Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 334 (2007) 368–381
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Symmetry-based solution of a model for a combination of a risky investment and a riskless investment

P.G.L. Leach [a,*], J.G. O'Hara [b], W. Sinkala [c]

[a] *School of Mathematical Sciences, Howard College, University of KwaZulu-Natal, Durban 4041, South Africa*
[b] *School of Statistics and Actuarial Science, Howard College, University of KwaZulu-Natal, Durban 4041, South Africa*
[c] *Department of Mathematics and Applied Mathematics, Faculty of Science and Engineering, Walter Sisulu University, Private Bag X1, Mthatha 5117, South Africa*

Received 15 June 2006

Available online 4 January 2007

Submitted by M. Peligrad

Abstract

Benth and Karlsen [F.E. Benth, K.H. Karlsen, A note on Merton's portfolio selection problem for the Schwartz mean-reversion model, Stoch. Anal. Appl. 23 (2005) 687–704] treated a problem of the optimisation of the selection of a portfolio based upon the Schwartz mean-reversion model. The resulting Hamilton–Jacobi–Bellman equation in $1+2$ dimensions is quite nonlinear. The solution obtained by Benth and Karlsen was very ingenious. We provide a solution of the problem based on the application of the Lie theory of continuous groups to the partial differential equation and its associated boundary and terminal conditions.

Keywords: Mean-reversion; Portfolio selection; Lie symmetry

1. Introduction

The use of symmetry reductions for the partial differential equations which arise in the modelling of various aspects of investment strategies in Financial Mathematics is not well developed.

* Corresponding author.
E-mail addresses: leachp@ukzn.ac.za (P.G.L. Leach), oharaj@ukza.ac.za (J.G. O'Hara), sinkala@getafix.utr.ac.za (W. Sinkala).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.11.056

In part this could be due to the relatively recent expansion in the mathematical modelling of various financial instruments such as options and other derivatives and in part to the general lack of appreciation of the value and applicability of symmetry analysis in the solution of differential equations once one moves from certain, relatively specialised, areas such as gas dynamics. Even in those areas of Physics which have been 'traditionally' imbued with the spirit of group theory such as Quantum Mechanics the group theoretic approach has been based in the main on second-order Casimir operators and applications in Financial Mathematics which flow from physical considerations have tended to follow suit (see, for example, [18]). One of the first, if not the first, application of the mathematical theory of Lie algebras to an evolution equation arising in Financial Mathematics was made by Gazizov and Ibragimov [9] to that most famous equation of this field, the Black–Scholes equation [6,7,21] which is an evolution equation particularly rich in symmetry. Some further applications are found in [4,29,35] and [36]. Recently Myeni and Leach [26,27] extended recent investigations of Complete Symmetry Groups in the area of ordinary differential equations [1–3,32] to some of the evolution equations of the type found in Financial Mathematics. It would be a fair comment to remark that the symmetry analysis of the equations obtained in the mathematical modelling of various financial instruments has only just begun.

In this paper we present a group theoretical analysis of an evolution equation discussed in a paper by Benth and Karlsen [5]. In their paper Benth and Karlsen consider the optimisation problem of an investor dividing his funds between an investment which is risky and one which is riskless. Benth and Karlsen provided the solution to the resultant terminal-boundary value problem by what appears to be an ad hoc method which nevertheless is very sophisticated in that an ingenious *Ansatz* for the structure of the solution leads to an elegant resolution of the problem. Here we do not rely on the availability of an ingenious *Ansatz*. Rather we follow the algorithmic procedure of the Lie analysis. It may come as no great surprise that the fortunate *Ansatz* of Benth and Karlsen coincides with the strategy for solution dictated by the methodology of the symmetry analysis.

The evolution equation derived by Benth and Karlsen is nonlinear and in our analysis of this equation for its Lie point symmetries we observe some interesting properties of the group structure of the equation and what can only be described as a very fortuitous coincidence of symmetry property and the conditions for the solution of the total problem of equation plus conditions. As in all analyses the successful resolution of the problem at hand opens an avenue for the further investigation of the properties of these evolution equations arising in the mathematical modelling of scenarii in Financial Mathematics having the nature of an Hamilton–Jacobi–Bellman equation [28,29].

2. The optimisation problem

Benth and Karlsen consider the classical problem of the optimisation of a portfolio proposed by Merton [19,20] in which the risky asset follows an exponential Ornstein–Uhlenbeck process—also known as the Schwartz mean-reversion dynamics. We give a flavour of the derivation of the equation.

Suppose that we have a complete probability space, $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P\}$, which satisfies the usual hypotheses of a complete probability space. Let B_t be a Brownian motion defined upon it. We take the time horizon to be finite, i.e. $T < \infty$. It is assumed that the dynamics of the price of the risky asset follow the Schwartz model [33], *videlicet*

$$\mathrm{d}S_t = \alpha(\mu - \log S_t)S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}B_t, \tag{2.1}$$

where S_0 is the price observed at time $t = 0$ and α and μ are constants. The speed of mean-reversion is measured by α (> 0) and its level by μ. The volatility, σ, is taken to be a positive constant. The logarithm of S_t, which we denote by X_t ($:= \log S_t$), is an Ornstein–Uhlenbeck process, i.e.

$$\mathrm{d}X_t = \alpha\left(\mu + \frac{\sigma^2}{2\alpha} - X_t\right)\mathrm{d}t + \sigma\,\mathrm{d}B_t. \tag{2.2}$$

The risk-free investment, for example a bond, has the usual dynamics of

$$\mathrm{d}R_t = rR_t\,\mathrm{d}t, \quad R_0 = 1, \tag{2.3}$$

where r (> 0) is the continuously compounded risk-free interest rate which we take to be a constant.

Initially the investor starts at a time $t \leqslant T$ with wealth $W_t = w$. The spot-price level is $S_t = s$. The investor allocates a fraction, π_u, $u \in [t, T]$, where π_u is a progressively measurable process, of the total wealth, W_u, in the risky asset. We assume that the investor has a risk preference described by the power utility function, $U(w) = \gamma^{-1}w^\gamma$, where $\gamma \in (0, 1)$. For the investor the optimisation problem is to find an admissible trading strategy, π_t, such that the utility derived from the wealth is maximised at the final time, T. The value function is defined as

$$v(s, t, w) = \sup_{\pi \in \mathcal{A}_t} E^{s,t,w}\big[U\big(W_T^\pi\big)\big], \tag{2.4}$$

where the supremum is taken over all admissible strategies, $\mathcal{A}_t$. The operator, $E^{s,t,w}$, represents the expectation conditioned on $W_t^\pi = w$ and $S_t = s$. Benth and Karlsen use the classical dynamic programming principle to derive the corresponding Hamilton–Jacobi–Bellman equation for the value function (2.4) to be

$$\begin{aligned} &v_t + rwv_w + \alpha(\mu - \log s)sv_s + \frac{1}{2}\sigma^2 s^2 v_{ss} - \frac{1}{2}\sigma^2 s^2 \frac{v_{ws}^2}{v_{ww}} \\ &\quad - \frac{[\alpha(\mu - \log s) - r]^2}{2\sigma^2}\frac{v_w^2}{v_{ww}} - \big[\alpha(\mu - \log s) - r\big]s\frac{v_w v_{ws}}{v_{ww}} = 0, \end{aligned} \tag{2.5}$$

which is a nonlinear $2 + 1$ partial differential evolution equation.

The full problem to be solved is Eq. (2.5) subject to the terminal and boundary conditions

$$v(s, T, w) = \gamma^{-1}w^\gamma \quad \text{and} \quad v(s, T, 0) = 0. \tag{2.6}$$

Before we commence our analysis we provide a brief summary of the method of analysis used by Benth and Karlsen. Firstly they reduce the Hamilton–Jacobi–Bellman equation, (2.5), to a $1 + 1$ evolution equation and remove the nonlinearities by means of a transformation of Hopf–Cole type recently introduced into the theory of portfolio optimisation by Zariphopoulou [37]. Following this transformation and inspired by the solution of Merton [19] Benth and Karlsen make an *Ansatz* for the solution of (2.5) of the form

$$v(s, t, w) = \gamma^{-1}w^\gamma g(s, t)^{1-\gamma}, \tag{2.7}$$

where $g(s, t)$ is to be determined from the solution of the $1 + 1$ evolution equation

$$\begin{aligned} &g_t + \frac{1}{1-\gamma}\big[\alpha(\mu - \log s) - \gamma r\big]sg_s + \frac{1}{2}\sigma^2 s^2 g_{ss} \\ &\quad + \left\{\frac{r\gamma}{1-\gamma} + \frac{\gamma[\alpha(\mu - \log s) - r]^2}{2\sigma^2(1-\gamma)^2}\right\}g = 0 \end{aligned} \tag{2.8}$$

subject to the terminal condition

$$g(s,T)=1. \tag{2.9}$$

Benth and Karlsen make the further *Ansatz* that

$$g(s,t)=\exp\bigl[f_0(t)+f_1(t)\log s+f_2(t)\log^2 s\bigr] \tag{2.10}$$

and determine the three functions $f_i(t)$, $i=0,2$, by substitution into (2.8) and then specification of the constants of integration through satisfaction of (2.9).

One can only marvel at the ingenuity of the process of solution devised by the apparently ad hoc considerations which Benth and Karlsen used.

3. Symmetry analysis of (2.5)

In terms of the variable, s, (2.5) has the structure of an equation of Euler type. We introduce the change of variables and notation $v(s,t,w)\to w(t,x,y)$, where t remains unchanged, w is replaced by x and $y=\mu-\log s$. Equation (2.5) is now

$$\begin{aligned} &w_t+rxw_x-\alpha yw_y+\frac{1}{2}\sigma^2(w_{yy}+w_y)\\ &\quad-\frac{1}{2}\sigma^2\frac{w_{xy}^2}{w_{xx}}-\frac{(\alpha y-r)^2}{2\sigma^2}\frac{w_x^2}{w_{xx}}+(\alpha y-r)\frac{w_xw_{xy}}{w_{xx}}=0. \end{aligned} \tag{3.1}$$

We make use of Program LIE [10,34] to calculate the Lie point symmetries of (3.1).[1] We obtain the symmetries

$$\Gamma_1=\partial_t, \tag{3.2}$$

$$\Gamma_2=x\partial_x, \tag{3.3}$$

$$\Gamma_3=w\partial_w, \tag{3.4}$$

$$\Gamma_4=f(t,y)\partial_w, \tag{3.5}$$

$$\Gamma_5=g(t,y)\partial_x, \tag{3.6}$$

where $f(t,y)$ and $g(t,y)$ are solutions of the linear parabolic equations

$$\frac{\partial f}{\partial t}+\frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial y^2}+\left(\frac{1}{2}\sigma^2-\alpha y\right)\frac{\partial f}{\partial y}=0 \tag{3.7}$$

and

$$\frac{\partial g}{\partial t}+\frac{1}{2}\sigma^2\frac{\partial^2 g}{\partial y^2}+\left(\frac{1}{2}\sigma^2-r\right)\frac{\partial g}{\partial y}-rg=0, \tag{3.8}$$

respectively.

This result is rather unexpected for an evolution equation in that usually one finds at most a single family of an infinite class of symmetries. Furthermore this infinite class is normally associated with a linear equation although for an exception see Leach et al. [16]. The interesting point for (3.1) is that the two infinite families are based upon different variables. The infinite class of

[1] Equally one could use the very effective code of Nucci [30,31]. Other packages are available and the reader is referred to the review papers of Hereman [11,12] for an assessment of their capabilities and characteristics.

symmetries associated with the dependent variable w, *videlicet* Γ_4, is moderately normal and one could ascribe its existence to the homogeneity of the equation in w. On the other hand the infinite class of symmetries associated with the independent variable x, *videlicet* Γ_5, is most unexpected although we note that (3.1) is also homogeneous in x. When one looks at the remaining symmetries, the impression of an equivalence between x and w is deepened. It is almost as if we have a joining of two $1+1$ evolution equations with the only nongeneric symmetry being due to their autonomy.

For the present sake of completeness we list the Lie point symmetries of (3.7) and (3.8). They are in turn

$$\Delta_1 = \exp[\alpha t]\left\{\frac{1}{2}\sigma^2\partial_y - \left(\frac{1}{2}\sigma^2 - \alpha y\right) f\partial_f\right\}, \tag{3.9}$$

$$\Delta_2 = \exp[-\alpha t]\partial_y, \tag{3.10}$$

$$\Delta_3 = \partial_t, \tag{3.11}$$

$$\Delta_4 = \exp[2\alpha t]\left\{\frac{1}{2}\sigma^2\partial_t + \frac{1}{2}\sigma^2\left(\frac{1}{2}\sigma^2 - \alpha y\right)\partial_y + \left[\frac{1}{2}\sigma^2\alpha - \left(\frac{1}{2}\sigma^2 - \alpha y\right)^2\right] f\partial_f\right\}, \tag{3.12}$$

$$\Delta_5 = \exp[-2\alpha t]\left\{\partial_t + \left(\frac{1}{2}\sigma^2 - \alpha y\right)\partial_y\right\}, \tag{3.13}$$

$$\Delta_6 = f\partial_f, \tag{3.14}$$

$$\Delta_7 = p(t, y)\partial_f, \tag{3.15}$$

where $p(t, y)$ is a solution of (3.7), and

$$\Sigma_1 = \partial_y, \tag{3.16}$$

$$\Sigma_2 = \sigma^2 t\partial_y - \left[\left(\frac{1}{2}\sigma^2 - r\right)t - y\right] g\partial_g, \tag{3.17}$$

$$\Sigma_3 = \partial_t, \tag{3.18}$$

$$\Sigma_4 = 2t\partial_t + \left[\left(\frac{1}{2}\sigma^2 - r\right)t + y\right]\partial_y + 2rtg\partial_g, \tag{3.19}$$

$$\Sigma_5 = 2\sigma^2 t^2\partial_t + 2\sigma^2 ty\partial_y + \left\{\left[\left(\frac{1}{2}\sigma^2 - r\right)t - y\right]^2 - \sigma^2 t\right\} g\partial_g, \tag{3.20}$$

$$\Sigma_6 = g\partial_g, \tag{3.21}$$

$$\Sigma_7 = q(t, y)\partial_g, \tag{3.22}$$

where $q(t, y)$ is a solution of (3.8), respectively.

It is obvious from the symmetries that each of (3.7) and (3.8) is related to the classical heat equation by means of a point transformation just as is the case with the Black–Scholes equation. The algebra of the symmetries in each case is $\{sl(2, R) \oplus_s W\} \oplus_s \infty A_1$, where W is the Weyl–Heisenberg algebra comprising the symmetries 1, 2 and 6, which is also denoted as $A_{3,1}$ in the Mubarakzyanov classification scheme [22–25], and ∞A_1 is the infinite-dimensional algebra of solution symmetries represented by symmetry 7.

4. The first similarity reduction

In terms of the variables of Section 3 the problem under consideration, (2.5) subject to (2.6), is

$$w_t + rxw_x - \alpha y w_y + \frac{1}{2}\sigma^2(w_{yy} + w_y) \\ - \frac{1}{2}\sigma^2 \frac{w_{xy}^2}{w_{xx}} - \frac{(\alpha y - r)^2}{2\sigma^2}\frac{w_x^2}{w_{xx}} + (\alpha y - r)\frac{w_x w_{xy}}{w_{xx}} = 0 \tag{4.1}$$

subject to

$$w(T, x, y) = \gamma^{-1}x^\gamma \quad \text{and} \quad w(T, 0, y) = 0. \tag{4.2}$$

The symmetries listed in (3.2)–(3.6) are symmetries for Eq. (4.1). They are not necessarily compatible with the conditions (4.2). Indeed singly they are not. We recall that the derivation of the Lie point symmetries of a differential equation results in the presentation of a (hopefully!) multiparameter differential operator, *the symmetry*, which leaves an equation invariant when the equation is taken into account.[2] The practice of breaking the single multiparameter symmetry into many single-parameter symmetries is dictated by the desire to elucidate the different types of infinitesimal transformation under which the partial differential equation is invariant and to understand the classification of the equation through the algebra represented by the several one-parameter transformation groups. The decomposition is not unique and can even be subject to the personal preferences of the writer, for example a preference for $sl(2, R)$ over $so(2, 1)$. When it comes to the consideration of the admissibility of the boundary/terminal conditions, as in (4.2), it is better to take the original form of the symmetry, i.e. the generator of a multiparameter infinitesimal transformation.

We write

$$\Lambda = \sum_{i=1}^{5} a_i \Gamma_i = a_1\partial_t + (a_2 x + a_5 g)\partial_x + (a_3 w + a_4 f)\partial_w \tag{4.3}$$

and apply Λ to the conditions (4.2) expressed as

$$t - T = 0, \qquad w - \gamma^{-1}x^\gamma = 0 \tag{4.4}$$

and

$$x = 0, \qquad w = 0. \tag{4.5}$$

For (4.4) we have $a_1 = 0$ and

$$a_3 w + a_4 f(T, y) - \gamma^{-1}\big(\gamma a_2 x^\gamma + \gamma a_5 g(T, y)x^{\gamma-1}\big) = 0$$

so that

$$(a_3 - \gamma a_2)\gamma^{-1}x^\gamma + a_4 f(T, y) - a_5 g(T, y)x^{\gamma-1} = 0 \tag{4.6}$$

from which it is evident that

$$a_3 = \gamma a_2, \quad a_4 = 0 \quad \text{and} \quad a_5 = 0, \tag{4.7}$$

[2] Here we do not enter into a discussion of the different types of symmetry or the object of which they are symmetries. Rather we confine our attention to the matter at hand which is the existence of symmetry in a given partial differential equation.

since (4.6) is an identity for all x and y, in addition to $a_1 = 0$. The requirement (4.5) makes no additional demands. Hence we have that the symmetry consistent with both (4.1) and (4.2) is the one-parameter generator

$$\Lambda_a = x\partial_x + \gamma w\partial_w, \tag{4.8}$$

where we drop the single parameter, a_2, as being inessential since it is a common multiplier.[3]

We have a single Lie point symmetry, (4.8), consistent with both the partial differential equation (4.1) and the associated conditions (4.2). We determine the invariants for the reduction from a $2+1$ equation to a $1+1$ equation from the associated Lagrange's system

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{x} = \frac{\mathrm{d}y}{0} = \frac{\mathrm{d}w}{\gamma w}. \tag{4.9}$$

The invariants are t, y and $wx^{-\gamma}$ so that the reduction is achieved by writing

$$w(t,x,y) = \gamma^{-1}x^{\gamma}u(t,y) \tag{4.10}$$

in which we immediately recognise a theoretical basis for the *Ansatz*, (2.7), of Benth and Karlsen.[4]

Under the transformation (4.10) the system (4.1), (4.2) becomes

$$\begin{aligned}\frac{\partial u}{\partial t} &+ \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial y^2} - \frac{1}{2}\sigma^2\frac{\gamma}{\gamma-1}\frac{1}{u}\left(\frac{\partial u}{\partial y}\right)^2 + \left[\frac{1}{2}\sigma^2 - \alpha y + \frac{\gamma}{\gamma-1}(\alpha y - r)\right]\frac{\partial u}{\partial y}\\ &+ \left[\gamma r - \frac{(\alpha y - r)^2}{2\sigma^2}\frac{\gamma}{\gamma-1}\right]u = 0\end{aligned} \tag{4.11}$$

and

$$u(T,y) = 1 \tag{4.12}$$

as the sole remaining condition since the transformation (4.10) automatically provided satisfaction of (4.2).[5] Equation (4.11) is still nonlinear, but of a form which can be rendered linear by means of a simple transformation of the type introduced by Zariphopoulou [37]. We derive the linearisation transformation via an algorithm of Lie symmetry analysis. With the help of Program LIE and/or other packages one can easily show that Eq. (4.11) admits seven Lie point symmetries including the generator of an infinite-parameter Lie group of point transformations,

$$\Upsilon_\zeta = u^{\gamma/(\gamma-1)}\zeta(t,y)\partial_u, \tag{4.13}$$

where $\zeta(t,y)$ is any solution of the linear equation

$$\frac{\partial \zeta}{\partial t} + \frac{1}{2}\sigma^2\left(\frac{\partial^2\zeta}{\partial y^2} + \frac{\partial\zeta}{\partial y}\right) - \frac{1}{1-\gamma}(\alpha y - \gamma r)\frac{\partial\zeta}{\partial y} + \frac{\gamma}{1-\gamma}\left[r + \frac{(\alpha y - r)^2}{2\sigma^2(1-\gamma)}\right]\zeta = 0. \tag{4.14}$$

[3] Given that the 'solution' symmetries are virtually doomed to make no contribution the existence of even a one-parameter group is remarkable since the partial differential equation (4.1) is not generously endowed with nongeneric Lie point symmetries. The possibility of a richer result could occur if the 'solution symmetries' were simply functions of t for then (4.6) would not contain terms with an independent dependence upon y.

[4] The two expressions are not precisely identical. The reduction of Benth and Karlsen has greater internal structure than the form the symmetry (4.8) conveys. For the moment it suffices that *Ansatz* and algorithm—if not precisely *Satz*—move in parallel. However, we have included the numerical factor, γ^{-1}, somewhat gratuitously simply to keep as close a parallel with the reduction of Benth and Karlsen and yet at the same time to avoid asymmetric assumptions.

[5] The given domain of γ is particularly convenient in this respect!

By appealing to Theorems 6.4.4-1, 6.4.4-2 of [8] we see that there exists an invertible transformation that maps (4.11) into (4.14) of the form

$$\zeta = \varphi(t, y, u), \tag{4.15}$$

where φ is governed by the first-order partial differential equation

$$u^{\gamma/(\gamma-1)}\frac{\partial\varphi}{\partial u} = 1. \tag{4.16}$$

A particular solution of (4.16) is

$$\varphi(t, y, u) = (1-\gamma)u^{1/(1-\gamma)}. \tag{4.17}$$

From (4.15) and (4.17), and after scaling away the unnecessary constant $(1-\gamma)^{\gamma-1}$, we have that the invertible transformation that maps the nonlinear equation (4.11) into the linear equation (4.14) is

$$u(t, y) = \zeta^{1-\gamma}(t, y). \tag{4.18}$$

The terminal condition, (4.12), is essentially unaltered under the transformation (4.18) since now $\zeta(T, y) = 1$.

We note that (4.14) becomes (3.7) when we set $\gamma = 0$. However, we recall that the model is proposed for $\gamma \in (0, 1)$ and so $\gamma = 0$ is outside the range of the model.[6]

In the reduction of the $2+1$ equation, (3.1), to the $1+1$ equation, (4.14), by means of the Lie point symmetries of (3.1) we have recovered the transformation (2.7) employed by Benth and Karlsen. One can only marvel at the prescience of their *Ansatz*!

5. Solution of the reduced system

The mathematical model for the problem of the optimisation of a portfolio with the utility function assumed has been reduced to the $1+1$ linear evolution equation

$$\frac{\partial\zeta}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2\zeta}{\partial y^2} + \left[\frac{1}{2}\sigma^2 - \frac{1}{1-\gamma}(\alpha y - \gamma r)\right]\frac{\partial\zeta}{\partial y} + \frac{\gamma}{1-\gamma}\left[r + \frac{(\alpha y - r)^2}{2\sigma^2(1-\gamma)}\right]\zeta = 0 \tag{5.1}$$

subject to the terminal condition

$$\zeta(T, y) = 1. \tag{5.2}$$

Equation (5.1) is a linear $1+1$ evolution equation and as such has $n+1+\infty$, $n = 0, 1, 3, 5$, Lie point symmetries where the precise value of n depends upon the coefficients in the equation [4,29]. Since (5.1) is autonomous, n is at least one. The potential reduction of symmetry from the maximal value comes from the term in $\partial\zeta/\partial y$. To identify the subset to which (5.1) belongs we may perform a standard change of variables by writing

$$\zeta(t, y) = \phi(t, y)\exp\left[at + by + cy^2\right], \tag{5.3}$$

where a, b and c are parameters to be determined. When we make this substitution, (5.1) becomes

6 This does not mean that (3.7) is irrelevant to the solution of (4.14). A similar situation has been reported in the analysis of the Riemann formulation of the isentropic, unsteady, flow of a compressible gas in which the critical parameter is a function of the ratio of specific heats. When the ratio is a very nonphysical infinity, the resulting value of the parameter gives an equation from which other solutions follow [15].

$$
\begin{aligned}
&\frac{\partial\phi}{\partial t}+\frac{1}{2}\sigma^2\frac{\partial^2\phi}{\partial y^2}+\left\{\left(\frac{1}{2}+b+2cy\right)\sigma^2-\frac{\alpha y-\gamma r}{1-\gamma}\right\}\frac{\partial\phi}{\partial y}\\
&\quad+\left\{a+c\sigma^2+(b+2cy)\left[\frac{1}{2}(1+b+2cy)\sigma^2-\frac{\alpha y-\gamma r}{1-\gamma}\right]\right.\\
&\quad\left.+\frac{\gamma r}{1-\gamma}+\frac{\gamma}{(1-\gamma)^2}\frac{(\alpha y-r)^2}{2\sigma^2}\right\}\phi=0.
\end{aligned}
\tag{5.4}
$$

We remove the term in $\partial\phi/\partial y$ by setting

$$c=\frac{\alpha}{2\sigma^2(1-\gamma)}\quad\text{and}\quad b=-\frac{1}{2}-\frac{\gamma r}{\sigma^2(1-\gamma)}. \tag{5.5}$$

The coefficient of ϕ is rendered reasonably compact if we set

$$a=-\frac{\alpha}{2(1-\alpha)}-\frac{\gamma}{2\sigma^2(1-\gamma)}\left(r+\frac{1}{2}\sigma^2\right)^2 \tag{5.6}$$

and we obtain

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}\sigma^2\frac{\partial^2\phi}{\partial y^2}-\frac{\alpha^2}{2\sigma^2(1-\gamma)}\left(y-\frac{\sigma^2}{2\alpha}\right)^2\phi=0. \tag{5.7}$$

The form of (5.7) is made more obvious by the cosmetic changes of variables

$$t=\frac{\tau}{\sigma^2},\qquad y-\frac{\sigma^2}{2\alpha}=\rho\quad\text{and}\quad\phi(t,y)=\psi(\tau,\rho) \tag{5.8}$$

so that we now have

$$\frac{\partial\psi}{\partial\tau}+\frac{1}{2}\sigma^2\frac{\partial^2\psi}{\partial\rho^2}-\frac{1}{2}\Omega\rho^2\psi=0,\quad \Omega^2=\frac{\alpha^2}{\sigma^4(1-\gamma)}\neq 0 \tag{5.9}$$

and this is a form for which the value of n is manifestly 5. Consequently we have the reasonable expectation that at least one symmetry compatible with the terminal condition, (5.2), exists.

We use LIE 51 to compute the symmetries (5.9). They are

$$
\begin{aligned}
&\Sigma_1=\exp[\Omega\tau][\partial_\rho+\Omega\rho\psi\partial_\psi],\\
&\Sigma_2=\exp[-\Omega\tau][\partial_\rho-\Omega\rho\psi\partial_\psi],\\
&\Sigma_3=\partial_\tau,\\
&\Sigma_4=\exp[2\Omega\tau]\left[\partial_\tau+\Omega\rho\partial_\rho+\left(\Omega^2\rho^2-\frac{1}{2}\Omega\tau\right)\psi\partial_\psi\right],\\
&\Sigma_5=\exp[-2\Omega\tau]\left[\partial_\tau-\Omega\rho\partial_\rho+\left(\Omega^2\rho^2+\frac{1}{2}\Omega\tau\right)\psi\partial_\psi\right],\\
&\Sigma_6=\psi\partial_\psi,\\
&\Sigma_7=f(\tau,\rho)\partial_\psi,
\end{aligned}
\tag{5.10}
$$

where $f(\tau,\rho)$ is a solution of (5.9) and provides the infinite subalgebra of solution symmetries.

We rewrite the symmetries listed in (5.10) in a form appropriate for (5.7) as

$$\Delta_1=\exp\left[\sigma^2\Omega t\right]\left(\partial_y+\Omega\left(y-\frac{\sigma^2}{2\alpha}\right)\phi\partial_\phi\right),$$

$$\begin{aligned}
&\Delta_2 = \exp\left[-\sigma^2\Omega t\right]\left(\partial_y - \Omega\left(y - \frac{\sigma^2}{2\alpha}\right)\phi\partial_\phi\right),\\
&\Delta_3 = \partial_t,\\
&\Delta_4 = \exp\left[2\sigma^2\Omega t\right]\left[\partial_t + \sigma^2\Omega\left(y - \frac{\sigma^2}{2\alpha}\right)\partial_y + \left(\sigma^2\Omega^2\left(y - \frac{\sigma^2}{2\alpha}\right)^2 - \frac{1}{2}\sigma^4\Omega t\right)\phi\partial_\phi\right],\\
&\Delta_5 = \exp\left[-2\sigma^2\Omega t\right]\left[\partial_t - \sigma^2\Omega\left(y - \frac{\sigma^2}{2\alpha}\right)\partial_y + \left(\sigma^2\Omega^2\left(y - \frac{\sigma^2}{2\alpha}\right)^2 + \frac{1}{2}\sigma^2\Omega t\right)\phi\partial_\phi\right],\\
&\Delta_6 = \phi\partial_\phi,\\
&\Delta_7 = f\left(\sigma^2 t, y - \frac{\sigma^2}{2\alpha}\right)\partial_\phi
\end{aligned} \tag{5.11}$$

in which there has been a certain amount of cosmetic rearrangement of constants.

Our task is now to find such linear combinations of the symmetries Δ_1 to Δ_6 which are consistent with the terminal condition

$$t = T \quad \text{and} \quad \phi(T, y) = \exp\left[-aT - by - cy^2\right]. \tag{5.12}$$

To determine the combinations we write a general symmetry as

$$\Delta = \sum_{i=1}^{6} k_i \Delta_i \tag{5.13}$$

and apply it to the conditions in (5.12) as an identity. To maintain a modicum of compactness in the notation we write

$$\begin{aligned}
&k_1 \exp\left[\sigma^2\Omega T\right] = K_1, \qquad k_2 \exp\left[-\sigma^2\Omega T\right] = K_2,\\
&k_4 \exp\left[2\sigma^2\Omega T\right] = K_4, \qquad k_5 \exp\left[-2\sigma^2\Omega T\right] = K_5.
\end{aligned} \tag{5.14}$$

Although the value of T is fixed, that of y is not. Thus we obtain four linear equations for the six parameters, k_i, $i = 1, 6$. Commencing with the condition on the time the equations are

$$k_3 + K_4 + K_5 = 0, \tag{5.15}$$

$$K_4\left[\sigma^2\Omega^2 + \frac{\Omega\alpha}{1-\gamma}\right] + K_5\left[\sigma^2\Omega^2 - \frac{\Omega\alpha}{1-\gamma}\right] = 0, \tag{5.16}$$

$$\begin{aligned}
&K_1\left[\frac{\alpha}{\sigma^2(1-\gamma)} + \Omega\right] + K_2\left[\frac{\alpha}{\sigma^2(1-\gamma)} - \Omega\right]\\
&\quad + K_4\frac{\Omega\gamma(\sigma^2 - 2r)}{2(1-\gamma)} - K_5\frac{\Omega\gamma(\sigma^2 - 2r)}{2(1-\gamma)} = 0,
\end{aligned} \tag{5.17}$$

$$K_1\frac{\gamma(\sigma^2 - 2r)}{2\sigma^2(1-\gamma)} + K_2\frac{\gamma(\sigma^2 - 2r)}{2\sigma^2(1-\gamma)} - \frac{1}{4}K_4\sigma^4\Omega T - \frac{1}{4}K_5\sigma^4\Omega T + k_6 = 0. \tag{5.18}$$

It should be quite apparent that k_3 and k_6 are suitable to be taken as the independent parameters. Consequently there are two symmetries compatible with the terminal condition. One is dependent upon the parameter k_3 and the other upon k_6. We need use only one of them since the uniqueness of the solution of (5.7) subject to the terminal condition (5.12) follows from the Fokker–Planck Theorem. From (5.15) and (5.16) it is evident that k_4 and k_5 are zero if one takes $k_3 = 0$. This makes the equations for k_1 and k_2 simpler and the resulting symmetry does not contain ∂_t. The symmetries listed in (5.11), apart from Δ_7, divide naturally into two three-dimensional subalgebras. One subalgebra is $sl(2,R)$ and the other is the Weyl–Heisenberg algebra denoted by $A_{3,1}$ in the Mubarakzyanov classification scheme. The elements of the former combine to make the symmetry based on a nonzero value of k_3 whereas the elements of the latter are found in the symmetry based on a nonzero value of k_6. Since the combined algebra is $sl(2,R)\oplus_s A_{3,1}$, not only is the symmetry based upon k_6 simpler in form it is also the normal subgroup for the reduction of partial differential equation to an ordinary differential equation. Even though we do not use the symmetry based on k_3, its persistence in the reduction facilitates the solution of the resulting ordinary differential equation. The greater ease of finding a solution due to its presence against the situation in which there is only one symmetry has been noted [29].

We now use the symmetry,

$$\begin{aligned}\Delta_W = {} & \big\{\alpha\sinh\big[\sigma^2\Omega(t-T)\big]-\sigma^2(1-\gamma)\cosh\big[\sigma^2\Omega(t-T)\big]\big\}\partial_y \\ & + \left\{\Omega\left(y-\frac{\sigma^2}{2\alpha}\right)\big[\alpha\cosh\big[\sigma^2\Omega(t-T)\big]-\Sigma^2(1-\gamma)\sinh\big[\sigma^2\Omega(t-T)\big]\big]\right. \\ & \left.+\frac{1}{2}\gamma\big(\sigma^2-2r\big)\right\}\phi\partial_\phi + A(t)\partial_y + \left[B+\Omega\left(y-\frac{\sigma^2}{2\alpha}\right)C(t)\right]\phi\partial_\phi, \end{aligned} \tag{5.19}$$

in which we have introduced the constant B and functions $A(t)$ and $C(t)$ to maintain some compactness of expression, to calculate the similarity solution of (5.7). The associated Lagrange's system of Δ_W is

$$\frac{\mathrm{d}t}{0}=\frac{\mathrm{d}y}{A}=\frac{\mathrm{d}\phi}{[B+\Omega(y-\frac{\sigma^2}{2\alpha})C]\phi}. \tag{5.20}$$

The first term gives the obvious similarity variable t and the second and third terms give the second characteristic as

$$\eta=\phi\exp\left\{-\left[\frac{B}{A}\left(y-\frac{\sigma^2}{2\alpha}\right)+\frac{1}{2}\Omega\left(y-\frac{\sigma^2}{2\alpha}\right)^2\frac{C}{A}\right]\right\}. \tag{5.21}$$

When we set

$$\phi=F(t)\exp\left\{\left[\frac{B}{A}\left(y-\frac{\sigma^2}{2\alpha}\right)+\frac{1}{2}\Omega\left(y-\frac{\sigma^2}{2\alpha}\right)^2\frac{C}{A}\right]\right\}, \tag{5.22}$$

Eq. (5.7) reduces to

$$\frac{\dot F}{F}=\frac{\alpha^2}{2\sigma^2(1-\gamma)}-\frac{1}{2}\frac{\dot A}{A}-\frac{1}{2}\left(\frac{1}{2}\gamma\big(\sigma^2-2r\big)\right)^2\frac{\sigma^2}{A^2}. \tag{5.23}$$

The performance of the quadrature is trivial. After we substitute the terminal condition to evaluate the constant of integration, we obtain

$$\begin{aligned}\phi(t,y) = {} & \left[\cosh\big[\sigma^2\Omega(t-T)\big]-\frac{\alpha}{\sigma^2(1-\gamma)}\right]^{-\frac{1}{2}} \\ & \times\exp\left[\frac{\alpha^2(t-T)}{2\sigma^2(1-\gamma)}-\frac{\alpha\gamma^2(\sigma^2-2r)^2\tanh[\sigma^2\Omega(t-T)]}{\sigma^2\Omega(1-\gamma)[\sigma^2(1-\gamma)-\alpha\tanh[\sigma^2\Omega(t-T)]]}\right] \\ & -\frac{\frac{1}{2}\gamma(\sigma^2-2r)}{\sigma^2(1-\gamma)\cosh[\sigma^2\Omega(t-T)]-\alpha\sinh[\sigma^2\Omega(t-T)]}\left(y-\frac{\sigma^2}{2\alpha}\right)\end{aligned}$$

$$-\frac{1}{2}\Omega\frac{\alpha\cosh[\sigma^2\Omega(t-T)]-\sigma^2(1-\gamma)\sinh[\sigma^2\Omega(t-T)]}{\sigma^2(1-\gamma)\cosh[\sigma^2\Omega(t-T)]-\alpha\sinh[\sigma^2\Omega(t-T)]}\left(y-\frac{\sigma^2}{2\alpha}\right). \quad (5.24)$$

The solution to the original problem follows from (5.24).

6. Conclusion

In this paper we have taken the model proposed by Benth and Karlsen for the optimisation of a portfolio, which gives rise to a highly nonlinear evolution partial differential equation in $2+1$ variables, and subjected it to a symmetry analysis in the tradition of Lie. That this equation, (2.5), had as much symmetry as the analysis revealed was remarkable enough. One of the major obstacles to the solution of partial differential equations using symmetries[7] is the concomitant requirement to satisfy the initial/boundary/terminal conditions. In this respect the equations of Financial Mathematics having the nature of a Hamilton–Jacobi–Bellman equation seem to be susceptible to analysis from the approach of symmetry. The Black–Scholes equation [9] was the original exemplar. As a linear $1+1$ evolution equation which can be transformed to the classical heat equation its possession of the full complement of Lie point symmetries for a $1+1$ evolution equation [17] was one thing. That these symmetries were compatible with the terminal condition which is a standard feature of the class of problems being modelled could be imagined as being fortuitous. One could not be optimistic that a similarly complete result would apply to the other equations devised in the mathematical modelling of a multitude of financial scenarii of the most diverse variety [7,14]. This is particularly the case for nonlinear equations which generally do not enjoy the fullness of the complement of symmetries of linear systems. Naturally it would seem that there are exceptions! This has been the case of the model analysed here. The symmetries were compatible with the boundary condition (the second of (2.6)) so that a symmetry-based reduction from a $2+1$ to a $1+1$ equation was possible within the context of the problem. The reduced equation was still nonlinear, but could be reduced to a linear equation by means of a cosmetic transformation that was derived using the symmetry of the equation. The linear equation was not obviously well-endowed with Lie point symmetries, but some standard transformations, conflated into the single (5.3), made the $1+1$ equation one of evident maximal symmetry. The remaining condition, that of the value of the function at the terminal time T, reduced the number of Lie point symmetries to just two. One would have been sufficient to reduce the $1+1$ equation to an ordinary differential equation in terms of the similarity variables. The additional symmetry made the resulting differential equation just that much simpler to solve.

It seems to be a peculiarity of Hamilton–Jacobi–Bellman equations that the amount of symmetry revealed by the Lie analysis is often significant and so helpful to the determination of solutions. This makes the application of symmetry methods to the equations of Mathematical Finance a technique to be used as a component of the standard repertoire of analysis. We observed above the very nice analysis of Benth and Karlsen which was not based upon the methods of Lie. Their feeling and intuition can only be admired. Not everyone is so endowed with such insight. For these latter persons the use of the Lie analysis can be expected to aid the search for insight into some of these complex and nonlinear equations in the Mathematics of Finance.

[7] In principle this can also be an obstacle for ordinary differential equations, but it is nothing like the same order of difficulty. For a recent contribution to this aspect of ordinary differential equations with initial/boundary conditions see Hydon [13].

Acknowledgment

PGLL thanks the University of KwaZulu-Natal for its continued support.

References

[1] K. Andriopoulos, P.G.L. Leach, G.P. Flessas, Complete symmetry groups of ordinary differential equations and their integrals: Some basic considerations, J. Math. Anal. Appl. 262 (2001) 256–273.
[2] K. Andriopoulos, P.G.L. Leach, The economy of complete symmetry groups for linear higher-dimensional systems, J. Nonlinear Math. Phys. 9 (Second Suppl.) (2002) 10–23.
[3] K. Andriopoulos, P.G.L. Leach, The complete symmetry group of the generalised hyperladder problem, J. Math. Anal. Appl. 293 (2004) 633–644.
[4] K. Andriopoulos, P.G.L. Leach, Newtonian economics, in: N.H. Ibragimov, C. Sophocleous, P.A. Damianou (Eds.), Group Analysis of Differential Equations, University of Cyprus, Nicosia, 2005, pp. 134–142.
[5] F.E. Benth, K.H. Karlsen, A note on Merton's portfolio selection problem for the Schwartz mean-reversion model, Stoch. Anal. Appl. 23 (2005) 687–704.
[6] Fischer Black, Myron Scholes, The valuation of option contracts and a test of market efficiency, J. Finance 27 (1972) 399–417.
[7] Fischer Black, Myron Scholes, The pricing of options and corporate liabilities, J. Political Economy 81 (1973) 637–659.
[8] G.W. Bluman, S. Kumei, Symmetries and Differential Equations, Appl. Math. Sci., vol. 81, Springer-Verlag, New York, 1989.
[9] R.K. Gazizov, N.H. Ibragimov, Lie symmetry analysis of differential equations in finance, Nonlinear Dynam. 17 (1998) 387–407.
[10] A.K. Head, LIE, a PC program for Lie analysis of differential equations, Comput. Phys. Comm. 77 (1993) 241–248.
[11] W. Hereman, Review of symbolic software for the computation of Lie symmetries of differential equations, Euromath. Bull. 1 (1994) 45–79.
[12] W. Hereman, in: N.H. Ibragimov (Ed.), New Trends in Theoretical Development and Computational Methods, in: CRC Handbook of Lie Group Analysis of Differential Equations, vol. 3, CRC Press, Boca Raton, 1996 (Chapter 13).
[13] P.E. Hydon, Symmetry analysis of initial-value problems, J. Math. Anal. Appl. 309 (2005) 103–116.
[14] L.K. Kwok, Mathematical Models of Financial Derivatives, Springer, New York, 1988.
[15] P.G.L. Leach, 'Air' polynomials, Lie point symmetries and an hyperbolic equation, J. Phys. A: Math. Gen. 39 (2006) 5037–5052.
[16] P.G.L. Leach, K. Andriopoulos, A.N. Yannacopoulos, Nonlinear equations in financial mathematics: Symmetries and linearisation, preprint, Research Group in Mathematical Physics, Department of Communication and Systems Engineering, School of Sciences, University of the Aegean, Karlovassi 83 200, Greece, 2005.
[17] R.L. Lemmer, P.G.L. Leach, A classical viewpoint on quantum chaos, Arab J. Math. Sci. 5 (1999/1420) 1–17.
[18] C.F. Lo, C.H. Hui, Lie algebraic approach for pricing moving carrier options with time-dependent parameters, J. Math. Anal. Appl. 323 (2006) 1455–1464.
[19] R.C. Merton, Lifetime portfolio selection under uncertainty: The continuous time case, Rev. Econom. Statist. 51 (1969) 247–257.
[20] R.C. Merton, Consumption and portfolio rules in a continuous time model, J. Econom. Theory 3, 4 (1971) 373–413.
[21] C. Robert Merton, Theory of rational option pricing, Bell J. Econom. Management Sci. 4 (1973) 141–183.
[22] V.V. Morozov, Classification of six-dimensional nilpotent Lie algebras, Izv. Vyssh. Uchebn. Zaved. Mat. 5 (1958) 161–171.
[23] G.M. Mubarakzyanov, On solvable Lie algebras, Izv. Vyssh. Uchebn. Zaved. Mat. 32 (1963) 114–123.
[24] G.M. Mubarakzyanov, Classification of real structures of five-dimensional Lie algebras, Izv. Vyssh. Uchebn. Zaved. Mat. 34 (1963) 99–106.
[25] G.M. Mubarakzyanov, Classification of solvable six-dimensional Lie algebras with one nilpotent base element, Izv. Vyssh. Uchebn. Zaved. Mat. 35 (1963) 104–116.
[26] S.M. Myeni, P.G.L. Leach, The complete symmetry group of evolution equations, preprint, School of Mathematical Sciences, University of KwaZulu-Natal, 2005.
[27] S.M. Myeni, P.G.L. Leach, Nonlocal symmetries and the complete symmetry group of $1+1$ evolution equations, J. Nonlinear Math. Phys. 13 (2006) 377–392.
[28] V. Naicker, Diffusion, Black–Scholes and quantum mechanics, Honours project, School of Mathematical Sciences, Howard College, University of KwaZulu-Natal, 2004.

[29] V. Naicker, K. Andriopoulos, P.G.L. Leach, Symmetry reduction of some nonlinear Hamilton–Jacobi–Bellman equations arising in the mathematics of finance, J. Nonlinear Math. Phys. 12 (2005) 264–274.
[30] M.C. Nucci, Interactive REDUCE programs for calculating classical, nonclassical and Lie–Bäcklund symmetries for differential equations, preprint, Georgia Institute of Technology, Math 062090-051, 1990.
[31] M.C. Nucci, Interactive REDUCE programs for calculating Lie point, nonclassical, Lie–Bäcklund, and approximate symmetries of differential equations: Manual and floppy disk, in: N.H. Ibragimov (Ed.), New Trends in Theoretical Development and Computational Methods, in: CRC Handbook of Lie Group Analysis of Differential Equations, vol. III, CRC Press, Boca Raton, 1996.
[32] M.C. Nucci, P.G.L. Leach, Jacobi's last multiplier and the complete symmetry group of the Euler–Poinsot system, J. Nonlinear Math. Phys. 9 (Second Suppl.) (2002) 110–121.
[33] E.S. Schwartz, The stochastic behaviour of commodity prices: Implications for valuation and hedging, J. Finance 52 (1997) 923–973.
[34] J. Sherring, A.K. Head, G.E. Prince, Dimsym and LIE: Symmetry determining packages, Math. Comput. Modelling 25 (1997) 153–164.
[35] W. Sinkala, P.G.L. Leach, J.G. O'Hara, Invariant properties of a general bond-pricing equation, preprint, Department of Mathematics and Applied Mathematics, Faculty of Science and Engineering, Walter Sisulu University, Private Bag X1, Mthatha 5117, Republic of South Africa, 2005.
[36] W. Sinkala, P.G.L. Leach, J.G. O'Hara, Optimal system and group-invariant solutions of the Cox–Ingersoll–Ross pricing equation, preprint, Department of Mathematics and Applied Mathematics, Faculty of Science and Engineering, Walter Sisulu University, Private Bag X1, Mthatha 5117, Republic of South Africa, 2005.
[37] T. Zariphopoulou, Optimal investment and consumption models with nonlinear stock dynamics, Math. Methods Oper. Res. 50 (1999) 271–296.