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Maximal regularity for perturbed integral equations on periodic Lebesgue spaces

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ABSTRACT

We characterize the maximal regularity of periodic solutions for an additive perturbed integral equation with infinite delay in the vector-valued Lebesgue spaces. Our method is based on operator-valued Fourier multipliers. We also study resonances, characterizing the existence of solutions in terms of a compatibility condition on the forcing term.

1. Introduction

Maximal regularity is a very useful tool in the study of linear and nonlinear evolution equations and results in this direction have been studied extensively in recent years (see, for example, [2,4,5,12,14,26,28,30] and the bibliography therein). Indeed, besides of giving optimal results under minimal regularity assumptions on the coefficients of differential operators (cf. [1]), in the investigation of nonlinear equations it facilitates the application of linearization techniques based on the implicit function theorem (see, e.g., [18]).

However, in concrete situations it is no easy task to verify that a given equation possesses the property of maximal regularity. Therefore it is important to have at our disposal general theorems which allow to derive the desired property for large classes of equations.

In this paper we characterize the maximal regularity on periodic Lebesgue spaces for the following integral equation with infinite delay

$$u(t) = A\int_{-\infty}^{t} a(t-s)u(s)\,ds + B\int_{-\infty}^{t} b(t-s)u(s)\,ds + f(t), \tag{1.1}$$

where $a(\cdot), b(\cdot) \in L^1(\mathbb{R}_+)$ are scalar-valued kernels, A and B are closed linear operators defined on a *UMD* space X, such that $D(A) \cap D(B) \neq \{0\}$. Equations of the form (1.1) has been motived by Pugliese [32] and Prüss [31, p. 235].

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In contrast with many papers on the subject of maximal regularity for integral equations, in this work we will study directly the full problem (1.1) by a method based on operator-valued Fourier multiplier theorems, which was initiated by L. Weis in [35] in the investigation of maximal regularity for abstract equations (see also [7] and [34]). The specific operator-valued Fourier multiplier theorems which we use are those established by Arendt and Bu in [8, Theorem 1.3].

Maximal regularity on periodic Lebesgue, Besov and Triebel spaces for the subject of integrodifferential equations, and by use of operator-valued Fourier multiplier theorems, have been studied recently in [11,21–24,29]. Our case is more difficult to handle because of the presence of the perturbing operator B. We are able to obtain necessary and sufficient conditions for maximal regularity in terms of R-boundedness of

$$\left\{\left(I-\tilde{b}(ik)B-\tilde{a}(ik)A\right)^{-1}\right\}_{k\in\mathbb{Z}} \quad \text{and} \quad \tilde{b}(ik)B\left\{\left(I-\tilde{b}(ik)B-\tilde{a}(ik)A\right)^{-1}\right\}_{k\in\mathbb{Z}}. \tag{1.2}$$

We do not make in this paper any parabolicity assumption on the operators, not even that A generates a semigroup. In fact, we give examples showing that the condition that A be the generator of a semigroup is not necessary. We study, in particular, the special case $B = A^{\varepsilon}$. Under analogous assumptions as in [13] we prove that if $\varepsilon \in (1/2, 1]$, then an integral version of the problem

$$u''(t) + A^{\varepsilon}u'(t) + Au(t) = f(t)$$

has a periodic solution under the presence of resonances, provided a compatibility condition in terms of the forcing term f is satisfied.

The paper is organized as follows. In Section 2 we collect some results about operator-valued Fourier multipliers and R-bounded families. In Section 3 we characterize R-boundedness of (1.2) in terms of L^p-multipliers (Theorem 3.1). We obtain our main result which characterizes the maximal regularity of (1.1) on periodic Lebesgue spaces (Theorem 3.5). We remark that in this case, Fejer's theorem can be used to construct the solution.

Section 4 is devoted to maximal regularity in the periodic Lebesgue spaces $L^p_{2\pi}(\mathbb{R}; X)$ in case that (1.1) has resonances. A similar case was studied in [16] but under the condition that A generates an analytic semigroup. We show that under essentially the same conditions established in Section 3, Eq. (1.1) has L^p periodic solution if and only if f satisfies suitable compatibility conditions (Theorem 4.4). Also in this case we give a representation formula for all the solutions, which allows the study of their qualitative properties.

2. Preliminaries

Let X, Y be Banach spaces. We denote by $\mathcal{B}(X, Y)$ the space of all bounded linear operators from X to Y. When $X = Y$, we write simply $\mathcal{B}(X)$. As usual, we identify the spaces of (vector- or operator-valued) functions defined on $[0, 2\pi]$ to their periodic extensions to $\mathbb{R}$. Thus, in this paper, we consider the space $L^p_{2\pi}(\mathbb{R}; X)$ (denoted also $L^p((0, 2\pi); X)$, $1 \leqslant p \leqslant \infty$) of all 2π-periodic Bochner measurable X-valued functions f such that the restriction of f to $[0, 2\pi]$ is p-integrable (usual modification in case $p = \infty$). For a function $f \in L^1_{2\pi}(\mathbb{R}; X)$, denote by $\hat{f}(k)$, for $k \in \mathbb{Z}$, the kth Fourier coefficient of f, that is,

$$\hat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} e_{-k}(t)f(t)\,dt,$$

where $e_k(t) = e^{ikt}$, with $t \in \mathbb{R}$.

We begin with preliminaries about operator-valued Fourier multipliers. More information may be found in Arendt and Bu [8] for the periodic case and Amann [4], Weis [34] for the non-periodic case.

Definition 2.1. For $1 \leqslant p \leqslant \infty$, we say that a sequence $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X, Y)$ is an $L^p_{X,Y}$-multiplier, if for each $f \in L^p_{2\pi}(\mathbb{R}; X)$ there exists $u \in L^p_{2\pi}(\mathbb{R}; Y)$ such that

$$\hat{u}(k) = M_k \hat{f}(k) \quad \text{for all } k \in \mathbb{Z}.$$

It follows from the uniqueness theorem of Fourier series that u is uniquely determined by f. If a sequence $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X, Y)$ is an L^p-multiplier, then and only then, there exists a unique bounded operator $\mathcal{M}: L^p_{2\pi}(\mathbb{R}; X) \to L^p_{2\pi}(\mathbb{R}; Y)$ such that

$$\widehat{(\mathcal{M}f)}(k) = M_k \hat{f}(k),$$

for all $k \in \mathbb{Z}$ and all $f \in L^p_{2\pi}(\mathbb{R}; X)$. The set of Fourier multipliers is a vector space. Moreover, it is clear from the definition that if X, Y, Z are Banach spaces and $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X, Y)$ and $\{N_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(Y, Z)$ are Fourier multipliers then $\{N_k M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X, Z)$ is a Fourier multiplier as well. When $X = Y$, the space of Fourier multipliers is an operator algebra.

For $j \in \mathbb{N}$, denote by r_j the jth Rademacher function on $[0, 1]$, i.e. $r_j(t) = sgn(\sin(2^j \pi t))$. For $x \in X$ we denote by $r_j \otimes x$ the vector-valued function $t \to r_j(t)x$.

Definition 2.2. A family $\mathbf{T} \subset \mathcal{B}(X,Y)$ is called R-bounded if there exists $c_p \geqslant 0$ such that

$$\left\| \sum_{j=1}^{n} r_j \otimes T_j x_j \right\|_{L^p(0,1;Y)} \leqslant c_p \left\| \sum_{j=1}^{n} r_j \otimes x_j \right\|_{L^p(0,1;X)} \tag{2.1}$$

for all $T_1, \dots, T_n \in \mathbf{T}$, $x_1, \dots, x_n \in X$ and $n \in \mathbb{N}$, where $1 \leqslant p < \infty$. We denote by $R_p(\mathbf{T})$ the smallest constant c_p such that (2.1) holds.

The notion of R-boundedness goes back to Bourgain [10], Berkson and Gillespie [9], and Clément, de Pagter, Sukochev and Witvliet [15]. Several properties of R-bounded families can be founded in the recent monograph of Denk, Hieber and Prüss [17]. For the reader's convenience, we summarize here from [17, Section 3] some results.

Remark 2.3.

(a) Any finite family $\mathcal{T} \subset \mathcal{B}(X,Y)$ is R-bounded.
(b) If $\mathcal{T} \subset \mathcal{B}(X,Y)$ is R-bounded then it is uniformly bounded, with

$$\sup\{\|T\|:\ T \in \mathcal{T}\} \leqslant R_p(\mathcal{T}).$$

(c) The definition of R-boundedness is independent of $p \in [1,\infty)$.
(d) When X and Y are Hilbert spaces, $\mathcal{T} \subset \mathcal{B}(X,Y)$ is R-bounded if and only if $\mathcal{T}$ is uniformly bounded.
(e) Let X, Y be Banach spaces and $\mathcal{T}, \mathcal{S} \subset \mathcal{B}(X,Y)$ be R-bounded. Then

$$\mathcal{T} + \mathcal{S} = \{T + S:\ T \in \mathcal{T},\ S \in \mathcal{S}\}$$

is R-bounded as well, and $R_p(\mathcal{T} + \mathcal{S}) \leqslant R_p(\mathcal{T}) + R_p(\mathcal{S})$.
(f) Let X, Y, Z be Banach spaces, and $\mathcal{T} \subset \mathcal{B}(X,Y)$ and $\mathcal{S} \subset \mathcal{B}(Y,Z)$ be R-bounded. Then

$$\mathcal{S}\mathcal{T} = \{ST:\ T \in \mathcal{T},\ S \in \mathcal{S}\}$$

is R-bounded, and $R_p(\mathcal{S}\mathcal{T}) \leqslant R_p(\mathcal{S})R_p(\mathcal{T})$.
(g) Let X, Y be Banach spaces and $\mathcal{T} \subset \mathcal{B}(X,Y)$ be R-bounded. By contraction principle of Kahane, see [17,25], we have that, if $\{\alpha_k\}_{k\in\mathbb{Z}}$ is a bounded sequence then

$$\{\alpha_k T:\ T \in \mathcal{T},\ k \in \mathbb{Z}\}$$

is R-bounded.

Remark 2.4. Let $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Y)$ be an $L^p_{X,Y}$-multiplier, where $1 \leqslant p < \infty$. An inspection of the proof of [8, Proposition 1.11] shows that the set $\{M_k\}_{k\in\mathbb{Z}}$ is R-bounded.

In order to present the conditions that we will need later we introduce some notation. Let $\{a_k\}_{k\in\mathbb{Z}}$ be a sequence of complex numbers. We set

$$\Delta^0 a_k = a_k, \qquad \Delta a_k = \Delta^1 a_k = a_{k+1} - a_k$$

and for $n = 2, 3, \dots$

$$\Delta^n a_k = \Delta\left(\Delta^{n-1} a_k\right).$$

The following concept of n-regularity introduced in [23] is the discrete analog for the notion of n-regularity related to Volterra integral equations (see [31, Chapter I, Section 3.2]).

Definition 2.5. A sequence $\{a_k\}_{k\in\mathbb{Z}} \subseteq \mathbb{C} \setminus \{0\}$ is called n-regular ($n \in \mathbb{N}$) if

$$\sup_{1\leqslant l\leqslant n} \sup_{k\in\mathbb{Z}} \left\| k^l \left(\Delta^l a_k\right)/a_k \right\| < \infty. \tag{2.2}$$

Remark 2.6. Note that if $\{a_k\}_{k\in\mathbb{Z}}$ is 1-regular then $\lim_{|k|\to\infty} a_{k+1}/a_k = 1$. Observe that an n-regular sequence need not be bounded. We cite here from [23] some useful properties of n-regular sequences valid for $n \leqslant 3$.

(i) If $\{a_k\}_{k\in\mathbb{Z}}$ and $\{b_k\}_{k\in\mathbb{Z}}$ are n-regular sequences such that $\sup_k |\frac{a_k}{a_k+b_k}| < \infty$, then the sequence $\{a_k + b_k\}_{k\in\mathbb{Z}}$ is n-regular.
(ii) If the sequences $\{a_k\}_{k\in\mathbb{Z}}$ and $\{b_k\}_{k\in\mathbb{Z}}$ are n-regular, then the sequence $\{a_k b_k\}_{k\in\mathbb{Z}}$ is n-regular.
(iii) The sequence $\{a_k\}_{k\in\mathbb{Z}}$ is n-regular if and only if the sequence $\{\frac{1}{a_k}\}_{k\in\mathbb{Z}}$ is n-regular.
(iv) If the sequences $\{a_k\}_{k\in\mathbb{Z}}$ and $\{b_k\}_{k\in\mathbb{Z}}$ are n-regular, then the sequence $\{a_k/b_k\}_{k\in\mathbb{Z}}$ is n-regular.

Remark 2.7. We recall that those Banach spaces X for which the Hilbert transform defined by

$$(Hf)(t) = \lim_{\substack{\epsilon\to 0\\ R\to\infty}} \frac{1}{\pi} \int\limits_{\epsilon\leqslant|s|\leqslant R} \frac{f(t-s)}{s}\,ds \tag{2.3}$$

is bounded on $L^p(\mathbb{R}, X)$ for some $p \in (1,\infty)$ are called *UMD*-spaces. The limit in the above formula is to be understood in the L^p sense. For more information and details on the Hilbert transform and the *UMD* Banach spaces we refer to [4, Sections III.4.3–III.4.5]. The definition of Banach spaces with the unconditional martingale difference property *UMD* was introduced by D.L. Burkholder. Examples of *UMD* spaces include Hilbert spaces, Sobolev spaces $W_p^s(\Omega)$, $1 < p < \infty$ (see [3]) and Lebesgue spaces $L^p(\Omega,\mu)$, $1 < p < \infty$, $L^p(\Omega,\mu; X)$, $1 < p < \infty$, when X is a *UMD* space.

The following theorem, due to Arendt and Bu [8, Theorem 1.3], is the discrete analogue of the operator-valued version of Mikhlin's theorem due to Weis [34] and play a central role in this paper.

Theorem 2.8. *Let X, Y be UMD spaces and let $\{M_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{B}(X,Y)$. If the sets $\{M_k\}_{k\in\mathbb{Z}}$ and $\{k(M_{k+1}-M_k)\}_{k\in\mathbb{Z}}$ are R-bounded, then $\{M_k\}_{k\in\mathbb{Z}}$ is an $L^p_{X,Y}$-multiplier for $1 < p < \infty$.*

3. A characterization

We first consider the relation between multipliers and R-boundedness for special cases of sequences of operators. Analogous results were established in [11,21] and [22] in case of a unique closed linear operator A. In contrast, in what follows, we assume that A and B are closed linear operators with $D(A)\cap D(B) \neq \{0\}$. We denote by $[D(A)\cap D(B)]$ the domain of $A+B$ endowed with the graph norm, that is: $\|x\|_{[D(A)\cap D(B)]} = \|x\| + \|Ax\| + \|Bx\|$, so that it becomes a Banach space. The following theorem corresponds to an extension of [21, Proposition 2.8] and is the key in the study of characterizations of maximal regularity.

Theorem 3.1. *Let $\{a_k\}_{k\in\mathbb{Z}}$ and $\{b_k\}_{k\in\mathbb{Z}}$ be a 1-regular sequences with $\{b_k\}$ bounded. Let A and B be a closed linear operators defined on a UMD space X. Assume that the operator $I - a_kA - b_kB$ with domain $D(A)\cap D(B)$ is invertible and $(I - a_kA - b_kB)^{-1} \in \mathcal{B}(X, [D(A)\cap D(B)])$, then the following assertions are equivalent*:

(i) $\{(I - a_kA - b_kB)^{-1}\}_{k\in\mathbb{Z}}$ *and* $\{b_kB(I - a_kA - b_kB)^{-1}\}_{k\in\mathbb{Z}}$ *are* $L^p_{X,X}$*-multipliers,* $1 < p < \infty$.
(ii) $\{(I - a_kA - b_kB)^{-1}\}_{k\in\mathbb{Z}}$ *and* $\{b_kB(I - a_kA - b_kB)^{-1}\}_{k\in\mathbb{Z}}$ *are R-bounded.*

Proof. (ii) $\Rightarrow$ (i). Let $M_k = (I - a_kA - b_kB)^{-1}$. In order to prove (i) is sufficient to show, by Theorem 2.8, that the sets $\{k(M_{k+1}-M_k)\}_{k\in\mathbb{Z}}$ and $\{k(b_{k+1}BM_{k+1} - b_kBM_k)\}$ are R-bounded. In fact, a computation gives the following identity

$$\begin{aligned} k[M_{k+1}-M_k] &= k\frac{a_{k+1}-a_k}{a_k}M_{k+1}M_k + k\frac{b_{k+1}-b_k}{b_k}\frac{a_{k+1}}{a_k}M_{k+1}b_kBM_k \\ &\quad - k\frac{a_{k+1}-a_k}{a_k}\frac{b_{k+1}}{b_k}M_{k+1}b_kBM_k - k\frac{a_{k+1}-a_k}{a_k}M_{k+1}. \end{aligned} \tag{3.1}$$

By 1-regularity, it follows that the sequences $\{a_{k+1}/a_k\}$ and $\{b_{k+1}/b_k\}$ are bounded. Hence from (3.1) and Remark 2.3 we obtain that $\{k(M_{k+1}-M_k)\}_{k\in\mathbb{Z}}$ is R-bounded. On the other hand, we have the identity

$$\begin{aligned} k[b_{k+1}BM_{k+1} - b_kBM_k] &= k\frac{b_{k+1}-b_k}{b_k}b_kBM_k + kb_{k+1}B(M_{k+1}-M_k) \\ &= k\frac{b_{k+1}-b_k}{b_k}b_kBM_k + k\frac{a_{k+1}-a_k}{a_k}b_{k+1}BM_{k+1}M_k \\ &\quad + k\frac{b_{k+1}-b_k}{b_k}\frac{a_{k+1}}{a_k}b_{k+1}BM_{k+1}b_kBM_k \\ &\quad - k\frac{a_{k+1}-a_k}{a_k}\frac{b_{k+1}}{b_k}b_{k+1}BM_{k+1}b_kBM_k - k\frac{a_{k+1}-a_k}{a_k}b_{k+1}BM_{k+1}, \end{aligned}$$

and hence again by Remark 2.3 we obtain the desired assertion. Finally, (i) $\Rightarrow$ (ii) follows from Remark 2.4. $\square$

In what follows, we are concerned with the following integral equation with infinite delay

$$\begin{cases} u(t) = A\displaystyle\int_{-\infty}^{t} a(t-s)u(s)\,ds + B\int_{-\infty}^{t} b(t-s)u(s)\,ds + f(t), \\ u(0) = u(2\pi) \end{cases} \tag{3.2}$$

where $a, b \in L^1(\mathbb{R}_+)$ are scalar kernels, and A, B are closed linear operators defined on a *UMD* space X, such that $D(A) \cap D(B) \neq \{0\}$. Our objective is to provide necessary and sufficient conditions for the existence and uniqueness of periodic solutions to Eq. (3.2) in vector-valued Lebesgue spaces.

Denote by $\tilde{a}(\lambda)$, $\tilde{b}(\lambda)$ the Laplace transforms of a and b respectively. In the following we will assume that $\tilde{a}(ik)$, $\tilde{b}(ik)$ exist for all $k \in \mathbb{Z}$. We suppose that $\lambda \to \tilde{a}(\lambda)$ (resp. $\tilde{b}(\lambda)$) admits an analytical extension to a sector containing the imaginary axis, and still denote this extension by $\tilde{a}$ (resp. $\tilde{b}$) and by $\tilde{a}_k$ (resp. $\tilde{b}_k$) the Laplace transform $\tilde{a}(ik)$ (resp. $\tilde{b}(ik)$).

We define

$$\rho_{a,b}(A, B) = \big\{\lambda \in \mathbb{C}\colon\ I - a(\lambda)A - b(\lambda)B \text{ is invertible and } \big(I - a(\lambda)A - b(\lambda)B\big)^{-1} \in \mathcal{B}\big(X, \big[D(A) \cap D(B)\big]\big)\big\},$$

and denote by $\sigma_{a,b}(A, B)$ the complementary set $\mathbb{C} \setminus \rho_{a,b}(A, B)$.

Definition 3.2. Let $1 < p < \infty$. A function u is called a mild L^p-solution of (3.2) if $a * u \in L^p_{2\pi}(\mathbb{R}; [D(A)])$, $b * u \in L^p_{2\pi}(\mathbb{R}; [D(B)])$ and Eq. (3.2) holds for almost all $t \in [0, 2\pi]$.

Remark 3.3. We note that the above definition differs of the notion of strong L^p-solution, which considers instead of (3.2) the equation $u(t) = (a * Au)(t) + (b * Bu)(t) + f(t)$. In general not every mild solution is a strong solution. Mild solutions for integral equations in case $B \equiv 0$ has been studied previously in the literature (see e.g. [31]).

Definition 3.4. Let $1 \leqslant p < \infty$. We say that the problem (3.2) has L^p-maximal regularity (or is well posed) if for every $f \in L^p_{2\pi}(\mathbb{R}; X)$ there exists a unique mild L^p-solution of (3.2).

We recall that a pair (A, B) is called coercive if $A + tB$ with domain $D(A) \cap D(B)$ is closed for all $t > 0$ and there is a constant $M > 0$ such that

$$\|Ax\| + t\|Bx\| \leqslant M\|Ax + tBx\|$$

for all $x \in D(A) \cap D(B)$, $t > 0$. For further information we refer [31] and [33]. The following is the main result of this section.

Theorem 3.5. *Let $a, b \in L^1(\mathbb{R}_+)$ be functions such that $\{\tilde{a}_k\}$ and $\{\tilde{b}_k\}$ are 1-regular sequences. Let A and B be closed linear operators defined on a UMD space X such that $(\tilde{a}_k A, \tilde{b}_k B)$ is a coercive pair. Then the following assertions are equivalent for $1 < p < \infty$.*

(i) *Problem (3.2) has L^p-maximal regularity.*
(ii) *$\{ik\}_{k\in\mathbb{Z}} \in \rho_{\tilde{a},\tilde{b}}(A, B)$, $\{(I - \tilde{a}_k A - \tilde{b}_k B)^{-1}\}_{k\in\mathbb{Z}}$, and $\{\tilde{b}_k B(I - \tilde{a}_k A - \tilde{b}_k B)^{-1}\}_{k\in\mathbb{Z}}$ are R-bounded.*

Proof. (ii) $\Rightarrow$ (i). Denote by $M_k = (I - \tilde{a}_k A - \tilde{b}_k B)^{-1}$ and let $f \in L^p_{2\pi}(\mathbb{R}, X)$. By Proposition 3.1, there is $u \in L^p_{2\pi}(\mathbb{R}; X)$ such that

$$\hat{u}(k) = M_k \hat{f}(k), \quad \text{for all } k \in \mathbb{Z}. \tag{3.3}$$

We conclude that $\hat{u}(k) \in D(A) \cap D(B)$.

On the other hand, since $\{\tilde{b}_k I\}$ and $\{M_k\}$ are R-bounded sets, an easy computation proves that $\{\tilde{b}_k M_k\}$ is an $L^p_{X,X}$-multiplier. Hence, for $f \in L^p_{2\pi}(\mathbb{R}, X)$ there is $h \in L^p_{2\pi}(\mathbb{R}; X)$ such that

$$\hat{h}(k) = \tilde{b}_k M_k \hat{f}(k) = \tilde{b}_k \hat{u}(k), \quad \text{for all } k \in \mathbb{Z}.$$

We conclude that $h(t) = (b * u)(t)$ and hence $(b * u) \in L^p_{2\pi}(\mathbb{R}; X)$.

By Theorem 3.1 we have that $\{\tilde{b}_k B M_k\}$ is an $L^p_{X,X}$-multiplier. Then, for $f \in L^p_{2\pi}(\mathbb{R}, X)$ there is $v \in L^p_{2\pi}(\mathbb{R}; X)$ such that

$$\hat{v}(k) = \tilde{b}_k B M_k \hat{f}(k), \quad \text{for all } k \in \mathbb{Z}.$$

By (3.3) we obtain $\hat{v}(k) = B\tilde{b}_k \hat{u}(k)$, for all $k \in \mathbb{Z}$. By Lemma 3.1 in [8] we obtain that $(b * u)(t) \in D(B)$ and $v(t) = B(b * u)(t)$. Since $b * u \in L^p_{2\pi}(\mathbb{R}; X)$ and $B(b * u) \in L^p_{2\pi}(\mathbb{R}; X)$ using Hölder's inequality, we obtain that $b * u \in L^p_{2\pi}(\mathbb{R}; [D(B)])$.

Now, from (3.3) we have that $\tilde{a}_k A\hat{u}(k) = \hat{u}(k) - \tilde{b}_k B\hat{u}(k) - \hat{f}(k)$ or $A\tilde{a}_k \hat{u}(k) = \hat{u}(k) - \hat{h}(k) - \hat{f}(k)$. By Lemma 3.1 in [8] it follows that $(a * u)(t) \in D(A)$ and

$$A(a * u)(t) = u(t) - h(t) - f(t) = u(t) - B(b * u)(t) - f(t) \tag{3.4}$$

(cf. [21, Eq. (2.1)]). Note that $A(a * u) \in L^p_{2\pi}(\mathbb{R}; X)$, hence $a * u \in L^p_{2\pi}(\mathbb{R}; [D(A)])$.

From (3.4) and the uniqueness theorem of Fourier coefficients that (3.2) holds for almost all $t \in [0, 2\pi]$. We have proved that u is a mild L^p-solution of (3.2). It remains to show uniqueness.

Let $a * u \in L^p_{2\pi}(\mathbb{R}; [D(A)])$ and $b * u \in L^p_{2\pi}(\mathbb{R}; [D(B)])$ be such that

$$u(t) - A\int_{-\infty}^{t} a(t-s)u(s) - B\int_{-\infty}^{t} b(t-s)u(s)\,ds = 0.$$

Taking Fourier transforms on both sides we obtain that $\hat{u}(k) \in D(A) \cap D(B)$ and $(I - (\tilde{a}_k A + \tilde{b}_k B))\hat{u}(k) = 0$. Since $\{ik\}_{k\in\mathbb{Z}} \in \rho_{\tilde{a},\tilde{b}}(A, B)$ this implies that $\hat{u}(k) = 0$ for all $k \in \mathbb{Z}$ and thus $u = 0$.

(i) $\Rightarrow$ (ii). Let $k \in \mathbb{Z}$ and $y \in X$. Define $f = e_k \otimes y$. There exists u such that $a * u \in L^p_{2\pi}(\mathbb{R}; [D(A)])$, $b * u \in L^p_{2\pi}(\mathbb{R}; [D(B)])$ and

$$u(t) = A\int_{-\infty}^{t} a(t-s)u(s) + B\int_{-\infty}^{t} b(t-s)u(s)\,ds + f(t).$$

Taking Fourier transforms on both sides we obtain that $\hat{u}(k) \in D(A) \cap D(B)$ and

$$\hat{u}(k) - \tilde{a}_k A\hat{u}(k) - \tilde{b}_k B\hat{u}(k) = \hat{f}(k) = y.$$

Thus, $(I - \tilde{a}_k A - \tilde{b}_k B)$ is surjective. Let $x \in D(A) \cap D(B)$. If $(I - \tilde{a}_k A - \tilde{b}_k B)x = 0$, that is, $Ax = \frac{I - \tilde{b}_k B}{\tilde{a}_k}x$, then $u(t) = e^{ikt}x$ defines a periodic solution of $u(t) - A\int_{-\infty}^{t} a(t-s)u(s) - B\int_{-\infty}^{t} b(t-s)u(s)\,ds = 0$. Hence $u = 0$ by the assumption of uniqueness and thus $x = 0$.

Finally, by hypothesis $(\tilde{a}_k A, \tilde{b}_k B)$ is coercive pair and then we have that $\tilde{a}_k A + \tilde{b}_k B$ is closed. Since $(I - \tilde{a}_k A - \tilde{b}_k B)$ is bijective we conclude that $\{ik\}_{k\in\mathbb{Z}} \in \rho_{\tilde{a},\tilde{b}}(A, B)$.

Next we show that $\{(I - \tilde{a}_k A - \tilde{b}_k B)^{-1}\}_{k\in\mathbb{Z}}$ is an $L^p_{X,X}$-multiplier. Let $f \in L^p_{2\pi}(\mathbb{R}, X)$. By hypothesis, there exist a unique u such that $a * u \in L^p_{2\pi}(\mathbb{R}, [D(A)])$, $b * u \in L^p_{2\pi}(\mathbb{R}, [D(B)])$ and

$$u(t) = A\int_{-\infty}^{t} a(t-s)u(s) + B\int_{-\infty}^{t} b(t-s)u(s)\,ds + f(t).$$

Taking Fourier transforms on both sides of the equality, we obtain that $\hat{u}(k) \in D(A) \cap D(B)$ and

$$\hat{u}(k) = (I - \tilde{a}_k A - \tilde{b}_k B)^{-1}\hat{f}(k).$$

Since $\hat{u}(k) \in D(B)$ we have $B\tilde{b}_k\hat{u}(k) = B\tilde{b}_k(I - \tilde{a}_k A - \tilde{b}_k B)^{-1}\hat{f}(k)$. Note that if $b * u \in L^p_{2\pi}(\mathbb{R}, [D(B)])$ then $B(b * u) \in L^p_{2\pi}(\mathbb{R}, X)$. Let $v = B(b * u)$. By Lemma 3.1 in [8] we obtain $B\tilde{b}_k\hat{u}(k) = \hat{v}(k)$ hence $\hat{v}(k) = B\tilde{b}_k(I - \tilde{a}_k A - \tilde{b}_k B)^{-1}\hat{f}(k)$. Then, by definition, $\{B\tilde{b}_k(I - \tilde{a}_k A - \tilde{b}_k B)^{-1}\}_{k\in\mathbb{Z}}$ is an $L^p_{X,X}$-multiplier. Analogously we show that $\{A\tilde{a}_k(I - \tilde{a}_k A - \tilde{b}_k B)^{-1}\}_{k\in\mathbb{Z}}$ is an $L^p_{X,X}$-multiplier. Finally, from the identity

$$(I - \tilde{a}_k A - \tilde{b}_k B)^{-1} = I + B\tilde{b}_k(I - \tilde{a}_k A - \tilde{b}_k B)^{-1} + A\tilde{a}_k(I - \tilde{a}_k A - \tilde{b}_k B)^{-1},$$

we obtain that $(I - \tilde{a}_k A - \tilde{b}_k B)^{-1}$ is an $L^p_{X,X}$-multiplier. The assertion (i) then follows from Theorem 3.1. $\square$

Remark 3.6.

(i) Fejer's theorem (see [6, Theorem 4.2.19]) allows us to construct the solution $u(\cdot)$ given by the above theorem. In fact, we have

$$u(t) = \lim_{n\to\infty}\sum_{k=-n}^{n}\left(1 - \frac{|k|}{n+1}\right)e^{ikt}(I - \tilde{a}_k A - \tilde{b}_k B)^{-1}\hat{f}(k)$$

with convergence in $L^p_{2\pi}(\mathbb{R}; X)$.

(ii) In the context of Theorem 3.5 we have $A(a * u), B(b * u) \in L^p_{2\pi}(\mathbb{R}, X)$. Moreover, by the Closed Graph Theorem there exists a constant $C > 0$, independent of $f \in L^p_{2\pi}(\mathbb{R}, X)$ such that

$$\|u\|_{L^p_{2\pi}(\mathbb{R},X)} + \|Aa * u\|_{L^p_{2\pi}(\mathbb{R},X)} + \|Bb * u\|_{L^p_{2\pi}(\mathbb{R},X)} \leqslant C\|f\|_{L^p_{2\pi}(\mathbb{R},X)}.$$

(iii) The R-boundedness condition in Theorem 3.5(i) can be satisfied in special cases of operators A appearing in applications. For example, if A is sectorial and B is such that $D(A) \subset D(B)$ and $\|Bx\| \leqslant \alpha\|Ax\|$, $x \in D(A)$, for some $\alpha > 0$ then we can make use of [17, Proposition 4.2]. In the case of relatively bounded perturbations i.e. $\|Bx\| \leqslant \alpha\|Ax\| + \beta\|x\|$, $x \in D(A)$ with small relative bound α, we can modify the arguments of [17, Proposition 4.2] to satisfy the hypothesis of Theorem 3.5.

(iv) We note that the coercivity condition on the pair $(\tilde{a}_k A, \tilde{b}_k B)$ is used only in the implication (ii) $\to$ (i).

Example 3.7. Given $\gamma > 0$, we consider the following particular case of (3.2):

$$\begin{cases} u(t) = -A \int_{-\infty}^{t} e^{-\gamma(t-s)}(t-s)u(s)\,ds - A^{\varepsilon} \int_{-\infty}^{t} e^{-\gamma(t-s)}u(s)\,ds + f(t), \\ u(0) = u(2\pi), \end{cases} \tag{3.5}$$

where A is closed linear operator defined on *UMD* space X. We note that Eq. (3.5) can be considered as an integral version of the second order Cauchy problem

$$u''(t) + A^{\varepsilon}u'(t) + Au(t) = f(t), \tag{3.6}$$

since, formally, it corresponds to the second derivative of (3.5) in the limit case $\gamma = 0$. L^p-maximal regularity for Eq. (3.6) with initial conditions $u(0) = u'(0) = 0$ was recently proved in [13] in case A is sectorial and admits a bounded RH^{∞}-calculus of angle less than $\pi/2\varepsilon$. There was also proved that for $\varepsilon \in (0, 1/2)$ one cannot expect L^p-maximal regularity in general, even if A is a selfadjoint operator on a Hilbert space. The proof of the main result in [13] relies on the problem of estimating some concrete scalar holomorphic functions. We use these estimates to prove our main result concerning now Eq. (3.5).

For the kernels $a(t) = -te^{-\gamma t}$ and $b(t) = -e^{-\gamma t}$ we easily check the 1-regularity of $\tilde{a}_k = \frac{-1}{(ik+\gamma)^2}$ and $\tilde{b}_k = \frac{-1}{(ik+\gamma)}$. Now, we assume that A is densely defined and sectorial of angle $\beta \in (0, \pi)$, that is, $\sigma(A) \subset \overline{\sum_{\beta}}$, and for every $\beta' \in (\beta, \pi)$

$$\sup_{z \in \mathbb{C} \setminus \sum_{\beta'}} \left\| z(z-A)^{-1} \right\| < \infty,$$

with $\sum_{\beta} = \{z \in \mathbb{C}\colon\ |\arg z| < \beta\}$. In this condition, for A, is possible to define a functional calculus from $H_0^{\infty}(\sum_{\beta'})$ into $\mathcal{B}(X)$. This H^{∞} functional calculus may be extended in a natural way in order to define the fractional powers A^{ε} for every $\varepsilon > 0$ (see [19,27]).

In order to obtain L^p-maximal regularity of (3.5) we need to establish conditions such that $\{(I - \tilde{b}_k A^{\varepsilon} - \tilde{a}_k A)^{-1}\}_{k \in \mathbb{Z}}$ and $\{B\tilde{b}_k(I - \tilde{b}_k A^{\varepsilon} - \tilde{a}_k A)^{-1}\}_{k \in \mathbb{Z}}$ are R-bounded sequences. Note that in this case we have

$$\left(I - \tilde{b}_k A^{\varepsilon} - \tilde{a}_k A\right)^{-1} = \left(I + \frac{1}{(ik+\gamma)}A^{\varepsilon} + \frac{1}{(ik+\gamma)^2}A\right)^{-1} = (ik+\gamma)^2\left((ik+\gamma)^2 I + (ik+\gamma)A^{\varepsilon} + A\right)^{-1},$$

and

$$A^{\varepsilon}b_k\left(I - \tilde{b}_k A^{\varepsilon} - \tilde{a}_k A\right)^{-1} = -(ik+\gamma)A^{\varepsilon}\left((ik+\gamma)^2 I + (ik+\gamma)A^{\varepsilon} + A\right)^{-1}.$$

We denote by $M(\lambda) = (\lambda^2 I + \lambda A^{\varepsilon} + A)^{-1}$ whenever exists. For the notion of RH^{∞} functional calculus and R-boundedness in the following result, we refer to [17,20,34].

Proposition 3.8. *Let A be a sectorial operator which admits a bounded RH^{∞} functional calculus of angle $\beta \in (0, \frac{\pi}{3})$. Then* (3.5) *with $\varepsilon = 1/2$ has L^p-maximal regularity.*

Proof. For all $\lambda \in \mathbb{C}$ and $z \in \mathbb{C} \setminus (-\infty, 0]$, we define $M_1(\lambda, z) := \lambda^2(\lambda^2 I + \lambda z^{\varepsilon} + z)^{-1} \in \mathbb{C} \cup \{\infty\}$ and $M_2(\lambda, z) := \lambda z^{\varepsilon}(\lambda^2 I + \lambda z^{\varepsilon} + z)^{-1} \in \mathbb{C} \cup \{\infty\}$. Choose $\beta' > \beta$ and $\delta > 0$ such that $\beta'/2 < \pi/6 - \delta$. By the proof of [13, Lemma 4.1(b)], there exist a constant $M \geqslant 0$ independent of $\lambda \in \sum_{\delta+\pi/2}$ and $z \in \sum_{\beta'}$ such that $\{M_j(\lambda, z)\} \subset H^{\infty}(\sum_{\beta'})$ is uniformly bounded for $j = 1, 2$. Since A admits a bounded RH^{∞}-calculus of angle β we conclude from [17, Proposition 4.10] that the set $\mathcal{M} := \{M_j(\lambda, A)\colon\ \lambda \in \sum_{\delta+\pi/2}\}$ is R-bounded in $\mathcal{B}(X)$ for $j = 1, 2$. In particular, the sets $(ik+\gamma)^2 M(ik+\gamma)$ and $(ik+\gamma)A^{1/2}M(ik+\gamma)$ are R-bounded. The conclusion follows from Theorem 3.5 and Remark 3.6(iv). □

Examples of concrete operators satisfying the condition of the above proposition can be obtained from [17, Theorem 4.11]. Now, suppose $\varepsilon \in (\frac{1}{2}, 1]$ in (3.5). Then again from the proof of [13, Lemma 4.1(c)] we get that there exist constants $\omega_1 \geqslant 0$ and $M_1 \geqslant 0$ such that for every $\lambda \in \omega_1 + \sum_{\delta+\pi/2}$ and every $z \in \sum_{\beta'}$ the sets $\{M_j(\lambda, z)\} \subset H^{\infty}(\sum_{\beta'})$ are uniformly bounded for $j = 1, 2$. We have to distinguish two cases:

Case 1. $\omega_1 > \gamma$. Then the same conclusion of Proposition 3.8 holds.

Case 2. $\omega_1 \leqslant \gamma$. Then possibly $ik \in \sigma_{\tilde{a},\tilde{b}}(A, B)$ for at most a finite number of $k \in \mathbb{Z}$. We are in this case in the presence of *resonances* which will be the subject of the next section.

Example 3.9. Let $X = l^2(\mathbb{Z})$ and consider the system

$$u_n = (n - i\beta)a * u_n + f_n, \quad n \in \mathbb{Z},\ 0 < \beta < 1. \tag{3.7}$$

This problem is of the form (3.2) with

$$(Au)_n = (n - i\beta)u_n, \qquad D(A) = \{(u_n) \in l^2(\mathbb{Z}):\ (n \cdot u_n) \in l^2(\mathbb{Z})\},$$

and $b(t) = 0$ for all $t \in \mathbb{R}_+$. Note that A does not generate a C_0-semigroup since $\sigma(A) = \{n - i\beta:\ n \in \mathbb{Z}\}$ is not contained in any left halfplane. Define $a(t) = e^{-\alpha t}$, $\alpha > 0$. Clearly the sequence $\tilde{a}(ik) = \frac{1}{ik+\alpha}$ is 1-regular and $\{ik + \alpha\}_{k\in\mathbb{Z}} \subset \rho(A)$. Moreover, for each $x = (x_n) \in l^2(\mathbb{Z})$ we have

$$\begin{aligned}
\left\|\left(I - \tilde{a}(ik)A\right)^{-1}x\right\| &= \left\|(ik+\alpha)(ik+\alpha-A)^{-1}x\right\| \\
&= \sum_{n\in\mathbb{Z}} \left|\frac{ik+\alpha}{ik+\alpha-n+i\beta}x_n\right|^2 \\
&\leqslant \sum_{n\in\mathbb{Z}} \frac{k^2+\alpha^2}{(\alpha-n)^2+(\beta+k)^2}|x_n|^2 \\
&\leqslant \sum_{n\in\mathbb{Z}} \frac{k^2+\alpha^2}{(k+\beta)^2}|x_n|^2.
\end{aligned}$$

Since $0 < \beta < 1$, we obtain for all $k \in \mathbb{Z}$

$$\left\|\left(I - \tilde{a}(ik)A\right)^{-1}x\right\| \leqslant \max\left\{\frac{\alpha^2}{\beta^2}, \frac{\alpha^2+1}{(\beta-1)^2}\right\} \sum_{n\in\mathbb{Z}} |x_n|^2 =: M\|x\|,$$

where, as indicated, the constant M depends only on α and β. Then, the hypothesis of Theorem 3.5 is satisfied (cf. Remark 2.3(d)) and we conclude that for every $f \in L^p_{2\pi}(\mathbb{R}, l^2(\mathbb{Z}))$ there exists a unique mild L^p-solution of (3.7).

4. A resonance case

In the previous section we have assumed that no element of $\sigma_{\tilde{a},\tilde{b}}(A, B)$ lies in the set $i\mathbb{Z}$ and we characterize the property that, for every $f \in L^p_{2\pi}(\mathbb{R}, X)$ there exists a unique mild L^p-solution of (3.2). Now, following [16] we consider a resonance case. In what follows, we assume that there are $k_1, \ldots, k_N \in \mathbb{Z}$ such that

$$\begin{cases} \text{(i) } ik_j \in \sigma_{\tilde{a},\tilde{b}}(A, B) & \text{for } j = 1, \ldots, N, \\ \text{(ii) } ik \notin \sigma_{\tilde{a},\tilde{b}}(A, B) & \text{for } k \in \mathbb{Z},\ k \neq k_1, \ldots, k_N, \\ \text{(iii) } ik_j \text{ is a simple pole of } F(\cdot) & \text{for } j = 1, \ldots, N \end{cases} \tag{4.1}$$

where $F : \rho_{\tilde{a},\tilde{b}}(A, B) \to \mathcal{B}(X, [D(A) \cap D(B)])$ is defined by $F(\lambda) = (I - \tilde{a}(\lambda)A - \tilde{b}(\lambda)B)^{-1}$. We begin with some preliminary results about the solvability of the equation

$$\left(I - \tilde{a}(\lambda_0)A - \tilde{b}(\lambda_0)B\right)x = y, \tag{4.2}$$

where λ_0 is a simple pole of $F(\cdot)$ and $x \in D(A) \cap D(B)$. Denote by Q the residue of $F(\cdot)$ at λ_0, that is,

$$Q = \lim_{\lambda\to\lambda_0} (\lambda - \lambda_0)F(\lambda) = \frac{1}{2\pi i} \int_{B(\lambda_0,\varepsilon)} F(\lambda)\, d\lambda, \tag{4.3}$$

where $\varepsilon > 0$ and $B(\lambda_0, \varepsilon) := \{\lambda \in \mathbb{C}:\ |\lambda - \lambda_0| < \varepsilon\}$. We define

$$G(\lambda) = \begin{cases} (\lambda - \lambda_0)F(\lambda), & 0 < |\lambda - \lambda_0| < \varepsilon, \\ Q, & \lambda = \lambda_0. \end{cases} \tag{4.4}$$

Note that $Q \in \mathcal{B}(X, [D(A) \cap D(B)])$ is a non-zero operator which verifies the properties established in the following lemma and proposition. The proofs follow essentially the same steps contained in [16, Lemma 1.5 and Proposition 1.6] and therefore are omitted.

Lemma 4.1. *With the notations as above, we have*

$$Q = Q\left[-\tilde{a}'(\lambda_0)A - \tilde{b}'(\lambda_0)B\right]Q.$$

The following result is the key for results on existence of solutions in the resonance case.

Proposition 4.2. *Let λ_0 be a simple pole of $F(\cdot)$ and let Q be defined by* (4.3)*. Then*

$$\operatorname{Ker}\big(I-\tilde{a}(\lambda_0)A-\tilde{b}(\lambda_0)B\big)=Q(X). \tag{4.5}$$

Moreover, for any $y\in X$ such that $Qy=0$, all solutions of (4.2) *are given by*

$$x=G'(\lambda_0)y-QA(\tilde{a}'G)'(\lambda_0)y-QB(\tilde{b}'G)'(\lambda_0)y. \tag{4.6}$$

Remark 4.3. The first lemma follows from the identity

$$G(\lambda)\frac{\mu-\lambda_0}{\lambda-\mu}-G(\mu)\frac{\lambda-\lambda_0}{\lambda-\mu}=G(\lambda)\left[\frac{\tilde{a}(\lambda)-\tilde{a}(\mu)}{\lambda-\mu}A+\frac{\tilde{b}(\lambda)-\tilde{b}(\mu)}{\lambda-\mu}B\right]G(\mu),$$

whereas the proposition is proved examining the identity

$$\begin{aligned}G'(\lambda)&=F(\lambda)\big[1+\big(\tilde{a}'(\lambda_0)A+\tilde{b}'(\lambda_0)B\big)Q\big]+(\lambda-\lambda_0)F(\lambda)A\frac{\tilde{a}'(\lambda)G(\lambda)-\tilde{a}'(\lambda_0)G(\lambda_0)}{\lambda-\lambda_0}\\&\quad+(\lambda-\lambda_0)F(\lambda)B\frac{\tilde{b}'(\lambda)G(\lambda)-\tilde{b}'(\lambda_0)G(\lambda_0)}{\lambda-\lambda_0}.\end{aligned}$$

Now, arguing as in the proof of Theorem 3.5, we find that, if $f\in L^p_{2\pi}(\mathbb{R},X)$ and u is a mild L^p-solution of (3.2) then $a*u\in L^p_{2\pi}(\mathbb{R},[D(A)])$, $b*u\in L^p_{2\pi}(\mathbb{R},[D(B)])$, Eq. (3.2) holds and we have

$$(I-\tilde{a}_kA-\tilde{b}_kB)\hat{u}(k)=\hat{f}(k),\quad k\in\mathbb{Z}. \tag{4.7}$$

For each $k\neq k_n$, $n=1,\dots,N$, Eq. (4.7) can be uniquely solved, with

$$\hat{u}(k)=(I-\tilde{a}_kA-\tilde{b}_kB)^{-1}\hat{f}(k).$$

For k_n, $n=1,\dots,N$, by Proposition 4.2 Eq. (4.7) is solvable if and only if

$$Q_n\hat{f}(k_n)=0, \tag{4.8}$$

where Q_n is the residue of $F(\cdot)$ at $\lambda=ik_n$. If (4.8) holds, then by (4.6), the Fourier coefficients of the solution to (4.7) in k_n, $n=1,\dots,N$ are given by

$$\hat{u}(k_n)=\big[G_n'(ik_n)-Q_nA(\tilde{a}'G_n)'(ik_n)-Q_nB(\tilde{b}'G_n)'(ik_n)\big]\hat{f}(k_n), \tag{4.9}$$

where G_n is an analytic function defined by

$$G_n(\lambda)=\begin{cases}(\lambda-ik_n)F(\lambda), & 0<|\lambda-ik_n|<\varepsilon,\\ Q_n, & \lambda=ik_n\end{cases} \tag{4.10}$$

for any $\varepsilon>0$ sufficiently small.

Now, define the following sequence of operators

$$M_k=\begin{cases}(I-\tilde{a}_kA-\tilde{b}_kB)^{-1}, & k\in\mathbb{Z}\setminus\{k_1,\dots,k_N\},\\ G_j'(ik_j)-Q_jA(\tilde{a}'G_j)'(ik_j)-Q_jB(\tilde{b}'G_j)'(ik_j), & k\in\{k_1,\dots,k_N\}.\end{cases} \tag{4.11}$$

Since $ik\in\rho_{\tilde{a},\tilde{b}}(A,B)$ for all $k\in\mathbb{Z}\setminus\{k_j\}_{j=1,\dots,N}$ we have $\{M_k\}_{k\in\mathbb{Z}}\subset\mathcal{B}(X,[D(A)\cap D(B)])$.

With the above preliminaries, we are in position to prove the main theorem of this section, which gives compatibility conditions on f which are necessary and sufficient for the existence of a mild L^p-solution to Eq. (3.2). We note that compared with [16], where a similar integral equation with $B\equiv 0$ was studied, our hypothesis on the operators A and B is weaker.

Theorem 4.4. *Let $a,b\in L^1(\mathbb{R}_+)$ be functions such that $\tilde{a}_k$ and $\tilde{b}_k$ are 1-regular sequences and suppose that* (4.1) *holds. Let A and B be closed linear operators defined on a UMD space X. If $\{M_k\}_{k\in\mathbb{Z}\setminus\{k_j\}_{j=1,\dots,N}}$ and $\{\tilde{b}_kBM_k\}_{k\in\mathbb{Z}\setminus\{k_j\}_{j=1,\dots,N}}$ are R-bounded sequences, then for every $f\in L^p_{2\pi}(\mathbb{R},X)$ Eq.* (3.2) *has a mild L^p-solution if and only if $Q_j\hat{f}(k_j)=0$, for every $j=1,\dots,N$.*

In addition, in this case, all mild solutions of (3.2) *are given by*

$$\begin{aligned}u(t)&=\lim_{n\to\infty}\sum_{\substack{k=-n\\k\neq k_1,\dots,k_N}}^{n}\left(1-\frac{|k|}{n+1}\right)e^{ikt}(I-\tilde{a}_kA-\tilde{b}_kB)^{-1}\hat{f}(k)\\&\quad+\sum_{j=1}^{N}e^{ik_jt}\big[G_j'(ik_j)-Q_jA(\tilde{a}'G_j)'(ik_j)-Q_jB(\tilde{b}'G_j)'(ik_j)\big]\hat{f}(k_j).\end{aligned} \tag{4.12}$$

Proof. We assume that for every $f \in L^p_{2\pi}(\mathbb{R}, X)$ there exist v such that $a * v \in L^p_{2\pi}(\mathbb{R}, [D(A)])$, $b * v \in L^p_{2\pi}(\mathbb{R}, [D(B)])$ and (3.2) is satisfied. Taking Fourier transforms on both sides in (3.2) we obtain that $\hat{v}(k) \in D(A) \cap D(B)$ and

$$(I - \tilde{a}_k A - \tilde{b}_k B)\hat{v}(k) = \hat{f}(k), \quad \text{for all } k \in \mathbb{Z}.$$

For $\lambda \in \rho_{\tilde{a},\tilde{b}}(A, B)$, and $k_1, k_2, \ldots, k_N$ from the identity

$$(\lambda - ik_n)F(\lambda)\big(I - \tilde{a}(\lambda)A - \tilde{b}(\lambda)B\big)\hat{v}(k_n) = (\lambda - ik_n)\hat{v}(k_n),$$

it follows that

$$\lim_{\lambda \to ik_n} (\lambda - ik_n)F(\lambda)\big(I - \tilde{a}(\lambda)A - \tilde{b}(\lambda)B\big)\hat{v}(k_n) = 0.$$

Since the limits $\lim_{\lambda \to ik_n}(\lambda - ik_n)F(\lambda)$ and $\lim_{\lambda \to ik_n}(I - \tilde{a}(\lambda)A - \tilde{b}(\lambda)B)\hat{v}(k_n)$ both exist, we obtain that

$$Q_n\big(I - \tilde{a}(ik_n)A - \tilde{b}(ik_n)B\big)\hat{v}(k_n) = 0,$$

or, equivalently, $Q_j\hat{f}(k_j) = 0$, for all k_j, $j = 1, \ldots, N$. Hence by Proposition 4.2 Eq. (4.7) is solvable and

$$\hat{v}(k) = \begin{cases} (I - \tilde{a}_k A - \tilde{b}_k B)^{-1}\hat{f}(k), & k \in \mathbb{Z} \setminus \{k_1, \ldots, k_N\}, \\ [G'_j(ik_j) - Q_j A(\tilde{a}' G_j)'(ik_j) - Q_j B(\tilde{b}' G_j)'(ik_j)]\hat{f}(k_j), & j = 1, \ldots, N, \end{cases}$$

from which (4.12) follows.

Conversely, assume that $f \in L^p_{2\pi}(\mathbb{R}, X)$ and $Q_j\hat{f}(k_j) = 0$. We define $u(t)$ by (4.12). Then

$$\hat{u}(k) = M_k\hat{f}(k), \tag{4.13}$$

for all $k \in \mathbb{Z}$, where M_k is defined by (4.11). Note that $\hat{u}(k) \in D(A) \cap D(B)$ for all $k \in \mathbb{Z}$.

In order to simplify the notation we write $F_k := (I - \tilde{a}_k A - \tilde{b}_k B)^{-1} = F(ik)$, for all $k \in \mathbb{Z}$, $k \neq k_1, \ldots, k_N$ and $H_j := [G'_j(ik_j) - Q_j A(\tilde{a}' G_j)'(ik_j) - Q_j B(\tilde{b}' G_j)'(ik_j)]$ for k_j, $j = 1, \ldots, N$.

Since $\{M_k\}_{k\in\mathbb{Z}}$ is R-bounded, we claim that $\{k(M_{k+1} - M_k)\}_{k\in\mathbb{Z}}$ is also R-bounded. In fact, note that any finite family of operators is R-bounded, and for all $k \neq k_1, \ldots, k_N$ we have

$$k(M_{k+1} - M_k) = k\frac{\tilde{b}_{k+1} - \tilde{b}_k}{\tilde{b}_k}\tilde{b}_k F_{k+1} B F_k + k\frac{\tilde{a}_{k+1} - \tilde{a}_k}{\tilde{a}_k} F_{k+1}\big[(I - \tilde{b}_k B)F_k - I\big].$$

Since $\{F_k\}$ and $\{\tilde{b}_k B F_k\}$ are R-bounded sets and $\tilde{a}_k$ and $\tilde{b}_k$ are 1-regular sequences the claim follows by Remark 2.3. From Theorem 2.8 we conclude that $\{M_k\}_{k\in\mathbb{Z}}$ is an $L^p_{X,X}$-multiplier. So we obtain that there exists $v \in L^p_{2\pi}(\mathbb{R}; X)$ such that $\hat{v}(k) = M_k\hat{f}(k)$ for all $k \in \mathbb{Z}$. Note that a similar argument as in the proof of Theorem 3.5 shows that $a * v \in L^p_{2\pi}(\mathbb{R}; [D(A)])$ and $b * v \in L^p_{2\pi}(\mathbb{R}; [D(B)])$. Hence the uniqueness theorem shows that $u = v$ for t a.e. It remains to show that u satisfies Eq. (3.2).

Using the identity $F_k = I + \tilde{a}_k A F_k + \tilde{b}_k B F_k$, valid for all $k \in \mathbb{Z} \setminus \{k_j\}$, $j = 1, \ldots, N$, we obtain

$$\begin{aligned}
u(t) &= \sum_{\substack{k=-n \\ k\neq k_i}}^{n}\left(1 - \frac{|k|}{n+1}\right)e^{ikt}F_k\hat{f}(k) + \sum_{j=1}^{N} e^{ik_jt}H_j\hat{f}(k_j) \\
&= \lim_{n\to\infty}\sum_{\substack{k=-n \\ k\neq k_i}}^{n}\left(1 - \frac{|k|}{n+1}\right)e^{ikt}\hat{f}(k) + \lim_{n\to\infty}\sum_{\substack{k=-n \\ k\neq k_i}}^{n}\left(1 - \frac{|k|}{n+1}\right)e^{ikt}\tilde{a}_k A F_k\hat{f}(k) \\
&\quad + \lim_{n\to\infty}\sum_{\substack{k=-n \\ k\neq k_i}}^{n}\left(1 - \frac{|k|}{n+1}\right)e^{ikt}\tilde{b}_k B F_k\hat{f}(k) + \sum_{j=1}^{N} e^{ik_jt}H_j\hat{f}(k_j) \\
&= \lim_{n\to\infty}\left\{\sum_{\substack{k=-n \\ k\neq k_i}}^{n}\left(1 - \frac{|k|}{n+1}\right)e^{ikt}\hat{f}(k) + \sum_{j=1}^{N}\left(1 - \frac{|k_j|}{n+1}\right)e^{ik_jt}\hat{f}(k_j)\right\} - \lim_{n\to\infty}\sum_{j=1}^{N}\left(1 - \frac{|k_j|}{n+1}\right)e^{ik_jt}\hat{f}(k_j) \\
&\quad + \lim_{n\to\infty}\left\{\sum_{\substack{k=-n \\ k\neq k_i}}^{n}\left(1 - \frac{|k|}{n+1}\right)e^{ikt}\tilde{a}_k A\hat{u}(k) + \sum_{j=1}^{N}\left(1 - \frac{|k_j|}{n+1}\right)e^{ik_jt}\tilde{a}(ik_j)A\hat{u}(k_j)\right\} \\
&\quad - \lim_{n\to\infty}\sum_{j=1}^{N}\left(1 - \frac{|k_j|}{n+1}\right)e^{ik_jt}\tilde{a}(ik_j)A\hat{u}(k_j)
\end{aligned}$$

$$
\begin{aligned}
&+\lim_{n\to\infty}\left\{\sum_{\substack{k=-n\\k\neq k_i}}^{n}\left(1-\frac{|k|}{n+1}\right)e^{ikt}\tilde{b}_kB\hat{u}(k)+\sum_{j=1}^{N}\left(1-\frac{|k_j|}{n+1}\right)e^{ik_jt}\tilde{b}(ik_j)B\hat{u}(k_j)\right\}\\
&-\lim_{n\to\infty}\sum_{j=1}^{N}\left(1-\frac{|k_j|}{n+1}\right)e^{ik_jt}\tilde{b}(ik_j)B\hat{u}(k_j)+\sum_{j=1}^{N}e^{ik_jt}H_j\hat{f}(k_j)\\
=&\lim_{n\to\infty}\sum_{k=-n}^{n}\left(1-\frac{|k|}{n+1}\right)e^{ikt}\hat{f}(k)+\lim_{n\to\infty}\sum_{k=-n}^{n}\left(1-\frac{|k|}{n+1}\right)e^{ikt}\tilde{a}_kA\hat{u}(k)\\
&+\lim_{n\to\infty}\sum_{k=-n}^{n}\left(1-\frac{|k|}{n+1}\right)e^{ikt}\tilde{b}_kB\hat{u}(k)-\sum_{j=1}^{N}e^{ik_jt}\hat{f}(k_j)\\
&-\sum_{j=1}^{N}e^{ik_jt}\big[\tilde{a}(ik_j)A+\tilde{b}(ik_j)B\big]\hat{u}(k_j)+\sum_{j=1}^{N}e^{ik_jt}\hat{u}(k_j)\\
=&f(t)+A(a*u)(t)+B(b*u)(t)-\sum_{j=1}^{N}e^{ik_jt}\hat{f}(k_j)+\sum_{j=1}^{N}e^{ik_jt}\big[I-\tilde{a}(ik_j)A-\tilde{b}(ik_j)B\big]\hat{u}(k_j).
\end{aligned}
$$

Since $Q_k\hat{f}(k_j)=0,\ j=1,\dots,N$, it then follows from Proposition 4.2, equalities (4.8) and (4.9) that

$$\big[I-\tilde{a}(ik_j)A-\tilde{b}(ik_j)B\big]\big[G_j'(ik_j)-Q_jA(\tilde{a}'G_j)'(ik_j)-Q_jB(\tilde{b}'G_j)'(ik_j)\big]\hat{f}(k_j)=\hat{f}(k_j).$$

Hence

$$u(t)=f(t)+A(a*u)(t)+B(b*u)(t),$$

proving the claim and the theorem. □

Example 4.5. Let $X=l^2(\mathbb{Z})$ and define $Ax_n=(n+in)x_n$ with maximal domain. Clearly A does not generate a C_0-semigroup. We take $b(t)\equiv 0$ and $a(t)=e^{-t}$ in Eq. (3.2). Clearly $\tilde{a}(ik)=\frac{1}{ik+1}$ is 1-regular and $\frac{1}{\tilde{a}(ik)}=ik+1\in\rho(A)$ for all $k\in\mathbb{Z}\setminus\{1\}$. Moreover $\lambda_0=i$ is a simple pole of $F(\lambda)=(I-\tilde{a}(\lambda)A)^{-1}$. It remains to show that the set $\{(I-\tilde{a}(ik)A)^{-1}\}_{k\in\mathbb{Z}\setminus\{1\}}$ is bounded. In fact, for each $x=(x_n)\in l^2(\mathbb{Z})$ and $k\in\mathbb{Z}\setminus\{1\}$ we have

$$
\begin{aligned}
\big\|\big(I-\tilde{a}(ik)A\big)^{-1}x\big\|^2=\big\|(ik+1)(ik+1-A)^{-1}x\big\|^2&=\sum_{n\in\mathbb{Z}}\left|\frac{ik+1}{ik+1-n-in}x_n\right|^2\\
&\leqslant\sum_{n\in\mathbb{Z}}\frac{k^2+1}{(1-n)^2+(k-n)^2}|x_n|^2\leqslant\sum_{n\in\mathbb{Z}}2\frac{k^2+1}{(k-1)^2}|x_n|^2,
\end{aligned}
$$

then we obtain

$$\sup_{k\in\mathbb{Z}\setminus\{1\}}\big\|\big(I-\tilde{a}(ik)A\big)^{-1}\big\|\leqslant 10.$$

We conclude by Theorem 4.4 that for every $f\in L^p_{2\pi}(\mathbb{R},l^2(\mathbb{Z}))$ the equation

$$u(t,x)=\frac{\partial}{\partial x}\int_{-\infty}^{t}e^{-(t-s)}\big(u(s,x)-iu(s,x)\big)\,ds+f(t,x),\quad x\in[0,2\pi],\ t\geqslant 0,$$

with boundary values $u(t,0)=u(t,2\pi)$, has a mild L^p-solution if and only if $Q_1\hat{f}(1)=0$.

To compute Q_1 we note that $F(\lambda)x_n=\frac{\lambda+1}{\lambda+1-n-in}x_n$ and hence

$$(\lambda-i)F(\lambda)x_n=\frac{(\lambda-i)(\lambda+1)}{(\lambda-in)+(1-n)}x_n=\begin{cases}(\lambda+1)x_1, & n=1,\\ \frac{(\lambda-i)(\lambda+1)}{(\lambda-in)+(1-n)}x_n, & n\neq 1.\end{cases}$$

Then

$$Q_1x_n:=\lim_{\lambda\to i}(\lambda-i)F(\lambda)x_n=\begin{cases}(i+1)x_1, & n=1,\\ 0, & n\neq 1.\end{cases}$$

Therefore if $f(t) = (f_n(t))$, then $Q_1 \hat{f}(1) = 0$ if and only if

$$\int_0^{2\pi} e^{-it} f_1(t)\,dt = 0.$$

Finally, concerning the remark after the proof of Proposition 3.8 in Section 3, we obtain the following result. It gives a criterion for the case of resonances to the problem (3.5). As remarked in Section 3, it also corresponds in some sense to the study of resonances for the second order Cauchy problem studied in [13] in case $A = B^{\varepsilon}$.

Corollary 4.6. *Let $\varepsilon \in (\frac{1}{2}, 1]$ and A be a sectorial operator which admits a bounded RH^{∞} functional calculus of angle $\beta \in (0, \frac{\pi}{2\varepsilon})$. Suppose that $\{ik\}_{k=0,\pm1,\ldots,\pm m} \in \sigma_{\tilde{a},\tilde{b}}(A, B)$ are simple poles of $F(\lambda) = \lambda^2(\lambda^2 + \lambda A^{\varepsilon} + A)^{-1}$ and let $f \in L^p_{2\pi}(\mathbb{R}, X)$. Then (3.5) has a mild L^p-solution if and only if $Q_k \hat{f}(k) = 0$ for all $k = \pm1, \ldots, \pm m$ where $Q_k = \lim_{\lambda \to ik}(\lambda - ik)F(\lambda)$.*

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