}{\mu(s)} & \text{if } \mu(s) > 0, \\ m\sin(\pi(n+1)/(2n)) & \text{if } \mu(s) = 0. \end{cases}$$

As we have noted in Remark 5.2, when μ is a constant, $F_{A,B}$ is also a constant. This fact permits us to calculate the exact value of the integrates of F_m and give explicitly the values $\bar{m}$ and $\underline{m}$. In fact the following estimates are given in [5, Lemmas 2.4 and 2.5] and in [7, Lemmas 2.3 and 2.4], here they are obtained as corollary of the two previous results.

Corollary 5.1. *Operator $L_n^+[M]$ is inverse positive on W_n if one of the following situations hold*:

(1) $n = 4p$, $p \in \{1, 2, \ldots\}$, $\mu \equiv 1$, $a = 0$, $b = N - 1$ *and*

$$M \leqslant \left[\frac{\tan(\pi/N)}{(1 + \tan(\pi/N))\cos(\pi/n)}\right]^n.$$

(2) *$n = 2 + 4p$, $p \in \{0, 1, \ldots\}$, $\mu \equiv 1$, $a = 0$, $b = N - 1$ and*

$$M \leqslant \left[\frac{\tan(\pi/N)}{1 + \tan(\pi/N)\cos(\pi/n)}\right]^n.$$

(3) *n odd, $\mu \equiv 1$, $a = 0$, $b = N - 1$ and*

$$M \leqslant \left[\frac{\tan(\pi/N)}{\tan(\pi/N)\cos(2\pi/n) + \cos(\pi/(2n))}\right]^n.$$

(4) *$n = 4p$, $p \in \{1, 2, \ldots\}$, $\mu \equiv 0$ and*

$$M \leqslant \left[\frac{\pi}{(b-a)\sin(\pi(n+2)/(2n))}\right]^n.$$

(5) *If $n = 2 + 4p$, $p \in \{1, 2, \ldots\}$, $\mu \equiv 0$ and*

$$M \leqslant \left[\frac{\pi}{b-a}\right]^n.$$

(6) *n odd, $\mu \equiv 0$ and*

$$M \leqslant \left[\frac{\pi}{(b-a)\sin(\pi(n+1)/(2n))}\right]^n.$$

Corollary 5.2. *Operator $L_n^-[M]$ is inverse negative on W_n if one of the following situations hold*:

(1) *$n = 4p$, $p \in \{1, 2, \ldots\}$, $\mu \equiv 1$, $a = 0$, $b = N - 1$ and*

$$M \leqslant \left[\frac{\tan(\pi/N)}{1 + \tan(\pi/N)\cos(2\pi/n)}\right]^n.$$

(2) *$n = 2 + 4p$, $p \in \{0, 1, \ldots\}$, $\mu \equiv 1$, $a = 0$, $b = N - 1$ and*

$$M \leqslant \left[\frac{\tan(\pi/N)}{\cos(\pi/n) + \tan(\pi/N)\cos(2\pi/n)}\right]^n.$$

(3) *n odd, $\mu \equiv 1$, $a = 0$, $b = N - 1$ and*

$$M \leqslant \left[\frac{\tan(\pi/N)}{\tan(\pi/N)\cos(\pi/n) + \cos(\pi/(2n))}\right]^n.$$

(4) *$n = 4p$, $p \in \{1, 2, \ldots\}$, $\mu \equiv 0$ and*

$$M \leqslant \left[\frac{\pi}{b-a}\right]^n.$$

(5) *$n = 2 + 4p$, $p \in \{1, 2, \ldots\}$, $\mu \equiv 0$ and*

$$M \leqslant \left[\frac{\pi}{(b-a)\sin(\pi(n+2)/(2n))}\right]^n.$$

(6) *n odd, $\mu \equiv 0$ and*

$$M \leqslant \left[\frac{\pi}{(b-a)\sin(\pi(n+1)/(2n))}\right]^n.$$

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