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Long-time behavior for a nonlinear plate equation with thermal memory

Hao Wu

School of Mathematical Sciences, Fudan University, No. 220 Handan, 200433 Shanghai, PR China

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ABSTRACT

We consider a nonlinear plate equation with thermal memory effects due to non-Fourier heat flux laws. First we prove the existence and uniqueness of global solutions as well as the existence of a global attractor. Then we use a suitable Łojasiewicz–Simon type inequality to show the convergence of global solutions to single steady states as time goes to infinity under the assumption that the nonlinear term f is real analytic. Moreover, we provide an estimate on the convergence rate.

1. Introduction

In this paper, we consider the following nonlinear plate equation with thermal memory effects due to non-Fourier heat flux laws

$$\begin{cases} \theta_t - \Delta u_t + \displaystyle\int_0^\infty \kappa(s)\big[-\Delta\theta(t-s)\big]\,ds = 0, \\ u_{tt} - \Delta u_t + \Delta(\Delta u + \theta) + f(u) = 0, \end{cases} \tag{1.1}$$

for $(t,x) \in \mathbb{R}^+ \times \Omega$, subject to the boundary conditions

$$\begin{cases} u(t) = \Delta u(t) = 0, \quad t \geqslant 0,\ x \in \Gamma, \\ \theta(t) = 0, \quad t \in \mathbb{R},\ x \in \Gamma, \end{cases} \tag{1.2}$$

and initial conditions

$$\begin{aligned} &u(0) = u_0, \qquad u_t(0) = v_0, \qquad \theta(0) = \theta_0, \quad x \in \Omega, \\ &\theta(-s) = \phi(s), \quad (s,x) \in \mathbb{R}^+ \times \Omega. \end{aligned} \tag{1.3}$$

Here, $\Omega \in \mathbb{R}^2$ is a bounded domain with smooth boundary Γ, θ represents the temperature variation from the equilibrium reference value while u is the vertical displacement of the plate. Function $\phi : \mathbb{R}^+ \times \Omega \mapsto \mathbb{R}$ is called the initial past history of temperature. The memory kernel $\kappa : \mathbb{R}^+ \mapsto \mathbb{R}$ is assumed to be a positive bounded convex function vanishing at infinity. For the sake of simplicity, we set all the physical constants to be one.

Recently, evolution equations under various non-Fourier heat flux laws have attracted interests of many mathematicians (cf. [1,2,4,8,11–17,19,33,34] and references cited therein). Let $\mathbf{q}$ be the heat flux vector. According to the Gurtin–Pinkin theory [21], the linearized constitutive equation of $\mathbf{q}$ is given by

E-mail address: haowufd@yahoo.com.

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$$\mathbf{q}(t) = -\int_0^\infty \kappa(s)\nabla\theta(t-s)\,ds, \tag{1.4}$$

where κ is the heat conductivity relaxation kernel. The presence of convolution term in (1.4) entails finite propagation speed of thermal disturbances, so that in this case the corresponding equation is of hyperbolic type. It is easy to see that (1.4) can be reduced to the classical Fourier law $\mathbf{q} = -\nabla\theta$ if κ is the Dirac mass at zero. Besides, if we take

$$\kappa(s) = \frac{1}{\sigma}e^{-\frac{s}{\sigma}}, \quad \sigma > 0, \tag{1.5}$$

and differentiate (1.4) with respect to t, we can (formally) arrive at the so-called Cattaneo–Fourier law (cf. [24,29,30])

$$\sigma\mathbf{q}_t(t) + \mathbf{q}(t) = -\nabla\theta(t). \tag{1.6}$$

On the other hand, evolution equations under Colemann–Gurtin theory for the heat conduction (cf. [6]) have also been studied extensively (see, for instance [1,2,10,17]). There the heat flux $\mathbf{q}$ depends on both the past history and on the instantaneous of the gradient of temperature:

$$\mathbf{q}(t) = -K_I\nabla\theta(t) - \int_0^\infty \kappa(s)\nabla\theta(t-s)\,ds, \tag{1.7}$$

where $K_I > 0$ is the instantaneous diffusivity coefficient.

There are a lot of work on thermoelastic plate equations in the literature. For linear thermoelastic plate equations without memory effects in heat conduction, exponential stability of the associated C_0-semigroups has been proven under different boundary conditions (cf. [32, Section 2.5], [35,36]). On the other hand, when the heat flux is modeled by non-Fourier laws, well-posedness and stability for the corresponding linear thermoelastic plate equations have been investigated in several recent papers (cf. [12,14] and references cited therein).

In this paper, we consider the nonlinear problem (1.1)–(1.3). Asymptotic behavior of global solutions to nonlinear isothermal plate equations has been considered before. We may refer to [22,25], where convergence to equilibrium as $t \to \infty$ was obtained by the well-known Łojasiewicz–Simon approach under the assumption that the nonlinearity is real analytic. However, to the best of our knowledge, there are few results on the long-time behavior of global solutions to nonlinear plate equations with thermal memory like (1.1)–(1.3). This is just the main goal of the present paper. First, we prove the existence and uniqueness of global solutions to (1.1)–(1.3). Then we derive some uniform estimates which yields the precompactness of the solutions and furthermore the existence of a global attractor. Finally, combining some techniques for evolution equations with memory and for plate equations, we are able to prove the convergence of global solutions to single steady states as time goes to infinity via a suitable Łojasiewicz–Simon type inequality. Moreover, we obtain some estimates on convergence rate. Further investigations concerning the infinite dimensional system associated with our problem such as existence of exponential attractors, etc., can be made by adapting the arguments in recent papers [10,34].

Our problem (1.1)–(1.3) is an evolution system with memory. It is well known in the literature that it would be more convenient to work in the history space setting by introducing a new variable η called summed past history of θ. This approach has been proven to be very effective in analyzing such kind of evolution systems (cf. [1,2,10,11,13,14,16,17,19,33, 34]). On the other hand, it has been pointed out in the previous literature that when memory effects are present, the additional variable η does not enjoy any regularizing effect. As a result, to ensure the precompactness of the trajectory, we have to make suitable decomposition of the solution which is typical for dissipative systems. To overcome the lack of compactness of the history space $\mathcal{M}$ in which the variable η exists, an *ad hoc* compactness lemma will be used (cf. [10,11,19]).

Comparing with the Colemann–Gurtin law (cf. [1,2,17]), the dissipation in temperature θ for our system is only due to the memory effect, which is rather weak. The stronger dissipation provided in the Colemann–Gurtin law would make the problem easier to be dealt with. For instance, we can refer to [1] in which the authors considered a nonisothermal phase-field system with (1.7) and proved convergence to equilibrium for global solutions by the Łojasiewicz–Simon approach (see also [2] for a conserved phase-field model). To overcome the difficulty due to such a weaker dissipation under the Gurtin–Pinkin law (1.4), it is necessary to introduce a suitable additional functional which may vary from problem to problem to produce some new dissipations (cf. [10,11,13,19,33] and references cited therein). By using this idea, convergence to equilibrium for a nonisothermal Cahn–Hilliard equation was proven in [33] and in [13] a nonconserved phase-field model of Caginalp type consisting of two coupled integro-partial differential equations was successfully treated. Besides, in order to prove the convergence result for our problem, we have to make use of an extended Łojasiewicz–Simon type inequality associated with a fourth order operator, which can be derived from the abstract result in [22]. Due to the structure of (1.2), the standard Łojasiewicz–Simon approach used in the parabolic case must be modified by introducing an appropriate auxiliary functional (see Section 5) which usually depends on the problem under consideration (cf. [13,22,33,34,40] and references therein). In our case, the required auxiliary functional is formed by adding two perturbations to the original Lyapunov functional of system (1.1)–(1.3) and coefficients of those perturbations should be chosen properly. As far as the convergence rate is concerned, it is known that an estimate in certain (lower order) norm can usually be obtained directly from the

Łojasiewicz–Simon approach (see, e.g., [18,23,42]). Then one straightforward way to get estimates in higher order norms is using interpolation inequalities (cf. [15,23,33]) and, consequently, the decay exponent deteriorates. We shall show that by using suitable energy estimates and constructing proper differential inequalities, it is possible to obtain the same estimates on convergence rate in both higher and lower order norms. In particular, we find that as long as uniform estimates in certain norm can be obtained, we are able to prove convergence rate in the corresponding norm without loss in the decay exponent. In our case, we can also avoid using the decomposition argument used in [33] for this purpose. This technique has been successfully applied to other problems as well (cf. [9,20,28,39,40]) and it could be used to improve some previous results in the literature (e.g., [13,15,27,33,41]). At last we show that actually better results on convergence rate for problem (1.1)–(1.3) can be obtained if we use the decomposition of the trajectory $z = z_D + z_C$ (see Section 4). More precisely, the decay part z_D converges to zero exponentially fast while the compact part z_C converges to equilibrium in a higher order norm with the same rate as for the whole trajectory.

The remaining part of this paper is organized as follows. In Section 2, we introduce the functional setting, the main results of this paper and some technical lemmas. Well-posedness of problem (1.1)–(1.3) is proven in Section 3. Section 4 is devoted to the uniform estimates and precompactness as well as the existence of a global attractor. In the final Section 5, we prove the convergence of global solutions to single steady states as time goes to infinity and obtain an estimate on convergence rate.

2. Preliminaries and main results

We shall work under the functional settings used in e.g. [14]. Consider the positive operator A on $L^2(\Omega)$ defined by $A = -\Delta$ with domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Consequently, for $r \in \mathbb{R}$ we can introduce the Hilbert spaces $V^r = D(A^{r/2})$, endowed with the inner products

$$\langle w_1, w_2 \rangle_{V^r} = \langle A^{r/2} w_1, A^{r/2} w_2 \rangle, \quad \forall w_1, w_2 \in V^r,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\Omega)$. It is easy to see that the embedding $V^{r_1} \hookrightarrow V^{r_2}$ is compact for $r_1 > r_2$. In what follows, we shall denote the norm in $L^2(\Omega)$ by $\| \cdot \|$ for the sake of simplicity.

We suppose that κ is vanishing at ∞. Moreover, denoting

$$\mu(s) = -\kappa'(s),$$

we make the following assumptions on μ.

(H1) $\mu \in W^{1,1}(\mathbb{R}^+)$,
(H2) $\mu(s) \geqslant 0$, $\mu'(s) \leqslant 0$, $\forall s \in \mathbb{R}^+$,
(H3) $\mu'(s) + \delta \mu(s) \leqslant 0$, for some $\delta > 0$, $\forall s \in \mathbb{R}^+$,
(H4) $\kappa(0) = \int_0^\infty \mu(s)\, ds := \kappa_0 > 0$.

From recent work [4,13,37] and references cited therein, assumptions on μ might be properly weakened and our results still hold. Our results also hold under the assumptions made in [33,34] where (H1) (cf. [13]) is replaced by $\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$. In that case, μ is allowed to be unbounded in a right neighborhood of 0 and this can be handled by introducing a "cut-off" function near the origin.

For the nonlinear term f, we assume that

(F1) $f(s) \in C^2(\mathbb{R})$.

(F2)
$$\liminf_{|s| \to +\infty} \frac{f(s)}{s} > -\frac{1}{C_\Omega},$$

where C_Ω is the best constant depending only on Ω such that

$$\|w\|^2_{L^2(\Omega)} \leqslant C_\Omega \|Aw\|^2_{L^2(\Omega)}.$$

In order to prove the convergence to steady states, instead of (F1), we assume

(F1)′ $f(s)$ is real analytic in $s \in \mathbb{R}$.

We will also make use of the Poincaré inequality

$$\|w\| \leqslant C_P \|\nabla w\|, \quad w \in H_0^1(\Omega),$$

where C_P is a positive constant depending only on Ω.

In view of (H1), (H2), we introduce the weighted Hilbert spaces for $r \in \mathbb{R}$,

$$\mathcal{M}^r = L^2_\mu(\mathbb{R}^+; V^r),$$

with inner products

$$\langle\eta_1,\eta_2\rangle_{\mathcal{M}^r}=\int_0^\infty \mu(s)\big\langle A^{r/2}\eta_1(s),A^{r/2}\eta_2(s)\big\rangle\,ds.$$

Here we notice that the embeddings $\mathcal{M}^{r_1}\hookrightarrow\mathcal{M}^{r_2}$, for $r_1>r_2$, are continuous but not compact (cf. [10,11]).

Finally, we define the product Hilbert spaces

$$\mathcal{V}^r=V^{2+r}\times V^r\times V^r\times\mathcal{M}^{1+r},\quad r\in\mathbb{R},$$

with norm

$$\|z\|^2_{\mathcal{V}^r}=\big\|A^{(2+r)/2}z_1\big\|^2+\big\|A^{r/2}z_2\big\|^2+\big\|A^{r/2}z_3\big\|^2+\|z_4\|^2_{\mathcal{M}^{1+r}},$$

for all $z=(z_1,z_2,z_3,z_4)^T\in\mathcal{V}^r$.

It is convenient to work in the history space setting by introducing the so-called summed past history of θ which is defined as follows (cf. [7,12,14]),

$$\eta^t(s)=\int_0^s\theta(t-y)\,dy,\quad (t,s)\in[0,\infty)\times\mathbb{R}^+. \tag{2.1}$$

The variable η^t (formally) satisfies the linear equation

$$\eta^t_t(s)+\eta^t_s(s)=\theta(t),\quad \text{in } \Omega,\ (t,s)\in\mathbb{R}^+\times\mathbb{R}^+, \tag{2.2}$$

subject to the boundary and initial conditions

$$\eta^t(0)=0,\quad \text{in } \Omega,\ t\geqslant 0, \tag{2.3}$$

$$\eta^0(s)=\eta_0(s)=\int_0^s\phi(y)\,dy,\quad \text{in } \Omega,\ s\in\mathbb{R}^+. \tag{2.4}$$

We introduce a linear operator T on $\mathcal{M}^1$ defined by

$$T\eta=-\eta_s,\quad \eta\in D(T), \tag{2.5}$$

with domain

$$D(T)=\big\{\eta\in\mathcal{M}^1\ \big|\ \eta_s\in\mathcal{M}^1,\ \eta(0)=0\big\}, \tag{2.6}$$

here and in above η_s is the distributional derivative of η with respect to internal variable s.

As in [13,19], we notice that an integration by parts in time of the convolution products appearing in the equation for θ leads to

$$\begin{cases}\theta_t-\Delta u_t-\displaystyle\int_0^\infty\mu(s)\Delta\eta^t(s)\,ds=0,\\ u_{tt}-\Delta u_t+\Delta(\Delta u+\theta)+f(u)=0.\end{cases} \tag{2.7}$$

Let us now introduce the vector

$$z(t)=\big(u(t),v(t),\theta(t),\eta^t\big)^T,$$

and denote the initial data by

$$z_0=(u_0,v_0,\theta_0,\eta_0)^T\in\mathcal{V}^0.$$

Our problem (2.7), (1.2), (1.3) can be translated into the nonlinear abstract evolution equation in $\mathcal{V}^0$,

$$\begin{cases}z_t=Lz+G(z),\\ z(0)=z_0,\end{cases} \tag{2.8}$$

with

$$G(z)=\big(0,-f(u),0,0\big)^T. \tag{2.9}$$

Here the linear operator L is defined as

$$L\begin{pmatrix} u \\ v \\ \theta \\ \eta \end{pmatrix} = \begin{pmatrix} v \\ -Av - A(Au-\theta) \\ -Av - \int_0^\infty \mu(s) A\eta^t(s)\,ds \\ \theta + T\eta \end{pmatrix}, \tag{2.10}$$

with domain

$$D(L) = \left\{ z \in \mathcal{V}^0 \;\middle|\; \begin{array}{l} v,\ Au-\theta \in V^2 \\ \theta \in \mathcal{M}^1 \\ \int_0^\infty \mu(s) A\eta^t(s)\,ds \in V^0 \\ \eta \in D(T) \end{array} \right\}. \tag{2.11}$$

Remark 2.1. System (2.8) is obtained through formal integration by parts, however one can show that it is in fact equivalent to the original problem (1.1)–(1.3) (cf. [14]).

Now we are ready to state the main results of this paper

Theorem 2.1. *Let* (H1)–(H4) *and* (F1), (F2) *hold. The semigroup associated with problem* (2.8) *in* $\mathcal{V}^0$ *possesses a compact global attractor* $\mathcal{A}$ *in* $\mathcal{V}^0$.

Theorem 2.2. *Let* (H1)–(H4) *and* (F1)′, (F2) *hold. Then for any* $z_0 = (u_0, v_0, \theta_0, \eta_0)^T \in \mathcal{V}^0$, *there exists* u_∞ *being a solution to the following equation*

$$\begin{cases} A^2 u_\infty + f(u_\infty) = 0, & x \in \Omega, \\ u_\infty = \Delta u_\infty = 0, & x \in \Gamma, \end{cases} \tag{2.12}$$

such that as $t \to \infty$,

$$u(t) \to u_\infty, \quad \text{in } V^2, \tag{2.13}$$

$$v(t) \to 0, \qquad \theta(t) \to 0, \quad \text{in } L^2(\Omega), \tag{2.14}$$

$$\eta^t \to 0, \quad \text{in } \mathcal{M}^1. \tag{2.15}$$

Moreover, there exists a positive constant C *depending on the initial data such that*

$$\|u(t) - u_\infty\|_{V^2} + \|v(t)\| + \|\theta(t)\| + \|\eta^t\|_{\mathcal{M}^1} \leqslant C(1+t)^{-\frac{\rho}{(1-2\rho)}}, \quad \forall t \geqslant 0, \tag{2.16}$$

with $\rho \in (0, 1/2)$ *being the same constant as in the Łojasiewicz–Simon inequality* (*see Lemma* 5.3).

Remark 2.2. With minor modifications, corresponding results can be proven for equations under various other type of non-Fourier heat conduction laws:

$$\theta_t + c_1\theta - c_2\Delta\theta - \Delta u_t + \int_0^\infty \kappa(s)\big[c_3\theta(t-s) - \Delta\theta(t-s)\big]\,ds = 0, \tag{2.17}$$

with c_1, c_2, c_3 being nonnegative constants. When $c_1 > 0$, we have a (dissipative) term $c_1\theta$ in (2.17), which is arising from the assumption that besides the heat flux, the thermal power depends on the past history of θ (cf. [12]). The case $c_2 > 0$ corresponds to the Colemann–Gurtin theory as mentioned before. Moreover, we may refer to [12,31] for the case $c_3 > 0$. Although there might be additional terms like $c_1\theta$, $-c_2\Delta\theta$ and $\int_0^\infty \kappa(s)c_3\theta(t-s)\,ds$ in the equation, these terms provide stronger dissipations on θ from the mathematical point of view, which make the extensions of our results possible.

For reader's convenience we report below some helpful technical lemmas which will be used in this paper. The first one is a frequently used compactness lemma for the spaces $\mathcal{M}^r$ (cf. [14, Lemma 2.1]).

Lemma 2.1. *Let* $\mathbb{T}_\eta(y)$ *be defined as follows*

$$\mathbb{T}_\eta(y) = \int_{(0,1/y)\cup(y,\infty)} \mu(s)\|A^{1/2}\eta(s)\|^2\,ds, \quad y \geqslant 1. \tag{2.18}$$

If $\mathcal{C} \subset \mathcal{M}^1$ *satisfies*

(i) $\sup_{\eta\in\mathcal{C}} \|\eta\|_{\mathcal{M}^2} < \infty$,

(ii) $\sup_{\eta\in\mathcal{C}} \|T\eta\|_{\mathcal{M}^1} < \infty$,
(iii) $\lim_{y\to\infty}(\sup_{\eta\in\mathcal{C}} \mathbb{T}_\eta(y)) = 0$,

then $\mathcal{C}$ is relatively compact in $\mathcal{M}^1$.

The following lemma can be found in [3].

Lemma 2.2. *Let X be a Banach space and $Z \in C([0,\infty), X)$. Let $E : X \to \mathbb{R}$ be a function bounded from below such that $E(Z(0)) \leqslant M$ for $Z \in X$. If*

$$\frac{d}{dt}E\big(Z(t)\big) + \delta\big\|Z(t)\big\|_X^2 \leqslant k,$$

for some $\delta \geqslant 0$ and $k \geqslant 0$ independent of Z, then for all $\varepsilon > 0$ there is $t_0 = t_0(M,\varepsilon) > 0$ such that

$$E\big(Z(t)\big) \leqslant \sup_{\xi\in X}\big\{E(\xi)\colon \delta\|\xi\|_X^2 \leqslant k+\varepsilon\big\}, \quad \forall t \geqslant t_0.$$

3. Well-posedness

By using the semigroup approach, we are able to prove the existence and uniqueness of global solution to system (2.8).

Theorem 3.1. *Suppose that assumptions* (H1), (H2) *and* (F1), (F2) *hold. Then for any initial data $z_0 = (u_0, v_0, \theta_0, \eta_0)^T \in \mathcal{V}^0$, system* (2.8) *admits a unique global solution $z(t) \in C([0,+\infty), \mathcal{V}^0)$.*

Proof. We apply the semigroup theory (see, e.g., [42, Theorems 2.5.4, 2.5.5]).

Since

$$\langle T\eta, \eta\rangle_{\mathcal{M}^1} = \frac{1}{2}\int_0^\infty \mu'(s)\big\|A^{1/2}\eta(s)\big\|^2\,ds \leqslant 0, \quad \forall \eta \in D(T), \tag{3.1}$$

it is easy to see that

$$\langle Lz, z\rangle_{\mathcal{V}^0} = -\|\nabla v\|^2 + \langle T\eta, \eta\rangle_{\mathcal{M}^1} \leqslant 0, \quad \forall z \in D(L). \tag{3.2}$$

By a similar argument in [12, Section 3] (see also [14]) we can show that $I - L : D(L) \mapsto \mathcal{V}^0$ is onto. Thus L is an m-accretive operator. On the other hand, by the Sobolev embedding theorem, for any $z_1, z_2 \in \mathcal{V}^0$ with $\|z_1\|_{\mathcal{V}^0} \leqslant M$, $\|z_2\|_{\mathcal{V}^0} \leqslant M$, there exists a constant $L_M > 0$ depending on M such that

$$\big\|G(z_1) - G(z_2)\big\|_{\mathcal{V}^0} \leqslant L_M\|u_1 - u_2\|_{V^2} \leqslant L_M\|z_1 - z_2\|_{\mathcal{V}^0}.$$

Therefore, $G(z)$ is a nonlinear operator from $\mathcal{V}^0$ to $\mathcal{V}^0$ satisfying the local Lipschitz condition. Consequently, local existence of a unique mild solution $z(t) \in C([0,T], \mathcal{V}^0)$ follows from [42, Theorem 2.5.4].

Next we prove the global existence. Taking inner product of (2.8) and z in $\mathcal{V}^0$, we get

$$\frac{d}{dt}\left(\frac{1}{2}\big\|z(t)\big\|_{\mathcal{V}^0}^2 + \int_\Omega F(u)\,dx\right) + \|\nabla v\|^2 - \frac{1}{2}\int_0^\infty \mu'(s)\big\|A^{1/2}\eta^t(s)\big\|^2\,ds = 0, \tag{3.3}$$

where $F(u) = \int_0^u f(y)\,dy$.

Assumption (F2) implies that there exist constants $\delta \in (0,1)$ and $N = N(\delta) > 0$ such that (cf. [5])

$$F(s) \geqslant -\frac{1-\delta}{2C_\Omega}s^2, \quad \text{for } |s| \geqslant N.$$

To see this, let M be a positive constant such that $f(z)/z + \frac{1}{C_\Omega} \geqslant \frac{2\delta}{C_\Omega}$ for $|z| \geqslant M$ and certain $\delta \in (0,1)$. Then we have

$$F(s) + \frac{1}{2C_\Omega}s^2 = \int_0^M \left(\frac{f(z)}{z} + \frac{1}{C_\Omega}\right)z\,dz + \int_M^s \left(\frac{f(z)}{z} + \frac{1}{C_\Omega}\right)z\,dz \geqslant C + \frac{2\delta}{C_\Omega}\left(\frac{s^2}{2} - \frac{M^2}{2}\right) \geqslant \frac{\delta}{2C_\Omega}s^2, \tag{3.4}$$

for

$$s^2 \geqslant \max\left\{2M^2 - \frac{2C_\Omega C}{\delta}, 0\right\} := N^2.$$

For negative s one can repeat the same computation with M replaced by $-M$.

Now we have

$$\int_\Omega F(u)\,dx = \int_{|u|\leqslant N} F(u)\,dx + \int_{|u|>N} F(u)\,dx \geqslant -\frac{1-\delta}{2C_\Omega}\int_\Omega u^2\,dx + C\big(|\Omega|, f\big), \tag{3.5}$$

where $C(|\Omega|, f) = |\Omega| \min_{|s|\leqslant N} F(s)$.

By the definition of C_Ω in (F2) we can deduce

$$\int_\Omega F(u)\,dx \geqslant -\frac{1-\delta}{2}\|Au\|^2 + C\big(|\Omega|, f\big). \tag{3.6}$$

This implies that for any $\epsilon \in (0, \delta]$ there holds

$$\frac{1}{2}\big\|z(t)\big\|^2_{\mathcal{V}^0} + \int_\Omega F(u)\,dx = \frac{\epsilon}{2}\big\|z(t)\big\|^2_{\mathcal{V}^0} + \frac{1-\epsilon}{2}\big\|z(t)\big\|^2_{\mathcal{V}^0} + \int_\Omega F(u)\,dx \geqslant \frac{\epsilon}{2}\big\|z(t)\big\|^2_{\mathcal{V}^0} + C\big(|\Omega|, f\big). \tag{3.7}$$

As a result,

$$\frac{1}{2}\big\|z(t)\big\|^2_{\mathcal{V}^0} \leqslant \frac{1}{\epsilon}\bigg(\frac{1}{2}\big\|z(t)\big\|^2_{\mathcal{V}^0} + \int_\Omega F(u)\,dx - C\big(|\Omega|, f\big)\bigg). \tag{3.8}$$

Integrating (3.3) with respect to t, we infer from (3.8) that

$$\big\|z(t)\big\|^2_{\mathcal{V}^0} \leqslant C\big(\|z_0\|_{\mathcal{V}^0}, |\Omega|, f\big), \quad \forall t \geqslant 0. \tag{3.9}$$

This uniform estimate together with [42, Theorem 2.5.5] yields the global existence, i.e., $z(t) \in C([0, +\infty), \mathcal{V}^0)$. Moreover, it is not difficult to check that for any $z_{01}, z_{02} \in \mathcal{V}^0$, the corresponding global solutions $z_1(t), z_2(t)$ satisfy

$$\big\|z_1(t) - z_2(t)\big\|^2_{\mathcal{V}^0} \leqslant C_T\|z_{01} - z_{02}\|^2_{\mathcal{V}^0}, \quad 0 \leqslant t \leqslant T, \tag{3.10}$$

for all $T \geqslant 0$, where C_T is a constant depending on the norms of z_{01}, z_{02} in $\mathcal{V}^0$ and T.

The proof is complete. □

Remark 3.1. From the above theorem, we can see that the solution to our problem (2.8) defines a strongly continuous semigroup $S(t)$ on the phase space $\mathcal{V}^0$ such that $S(t)z_0 = z(t)$.

4. Precompactness of trajectories and global attractor

In this section, we will first prove (i) uniform estimate of the solution which also indicates the existence of an absorbing set, (ii) precompactness of trajectory $z(t)$. In what follows, we shall exploit some formal *a priori* estimates which can be justified rigorously by the standard density argument.

Lemma 4.1. *Let* (H1)–(H4) *and* (F1), (F2) *hold. There exists a positive constant R_0 such that the ball*

$$\mathcal{B}_0 := \big\{z \in \mathcal{V}^0 \;\big|\; \|z\|_{\mathcal{V}^0} \leqslant R_0\big\}$$

is an absorbing set. Namely, for any bounded set $\mathcal{B} \in \mathcal{V}^0$, there is $t_0 = t_0(\mathcal{B}) \geqslant 0$ such that $S(t)\mathcal{B} \subset \mathcal{B}_0$ for every $t \geqslant t_0$.

Proof. Multiplying the second equation in (1.1) by $\varepsilon^2 u$, integrating on Ω and adding the result to (3.3), we get

$$\begin{aligned}&\frac{d}{dt}\bigg(\frac{1}{2}\big\|z(t)\big\|^2_{\mathcal{V}^0} + \frac{\varepsilon^2}{2}\|\nabla u\|^2 + \int_\Omega F(u)\,dx + \varepsilon^2\int_\Omega uv\,dx\bigg) + \|\nabla v\|^2 - \varepsilon^2\|v\|^2 - \frac{1}{2}\int_0^\infty \mu'(s)\big\|A^{1/2}\eta^t(s)\big\|^2\,ds + \varepsilon^2\|Au\|^2 \\ &\quad = -\varepsilon^2\int_\Omega f(u)u\,dx + \varepsilon^2\int_\Omega \theta Au\,dx.\end{aligned} \tag{4.1}$$

In order to apply Lemma 2.2, we need more dissipation on the left-hand side of (4.1). To such an aim, we introduce the following functional (cf. [13,19,33] and references cited therein)

$$J(t) := -\int_0^\infty \mu(s)\big\langle\theta(t), \eta^t(s)\big\rangle\,ds. \tag{4.2}$$

It turns out from the Hölder inequality and (H1) that

$$\left|J(t)\right| \leqslant \left\|\theta(t)\right\| \int_0^\infty \mu(s)\left\langle \eta^t(s), \eta^t(s)\right\rangle^{\frac{1}{2}} ds \leqslant \frac{1}{2}\left\|\theta(t)\right\|^2 + \frac{1}{2}\left\|\eta^t(s)\right\|^2_{\mathcal{M}^0} \int_0^\infty \mu(s)\, ds \leqslant C\left\|z(t)\right\|^2_{\mathcal{V}^0}. \tag{4.3}$$

Besides, a direct calculation yields (cf. (2.2), (2.7))

$$\begin{aligned}\frac{d}{dt}J(t) &= -\int_0^\infty \mu(s)\left\langle \theta_t(t), \eta^t(s)\right\rangle ds - \int_0^\infty \mu(s)\left\langle \theta(t), \eta^t_t(s)\right\rangle ds \\ &= -\int_0^\infty \mu(s)\left\langle \Delta u_t(t), \eta^t(s)\right\rangle ds - \left\| \int_0^\infty \mu(s) A^{1/2}\eta^t(s)\, ds\right\|^2 - \kappa_0\|\theta\|^2 + \int_0^\infty \mu(s)\left\langle \theta(t), \eta^t_s(s)\right\rangle ds. \end{aligned} \tag{4.4}$$

Terms on the right-hand side of (4.4) can be controlled in the following way:

$$\left|-\int_0^\infty \mu(s)\left\langle \Delta u_t(t), \eta^t(s)\right\rangle ds\right| = \left|\int_0^\infty \mu(s)\left\langle \nabla v(t), \nabla\eta^t(s)\right\rangle ds\right| \leqslant \frac{1}{2}\|\nabla v\|^2 + \frac{\kappa_0}{2}\left\|\eta^t\right\|^2_{\mathcal{M}^1}, \tag{4.5}$$

$$\left\|\int_0^\infty \mu(s) A^{1/2}\eta^t(s)\, ds\right\|^2 \leqslant \int_0^\infty \mu(s)\, ds \int_0^\infty \mu(s)\left\langle A^{1/2}\eta^t, A^{1/2}\eta^t\right\rangle ds \leqslant \kappa_0\left\|\eta^t\right\|^2_{\mathcal{M}^1}, \tag{4.6}$$

$$\begin{aligned}\left|\int_0^\infty \mu(s)\left\langle \theta(t), \eta^t_s(s)\right\rangle ds\right| &= \left|-\int_0^\infty \mu'(s)\left\langle \theta(t), \eta^t(s)\right\rangle ds\right| \leqslant -\int_0^\infty \mu'(s)\left\|\theta(t)\right\|\left\|\eta^t(s)\right\| ds \\ &\leqslant \frac{\kappa_0}{2}\|\theta\|^2 - C_1\int_0^\infty \mu'(s)\left\|A^{1/2}\eta^t(s)\right\|^2 ds, \end{aligned} \tag{4.7}$$

where in (4.7) we use (H1) that μ' is integrable (this can be weakened as mentioned in the previous section) and $C_1 > 0$ depends on κ_0. Now we can conclude

$$\frac{d}{dt}J(t) + \frac{\kappa_0}{2}\|\theta\|^2 \leqslant \frac{1}{2}\|\nabla v\|^2 + C_2\left\|\eta^t\right\|^2_{\mathcal{M}^1} - C_1\int_0^\infty \mu'(s)\left\|A^{1/2}\eta^t\right\|^2 ds, \tag{4.8}$$

where $C_2 = \frac{3\kappa_0}{2} > 0$. Multiplying (4.8) by 2ε and adding it to (4.1) we obtain

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\left\|z(t)\right\|^2_{\mathcal{V}^0} + \frac{\varepsilon^2}{2}\|\nabla u\|^2 + \int_\Omega F(u)\, dx + \varepsilon^2\int_\Omega uv\, dx + 2\varepsilon J(t)\right) - \varepsilon^2\|v\|^2 \\ &\quad + (1-\varepsilon)\|\nabla v\|^2 + \varepsilon\kappa_0\|\theta\|^2 - \left(\frac{1}{2} - 2C_1\varepsilon\right)\int_0^\infty \mu'(s)\left\|A^{1/2}\eta^t(s)\right\|^2 ds + \varepsilon^2\|Au\|^2 \\ &\leqslant -\varepsilon^2\int_\Omega f(u)u\, dx + \varepsilon^2\int_\Omega \theta Au\, dx + 2C_2\varepsilon\left\|\eta^t\right\|^2_{\mathcal{M}^1}. \end{aligned} \tag{4.9}$$

Define

$$\Psi\big(z(t)\big) = \frac{1}{2}\left\|z(t)\right\|^2_{\mathcal{V}^0} + \frac{\varepsilon^2}{2}\left\|\nabla u(t)\right\|^2 + \int_\Omega F\big(u(t)\big)\, dx + \varepsilon^2\int_\Omega u(t)v(t)\, dx + 2\varepsilon J(t)$$

and

$$\mathcal{F}\big(\|u\|_{V^2}\big) = |\Omega| \max_{|y|\leqslant\|u\|_{V^2}}\left|F(y)\right|.$$

Due to (F1) and the Sobolev embedding theorem $V^2 \hookrightarrow L^\infty(\Omega)$, we can see that $\mathcal{F}(s)$ is bounded for $|s| \leqslant M$, $\forall M > 0$. It follows from this fact and (3.7) that for all $z \in \mathcal{V}^0$ and ε sufficiently small there holds

$$C_3\|z\|^2_{\mathcal{V}^0} - C_4 \leqslant \Psi \leqslant \|z\|^2_{\mathcal{V}^0} + \mathcal{F}\big(\|u\|_{V^2}\big), \tag{4.10}$$

where C_3, C_4 are positive constants independent of z.

(F2) implies that there exist constants $\sigma > 0$ and $N = N(\sigma) > 0$ satisfying

$$f(s)s \geqslant -\frac{1-\sigma}{C_\Omega}s^2, \quad \forall |s| \geqslant N.$$

As a result,

$$\int_\Omega f(u)u\,dx \geqslant -\frac{1-\sigma}{C_\Omega}\|u\|^2 + C\big(|\Omega|, f\big) \geqslant -(1-\sigma)\|Au\|^2 + C\big(|\Omega|, f\big), \tag{4.11}$$

where $C(|\Omega|, f) = |\Omega| \min_{|s| \leqslant N} f(s)s$.

Moreover, from (H3), we can see that

$$-\int_0^\infty \mu'(s)\big\|A^{1/2}\eta^t(s)\big\|^2\,ds \geqslant \delta \int_0^\infty \mu(s)\big\|A^{1/2}\eta^t(s)\big\|^2\,ds, \tag{4.12}$$

and by the Hölder inequality we have

$$\left|\varepsilon^2 \int_\Omega \theta A u\,dx\right| \leqslant \varepsilon \frac{\kappa_0}{2}\|\theta\|^2 + C_5 \varepsilon^3 \|Au\|^2. \tag{4.13}$$

In (4.9) and (4.13) we take ε small enough such that

$$0 < \varepsilon \leqslant \min\left\{\frac{1}{4}, \frac{1}{16C_1}, \frac{\delta}{16C_2}, \frac{\sigma}{2C_5}, \frac{1}{2C_P}\right\}.$$

Then it follows from (4.9)–(4.13) that

$$\frac{d}{dt}\Psi(t) + \delta_0 \big\|z(t)\big\|^2_{\mathcal{V}^0} \leqslant k, \tag{4.14}$$

where δ_0 is a constant depending on $\varepsilon, \sigma, \delta, \kappa_0$ and k is a certain positive constant depending on $\varepsilon, |\Omega|, f, \delta, \sigma, N(\sigma)$.

We infer from Lemma 2.2 that there is $t_0 = t_0(\mathcal{B}) > 0$ such that

$$\Psi\big(z(t)\big) \leqslant \sup_{\xi \in \mathcal{V}^0}\big\{\Psi(\xi):\ \delta_0\|\xi\|^2_{\mathcal{V}^0} \leqslant 1 + k\big\}, \quad \forall t \geqslant t_0,$$

which together with (4.10) implies the existence of absorbing set.

The proof is complete. □

Next we prove the precompactness of solutions to problem (2.8). Since our system (2.8) does not enjoy smooth property as parabolic equations, it suffices to show that the semigroup is asymptotically smooth (cf. [38]). To accomplish this, we make a decomposition of the flow into a uniformly stable part and a compact part (cf. [10,11,13,19,34]). Namely, we decompose the solution to (2.8) with initial data $z(0) = z_0 \in \mathcal{V}^0$ as

$$z(t) = z_D(t) + z_C(t),$$

where $z_D(t) = (u_D(t), v_D(t), \theta_D(t), \eta^t_D)^T$ and $z_C(t) = (u_C(t), v_C(t), \theta_C(t), \eta^t_C)^T$ satisfy

$$\begin{cases} \dfrac{d}{dt}z_D = Lz_D, \\ z_D(0) = z_0 \end{cases} \tag{4.15}$$

and

$$\begin{cases} \dfrac{d}{dt}z_C = Lz_C + G(z), \\ z_C(0) = 0. \end{cases} \tag{4.16}$$

Similar to Theorem 3.1, it is easy to check that system (4.15) admits a unique mild solution $z_D(t) \in C([0, +\infty), \mathcal{V}^0)$. Moreover, we have

Lemma 4.2. *There exist constants $C, \delta_1 > 0$ such that the solution z_D of (4.15) fulfills*

$$\big\|z_D(t)\big\|_{\mathcal{V}^0} \leqslant Ce^{-\frac{\delta_1}{2}t}, \quad \forall t \geqslant 0, \tag{4.17}$$

where $C > 0$ is a constant depending on $\|z_0\|_{\mathcal{V}^0}$.

Proof. Let $E_D(z) : \mathcal{V}^0 \mapsto \mathbb{R}$ be defined as follows

$$E_D(z_D) = \|z_D(t)\|^2_{\mathcal{V}^0} + \varepsilon^2\|\nabla u_D\|^2 + 2\varepsilon^2 \int_\Omega u_D v_D \, dx + 4\varepsilon J_D(t) \tag{4.18}$$

with

$$J_D(t) := -\int_0^\infty \mu(s)\langle\theta_D(t), \eta^t_D(s)\rangle \, ds. \tag{4.19}$$

It is easy to see that for $\varepsilon > 0$ sufficiently small

$$\frac{1}{2}\|z_D\|^2_{\mathcal{V}^0} \leqslant E_D(z_D) \leqslant 2\|z_D\|^2_{\mathcal{V}^0}. \tag{4.20}$$

Similar to the proof of Lemma 4.1, we can show that there exists $\delta_1 > 0$ such that

$$\frac{d}{dt}E_D(z_D) + 2\delta_1\|z_D\|^2_{\mathcal{V}^0} \leqslant 0. \tag{4.21}$$

As a consequence of (4.20), we have

$$\frac{d}{dt}E_D(z_D) + \delta_1 E_D(z_D) \leqslant 0 \tag{4.22}$$

which yields

$$E_D\big(z_D(t)\big) \leqslant E_D\big(z_D(0)\big)e^{-\delta_1 t}, \quad \forall t \geqslant 0. \tag{4.23}$$

(4.17) follows immediately from (4.20) and (4.23). □

Next we analyze z_C. For initial data $z_0 \in \mathcal{V}^0$, we can see that $z_C(t) = z(t) - z_D(t)$ belongs to a bounded set in $\mathcal{V}^0$ for $t \geqslant 0$. In what follows we will show that z_C is more regular and actually it is uniformly bounded in $\mathcal{V}^1$.

Lemma 4.3. *For all $z_0 \in \mathcal{V}^0$, there exists $C > 0$ depending on $\|z_0\|_{\mathcal{V}^0}$ such that*

$$\|z_C(t)\|_{\mathcal{V}^1} \leqslant C, \quad \forall t \geqslant 0. \tag{4.24}$$

Proof. Taking the inner product of (4.16) and Az_C in $\mathcal{V}^0$, we have

$$\frac{d}{dt}\left(\frac{1}{2}\|A^{1/2}z_C\|^2_{\mathcal{V}^0} + \int_\Omega f(u)Au_C \, dx\right) + \|Av_C\|^2 - \frac{1}{2}\int_0^\infty \mu'(s)\|A\eta^t_C(s)\|^2 \, ds = \int_\Omega f'(u)vAu_C \, dx. \tag{4.25}$$

Multiplying the second equation in (4.16) by Au_C and integrating on Ω we get

$$\frac{d}{dt}\left(\frac{1}{2}\|Au_C\|^2 + \int_\Omega v_C Au_C \, dx\right) + \|A^{3/2}u_C\|^2 - \|A^{1/2}v_C\|^2 = -\int_\Omega f(u)Au_C \, dx + \int_\Omega A\theta_C Au_C \, dx. \tag{4.26}$$

Let

$$J_C(t) := -\int_0^\infty \mu(s)\langle A^{1/2}\theta_C(t), A^{1/2}\eta^t_C(s)\rangle \, ds. \tag{4.27}$$

Then in analogy to the argument in Lemma 4.1, we have

$$|J_C(t)| \leqslant \frac{1}{2}\|A^{1/2}\theta_C(t)\|^2 + \frac{1}{2}\|\eta^t_C(s)\|^2_{\mathcal{M}^1}\int_0^\infty \mu(s) \, ds \leqslant C\|z_C(t)\|^2_{\mathcal{V}^1} \tag{4.28}$$

and

$$\frac{d}{dt}J_C(t) + \frac{\kappa_0}{2}\|A^{1/2}\theta_C\|^2 \leqslant \frac{1}{2}\|Av_C\|^2 + C_6\|\eta^t_C\|^2_{\mathcal{M}^2} - C_7\int_0^\infty \mu'(s)\|A\eta^t_C\|^2 \, ds, \tag{4.29}$$

here $C_6, C_7 > 0$ depend on κ_0. Introduce the functional

$$\Phi(t) = \|A^{1/2} z_C(t)\|^2_{\mathcal{V}^0} + 2\int_\Omega f(u) A u_C \, dx + 2\varepsilon^2 \int_\Omega v_C A u_C \, dx + \varepsilon^2 \|A u_C\|^2 + 4\varepsilon J_C(t) + k,$$

where $k \geqslant 0$ denotes a generic constant depending on $\|z_0\|_{\mathcal{V}^0}$.

It is easy to see that, if the constant k appearing in the definition of Φ is large enough and ε is small enough, there holds

$$\frac{1}{2}\|A^{1/2} z_C(t)\|^2_{\mathcal{V}^0} \leqslant \Phi \leqslant 2\|A^{1/2} z_C(t)\|^2_{\mathcal{V}^0} + C(\|z_0\|_{\mathcal{V}^0}) + k. \tag{4.30}$$

It follows from (4.25), (4.26), (4.29) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Phi(t) + (1-\varepsilon)\|A v_C\|^2 - \left(\frac{1}{2} - 2C_7\varepsilon\right)\int_0^\infty \mu'(s)\|A\eta^t_C(s)\|^2 \, ds - \varepsilon^2\|A^{1/2} v_C\|^2 + \varepsilon^2\|A^{3/2} u_C\|^2 + \varepsilon\kappa_0\|A^{1/2}\theta_C\|^2 \\
&\quad \leqslant \int_\Omega f'(u) v A u_C \, dx + 2C_6\varepsilon\|\eta^t_C\|^2_{\mathcal{M}^2} - \varepsilon^2\int_\Omega f(u) A u_C \, dx + \varepsilon^2 \int_\Omega A\theta_C A u_C \, dx \\
&\quad := I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{4.31}$$

The right-hand side of (4.31) can be estimated as follows. By (F2) and the Sobolev embedding theorem we get

$$|I_1| = \left|\int_\Omega f'(u) v A u_C \, dx\right| \leqslant \|f'(u)\|_{L^\infty}\|v\|\|A u_C\| \leqslant C(|\Omega|, \|Au\|)\|v\|\|A^{3/2}u_C\| \leqslant \frac{\varepsilon^2}{4}\|A^{3/2}u_C\|^2 + C. \tag{4.32}$$

Besides,

$$|I_3| = \left|-\varepsilon^2\int_\Omega f(u) A u_C \, dx\right| \leqslant \varepsilon^2\|f(u)\|_{L^\infty}\|A u_C\| \leqslant \frac{\varepsilon^2}{4}\|A^{3/2}u_C\|^2 + C, \tag{4.33}$$

$$|I_4| = \left|\varepsilon^2\int_\Omega A u_C A\theta_C \, dx\right| \leqslant \frac{\varepsilon\kappa_0}{2}\|A^{1/2}\theta_C\|^2 + C_8\varepsilon^3\|A^{3/2}u_C\|^2, \tag{4.34}$$

where in the above three estimates, $C > 0$ is a constant depending on $\|z_0\|_{\mathcal{V}^0}$ and $C_8 = \frac{1}{2\kappa_0}$.

In (4.31)–(4.34), we take ε small enough such that

$$0 < \varepsilon \leqslant \min\left\{\frac{1}{4}, \frac{1}{2C_P}, \frac{\delta}{16C_6}, \frac{1}{16C_7}, \frac{1}{4C_8}\right\}. \tag{4.35}$$

Consequently, we can obtain the following inequality

$$\frac{d}{dt}\Phi(t) + \delta_2\Phi(t) \leqslant C, \tag{4.36}$$

where $\delta_2 > 0$, $C \geqslant 0$ are constants depending on ε and $\|z_0\|_{\mathcal{V}^0}$.

(4.30) and (4.36) yield

$$\|A^{1/2} z_C(t)\|_{\mathcal{V}^0} \leqslant C, \quad \forall t \geqslant 0.$$

The proof is complete. □

In order to obtain the required compactness, we have to take care of the fourth component η^t. Embedding $\mathcal{V}^1 \hookrightarrow \mathcal{V}^0$ is not compact because embedding $\mathcal{M}^2 \hookrightarrow \mathcal{M}^1$ is not compact in general. However, we have the following lemma whose proof is becoming standard (cf. [11,16,19] and references therein). For the sake of completeness, we give a sketch of the proof.

Lemma 4.4. *Let $\mathcal{C} = \bigcup_{t\geqslant 0}\eta^t_C$. Then $\mathcal{C}$ is relatively compact in $\mathcal{M}^1$.*

Proof. It is obvious that $\mathcal{C} \subset \mathcal{M}^1$. According to Lemma 2.1, we need to verify

$$\|\eta^t_C\|_{\mathcal{M}^2} \leqslant C, \quad t \geqslant 0, \tag{4.37}$$

$$\|T\eta^t_C\|_{\mathcal{M}^1} \leqslant C, \quad t \geqslant 0, \tag{4.38}$$

$$\lim_{y\to\infty}\left(\sup_{t\geqslant 0}\mathbb{T}_{\eta^t_C}(y)\right) = 0. \tag{4.39}$$

(4.37) follows from Lemma 4.3 immediately. Since $z_C(0) = 0$, η_C has the following explicit representation formula

$$\eta_C^t(s) = \begin{cases} \int_0^s \theta_C(t-y)\,dy, & 0 < s \leqslant t, \\ \int_0^t \theta_C(t-y)\,dy, & s > t. \end{cases} \tag{4.40}$$

Differentiating it with respect to s yields

$$T\eta_C^t(s) = \begin{cases} -\theta_C(t-s), & 0 < s \leqslant t, \\ 0, & s > t. \end{cases} \tag{4.41}$$

Thanks to (H4) and Lemma 4.3, we have

$$\int_0^\infty \mu(s)\left\|A^{1/2}T\eta_C^t(s)\right\|^2 ds = \int_0^t \mu(s)\left\|A^{1/2}T\eta_C^t(s)\right\|^2 ds = \int_0^t \mu(s)\left\|A^{1/2}\theta_C(t-s)\right\|^2 ds \leqslant C, \tag{4.42}$$

which yields (4.38). This also implies that $\eta_C^t \in H_\mu^1(\mathbb{R}^+; V^1)$.

(4.40) and Lemma 4.3 imply that

$$\left\|A^{1/2}\eta_C^t(s)\right\|^2 \leqslant C\left(1+s^2\right), \quad \forall s > 0.$$

For $y \geqslant 1$, we define

$$I(y) = C \int_{(0,1/y)\cup(y,\infty)} \mu(s)\left(1+s^2\right) ds.$$

It is obvious that (see the definition of $\mathbb{T}$)

$$\mathbb{T}_{\eta_C^t}(y) \leqslant I(y).$$

Assumption (H3) implies the exponential decay of the memory kernel, hence we have

$$I(y) \leqslant C, \quad \text{for } y \geqslant 1,$$

and as a consequence

$$\lim_{y\to\infty} I(y) = 0,$$

which yields (4.39). The lemma is proved. □

Lemmas 4.3 and 4.4 yield the compactness result we need

Lemma 4.5. *For any $z_0 \in \mathcal{V}^0$, $\bigcup_{t\geqslant 0} z_C(t)$ is relatively compact in $\mathcal{V}^0$.*

Proof of Theorem 2.1. On account of Lemmas 4.1, 4.2, 4.5 and the classical result in dynamical system [38, Theorem I.1.1], we can prove the conclusion of Theorem 2.1, i.e., problem (1.1)–(1.3) possesses a compact global attractor $\mathcal{A}$ in $\mathcal{V}^0$. □

5. Convergence to equilibrium and convergence rate

In this section we prove the convergence of global solutions to single steady states as time tends to infinity. Let $\mathcal{S}$ be the set of steady states of $S(t)$,

$$\mathcal{S} = \left\{Z \in \mathcal{V}^0\colon\ S(t)Z = Z,\ \text{for all } t \geqslant 0\right\}. \tag{5.1}$$

It is clear that every steady state Z_∞ has the form $Z_\infty = (u_\infty, 0, 0, 0)^T$, where u_∞ solves the following equation

$$A^2 u_\infty + f(u_\infty) = 0, \quad x \in \Omega, \tag{5.2}$$

with boundary conditions

$$u_\infty = \Delta u_\infty = 0, \quad x \in \Gamma. \tag{5.3}$$

The total energy

$$E(t) = \frac{1}{2}\left\|z(t)\right\|_{\mathcal{V}^0}^2 + \int_\Omega F(u)\,dx, \tag{5.4}$$

with $F(u)=\int_0^u f(z)\,dz$ serves as a Lyapunov functional for problem (2.8). Namely, we have

$$\frac{d}{dt}E(t)=-\|\nabla v\|^2+\frac{1}{2}\int_0^\infty \mu'(s)\big\|A^{1/2}\eta^t(s)\big\|^2\,ds\leqslant 0,\quad \forall t>0. \tag{5.5}$$

For any initial data $z_0\in\mathcal{V}^0$, its ω-limit set is defined as follows:

$$\omega(z_0)=\big\{z_\infty=(u_\infty,v_\infty,\theta_\infty,\eta_\infty)^T\ \big|\ \exists\{t_n\}\text{ such that } z(t_n)\to z_\infty\in\mathcal{V}^0,\ \text{as } t_n\to+\infty\big\}.$$

Then we have

Lemma 5.1. *For any $z_0\in\mathcal{V}^0$, the ω-limit set of z_0 is a nonempty compact connected subset in $\mathcal{V}^0$. Furthermore,*

(i) *$\omega(z_0)$ is invariant under the nonlinear semigroup $S(t)$ defined by the solution $z(t,x)$, i.e., $S(t)\omega(z_0)=\omega(z_0)$ for all $t\geqslant 0$.*
(ii) *$E(t)$ is constant on $\omega(z_0)$. Moreover, $\omega(z_0)\subset\mathcal{S}$.*

Proof. Since our system has a continuous Lyapunov functional $E(t)$, the conclusion of the present lemma follows from Lemmas 4.3, 4.5 and the well-known results in dynamical system (see, e.g., [38, Lemma I.1.1]). $\square$

Remark 5.1. Since solutions to problem (5.2), (5.3) are smooth, points in $\omega(z_0)$ are smooth. In particular, $\omega(z_0)$ is contained in a bounded set in $\mathcal{V}^1$.

After the previous preparations, we are ready to finish the proof of Theorem 2.2.

5.1. Convergence to equilibrium

For any initial datum $z_0\in\mathcal{V}^0$, it follows from Lemmas 4.3 and 4.5 that there is an equilibrium $(u_\infty,0,0,0)^T\in\omega(z_0)$ and an increasing unbounded sequence $\{t_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{t_n\to+\infty}\big(\big\|u(t_n)-u_\infty\big\|_{V^2}+\big\|v(t_n)\big\|+\big\|\theta(t_n)\big\|+\big\|\eta^t(t_n)\big\|_{\mathcal{M}^1}\big)=0. \tag{5.6}$$

Actually, the convergence for v,θ,η^t can be proved directly as follows:

Lemma 5.2. *Under the assumptions in Theorem* 2.2*, we have*

$$v(t)\to 0,\qquad \theta(t)\to 0,\quad \textit{in } L^2(\Omega), \tag{5.7}$$

and

$$\eta^t\to 0,\quad \textit{in } \mathcal{M}^1, \tag{5.8}$$

as time goes to infinity.

Proof. Taking the inner product of (2.8) with z in $\mathcal{V}^{-1}$, we get

$$\frac{1}{2}\frac{d}{dt}\big\|z(t)\big\|^2_{\mathcal{V}^{-1}}=-\|v\|^2+\frac{1}{2}\int_0^\infty \mu'(s)\langle\eta^t,\eta^t\rangle ds-\big\langle f(u),v\big\rangle_{V^{-1}}\leqslant-\big\langle f(u),v\big\rangle_{V^{-1}}, \tag{5.9}$$

where in the last step we use (H2). Then the Hölder inequality, (3.9) and the Sobolev embedding theorem yield

$$\frac{1}{2}\frac{d}{dt}\big(\|v\|^2_{V^{-1}}+\|\theta\|^2_{V^{-1}}+\big\|\eta^t\big\|^2_{\mathcal{M}^0}\big)\leqslant-\langle u,v\rangle_{V^1}-\big\langle f(u),v\big\rangle_{V^{-1}}\leqslant C\|Au\|\|v\|+C\big\|f(u)\big\|\|v\|\leqslant C, \tag{5.10}$$

where C is a constant depending on $\|z_0\|_{\mathcal{V}^0}$.

Multiplying (4.8) by 2ε and adding it to (3.3) yields

$$\begin{aligned}&\frac{d}{dt}\bigg(\frac{1}{2}\big\|z(t)\big\|^2_{\mathcal{V}^0}+\int_\Omega F(u)\,dx+2\varepsilon J(t)\bigg)+(1-\varepsilon)\|\nabla v\|^2+\varepsilon\kappa_0\|\theta\|^2\\&\quad-\bigg(\frac{1}{2}-2C_1\varepsilon\bigg)\int_0^\infty \mu'(s)\big\|A^{1/2}\eta^t(s)\big\|^2\,ds\leqslant 2C_2\varepsilon\big\|\eta^t\big\|^2_{\mathcal{M}^1}.\end{aligned} \tag{5.11}$$

It follows from (H3) that

$$\frac{d}{dt}\left(\frac{1}{2}\|z(t)\|^2_{\mathcal{V}^0}+\int_\Omega F(u)\,dx+2\varepsilon J(t)\right)+(1-\varepsilon)\|\nabla v\|^2+\varepsilon\kappa_0\|\theta\|^2+\left[\delta\left(\frac{1}{2}-2C_1\varepsilon\right)-2C_2\varepsilon\right]\left\|\eta^t\right\|^2_{\mathcal{M}^1}\leqslant 0. \tag{5.12}$$

Taking ε sufficiently small and integrating (5.12) with respect to t, we get

$$\int_0^\infty\left(\|\nabla v\|^2+\|\theta\|^2+\left\|\eta^t\right\|^2_{\mathcal{M}^1}\right)dt<\infty. \tag{5.13}$$

Denote

$$h(t)=\|v\|^2_{V^{-1}}+\|\theta\|^2_{V^{-1}}+\left\|\eta^t\right\|^2_{\mathcal{M}^0}. \tag{5.14}$$

Then from the continuous embedding $V^1\hookrightarrow V^0\hookrightarrow V^{-1}$, $\mathcal{M}^1\hookrightarrow\mathcal{M}^0$, we can conclude from (5.13) that

$$h(t)\in L^1(0,\infty). \tag{5.15}$$

This and (5.10) imply

$$\lim_{t\to+\infty}h(t)=0. \tag{5.16}$$

Finally, (5.7) and (5.8) follow from (5.6) and (5.16). The proof is complete. □

In order to complete the proof of Theorem 2.2, it remains to show the convergence of u. This can be done by making use of a suitable Łojasiewicz–Simon type inequality. In our case, it would be convenient to apply the abstract version in [22]. Denote

$$\mathcal{E}(u)=\frac{1}{2}\int_\Omega|Au|^2\,dx+\int_\Omega F(u)\,dx. \tag{5.17}$$

Then we have

Lemma 5.3 *(Łojasiewicz–Simon type inequality). Suppose that assumptions* (F1)′, (F2) *are satisfied. Let ψ be a critical point of $\mathcal{E}(u)$. There exist constants $\rho\in(0,\frac{1}{2})$ and $\beta>0$ depending on ψ such that for any $u\in V^2$ satisfying $\|u-\psi\|_{V^2}<\beta$, there holds*

$$\left\|A^2u+f(u)\right\|_{V^{-2}}\geqslant\left|\mathcal{E}(u)-\mathcal{E}(\psi)\right|^{1-\rho}. \tag{5.18}$$

Remark 5.2. We note that a "smooth" version of Łojasiewicz–Simon inequality of similar type has been introduced in [25]. However, the solution to our problem no longer enjoys the smooth property as in [25].

We prove the convergence result following a simple argument introduced in [26] in which the key observation is that after certain time t_0, the solution u will fall into the small neighborhood of u_∞ and stay there forever. Unlike parabolic equations, in order to apply the Łojasiewicz–Simon approach to our problem we have to introduce an auxiliary functional which is usually a perturbation of the Lyapunov functional $E(t)$ due to the structure of (2.8) (cf. [22,25,33,40] and references cited therein).

Define

$$H(t)=\frac{1}{2}\left\|v(t)\right\|^2+\frac{1}{2}\left\|\theta(t)\right\|^2+\frac{1}{2}\left\|\eta^t\right\|^2_{\mathcal{M}^1}+\mathcal{E}\big(u(t)\big)-\alpha\int_0^\infty\mu(s)\big\langle\theta(t),\eta^t(s)\big\rangle ds+\varepsilon\big\langle A^2u(t)+f\big(u(t)\big),v(t)\big\rangle_{V^{-2}}, \tag{5.19}$$

where $\alpha>0$, $\varepsilon>0$ are two coefficients to be determined later. It is easy to check that $H(t)$ is well defined for $t\geqslant 0$. A direct calculation yields

$$\begin{aligned}\frac{dH}{dt}={}&-\|\nabla v\|^2+\frac{1}{2}\int_0^\infty\mu'(s)\left\|A^{1/2}\eta^t(s)\right\|^2ds-\alpha\int_0^\infty\mu(s)\big\langle\Delta u_t(t),\eta^t(s)\big\rangle ds-\alpha\kappa_0\|\theta\|^2-\alpha\left\|\int_0^\infty\mu(s)A^{\frac{1}{2}}\eta^t(s)\,ds\right\|^2\\&+\alpha\int_0^\infty\mu(s)\big\langle\theta(t),\eta^t_s(s)\big\rangle ds+\varepsilon\left[\left\|A^2u+f(u)\right\|^2_{V^{-2}}+\big\langle A^2u+f(u),Av-A\theta\big\rangle_{V^{-2}}+\big\langle A^2v+f'(u)v,v\big\rangle_{V^{-2}}\right].\end{aligned} \tag{5.20}$$

It follows from the Hölder inequality, the Poincaré inequality and the Sobolev embedding theorem that

$$\left|\langle A^2u + f(u), Av - A\theta\rangle_{V^{-2}}\right| \leqslant \frac{1}{2}\left\|A^2u + f(u)\right\|_{V^{-2}}^2 + \|v\|^2 + \|\theta\|^2 \leqslant \frac{1}{2}\left\|A^2u + f(u)\right\|_{V^{-2}}^2 + C_9\|\nabla v\|^2 + \|\theta\|^2, \tag{5.21}$$

$$\left|\langle A^2v + f'(u)v, v\rangle_{V^{-2}}\right| \leqslant \|v\|^2 + \left\|f'(u)\right\|_{L^\infty}\|v\|^2 \leqslant C_{10}\|\nabla v\|^2, \tag{5.22}$$

where $C_9 = C_P^2$ and $C_{10} > 0$ depends on C_P and $\|z_0\|_{\mathcal{V}^0}$. Recalling (4.8) and (H3), we deduce from (5.20)–(5.22) that

$$\begin{aligned}
\frac{dH}{dt} &\leqslant -\left[1 - \frac{1}{2}\alpha - (C_9 + C_{10})\varepsilon\right]\|\nabla v\|^2 + \left(\frac{1}{2} - C_1\alpha\right)\int_0^\infty \mu'(s)\left\|A^{1/2}\eta^t(s)\right\|^2 ds \\
&\quad - \left(\frac{\alpha\kappa_0}{2} - \varepsilon\right)\|\theta\|^2 + C_2\alpha\left\|\eta^t(s)\right\|_{\mathcal{M}^1}^2 - \frac{1}{2}\varepsilon\left\|A^2u + f(u)\right\|_{V^{-2}}^2 \\
&\leqslant -\left[1 - \frac{1}{2}\alpha - (C_9 + C_{10})\varepsilon\right]\|\nabla v\|^2 - \left[\left(\frac{1}{2} - C_1\alpha\right)\delta - C_2\alpha\right]\left\|\eta^t(s)\right\|_{\mathcal{M}^1}^2 \\
&\quad - \left(\frac{\alpha\kappa_0}{2} - \varepsilon\right)\|\theta\|^2 - \frac{1}{2}\varepsilon\left\|A^2u + f(u)\right\|_{V^{-2}}^2.
\end{aligned} \tag{5.23}$$

We take $\alpha > 0$ small enough such that

$$\left(\frac{1}{2} - C_1\alpha\right)\delta - C_2\alpha \geqslant \frac{1}{4}\delta \quad \text{and} \quad \frac{1}{2}\alpha \leqslant \frac{1}{4}, \tag{5.24}$$

namely,

$$0 < \alpha \leqslant \min\left\{\frac{\delta}{4(C_1\delta + C_2)}, \frac{1}{2}\right\}. \tag{5.25}$$

After fixing α, we take $\varepsilon > 0$ sufficiently small satisfying

$$0 < \varepsilon \leqslant \min\left\{\frac{1}{4(C_9 + C_{10})}, \frac{1}{4}\alpha\kappa_0\right\}. \tag{5.26}$$

As a result, there exists a positive constant γ such that

$$\frac{d}{dt}H(t) \leqslant -\gamma\left(\|\nabla v\|^2 + \left\|\eta^t(s)\right\|_{\mathcal{M}^1}^2 + \|\theta\|^2 + \left\|A^2u + f(u)\right\|_{V^{-2}}^2\right). \tag{5.27}$$

Thus $H(t)$ is decreasing on $[0, \infty)$. Because $H(t)$ is bounded from below, it has a finite limit as time goes to infinity. On the other hand, it follows from (5.6)–(5.8) that as $t_n \to \infty$,

$$H(t_n) \to E_\infty = \mathcal{E}(u_\infty). \tag{5.28}$$

From (5.27) we can infer that $H(t) \geqslant \mathcal{E}(u_\infty)$ for all $t > 0$, and the equality sign holds if and only if u is independent of t and solves problem (2.12) while $\theta = v = \eta^t = 0$.

We now consider all possibilities.

Case 1. If there is a $t_0 > 0$ such that at this time $H(t_0) = \mathcal{E}(u_\infty)$, then for all $t > t_0$, we deduce from (5.27) that

$$\|\nabla v\| \equiv 0. \tag{5.29}$$

Namely, u is independent of time for all $t > t_0$. Due to (5.6), we can see that (2.13) holds.

Case 2. For all $t > 0$, $H(t) > \mathcal{E}(u_\infty)$. In this case, there holds

$$-\frac{d}{dt}\left(H(t) - \mathcal{E}(u_\infty)\right)^\rho = -\rho\left(H(t) - \mathcal{E}(u_\infty)\right)^{\rho-1}\frac{d}{dt}H(t), \tag{5.30}$$

here $\rho \in (0, \frac{1}{2})$ is the exponent in Lemma 5.3. By the Hölder inequality, we obtain

$$\begin{aligned}
\left(H - \mathcal{E}(u_\infty)\right)^{1-\rho} &\leqslant C\big(\|v\|^{2(1-\rho)} + \|\theta\|^{2(1-\rho)} + \left\|\eta^t\right\|_{\mathcal{M}^1}^{2(1-\rho)} + \left|\mathcal{E}(u) - \mathcal{E}(u_\infty)\right|^{1-\rho} \\
&\quad + \|\theta\|^{1-\rho}\left\|\eta^t\right\|_{\mathcal{M}^1}^{1-\rho} + \left\|A^2u + f(u)\right\|_{V^{-2}}^{1-\rho}\|v\|^{1-\rho}\big).
\end{aligned} \tag{5.31}$$

Besides, the Young inequality yields

$$\left\|A^2u + f(u)\right\|_{V^{-2}}^{1-\rho}\|v\|^{1-\rho} \leqslant \left\|A^2u + f(u)\right\|_{V^{-2}} + C\|v\|^{\frac{1-\rho}{\rho}}. \tag{5.32}$$

Noting that $(1-\rho)/\rho > 1$ and $2(1-\rho) > 1$, by the uniform bounds obtained in previous section, we conclude

$$\big(H - \mathcal{E}(u_\infty)\big)^{1-\rho} \leqslant C\big(\|v\| + \|\theta\| + \big\|\eta^t\big\|_{\mathcal{M}^1} + \big\|A^2u + f(u)\big\|_{V^{-2}} + \big|\mathcal{E}(u) - \mathcal{E}(u_\infty)\big|^{1-\rho}\big). \tag{5.33}$$

It follows from (5.6) that there exists $N \in \mathbb{N}$ such that for any $n \geqslant N$, $\|u(t_n) - u_\infty\|_{V^2} < \beta$. Set

$$\bar{t}_n = \sup\big\{t > t_n \mid \big\|u(\tau) - u_\infty\big\|_{V^2} < \beta,\ \forall \tau \in [t_n, t]\big\}. \tag{5.34}$$

Observe that $\bar{t}_n > t_n$ for all $n \geqslant N$, due to the continuity of the orbit in $\mathcal{V}^0$. Now we have to deal with two subcases.

(a) There exists n_0 such that $\bar{t}_{n_0} = \infty$. By Lemma 5.3, (5.20), (5.30), (5.33) and the Poincaré inequality, we can conclude that

$$\begin{aligned}
-\frac{d}{dt}\big(H(t) - \mathcal{E}(u_\infty)\big)^\rho &\geqslant C\rho\gamma \frac{\|\nabla v\|^2 + \|\eta^t(s)\|^2_{\mathcal{M}^1} + \|\theta\|^2 + \|A^2u + f(u)\|^2_{V^{-2}}}{\|v\| + \|\theta\| + \|\eta^t\|_{\mathcal{M}^1} + \|A^2u + f(u)\|_{V^{-2}} + |\mathcal{E}(u) - \mathcal{E}(u_\infty)|^{1-\rho}} \\
&\geqslant C_{11}\big(\|\nabla v\| + \big\|\eta^t(s)\big\|_{\mathcal{M}^1} + \|\theta\| + \big\|A^2u + f(u)\big\|_{V^{-2}}\big).
\end{aligned} \tag{5.35}$$

Integrating from t_{n_0} to t, we obtain

$$\big(H(t) - \mathcal{E}(u_\infty)\big)^\rho + C_{11}\int_{t_{n_0}}^{t} \big(\|\nabla v\| + \big\|\eta^t(s)\big\|_{\mathcal{M}^1} + \|\theta\| + \big\|A^2u + f(u)\big\|_{V^{-2}}\big)\,d\tau \leqslant \big(H(t_{n_0}) - \mathcal{E}(u_\infty)\big)^\rho. \tag{5.36}$$

Recalling that $H(t) - \mathcal{E}(u_\infty) \geqslant 0$ for $t > 0$, we infer

$$\int_{t_{n_0}}^{t} \big\|v(\tau)\big\|_{V^1}\,d\tau < \infty, \quad \forall t \geqslant t_{n_0}. \tag{5.37}$$

Thus, $u(t)$ converges in V^1. Then by the precompactness property of $u(t)$ in V^2 (see Section 4), we can conclude (2.13).

(b) For all $n \in \mathbb{N}$, $\bar{t}_n < \infty$.

Since $H(t)$ is decreasing in $[0,\infty)$ and it has a finite limit $E_\infty = \mathcal{E}(u_\infty)$ as $t \to \infty$, then for any $\zeta \in (0,\beta)$ there exists an integer N such that when $n \geqslant N$, for all $t \geqslant t_n > 0$, there holds

$$\big(H(t_n) - \mathcal{E}(u_\infty)\big)^\rho - \big(H(t) - \mathcal{E}(u_\infty)\big)^\rho < \frac{C_{11}}{2}\zeta. \tag{5.38}$$

As a result, for $n \geqslant N$ there holds

$$\int_{t_n}^{\bar{t}_n} \big\|v(\tau)\big\|_{V^1}\,d\tau < \frac{\zeta}{2}. \tag{5.39}$$

Moreover, by choosing N sufficiently large we have

$$\big\|u(t_n) - u_\infty\big\|_{V^2} < \frac{\zeta}{2}, \quad \forall n \geqslant N. \tag{5.40}$$

These imply that

$$\big\|u(\bar{t}_n) - u_\infty\big\|_{V^1} \leqslant \big\|u(t_n) - u_\infty\big\|_{V^1} + \int_{t_n}^{\bar{t}_n} \big\|v(\tau)\big\|_{V^1}\,d\tau < \zeta, \quad \forall n \geqslant N. \tag{5.41}$$

Therefore,

$$\lim_{\bar{t}_n \to +\infty} \big\|u(\bar{t}_n) - u_\infty\big\|_{V^1} = 0. \tag{5.42}$$

On the other hand, the precompactness of u in V^2 implies that there exists a subsequence of $\{u(\bar{t}_n)\}$, still denoted by $\{u(\bar{t}_n)\}$, converging to u_∞ in V^2. Thus for n sufficiently large, we get

$$\big\|u(\bar{t}_n) - u_\infty\big\|_{V^2} < \beta, \tag{5.43}$$

which contradicts the definition of $\bar{t}_n$.

5.2. Convergence rate

For $t \geqslant t_0$ with t_0 sufficiently large, it follows from Lemma 5.3 and (5.33), (5.35) that

$$\frac{d}{dt}\big(H(t)-\mathcal{E}(u_\infty)\big)+C\big(H(t)-\mathcal{E}(u_\infty)\big)^{2(1-\rho)} \leqslant 0. \tag{5.44}$$

This yields (cf. [22,40,42])

$$H(t)-\mathcal{E}(u_\infty) \leqslant C(1+t)^{-1/(1-2\rho)}, \quad \forall t \geqslant t_0. \tag{5.45}$$

Integrating (5.33) on (t,∞), we have

$$\int_t^\infty \|v\|_{V^1}\,d\tau \leqslant C(1+t)^{-\rho/(1-2\rho)}, \quad \forall t \geqslant t_0. \tag{5.46}$$

By adjusting the constant C properly, we obtain

$$\big\|u(t)-u_\infty\big\|_{V^1} \leqslant C(1+t)^{-\rho/(1-2\rho)}, \quad \forall t \geqslant 0. \tag{5.47}$$

Based on this estimate for u in V^1 norm, we are able to obtain the estimates (in higher order norm) stated in Theorem 2.2.

By subtracting the evolution equations (2.7) and their corresponding stationary equations (2.12), we have

$$\begin{cases} \theta_t-\Delta u_t-\displaystyle\int_0^\infty \mu(s)\Delta\eta^t(s)\,ds=0, \\ u_{tt}-\Delta u_t+\Delta\theta+\Delta^2(u-u_\infty)+f(u)-f(u_\infty)=0. \end{cases} \tag{5.48}$$

Similar to (3.3), we can see that

$$\frac{d}{dt}\left(\frac{1}{2}\big\|u(t)-u_\infty\big\|_{V^2}^2+\frac{1}{2}\|v\|^2+\frac{1}{2}\|\theta\|^2+\frac{1}{2}\big\|\eta^t\big\|_{\mathcal{M}^1}^2+\int_\Omega F(u)\,dx-\int_\Omega F(u_\infty)\,dx-\int_\Omega f(u_\infty)(u-u_\infty)\,dx\right)$$
$$+\|\nabla v\|^2-\frac{1}{2}\int_0^\infty \mu'(s)\big\|A^{1/2}\eta^t(s)\big\|^2\,ds=0. \tag{5.49}$$

Multiplying the second equation in (5.48) by $u-u_\infty$ and integrating on Ω, we get

$$\frac{d}{dt}\left(\frac{1}{2}\|\nabla u-\nabla u_\infty\|^2+\int_\Omega v(u-u_\infty)\,dx\right)-\|v\|^2+\big\|A(u-u_\infty)\big\|^2$$
$$=-\int_\Omega \big(f(u)-f(u_\infty)\big)(u-u_\infty)\,dx+\int_\Omega \theta A(u-u_\infty)\,dx. \tag{5.50}$$

Multiplying (4.8) by 2ε and multiplying (5.50) by ε^2 respectively, then adding the resultants to (5.49) yield

$$\frac{d}{dt}\left(\frac{1}{2}\big\|u(t)-u_\infty\big\|_{V^2}^2+\frac{1}{2}\|v\|^2+\frac{1}{2}\|\theta\|^2+\frac{1}{2}\big\|\eta^t\big\|_{\mathcal{M}^1}^2+\frac{\varepsilon^2}{2}\big\|\nabla(u-u_\infty)\big\|^2+\int_\Omega F(u)\,dx\right.$$
$$\left.-\int_\Omega F(u_\infty)\,dx-\int_\Omega f(u_\infty)(u-u_\infty)\,dx+\varepsilon^2\int_\Omega v(u-u_\infty)\,dx+2\varepsilon J(t)\right)$$
$$+(1-\varepsilon)\|\nabla v\|^2+\varepsilon\kappa_0\|\theta\|^2+\left[\left(\frac{1}{2}-2C_1\varepsilon\right)\delta-2C_2\varepsilon\right]\big\|\eta^t\big\|_{\mathcal{M}^1}^2+\varepsilon^2\big\|A(u-u_\infty)\big\|^2$$
$$\leqslant \varepsilon^2\|v\|^2-\varepsilon^2\int_\Omega \big(f(u)-f(u_\infty)\big)(u-u_\infty)\,dx+\varepsilon^2\int_\Omega \theta A(u-u_\infty)\,dx. \tag{5.51}$$

We now estimate the three terms on the right-hand side of inequality (5.51),

$$\varepsilon^2\|v\|^2 \leqslant \varepsilon^2 C_P^2\|\nabla v\|^2, \tag{5.52}$$

$$\left|-\varepsilon^2\int_\Omega \big(f(u)-f(u_\infty)\big)(u-u_\infty)\,dx\right| \leqslant \varepsilon^2\|f'\|_{L^\infty}\|u-u_\infty\|^2 \leqslant C\varepsilon^2\|u-u_\infty\|^2, \tag{5.53}$$

$$\left|\varepsilon^2\int_\Omega \theta A(u-u_\infty)\,dx\right| \leqslant \frac{1}{4}\varepsilon^2\big\|A(u-u_\infty)\big\|^2+\varepsilon^2\|\theta\|^2. \tag{5.54}$$

On the other hand, by the Taylor's expansion, we have

$$F(u)=F(u_\infty)+f(u_\infty)(u-u_\infty)+f'(\xi)(u-u_\infty)^2, \tag{5.55}$$

where $\xi = au+(1-a)u_\infty$ with $a\in[0,1]$.

Then we deduce that

$$\begin{aligned}\left|\int_\Omega F(u)\,dx-\int_\Omega F(u_\infty)\,dx+\int_\Omega f(u_\infty)u_\infty\,dx-\int_\Omega f(u_\infty)u\,dx\right| &= \left|\int_\Omega f'(\xi)(u-u_\infty)^2\,dx\right| \\ &\leqslant \big\|f'(\xi)\big\|_{L^\infty}\|u-u_\infty\|^2 \leqslant C\|u-u_\infty\|^2. \end{aligned}\tag{5.56}$$

Let us define now, for $t\geqslant 0$,

$$\begin{aligned} y(t) &= \frac{1}{2}\big\|u(t)-u_\infty\big\|_{V^2}^2+\frac{1}{2}\|v\|^2+\frac{1}{2}\|\theta\|^2+\frac{1}{2}\big\|\eta^t\big\|_{\mathcal{M}^1}^2+\frac{\varepsilon^2}{2}\big\|\nabla(u-u_\infty)\big\|^2+\int_\Omega F(u)\,dx \\ &\quad -\int_\Omega F(u_\infty)\,dx-\int_\Omega f(u_\infty)(u-u_\infty)\,dx+\varepsilon^2\int_\Omega v(u-u_\infty)\,dx+2\varepsilon J(t). \end{aligned}\tag{5.57}$$

Taking ε sufficiently small, it follows from the Hölder inequality, (5.56) and (4.3) that there exist constants $\gamma_0,\gamma_1,\gamma_2>0$ such that

$$\gamma_0\|z-z_\infty\|_{\mathcal{V}^0}^2 \geqslant y(t) \geqslant \gamma_2\|z-z_\infty\|_{\mathcal{V}^0}^2-\gamma_1\|u-u_\infty\|_{V^1}^2. \tag{5.58}$$

Moreover, for small ε we can deduce from (5.51)–(5.54) and (5.58) that for certain $\gamma_3>0$, the following inequality holds

$$\frac{d}{dt}y(t)+\gamma_3 y(t) \leqslant C\|u-u_\infty\|_{V^1}^2. \tag{5.59}$$

The Gronwall inequality and (5.47) yield (see, e.g., [39,40])

$$y(t)\leqslant C(1+t)^{-2\rho/(1-2\rho)}, \quad \forall t\geqslant 0, \tag{5.60}$$

which together with (5.58) implies that

$$\|z-z_\infty\|_{\mathcal{V}^0}\leqslant C(1+t)^{-\rho/(1-2\rho)}, \quad \forall t\geqslant 0. \tag{5.61}$$

The proof of Theorem 2.2 is now complete.

Before ending this paper, we give a further remark on the estimate of convergence rate. As has been shown in the previous section, the solution $z(t)$ to our problem (2.8) with initial data $z_0\in\mathcal{V}^0$ can be decomposed into two parts $z(t)=z_D(t)+z_C(t)$, where $z_D(t)=(u_D(t),v_D(t),\theta_D(t),\eta_D^t)^T$ and $z_C(t)=(u_C(t),v_C(t),\theta_C(t),\eta_C^t)^T$ satisfy (4.15) and (4.16), respectively. It is also shown in Lemma 4.2 that $z_D(t)$ will decay to 0 in $\mathcal{V}^0$ exponentially fast. This convergence rate is obviously better than the rate for $z(t)$ obtained in Theorem 2.2. As a result, we can easily obtain the following result for the compact part $z_C(t)$ from Lemma 4.2 and Theorem 2.2.

Proposition 5.1. *Under the assumptions of Theorem 2.2, we have*

$$\big\|z_C(t)-z_\infty\big\|_{\mathcal{V}^0}\leqslant C(1+t)^{-\rho/(1-2\rho)}, \quad \forall t\geqslant 0, \tag{5.62}$$

where $C>0$ is a constant depending on $\|z_0\|_{\mathcal{V}^0}$ and $z_\infty=(u_\infty,0,0,0)^T$.

Proof. We notice that

$$\big\|z_C(t)-z_\infty\big\|_{\mathcal{V}^0}\leqslant \big\|z(t)-z_\infty\big\|_{\mathcal{V}^0}+\big\|z_D(t)\big\|_{\mathcal{V}^0}, \tag{5.63}$$

$$\lim_{t\to+\infty} e^{-\frac{\delta_1}{2}t}(1+t)^{\rho/(1-2\rho)}=0. \tag{5.64}$$

Then the conclusion (5.62) follows from Lemma 4.2 and (5.61) after the constant C is properly modified. □

Moreover, Lemma 4.3 provides a uniform estimate of z_C in $\mathcal{V}^1$. As a direct consequence, this fact and Proposition 5.1 imply the weak convergence of z_C such that

$$z_C(t) \rightharpoonup z_\infty, \quad \text{in } \mathcal{V}^1, \text{ as } t \to +\infty.$$

Based on the idea we used in the proof of Theorem 2.2, we are able to get a stronger result, namely

Theorem 5.1. *Under the assumptions of Theorem* 2.2*, we have*

$$\|z_C(t) - z_\infty\|_{\mathcal{V}^1} \leqslant C(1+t)^{-\rho/(1-2\rho)}, \quad \forall t \geqslant 0, \tag{5.65}$$

where $C > 0$ *is a constant depending on* $\|z_0\|_{\mathcal{V}^0}$ *and* $\|u_\infty\|_{V^3}$.

Proof. Subtracting (2.12) from (4.16) we have

$$\begin{cases} \dfrac{d}{dt}(z_C - z_\infty) = L(z_C - z_\infty) + \big(0, -f(u) + f(u_\infty), 0, 0\big)^T, \\ (z_C - z_\infty)|_{t=0} = z_\infty. \end{cases} \tag{5.66}$$

Taking the inner product of the resulting system (5.66) and $A(z_C - z_\infty)$ in $\mathcal{V}^0$, we get

$$\frac{d}{dt}\left(\frac{1}{2}\big\|A^{1/2}(z_C - z_\infty)\big\|^2_{\mathcal{V}^0} + \int_\Omega \big(f(u) - f(u_\infty)\big)A(u_C - u_\infty)\,dx\right) + \|Av_C\|^2 - \frac{1}{2}\int_0^\infty \mu'(s)\big\|A\eta_C^t(s)\big\|^2\,ds$$
$$= \int_\Omega f'(u)vA(u_C - u_\infty). \tag{5.67}$$

Next, multiplying the second equation in (5.66) by $A(u_C - u_\infty)$ and integrating on Ω, we have

$$\frac{d}{dt}\left(\int_\Omega v_C A(u_C - u_\infty)\,dx + \frac{1}{2}\big\|A(u_C - u_\infty)\big\|^2\right) + \big\|A^{3/2}(u_C - u_\infty)\big\|^2 - \big\|A^{1/2}v_C\big\|^2$$
$$= -\int_\Omega \big(f(u) - f(u_\infty)\big)A(u_C - u_\infty)\,dx + \int_\Omega A\theta_C A(u_C - u_\infty)\,dx. \tag{5.68}$$

Now we introduce the functional

$$\Upsilon(t) = \big\|A^{1/2}\big(z_C(t) - z_\infty\big)\big\|^2_{\mathcal{V}^0} + 2\int_\Omega \big(f(u) - f(u_\infty)\big)A(u_C - u_\infty)\,dx$$
$$+ 2\varepsilon^2\int_\Omega v_C A(u_C - u_\infty)\,dx + \varepsilon^2\big\|A(u_C - u_\infty)\big\|^2 + 4\varepsilon J_C(t). \tag{5.69}$$

It follows from Theorem 2.2, Proposition 5.1 and (4.28) that for $t \geqslant 0$,

$$\left|\int_\Omega \big(f(u) - f(u_\infty)\big)A(u_C - u_\infty)\,dx\right| \leqslant \|f'\|_{L^\infty}\|u - u_\infty\|\big\|A(u_C - u_\infty)\big\| \leqslant C(1+t)^{-2\rho/(1-2\rho)}, \tag{5.70}$$

$$\left|\int_\Omega v_C A(u_C - u_\infty)\,dx\right| \leqslant \|v_C\|\big\|A(u_C - u_\infty)\big\| \leqslant C(1+t)^{-2\rho/(1-2\rho)}, \tag{5.71}$$

$$\big|J_C(t)\big| \leqslant C\|z_C - z_\infty\|^2_{\mathcal{V}^1}. \tag{5.72}$$

As a result, after choosing $\varepsilon > 0$ sufficiently small, there is a constant $C > 0$ such that

$$\big\|z_C(t) - z_\infty\big\|^2_{\mathcal{V}^1} \leqslant 2\Upsilon(t) + C(1+t)^{-2\rho/(1-2\rho)}. \tag{5.73}$$

It follows from (5.67)–(5.68) and (4.29) that

$$\frac{1}{2}\frac{d}{dt}\Upsilon(t) + (1-\varepsilon)\|Av_C\|^2 - \left(\frac{1}{2} - 2C_7\varepsilon\right)\int_0^\infty \mu'(s)\big\|A\eta_C^t(s)\big\|^2\,ds$$
$$- \varepsilon^2\big\|A^{1/2}v_C\big\|^2 + \varepsilon^2\big\|A^{3/2}(u_C - u_\infty)\big\|^2 + \varepsilon\kappa_0\big\|A^{1/2}\theta_C\big\|^2$$
$$\leqslant \int_\Omega f'(u)vA(u_C - u_\infty)\,dx + 2C_6\varepsilon\big\|\eta_C^t\big\|^2_{\mathcal{M}^2} - \varepsilon^2\int_\Omega \big(f(u) - f(u_\infty)\big)A(u_C - u_\infty)\,dx + \varepsilon^2\int_\Omega A\theta_C A(u_C - u_\infty)\,dx. \tag{5.74}$$

The right-hand side of (5.74) can be estimated as follows

$$\left|\int_\Omega f'(u)vA(u_C-u_\infty)\,dx\right| \leqslant \|f'(u)\|_{L^\infty}\|v\|\|A(u_C-u_\infty)\| \leqslant C(1+t)^{-2\rho/(1-2\rho)}, \tag{5.75}$$

$$\left|\int_\Omega \big(f(u)-f(u_\infty)\big)A(u_C-u_\infty)\,dx\right| \leqslant \|f'\|_{L^\infty}\|u-u_\infty\|\|A(u_C-u_\infty)\| \leqslant C(1+t)^{-2\rho/(1-2\rho)}, \tag{5.76}$$

$$\left|\varepsilon^2\int_\Omega A\theta_C A(u_C-u_\infty)\,dx\right| \leqslant \frac{\varepsilon\kappa_0}{2}\|A^{1/2}\theta_C\|^2 + C\varepsilon^3\|A^{3/2}(u_C-u_\infty)\|^2, \tag{5.77}$$

where in the above estimates C is a constant depending on $\|z_0\|_{\mathcal{V}^0}$ at most. Similar to the previous section, we can choose $\varepsilon>0$ small enough and consequently there is a constant $\gamma_4>0$ such that

$$\frac{d}{dt}\Upsilon(t)+\gamma_4\Upsilon(t) \leqslant C(1+t)^{-2\rho/(1-2\rho)}. \tag{5.78}$$

As a result,

$$\Upsilon(t) \leqslant C(1+t)^{-2\rho/(1-2\rho)}, \quad \forall t \geqslant 0, \tag{5.79}$$

here $C>0$ is a constant depending on $\|z_0\|_{\mathcal{V}^0}$ and $\|u_\infty\|_{V^3}$ (see Remark 5.1).

The required estimate (5.65) follows from (5.79) and (5.73). We complete the proof. $\square$

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References

[1] S. Aizicovici, E. Feireisl, Long-time stabilization of solutions to a phase-field model with memory, J. Evol. Equ. 1 (2001) 69–84.
[2] S. Aizicovici, H. Petzeltová, Convergence of solutions of phase-field systems with a nonconstant latent heat, Dynam. Systems Appl. 14 (2005) 163–173.
[3] V. Belleri, V. Pata, Attractors for semilinear strongly damped wave equation on $\mathbb{R}^3$, Discrete Contin. Dyn. Syst. 7 (2001) 719–735.
[4] V.V. Chepyzhov, E. Mainini, V. Pata, Stability of abstract linear semigroups arising from heat conduction with memory, Asymptot. Anal. 50 (2006) 269–291.
[5] I. Chueshov, M. Eller, I. Lasiecka, On the attractor for a semilinear wave equation with critical exponent and nonlinear boundary dissipation, Comm. Partial Differential Equations 27 (2002) 1901–1951.
[6] B.D. Coleman, M.E. Gutin, Equipresence and constitutive equations for rigid heat conductors, Z. Angew. Math. Phys. 18 (1967) 199–208.
[7] C.M. Dafermos, Asymptotic stability in viscoelasticity, Arch. Ration. Mech. Anal. 37 (1970) 297–308.
[8] M. Fabrizio, B. Lazzari, J.E. Muñoz Rivera, Asymptotic behavior of thermoelastic plates of weakly hyperbolic type, Differential Integral Equations 13 (2000) 1347–1370.
[9] C.G. Gal, H. Wu, Asymptotic behavior of a Cahn–Hilliard equation with Wentzell boundary conditions and mass conversation, Discrete Contin. Dyn. Syst., in press.
[10] S. Gatti, M. Grasselli, V. Pata, Exponential attractors for a conserved phase-field system with memory, Phys. D 189 (2004) 31–48.
[11] C. Giorgi, M. Grasselli, V. Pata, Uniform attractors for a phase-field model with memory and quadratic nonlinearity, Indiana Univ. Math. J. 48 (1999) 1395–1445.
[12] C. Giorgi, V. Pata, Stability of linear thermoelastic systems with memory, Math. Models Methods Appl. Sci. 11 (2001) 627–644.
[13] M. Grasselli, On the large time behavior of a phase-field system with memory, Asymptot. Anal. 56 (2008) 229–249.
[14] M. Grasselli, J.E. Muñoz Rivera, V. Pata, On the energy decay of the linear thermoelastic plate with memory, J. Math. Anal. Appl. 309 (2005) 1–14.
[15] M. Grasselli, H. Petzeltová, G. Schimperna, Asymptotic behavior of a nonisothermal viscous Cahn–Hilliard equation with inertial term, J. Differential Equations 239 (2007) 38–60.
[16] M. Grasselli, V. Pata, Upper semicontinuous attractor for a hyperbolic phase-field model with memory, Indiana Univ. Math. J. 50 (2001) 1281–1308.
[17] M. Grasselli, V. Pata, F.M. Vegni, Longterm dynamics of a conserved phase-field model with memory, Asymptot. Anal. 33 (2003) 261–320.
[18] M. Grasselli, V. Pata, Asymptotic behavior of a parabolic–hyperbolic system, Commun. Pure Appl. Anal. 3 (2004) 849–881.
[19] M. Grasselli, V. Pata, Existence of a universal attractor for a fully hyperbolic phase-field system, J. Evol. Equ. 4 (2004) 27–51.
[20] M. Grasselli, H. Wu, S. Zheng, Asymptotic behavior of a non-isothermal Ginzburg–Landau model, Quart. Appl. Math., in press.
[21] M.E. Gurtin, A.C. Pipkin, A general theory of heat conduction with finite wave speeds, Arch. Ration. Mech. Anal. 31 (1968) 113–126.
[22] A. Haraux, M.A. Jendoubi, Convergence of bounded weak solutions of the wave equation with dissipation and analytic nonlinearity, Calc. Var. 9 (1999) 95–124.
[23] A. Haraux, M.A. Jendoubi, Decay estimates to equilibrium for some evolution equations with an analytic nonlinearity, Asymptot. Anal. 26 (2001) 21–36.
[24] L. Herrera, D. Pávon, Hyperbolic theories of dissipation: Why and when do we need them?, Phys. A 307 (2002) 121–130.
[25] K.-H. Hoffmann, P. Rybka, Analyticity of the nonlinear term forces convergence of solutions to two equations of continuum mechanics, Nonlinear Anal. 50 (2002) 409–424.
[26] M.A. Jendoubi, A simple unified approach to some convergence theorem of L. Simon, J. Funct. Anal. 153 (1998) 187–202.
[27] J. Jiang, Convergence to equilibrium for a parabolic–hyperbolic phase-field model with Cattaneo heat flux law, J. Math. Anal. Appl. 341 (2008) 149–169.
[28] J. Jiang, Convergence to equilibrium for a fully hyperbolic phase-field model with Cattaneo heat flux law, preprint, 2008.
[29] D.D. Joseph, L. Preziosi, Heat waves, Rev. Modern Phys. 61 (1989) 41–73.
[30] D.D. Joseph, L. Preziosi, Addendum to the paper: Heat waves, Rev. Modern Phys. 62 (1990) 375–391.
[31] J. Lagnese, Boundary Stabilization of Thin Plates, SIAM Stud. Appl. Math., vol. 10, SIAM, Philadelphia, 1989.

[32] Z. Liu, S. Zheng, Semigroups Associated with Dissipative Systems, Chapman & Hall/CRC Press, Boca Raton, FL, 1999.
[33] G. Mola, Convergence to equilibria for a three-dimensional conserved phase-field system with memory, Electron. J. Differential Equations 23 (2008) 1–16.
[34] G. Mola, Global and exponential attractors for a conserved phase-field system with Grutin–Pinkin heat conduction law, PhD thesis, Dipartimento di Matematica "F. Brioschi", Politecnico di Milano, 2007.
[35] J.E. Muñoz Rivera, R. Racke, Smoothing properties, decay and global existence of solutions to nonlinear coupled systems of thermoelastic type, SIAM J. Math. Anal. 26 (1995) 1547–1563.
[36] J.E. Muñoz Rivera, R. Racke, Large solutions and smoothing properties for nonlinear thermoelastic systems, J. Differential Equations 127 (1996) 454–483.
[37] V. Pata, Exponential stability in linear viscoelasticity, Quart. Appl. Math. 64 (2006) 499–513.
[38] R. Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Appl. Math. Sci., vol. 68, Springer-Verlag, New York, 1988.
[39] H. Wu, Convergence to equilibrium for a Cahn–Hilliard model with the Wentzell boundary condition, Asymptot. Anal. 54 (2007) 71–92.
[40] H. Wu, M. Grasselli, S. Zheng, Convergence to equilibrium for a parabolic–hyperbolic phase-field system with Neumann boundary conditions, Math. Models Methods Appl. Sci. 17 (2007) 1–29.
[41] H. Wu, M. Grasselli, S. Zheng, Convergence to equilibrium for a nonlinear parabolic–hyperbolic phase-field system with dynamic boundary condition, J. Math. Anal. Appl. 329 (2007) 948–976.
[42] S. Zheng, Nonlinear Evolution Equations, Pitman Ser. Monogr. Surv. Pure Appl. Math., vol. 133, Chapman & Hall/CRC Press, Boca Raton, FL, 2004.