



Periodic behavior for a degenerate fast diffusion equation [☆]

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ABSTRACT

This work deals with the study of periodic solutions to a degenerate fast diffusion equation. The existence of the periodic solution to an intermediate problem restraint to a period T is proved first and then the result is extended by the data periodicity to all time real space. The approach involves an appropriate approximating problem whose periodic solution is proved via a fixed point theorem. Next, a passing to the limit procedure leads to the existence of the solution to the original problem on a time period. Finally, the behavior at large time of the solution to a Cauchy problem with periodic data is characterized.

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1. Statement of the problem

Periodic problems for possibly degenerate equations of the type

$$\frac{d}{dt}(My(t)) + Ly(t) = f(t), \quad 0 \leq t \leq 1, \quad (1.1)$$

with the periodic condition

$$(My)(0) = (My)(1) \quad (1.2)$$

have been studied in the paper [2], for L and M two closed linear operators from a complex Banach space into itself, under the assumptions that the domain $D(L)$ of L is continuously embedded in $D(M)$ and L has a bounded inverse. Assuming suitable hypotheses on the modified resolvent $(\lambda M + L)^{-1}$, it has been proved that problem (1.1)–(1.2) admits one 1-periodic solution. Some examples of applications to partial differential equations and ordinary differential equations have been given. The latter case has been studied in the paper [3], too.

In this paper we shall approach a concrete PDE problem (1.1)–(1.2) where L is a nonlinear multivalued operator.

We consider Ω an open bounded subset of \mathbf{R}^N ($N \in \mathbf{N}^* = \{1, 2, \dots\}$), with the boundary $\Gamma := \partial\Omega$ of class C^1 and denote the space variable by $x := (x_1, \dots, x_N) \in \Omega$ and the time by $t \in \mathbf{R}$. We are concerned with the study of periodic solutions to a nonlinear model consisting in a degenerate diffusion equation with homogeneous Dirichlet boundary conditions

$$\begin{aligned} \frac{\partial(m(x)u)}{\partial t} - \Delta\beta^*(u) &\ni f \quad \text{in } \Omega \times \mathbf{R}, \\ u(x, t) &= 0 \quad \text{on } \Gamma \times \mathbf{R}, \end{aligned}$$

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$$u(x, t) = u(x, t + T) \quad \text{in } \Omega \times \mathbf{R}, \quad (1.3)$$

under the assumption of the T -periodicity of the function f ,

$$f(x, t) = f(x, t + T) \quad \text{for } (x, t) \in \Omega \times \mathbf{R}, \quad 0 < T < \infty. \quad (1.4)$$

In this problem $\beta^* : (-\infty, u_s] \rightarrow \mathbf{R}$ is a multivalued function defined as

$$\beta^*(r) := \begin{cases} \int_0^r \beta(\xi) d\xi, & \text{if } r < u_s, \\ [K_s^*, +\infty), & \text{if } r = u_s, \end{cases} \quad (1.5)$$

where $\beta : (-\infty, u_s) \rightarrow \mathbf{R}$ is a positive differentiable, monotonically increasing function, which blows up at $r = u_s$, but having the integral finite at this point. Namely we set

$$\beta(r) \geq \rho > 0, \quad \text{for each } r < u_s, \quad \beta(r) := \rho \quad \text{for } r \leq 0, \quad (1.6)$$

$$\lim_{r \nearrow u_s} \beta(r) = +\infty \quad \text{and} \quad \lim_{r \nearrow u_s} \int_0^r \beta(\xi) d\xi = K_s^*. \quad (1.7)$$

Consequently, β^* has the properties

$$(\beta^*(r) - \beta^*(\bar{r}))(r - \bar{r}) \geq \rho(r - \bar{r})^2, \quad \text{for every } r, \bar{r} \in (-\infty, u_s], \quad (1.8)$$

$$\lim_{r \rightarrow -\infty} \beta^*(r) = -\infty, \quad (1.9)$$

$$\lim_{r \nearrow u_s} \beta^*(r) = K_s^*. \quad (1.10)$$

In the above relationships ρ , u_s and K_s^* are positive known constants and the hypotheses (1.7) reveal the character of fast diffusion (see [1,4]).

We also notice that $(\beta^*)^{-1} : \mathbf{R} \rightarrow (-\infty, u_s]$ is single-valued, monotonically increasing on $(-\infty, K_s^*)$ and constant for $r \in [K_s^*, +\infty)$, i.e., $(\beta^*)^{-1}(r) = u_s$. Also, it follows that $(\beta^*)^{-1}$ is Lipschitz with the constant $\frac{1}{\rho}$.

We still assume that

$$m \in C^1(\overline{\Omega}), \quad 0 \leq m(x) \leq 1, \quad x \in \overline{\Omega}. \quad (1.11)$$

More exactly, we consider that the degeneration of the equation may occur on $\overline{\Omega_0}$, where Ω_0 is an open bounded subset of Ω , strictly contained in Ω . The upper bound of m can be taken any positive constant, but by rescaling, we may consider it equal to 1, without any loss of generality.

The model (1.3) with initial data $(m(x)u(x, 0) = v_0(x)$ instead of the periodic condition) was studied in [4] where it was proved that it has a unique weak solution in appropriate functional spaces. In fact, the model was introduced in [7] and it describes for example the water infiltration in a unsaturated porous medium in which saturation can occur. This event is mathematically modeled by both the blow-up of the function β at u_s and the multivalued function β^* . The function $m(x)$ characterizes the space variable porosity of the nonhomogeneous medium, while the vanishing of m indicates the existence of impermeable intrusions in the soil.

A study of the periodic solutions to fast diffusion equations with $m(x) = 1$ was done in [8] for the case with a nonlinear convection, in connection with some results given in [6].

The paper is organized as follows: first we shall prove that the problem

$$\begin{aligned} \frac{\partial(m(x)u)}{\partial t} - \Delta \beta^*(u) &\ni f \quad \text{in } \Omega \times \mathbf{R}, \\ u(x, t) &= 0 \quad \text{on } \Gamma \times \mathbf{R}, \\ m(x)(u(x, t) - u(x, t + T)) &= 0 \quad \text{in } \Omega \times \mathbf{R} \end{aligned} \quad (1.12)$$

has a unique solution.

In order to prove the existence for problem (1.12) we shall establish the existence for the solution to the problem on a time period

$$\begin{aligned} \frac{\partial(m(x)u)}{\partial t} - \Delta \beta^*(u) &\ni f \quad \text{in } Q := \Omega \times (0, T), \\ u(x, t) &= 0 \quad \text{on } \Sigma := \Gamma \times (0, T), \\ m(x)(u(x, 0) - u(x, T)) &= 0 \quad \text{in } \Omega. \end{aligned} \quad (1.13)$$

This will be done by a fixed point argument in Section 2. The result obtained for (1.13) will be extended by periodicity to all $t \in \mathbf{R}$ and the longtime behavior of the solution corresponding to a periodic f and a certain initial datum v_0 will be established in connection with the periodic solution to (1.12).

Finally, we shall show that the existence of the unique periodic solution to (1.12) implies the existence of the unique periodic solution to (1.3).

Functional framework and preliminaries. For approaching the problems previously specified we shall consider the Hilbert space $V = H_0^1(\Omega)$ with the usual Hilbertian norm and its dual $V' = H^{-1}(\Omega)$, endowed with the scalar product $(u, \bar{u})_{V'} := \langle u, \psi \rangle_{V', V}$, where $\psi \in V$ satisfies $-\Delta \psi = \bar{u}$, $\psi = 0$ on Γ , and $\langle u, \psi \rangle_{V', V}$ is the pairing between V' and V .

For simplicity, we shall denote by (\cdot, \cdot) and $\|\cdot\|$ the scalar product and the norm in $L^2(\Omega)$, respectively.

Definition 1.1. Let

$$m \in C^1(\bar{\Omega}), \quad f \in L^\infty(0, T; V'). \quad (1.14)$$

We call a *solution* to (1.13) a function u which satisfies

$$\begin{aligned} u &\in L^2(0, T; V), \quad u \leq u_s, \quad \text{a.e. } (x, t) \in Q, \\ mu &\in C([0, T]; L^2(\Omega)) \cap W^{1,2}(0, T; V'), \\ \zeta &\in L^2(0, T; V), \quad \zeta(x, t) \in \beta^*(u(x, t)), \quad \text{a.e. } (x, t) \in Q, \end{aligned} \quad (1.15)$$

the condition $m(x)(u(x, 0) - u(x, T)) = 0$ in Ω and the equation

$$\int_0^T \left\langle \frac{d(m(x)u)}{dt}(t), \phi(t) \right\rangle_{V', V} dt + \int_Q \nabla \zeta(x, t) \cdot \nabla \phi(x, t) dx dt = \int_0^T \langle f(t), \phi(t) \rangle_{V', V} dt, \quad \text{a.e. } t \in (0, T), \quad (1.16)$$

for each $\phi \in L^2(0, T; V)$, where $\zeta(x, t) \in \beta^*(u(x, t))$, a.e. $(x, t) \in Q$.

On the domain

$$D(A) := \{u \in L^2(\Omega); \text{ there exists } \eta \in V, \text{ such that } \eta(x) \in \beta^*(u(x)), \text{ a.e. } x \in \Omega\}$$

we define the multivalued operator $A : D(A) \subset V' \rightarrow V'$ by

$$\langle Au, \psi \rangle_{V', V} := \int_\Omega \nabla \eta \cdot \nabla \psi dx, \quad \text{for each } \psi \in V, \text{ where } \eta(x) \in \beta^*(u(x)), \text{ a.e. } x \in \Omega.$$

We remark that $u \in D(A)$ implies $u \in V$, due to the Lipschitz property of the inverse of β^* .

Next, we introduce the multiplication operator $M : D(A) \rightarrow L^2(\Omega)$, $Mu := mu$, whose inverse is multivalued. Thus, we can write the abstract problem

$$\frac{d}{dt}(Mu(t)) + Au(t) \ni f(t), \quad \text{a.e. } t \in (0, T), \quad (1.17)$$

$$M(u(0) - u(T)) = 0 \quad (1.18)$$

and notice that the solution to (1.17)–(1.18) is a solution to (1.13) in the sense of Definition 1.1.

Denoting $v(x, t) := m(x)u(x, t)$ we can rewrite (1.17)–(1.18) in terms of v as,

$$\begin{aligned} \frac{dv}{dt} + A_M v &\ni f, \quad \text{a.e. } t \in (0, T), \\ v(0) &= v(T), \end{aligned} \quad (1.19)$$

where $A_M = AM^{-1}$ and

$$D(A_M) := \left\{ v \in L^2(\Omega); \frac{v}{m} \in L^2(\Omega), \exists \zeta \in V, \zeta(x) \in \beta^*\left(\frac{v}{m}(x)\right), \text{ a.e. } x \in \Omega \right\}.$$

We easily see that $v \in D(A_M)$ if and only if $u = \frac{v}{m} \in D(A)$.

For a later use we define $j : \mathbf{R} \rightarrow (-\infty, +\infty]$ by

$$j(r) := \begin{cases} \int_0^r \beta^*(\xi) d\xi, & r \leq u_s, \\ +\infty, & r > u_s, \end{cases} \quad (1.20)$$

where the left limit of β^* at u_s was specified in (1.10). The function j is proper, convex, lower semicontinuous and

$$\partial j(r) = \begin{cases} \beta^*(r), & r < u_s, \\ [K_s^*, +\infty), & r = u_s, \\ \emptyset, & r > u_s \end{cases} \quad (1.21)$$

(see [7, p. 166]).

Also, we recall a result proved in [4] (see Theorem 3.2) related to the problem

$$\begin{aligned} \frac{\partial(m(x)u)}{\partial t} - \Delta \beta^*(u) &\ni f \quad \text{in } Q, \\ u(x, t) &= 0 \quad \text{on } \Sigma, \\ m(x)u(x, 0) &= v_0 \quad \text{in } \Omega. \end{aligned} \quad (1.22)$$

Theorem 1.2. *Let*

$$m \in C^1(\overline{\Omega}), \quad f \in L^2(0, T; V'), \quad \frac{v_0}{m} \in L^2(\Omega), \quad \frac{v_0}{m} \leq u_s, \quad \text{a.e. } x \in \Omega.$$

Then, the Cauchy problem (1.22) has a unique solution u , such that

$$\begin{aligned} mu &\in C([0, T]; L^2(\Omega)) \cap W^{1,2}(0, T; V'), \\ \beta^*(u) &\in L^2(0, T; V), \\ u &\in L^2(0, T; V), \quad u \leq u_s, \quad \text{a.e. } (x, t) \in Q. \end{aligned}$$

2. Existence on the time period $(0, T)$

In this section we shall study the existence of the solution to the problem (1.13) defined on the time period $(0, T)$. To this end we shall establish first an existence result for the approximate problem obtained by replacing m by

$$m_\varepsilon(x) := m(x) + \varepsilon, \quad \text{where } \varepsilon \leq m_\varepsilon(x) \leq 1 + \varepsilon$$

and β^* by the single-valued function $\beta_\varepsilon^* : \mathbf{R} \rightarrow \mathbf{R}$,

$$\beta_\varepsilon^*(r) := \begin{cases} \beta^*(r), & \text{if } r < u_s - \varepsilon, \\ \beta^*(u_s - \varepsilon) + \frac{K_s^* - \beta^*(u_s - \varepsilon)}{\varepsilon} [r - (u_s - \varepsilon)], & \text{if } r \geq u_s - \varepsilon, \end{cases} \quad (2.1)$$

for each positive ε . The function β_ε^* is continuous and monotonically increasing on \mathbf{R} , differentiable on $\mathbf{R} \setminus \{u_s - \varepsilon\}$, but with lateral finite derivatives at $u = u_s - \varepsilon$, satisfies (1.8) for any $r, \tilde{r} \in \mathbf{R}$ and

$$\lim_{r \rightarrow -\infty} \beta_\varepsilon^*(r) = -\infty, \quad \lim_{r \rightarrow +\infty} \beta_\varepsilon^*(r) = +\infty.$$

We denote by β_ε the derivative of β_ε^* defined as

$$\beta_\varepsilon(r) := \begin{cases} \beta(r), & \text{if } r < u_s - \varepsilon, \\ \frac{K_s^* - \beta^*(u_s - \varepsilon)}{\varepsilon}, & \text{if } r \geq u_s - \varepsilon \end{cases} \quad (2.2)$$

and remark that $\beta_\varepsilon(r) \geq \rho$ for any $r \in \mathbf{R}$.

Then we introduce $A_\varepsilon : D(A_\varepsilon) \subset V' \rightarrow V'$ by

$$\langle A_\varepsilon u, \psi \rangle_{V', V} := \int_{\Omega} \nabla \beta_\varepsilon^*(u) \cdot \nabla \psi \, dx, \quad \text{for every } \psi \in V,$$

$$D(A_\varepsilon) := \{u \in L^2(\Omega); \beta_\varepsilon^*(u) \in V\}$$

and consider the periodic approximating problem

$$\frac{d(m_\varepsilon u_\varepsilon)}{dt} + A_\varepsilon u_\varepsilon = f, \quad \text{a.e. } t \in (0, T), \quad (2.3)$$

$$m_\varepsilon(u_\varepsilon(0) - u_\varepsilon(T)) = 0 \quad (2.4)$$

which is equivalent with

$$\begin{aligned} \frac{dv_\varepsilon}{dt} + B_\varepsilon v_\varepsilon &= f, \quad \text{a.e. } t \in (0, T), \\ v_\varepsilon(0) &= v_\varepsilon(T), \end{aligned} \quad (2.5)$$

by the function replacement

$$v_\varepsilon = m_\varepsilon u_\varepsilon. \quad (2.6)$$

Here, $B_\varepsilon v_\varepsilon = A_\varepsilon(\frac{v_\varepsilon}{m_\varepsilon})$. We are going to prove the following existence result.

Theorem 2.1. Let ε be positive, fixed and

$$m \in C^1(\overline{\Omega}), \quad f \in L^\infty(0, T; V'). \quad (2.7)$$

Then, the periodic approximating problem (2.3)–(2.4) has a unique solution

$$u_\varepsilon \in C([0, T]; L^2(\Omega)) \cap W^{1,2}(0, T; V') \cap L^2(0, T; V), \quad (2.8)$$

$$\beta_\varepsilon^*(u_\varepsilon) \in L^2(0, T; V). \quad (2.9)$$

Moreover, the solution satisfies the estimates

$$\int_0^T \left\| \frac{d(m_\varepsilon u_\varepsilon)}{d\tau}(t) \right\|_{V'}^2 d\tau + \int_0^T \|\beta_\varepsilon^*(u_\varepsilon(t))\|_V^2 d\tau \leq 2 \int_0^T \|f(t)\|_{V'}^2 dt, \quad (2.10)$$

for each $t \in [0, T]$ and

$$\int_0^T \|m_\varepsilon(u_\varepsilon(\tau) - \bar{u}_\varepsilon(\tau))\|^2 d\tau \leq 2 \left(\frac{1+\varepsilon}{\rho} \right)^2 \int_0^T \|f(t) - \bar{f}(t)\|_{V'}^2 dt, \quad (2.11)$$

where u_ε and \bar{u}_ε are two periodic solutions corresponding to the periodic data f and \bar{f} , respectively.

Proof. Let ε be fixed. We shall apply a fixed point result and start by fixing in (2.4) $m_\varepsilon u_\varepsilon(0) = \theta_0 \in L^2(\Omega)$. We obtain thus the Cauchy problem

$$\begin{aligned} \frac{d(m_\varepsilon u_\varepsilon)}{dt} + A_\varepsilon u_\varepsilon &= f, \quad \text{a.e. } t \in (0, T), \\ m_\varepsilon u_\varepsilon(0) &= \theta_0 \end{aligned} \quad (2.12)$$

or, equivalently (by writing $v_\varepsilon = m_\varepsilon u_\varepsilon$)

$$\begin{aligned} \frac{dv_\varepsilon}{dt} + B_\varepsilon v_\varepsilon &= f, \quad \text{a.e. } t \in (0, T), \\ v_\varepsilon(0) &= \theta_0. \end{aligned} \quad (2.13)$$

In [4] it was proved that the Cauchy problem (2.13) has a unique solution $v_\varepsilon \in C([0, T]; L^2(\Omega)) \cap W^{1,2}(0, T; V') \cap L^2(0, T; V)$ with $\beta_\varepsilon^*(\frac{v_\varepsilon}{m_\varepsilon}) \in L^2(0, T; V)$, and a similar result holds for (2.12), too (this is a part of Theorem 3.1 in [4], corresponding to the initial datum in $L^2(\Omega)$).

Let us consider the set

$$M_\varepsilon := \left\{ \theta \in L^2(\Omega); \left\| \frac{\theta}{\sqrt{m_\varepsilon}} \right\| \leq R_\varepsilon, \text{ a.e. } x \in \Omega \right\}, \quad (2.14)$$

where R_ε is a large enough positive constant defined for each $\varepsilon > 0$ fixed. We define the mapping

$$\Psi_\varepsilon : M_\varepsilon \rightarrow M_\varepsilon, \quad \Psi_\varepsilon(\theta_0) = v_\varepsilon(T), \quad \text{for any } \theta_0 \in M_\varepsilon,$$

where $v_\varepsilon(t)$ is the solution to (2.13).

Since (2.13) (equivalently (2.12)) has a unique solution for $\theta_0 \in M_\varepsilon$, as specified before, we deduce that the mapping Ψ_ε is single-valued and we are going to show that it has a fixed point by the Schauder–Tychonoff theorem (see e.g., [5, p. 148]), working in the weak topology. In the subsequent part of the proof we shall show that the conditions of this theorem are verified.

(a) It is obvious that M_ε is a convex, bounded and strongly closed subset of $L^2(\Omega)$, hence it is weakly compact in $L^2(\Omega)$.

(b) Next, we have to show the inclusion $\Psi_\varepsilon(M_\varepsilon) \subset M_\varepsilon$.

Since $v_\varepsilon \in C([0, T]; L^2(\Omega))$ it follows that $v_\varepsilon(T) = m_\varepsilon(x)u_\varepsilon(T) \in L^2(\Omega)$. We test (2.13) at $\frac{v_\varepsilon}{m_\varepsilon} \in V$ and recalling (2.2) and (1.6) we get

$$\frac{1}{2} \frac{d}{dt} \left\| \frac{v_\varepsilon}{\sqrt{m_\varepsilon}}(t) \right\|^2 + \rho \left\| \frac{v_\varepsilon}{m_\varepsilon}(t) \right\|_V^2 \leq \|f(t)\|_{V'} \left\| \frac{v_\varepsilon}{m_\varepsilon}(t) \right\|_V.$$

We recall that $\|\cdot\|$ denotes the norm in $L^2(\Omega)$. Now, we apply the Poincaré inequality (with the constant c_P depending on N and the domain Ω) and obtain

$$\frac{d}{dt} \left\| \frac{v_\varepsilon}{\sqrt{m_\varepsilon}}(t) \right\|^2 + \frac{\rho}{c_P^2} \left\| \frac{v_\varepsilon}{m_\varepsilon}(t) \right\|^2 \leq \frac{1}{\rho} \|f\|_{L^\infty(0, T; V')}^2.$$

Since $m_\varepsilon(x) \leq 1 + \varepsilon$ we can still write

$$\frac{d}{dt} \left\| \frac{v_\varepsilon}{\sqrt{m_\varepsilon}}(t) \right\|^2 + \rho_\varepsilon \left\| \frac{v_\varepsilon}{\sqrt{m_\varepsilon}}(t) \right\|^2 \leq C_f$$

with $\rho_\varepsilon = \frac{\rho}{(1+\varepsilon)c_p^2}$ and $C_f = \frac{1}{\rho} \|f\|_{L^\infty(0,T;V')}^2$. Integrating on $(0, t)$ with $t \in [0, T]$ we get

$$\left\| \frac{v_\varepsilon}{\sqrt{m_\varepsilon}}(t) \right\|^2 \leq \left\| \frac{\theta_0}{\sqrt{m_\varepsilon}} \right\|^2 \exp(-\rho_\varepsilon t) + \frac{C_f}{\rho_\varepsilon} (1 - \exp(-\rho_\varepsilon t)).$$

If R_ε is large enough, $R_\varepsilon^2 \geq \frac{C_f}{\rho_\varepsilon}$ (for example this is ensured if $R_\varepsilon^2 \geq \frac{2c_p^2}{\rho} C_f$) and $\theta_0 \in M_\varepsilon$ it follows that $\|v_\varepsilon(t)\| \leq R_\varepsilon$, for any $t \in [0, T]$, hence $v_\varepsilon(T) = \Psi_\varepsilon(\theta_0) \in M_\varepsilon$. Therefore, we get that $\Psi_\varepsilon(M_\varepsilon)$ is weakly compact, too.

(c) Finally, we have to prove that the mapping Ψ_ε is weakly continuous.

For that we consider a sequence

$$\{\theta_0^n\}_{n \geq 1} \subset M_\varepsilon, \quad \theta_0^n \rightarrow \theta_0 \text{ weakly in } L^2(\Omega) \text{ as } n \rightarrow \infty, \quad (2.15)$$

and will show that

$$\Psi_\varepsilon(\theta_0^n) \rightarrow \Psi_\varepsilon(\theta_0) \text{ weakly in } L^2(\Omega) \text{ as } n \rightarrow \infty.$$

We introduce the approximating problem

$$\frac{dv_\varepsilon^n}{dt} + B_\varepsilon v_\varepsilon^n = f, \quad \text{a.e. } t \in (0, T), \quad (2.16)$$

$$v_\varepsilon^n(0) = \theta_0^n \quad (2.17)$$

and recall again Theorem 3.1 in [4] which asserts that (2.16)–(2.17) provides a sequence of solutions $\{v_\varepsilon^n\}_{n \geq 1}$, $v_\varepsilon^n \in C([0, T]; L^2(\Omega)) \cap W^{1,2}(0, T; V') \cap L^2(0, T; V)$ satisfying the estimates

$$\begin{aligned} & \int_\Omega m_\varepsilon(x) j_\varepsilon \left(\frac{v_\varepsilon^n}{m_\varepsilon}(t) \right) dx + \int_0^t \left\| \frac{dv_\varepsilon^n}{d\tau}(\tau) \right\|_{V'}^2 d\tau + \int_0^t \left\| \beta_\varepsilon^* \left(\frac{v_\varepsilon^n}{m_\varepsilon}(\tau) \right) \right\|_V^2 d\tau \\ & \leq 4(1 + \varepsilon) \left(\int_\Omega j_\varepsilon \left(\frac{\theta_0^n}{m_\varepsilon}(t) \right) dx + \int_0^T \|f(t)\|_{V'}^2 dt \right), \end{aligned} \quad (2.18)$$

$$\|v_\varepsilon^n(t)\|^2 \leq \frac{8(1 + \varepsilon)^2}{\rho} \left(\int_\Omega j_\varepsilon \left(\frac{\theta_0^n}{m_\varepsilon}(t) \right) dx + \int_0^T \|f(t)\|_{V'}^2 dt \right), \quad (2.19)$$

$$\|v_\varepsilon^n(t) - \bar{v}_\varepsilon^n(t)\|_{V'}^2 + \frac{\rho}{1 + \varepsilon} \int_0^t \|v_\varepsilon^n(\tau) - \bar{v}_\varepsilon^n(\tau)\|^2 d\tau \leq e^T \left(\|\theta_0^n - \bar{\theta}_0^n\|_{V'}^2 + \int_0^T \|f(t) - \bar{f}(t)\|_{V'}^2 dt \right), \quad (2.20)$$

all true for each $t \in [0, T]$. Here we have defined

$$j_\varepsilon(r) := \int_0^r \beta_\varepsilon^*(\xi) d\xi, \quad \text{for every } r \in \mathbf{R}, \quad (2.21)$$

and notice that

$$\partial j_\varepsilon(r) = \beta_\varepsilon^*(r), \quad \text{for every } r \in \mathbf{R} \quad (2.22)$$

and

$$j_\varepsilon(r) \geq \frac{\rho}{2} r^2, \quad \text{for each } r \in \mathbf{R}. \quad (2.23)$$

Moreover, by a straightforward computation starting from the definitions of j_ε and β_ε^* (see (2.1)) we get

$$\int_\Omega j_\varepsilon \left(\frac{\theta_0^n}{m_\varepsilon} \right) dx \leq \frac{K_s^* - \beta^*(u_s - \varepsilon)}{2\varepsilon} \left\| \frac{\theta_0^n}{m_\varepsilon} \right\|^2 \leq \frac{K_s^* - \beta^*(\theta_s - \varepsilon)}{2\varepsilon} \frac{R_\varepsilon^2}{\varepsilon^2},$$

so that the right-hand side in (2.18) is in fact independent of n . From here and (2.23) we still get

$$\|v_\varepsilon^n(t)\|^2 \leq \frac{8(1+\varepsilon)^2}{\rho} \left(\frac{R_\varepsilon^2(K_s^* - \beta^*(\theta_s - \varepsilon))}{2\varepsilon^3} + \int_0^T \|f(t)\|_{V'}^2 dt \right) \quad (2.24)$$

for any $t \in [0, T]$.

Therefore, on a subsequence (denoted in the same way) we get that

$$\beta_\varepsilon^* \left(\frac{v_\varepsilon^n}{m_\varepsilon} \right) \rightarrow \eta_\varepsilon \quad \text{weakly in } L^2(0, T; V) \text{ as } n \rightarrow \infty. \quad (2.25)$$

Since the inverse of β_ε^* is Lipschitz we deduce that

$$\frac{v_\varepsilon^n}{m_\varepsilon} \rightarrow \zeta_\varepsilon \quad \text{weakly in } L^2(0, T; V) \text{ as } n \rightarrow \infty. \quad (2.26)$$

Denoting $v_\varepsilon^n = m_\varepsilon \frac{v_\varepsilon^n}{m_\varepsilon}$ we obtain that $\{v_\varepsilon^n\}_{n \geq 1}$ lies in a bounded subset of $L^2(0, T; V)$, so that we can select a subsequence such that

$$v_\varepsilon^n \rightarrow v_\varepsilon \quad \text{weakly in } L^2(0, T; V) \text{ as } n \rightarrow \infty. \quad (2.27)$$

Consequently, we get that

$$v_\varepsilon^n \rightarrow v_\varepsilon := m_\varepsilon \zeta_\varepsilon \quad \text{weakly in } L^2(0, T; V) \text{ as } n \rightarrow \infty. \quad (2.28)$$

At the end we shall prove that ζ_ε is the solution to (2.12) and v_ε is in fact the solution to (2.13). By (2.18) we still have that

$$\frac{dv_\varepsilon^n}{dt} \rightarrow \frac{dv_\varepsilon}{dt} \quad \text{weakly in } L^2(0, T; V') \text{ as } n \rightarrow \infty \quad (2.29)$$

and using the Ascoli–Arzelà theorem we obtain

$$v_\varepsilon^n(t) \rightarrow v_\varepsilon(t) \quad \text{strongly in } V' \text{ for each } t \in [0, T], \text{ as } n \rightarrow \infty, \quad (2.30)$$

as proved further. Indeed, the family $\mathcal{M} = \{v_\varepsilon^n\}_n \subset C([0, T]; V')$ is bounded (this follows e.g., by (2.24)) and equi-uniformly continuous. To prove this, let $\varepsilon' > 0$ and consider that $\sigma(\varepsilon')$ exists such that $|t - s| \leq \sigma(\varepsilon')$, for $0 \leq s < t \leq T$. We have

$$\begin{aligned} \|v_\varepsilon^n(t) - v_\varepsilon^n(s)\|_{V'} &= \left\| \int_s^t \frac{dv_\varepsilon^n}{dt}(\tau) d\tau \right\|_{V'} \leq \int_s^t \left\| \frac{dv_\varepsilon^n}{dt}(\tau) \right\|_{V'} d\tau \leq |t - s|^{1/2} \left\| \frac{dv_\varepsilon^n}{dt} \right\|_{L^2(0, T; V')} \leq \varepsilon', \\ \text{for } \sigma(\varepsilon') &\leq \frac{\varepsilon'^2}{\gamma_0(R, \varepsilon)}, \quad \forall v_\varepsilon^n \in \mathcal{M} \end{aligned}$$

where $\gamma_0(R, \varepsilon)$ is the right-hand side in (2.18) which is independent of n . Still by (2.24) we get that the sequence $\{v_\varepsilon^n(t)\}_n$ is bounded in $L^2(\Omega)$ and since the injection of $L^2(\Omega)$ in V' is compact it follows that the sequence $\{v_\varepsilon^n(t)\}_n$ is compact in V' , for each $t \in [0, T]$. Hence the set \mathcal{M} is compact in $C([0, T]; V')$, i.e., $v_\varepsilon^n(t) \rightarrow v_\varepsilon(t) = m_\varepsilon \zeta_\varepsilon(t)$ strongly in V' , uniformly on $[0, T]$.

In particular,

$$v_\varepsilon^n(0) \rightarrow v_\varepsilon(0) := m_\varepsilon \zeta_\varepsilon(0) \quad \text{strongly in } V' \text{ as } n \rightarrow \infty \quad (2.31)$$

and

$$v_\varepsilon^n(T) \rightarrow v_\varepsilon(T) := m_\varepsilon \zeta_\varepsilon(T) \quad \text{strongly in } V' \text{ as } n \rightarrow \infty. \quad (2.32)$$

Further, we deduce by (2.28) and (2.29), using the Lions–Aubin theorem that

$$v_\varepsilon^n \rightarrow v_\varepsilon = m_\varepsilon \zeta_\varepsilon \quad \text{strongly in } L^2(0, T; L^2(\Omega)) \text{ as } n \rightarrow \infty. \quad (2.33)$$

From here and the continuity of β_ε^* we derive that

$$\beta_\varepsilon^* \left(\frac{v_\varepsilon^n}{m_\varepsilon} \right) \rightarrow \beta_\varepsilon^*(\zeta_\varepsilon) \quad \text{strongly in } L^2(0, T; L^2(\Omega)) \text{ as } n \rightarrow \infty,$$

because by a direct computation we have that

$$\left\| \beta_\varepsilon^* \left(\frac{v_\varepsilon^n}{m_\varepsilon} \right) - \beta_\varepsilon^*(\zeta_\varepsilon) \right\| \leq \frac{K_s^* - \beta_\varepsilon^*(u_s - \varepsilon)}{\varepsilon} \left\| \frac{v_\varepsilon^n}{m_\varepsilon} - \zeta_\varepsilon \right\|.$$

By (2.25) and the uniqueness of the limit we deduce that $\eta_\varepsilon = \beta_\varepsilon^*\left(\frac{v_\varepsilon^n}{m_\varepsilon}\right)$ a.e. in Q and therefore

$$\beta_\varepsilon^*\left(\frac{v_\varepsilon^n}{m_\varepsilon}\right) \rightarrow \beta_\varepsilon^*(\zeta_\varepsilon) \quad \text{weakly in } L^2(0, T; V) \text{ as } n \rightarrow \infty. \quad (2.34)$$

Similarly, j_ε being Lipschitz, we obtain that

$$j_\varepsilon\left(\frac{v_\varepsilon^n}{m_\varepsilon}\right) \rightarrow j_\varepsilon(\zeta_\varepsilon) \quad \text{as } n \rightarrow \infty. \quad (2.35)$$

By (2.24) we can extract a subsequence of $\{v_\varepsilon^n(T)\}_n$ weak-star convergent as $n \rightarrow \infty$ in $L^\infty(\Omega)$, and recalling (2.32) we get that

$$v_\varepsilon^n(T) \rightarrow v_\varepsilon(T) = m_\varepsilon \zeta_\varepsilon(T) \quad \text{weakly in } L^2(\Omega) \text{ as } n \rightarrow \infty. \quad (2.36)$$

Since v_ε^n is a solution to (2.16)–(2.17) with $v_\varepsilon^n(0) = \theta_0^n$, we can write

$$\int_0^T \left\langle \frac{dv_\varepsilon^n}{dt}(t), \phi(t) \right\rangle_{V', V} dt + \int_Q \nabla \beta_\varepsilon^*\left(\frac{v_\varepsilon^n}{m_\varepsilon}\right) \cdot \nabla \phi \, dx \, dt = \int_0^T \langle f(t), \phi(t) \rangle_{V', V} dt$$

for any $\phi \in L^2(0, T; V)$. Passing to the limit as $n \rightarrow \infty$ and taking into account (2.29) and (2.34) we obtain that

$$\int_0^T \left\langle \frac{dv_\varepsilon}{dt}(t), \phi(t) \right\rangle_{V', V} dt + \int_Q \nabla \beta_\varepsilon^*\left(\frac{v_\varepsilon}{m_\varepsilon}\right) \cdot \nabla \phi \, dx \, dt = \int_0^T \langle f(t), \phi(t) \rangle_{V', V} dt, \quad (2.37)$$

for any $\phi \in L^2(0, T; V)$. By (2.17) and (2.15) we have $v_\varepsilon^n(0) = \theta_0^n \rightarrow \theta_0$ strongly in $L^2(\Omega)$, which together with (2.31) prove that v_ε is the solution to (2.13) with the initial datum θ_0 . Obviously, $\zeta_\varepsilon = u_\varepsilon$ is the solution to (2.12) because by $v_\varepsilon = m_\varepsilon u_\varepsilon$ we have immediately that

$$\int_0^T \left\langle \frac{d(m_\varepsilon u_\varepsilon)}{dt}(t), \phi(t) \right\rangle_{V', V} dt + \int_Q \nabla \beta_\varepsilon^*(u_\varepsilon) \cdot \nabla \phi \, dx \, dt = \int_0^T \langle f(t), \phi(t) \rangle_{V', V} dt,$$

for any $\phi \in L^2(0, T; V)$.

Now, by (2.36) we have

$$\Psi_\varepsilon(\theta_0^n) = v_\varepsilon^n(T) \rightarrow v_\varepsilon(T) = \Psi_\varepsilon(\theta_0) \quad \text{weakly in } L^2(\Omega),$$

and because M_ε is weakly closed it follows that $v_\varepsilon(T) \in M_\varepsilon$.

This ends the checking of Schauder–Tychonoff theorem hypotheses so we can deduce that Ψ_ε has a fixed point, implying

$$v_\varepsilon(0) = \theta_0 = v_\varepsilon(T) \quad \text{or} \quad m_\varepsilon u_\varepsilon(0) = m_\varepsilon u_\varepsilon(T).$$

Consequently, we conclude that (2.3) has at least a solution.

In order to prove the first estimate we apply (2.37) for $\phi = \beta_\varepsilon^*\left(\frac{v_\varepsilon}{m_\varepsilon}\right) \in L^2(0, T; V)$ and get

$$\left\langle \frac{dv_\varepsilon}{dt}(t), \beta_\varepsilon^*\left(\frac{v_\varepsilon}{m_\varepsilon}\right) \right\rangle_{V', V} + \left\| \beta_\varepsilon^*\left(\frac{v_\varepsilon}{m_\varepsilon}\right) \right\|_V^2 \leq \|f(t)\|_{V'} \left\| \beta_\varepsilon^*\left(\frac{v_\varepsilon}{m_\varepsilon}\right) \right\|_V.$$

Then we integrate with respect to $t \in (0, T)$ using the relation

$$\left\langle \frac{dv_\varepsilon}{dt}(t), \beta_\varepsilon^*\left(\frac{v_\varepsilon}{m_\varepsilon}\right) \right\rangle_{V', V} = \frac{d}{dt} \left(m_\varepsilon j_\varepsilon\left(\frac{v_\varepsilon}{m_\varepsilon}(t)\right) \right) \quad (2.38)$$

and the fact that v_ε is periodic. We obtain

$$\int_0^T \left\| \beta_\varepsilon^*\left(\frac{v_\varepsilon}{m_\varepsilon}(t)\right) \right\|_V^2 dt \leq \int_0^T \|f(t)\|_{V'}^2 dt.$$

Next we multiply (2.3) scalarly in V' by $\frac{dv_\varepsilon}{dt}$ and get, using the definition of the scalar product in V'

$$\left\| \frac{dv_\varepsilon}{dt}(t) \right\|_{V'}^2 + \frac{d}{dt} \left(m_\varepsilon j_\varepsilon\left(\frac{v_\varepsilon}{m_\varepsilon}(t)\right) \right) \leq \langle f(t), \psi \rangle_{V', V} \leq \|f(t)\|_{V'} \|\psi\|_V,$$

where $-\Delta\psi = \frac{dv_\varepsilon}{dt}(t)$, $\psi = 0$ on Γ . Moreover, $\|\psi\|_V = \|\frac{dv_\varepsilon}{dt}(t)\|_{V'}$. We integrate over $(0, T)$ and obtain

$$\int_0^T \left\| \frac{dv_\varepsilon}{dt}(t) \right\|_{V'}^2 dt \leq \int_0^T \|f(t)\|_{V'}^2 dt.$$

By summing up these inequalities we get (2.10) as claimed.

Finally we multiply the difference of two equations (2.3) corresponding to the data f and \bar{f} by $(v_\varepsilon - \bar{v}_\varepsilon)$ scalarly in V' and get

$$\frac{1}{2} \frac{d}{dt} \|v_\varepsilon(t) - \bar{v}_\varepsilon(t)\|_{V'}^2 + \frac{\rho}{2(1+\varepsilon)} \|v_\varepsilon(t) - \bar{v}_\varepsilon(t)\|^2 \leq \frac{1+\varepsilon}{\rho} \|f(t) - \bar{f}(t)\|^2.$$

Integrating with respect to $t \in (0, T)$ we obtain (2.11), whence we can also deduce the uniqueness of the solution to (2.3). \square

Theorem 2.2. *Let*

$$m \in C^1(\bar{\Omega}), \quad f \in L^\infty(0, T; V'). \quad (2.39)$$

Then, the Cauchy problem (1.13) has a unique solution denoted u^ such that*

$$u^* \in L^2(0, T; V), \quad (2.40)$$

$$mu^* \in C([0, T]; L^2(\Omega)) \cap W^{1,2}(0, T; V'), \quad (2.41)$$

$$\zeta \in L^2(0, T; V), \quad \zeta(x, t) \in \beta^*(u^*(x, t)), \quad \text{a.e. } (x, t) \in Q, \quad (2.42)$$

$$u^* \leq u_s, \quad \text{a.e. } (x, t) \in Q. \quad (2.43)$$

The solution satisfies the estimate

$$\int_0^T \left\| \frac{d(mu^*)}{d\tau}(\tau) \right\|_{V'}^2 d\tau + \int_0^T \|\zeta(\tau)\|_V^2 d\tau \leq 2 \int_0^T \|f(t)\|_{V'}^2 dt, \quad (2.44)$$

for each $t \in [0, T]$, where $\zeta(x, t) \in \beta^(u^*(x, t))$ a.e. $(x, t) \in Q$.*

If u^ and \bar{u}^* are two solutions corresponding to the periodic data f and \bar{f} , respectively, then we have*

$$\int_0^T \|mu^*(\tau) - m\bar{u}^*(\tau)\|^2 d\tau \leq \frac{2}{\rho^2} \int_0^T \|f(t) - \bar{f}(t)\|_{V'}^2 dt. \quad (2.45)$$

Proof. Under the hypotheses, the approximating problem (2.3)–(2.4) has a unique solution given by Theorem 2.1, in which we shall let ε to tend to 0. Using (2.10) we get on a subsequence that

$$\beta_\varepsilon^*\left(\frac{v_\varepsilon}{m_\varepsilon}\right) \rightarrow \zeta \quad \text{weakly in } L^2(0, T; V) \text{ as } \varepsilon \rightarrow 0, \quad (2.46)$$

$$u_\varepsilon = \frac{v_\varepsilon}{m_\varepsilon} \rightarrow u \quad \text{weakly in } L^2(0, T; V) \text{ as } \varepsilon \rightarrow 0. \quad (2.47)$$

Now writing

$$v_\varepsilon = m_\varepsilon \frac{v_\varepsilon}{m_\varepsilon} \quad (2.48)$$

and since $m \in C^1(\bar{\Omega})$ we have that

$$\|v_\varepsilon\|_V \leq \text{constant independent of } \varepsilon, \quad (2.49)$$

hence

$$v_\varepsilon \rightarrow v \quad \text{weakly in } L^2(0, T; V) \text{ as } \varepsilon \rightarrow 0. \quad (2.50)$$

Next, the sequence $\{\frac{dv_\varepsilon}{dt}\}_\varepsilon$ is bounded in $L^2(0, T; V')$ and we get on a subsequence that

$$\frac{dv_\varepsilon}{dt} \rightarrow \frac{dv}{dt} \quad \text{weakly in } L^2(0, T; V') \text{ as } \varepsilon \rightarrow 0 \quad (2.51)$$

and by (2.3)

$$\Delta \beta_\varepsilon^* \left(\frac{v_\varepsilon}{m_\varepsilon} \right) \rightarrow \frac{dv}{dt} - f \quad \text{weakly in } L^2(0, T; V') \text{ as } \varepsilon \rightarrow 0. \quad (2.52)$$

By (2.47), (2.48) and (2.50) we deduce that

$$v = mu, \quad \text{a.e. on } Q \quad (2.53)$$

because $m_\varepsilon \rightarrow m$ uniformly on Ω . Obviously, $v \in C([0, T]; L^2(\Omega))$ by (2.50) and (2.51) and

$$v = 0, \quad \text{a.e. on } Q_0, \quad \text{where } Q_0 := \Omega_0 \times (0, T). \quad (2.54)$$

In the same way as before, by Ascoli–Arzelà theorem we can prove that $v_\varepsilon(t) \rightarrow v(t)$ strongly in V' (using (2.49) and (2.51)). Using (2.53) we can deduce by letting $\varepsilon \rightarrow 0$ in the second equation in (2.3) that

$$v(0) - v(T) = m(u(0) - u(T)) = 0. \quad (2.55)$$

By Lions–Aubin theorem we conclude also that $\{v_\varepsilon\}_\varepsilon$ is compact in $L^2(0, T; L^2(\Omega))$

$$v_\varepsilon \rightarrow v \quad \text{strongly in } L^2(0, T; L^2(\Omega)) \text{ as } \varepsilon \rightarrow 0. \quad (2.56)$$

We set now

$$\begin{aligned} \Omega_\delta &:= \{x \in \Omega; m(x) > \delta\}, & Q_\delta &:= \Omega_\delta \times (0, T), \quad \text{for } \delta > 0, \text{ arbitrary,} \\ \Omega_m &:= \{x \in \Omega; m(x) > 0\}, & Q_m &:= \Omega_m \times (0, T) \end{aligned} \quad (2.57)$$

and notice that Ω_δ and Ω_m are open because $m \in C^1(\overline{\Omega})$. We have

$$\frac{1}{m_\varepsilon} = \frac{1}{m + \varepsilon} < \frac{1}{\delta} \quad \text{on } \Omega_\delta,$$

so that, by (2.56) and (2.47) we can conclude that

$$u_\varepsilon = \frac{1}{m_\varepsilon} v_\varepsilon \rightarrow \frac{v}{m} := u \quad \text{strongly in } L^2(0, T; L^2(\Omega_\delta)), \quad (2.58)$$

so they are equal a.e. in Q_δ , for each $\delta > 0$. Still by (2.47) we have that

$$u_\varepsilon = \frac{v_\varepsilon}{m_\varepsilon} \rightarrow u \quad \text{weakly in } L^2(0, T; L^2(\Omega_m)).$$

We assert now that

$$\zeta(x, t) \in \beta^*(u(x, t)), \quad \text{a.e. on } Q_\delta, \quad (2.59)$$

where ζ is defined in (2.46), which also implies that $u(x, t) \leq u_s$ a.e. in Q_δ . These follow the by fact that β^* is the subdifferential of j and the arguments to sustain it are briefly presented below. We rely on the relationships

$$\lim_{\varepsilon \rightarrow 0} j_\varepsilon(z) = j(z) \quad \text{for any } z \in \mathbf{R}$$

proved by a direct calculus and

$$j(u) \leq \liminf_{\varepsilon \rightarrow 0} j_\varepsilon(u_\varepsilon)$$

which implies by Fatou's lemma (since $j_\varepsilon(u_\varepsilon) \geq 0$) that

$$\liminf_{\varepsilon \rightarrow 0} \int_{Q_\delta} j_\varepsilon(u_\varepsilon) dx dt \geq \int_{Q_\delta} \liminf_{\varepsilon \rightarrow 0} j_\varepsilon(u_\varepsilon) dx dt \geq \int_{Q_\delta} j(u) dx dt.$$

By (2.22) we have

$$j_\varepsilon(u_\varepsilon) \leq j_\varepsilon(z) + \beta_\varepsilon^*(u_\varepsilon)(u_\varepsilon - z) \quad \text{for any } z \in \mathbf{R}.$$

This relation, particularized for $z: \Omega_\delta \times (0, T) \rightarrow \mathbf{R}$, $z \in L^2(Q_\delta)$, leads to

$$\int_{Q_\delta} j_\varepsilon(u_\varepsilon) dx dt \leq \int_{Q_\delta} j_\varepsilon(z) dx dt + \int_{Q_\delta} \beta_\varepsilon^*(u_\varepsilon)(u_\varepsilon - z) dx dt \quad \text{for any } z \in L^2(Q_\delta).$$

Assume also that $z(x, t) \leq u_s$ a.e. on Q_δ and passing to the limit as $\varepsilon \rightarrow 0$ we obtain

$$\int_{Q_\delta} j(u) dx dt \leq \int_{Q_\delta} j(z) dx dt + \int_{Q_\delta} \zeta(u - z) dx dt \quad \text{for any } z \in L^2(Q_\delta).$$

This was also ensured by (2.46) and the strongly convergence (2.58). Next, it can be deduced that $\zeta \in \partial j(u)$ a.e. on Q_δ , which implies (2.59) as claimed. A detailed development of this proof can be found in [7, Theorem 3.1, p. 67, and Corollary 3.3, p. 174].

Now we are going to construct the solution on Q_0 .

We recall that u_ε is the solution to (2.3), equivalently written

$$\int_0^T \left\langle \frac{d(m_\varepsilon u_\varepsilon)}{dt}(t), \phi(t) \right\rangle_{V', V} dt + \int_Q \nabla \beta_\varepsilon^*(u_\varepsilon(t)) \cdot \nabla \phi(t) dx dt = \int_0^T \langle f(t), \phi(t) \rangle_{V', V} dt, \quad (2.60)$$

for every $\phi \in L^2(0, T; V)$ and pass to the limit as $\varepsilon \rightarrow 0$, using the previous convergencies. We obtain that

$$\int_0^T \left\langle \frac{d(mu)}{dt}(t), \phi(t) \right\rangle_{V', V} dt + \int_Q \nabla \zeta \cdot \nabla \phi dx dt = \int_0^T \langle f(t), \phi(t) \rangle_{V', V} dt, \quad (2.61)$$

where ζ is given by (2.46). Taking now $\phi \in L^2(0, T; H_0^1(\Omega_m))$ we deduce that

$$\int_0^T \left\langle \frac{d(mu)}{dt}(t), \phi(t) \right\rangle_{V', V} dt + \int_{Q_m} \nabla \zeta \cdot \nabla \phi dx dt = \int_0^T \langle f(t), \phi(t) \rangle_{V', V} dt, \quad (2.62)$$

for each $\phi \in L^2(0, T; H_0^1(\Omega_m))$, where $\zeta \in \beta^*(u)$ a.e. on Q_m .

Similarly, taking $\phi \in L^2(0, T; H_0^1(\Omega_0))$ we obtain the weak form of the equation on the subset where $m(x) = 0$,

$$\int_{Q_0} \nabla \zeta \cdot \nabla \phi dx dt = \int_0^T \langle f(t), \phi(t) \rangle_{V', V} dt, \quad \text{for each } \phi \in L^2(0, T; H_0^1(\Omega_0)), \quad (2.63)$$

where ζ is given by (2.46).

The relations (2.61)–(2.63) are the weak forms of the following problems:

$$\begin{aligned} \frac{d(mu)}{dt} - \Delta \zeta &= f \quad \text{in } \Omega \times (0, T), \\ \zeta &= 0 \quad \text{on } \Sigma \end{aligned} \quad (2.64)$$

(obtained also from (2.52)) and

$$\begin{aligned} \frac{d(mu)}{dt} - \Delta \zeta &= f \quad \text{in } \Omega_m \times (0, T), \\ \zeta &= 0 \quad \text{on } \Sigma, \\ -\Delta \zeta &= f \quad \text{in } \Omega_0 \times (0, T). \end{aligned} \quad (2.65)$$

Because $m \in C^1(\overline{\Omega})$ the common boundary of the domains Ω_m and Ω_0 (of equation $m(x) = 0$) is regular. Since $\zeta \in L^2(0, T; V)$, it follows that $\zeta(t)$ restrained to any line $\mathcal{L} \subset \Omega$, crossing the boundary $\partial\Omega_0$, belongs to V , a.e. $t \in (0, T)$, so that it is continuous on \mathcal{L} . Thus if we take $x_0 \in \partial\Omega_0$ then

$$\zeta^+(t) := \lim_{\substack{x \rightarrow x_0 \\ x \in \mathcal{L} \cap \Omega_m}} \zeta(t) = \lim_{\substack{x \rightarrow x_0 \\ x \in \mathcal{L} \cap \Omega_0}} \zeta(t) \quad \text{a.e. } t \in (0, T)$$

which implies that the trace of $\zeta(t) \in \beta^*(u(t))$ is continuous across the boundary $\partial\Omega_0$, a.e. $t \in (0, T)$. We take into account that $\zeta^+ \in \beta^*(u(t))$ a.e. on Q_m , hence ζ turns out to be the solution to the elliptic problem

$$\begin{aligned} -\Delta \zeta(t) &= f(t) \quad \text{in } \Omega_0, \\ \zeta(t) &= \zeta^+(t) \in \beta^*(u(t)) \quad \text{on } \partial\Omega_0, \quad \text{a.e. } t \in (0, T) \end{aligned} \quad (2.66)$$

for a.e. t fixed in $(0, T)$, where u is the solution to (1.13) in Q_m .

Then, we define the function

$$u^*(x, t) := \begin{cases} u(x, t), & \text{if } (x, t) \in Q_m, \\ (\beta^*)^{-1}(\zeta(x, t)), & \text{if } (x, t) \in Q_0 = \Omega_0 \times (0, T), \end{cases} \quad (2.67)$$

where ζ is the solution to (2.66) and show that it is the solution to (1.13) in the sense of Definition 1.1. Indeed, $\zeta(x, t) \in \beta^*(u^*(x, t))$ and $\zeta \in L^2(0, T; V)$, so it follows that $u^*(x, t) = (\beta^*)^{-1}(\zeta(x, t)) \in D(A)$, implying that $u^* \leq u_s$ a.e. on Q_0 . Then, mu^* belongs to the spaces specified in (1.15) (we take into account that $mu^* = 0$ on Q_0). Finally, we have to check that u^* satisfies Eq. (1.16) and this follows by a straightforward computation using (2.61)–(2.63).

This completes the proof of the solution existence.

To prove (2.44) and (2.45) we pass to the limit as $\varepsilon \rightarrow 0$ in (2.10) and (2.11), using the weakly lower semicontinuity property and taking into account (2.53), (2.51), (2.59), (2.67) and (2.56), respectively.

Now we can present the argument for the periodic solution uniqueness. We consider that there exist two solutions u_1^* and u_2^* to (1.12). By (2.45) and (2.53) it follows that $mu_1^* = mu_2^*$ a.e. on Q . Now we subtract Eqs. (2.64) corresponding to u_1^* and u_2^* and get

$$\begin{aligned} -\Delta(\zeta_1 - \zeta_2) &= 0 \quad \text{in } \Omega \times (0, T), \\ \zeta_1 - \zeta_2 &= 0 \quad \text{on } \Sigma, \end{aligned}$$

where $\zeta_1 \in \beta^*(u_1^*)$, $\zeta_2 \in \beta^*(u_2^*)$ a.e. on Q . Hence $\zeta_1 = \zeta_2$ and since $(\beta^*)^{-1}$ is single valued then $u_1^* = u_2^*$ a.e. on Q . \square

3. Longtime behavior of the solution

We extend first the previous result to $t \in \mathbf{R}$ and then establish the solution longtime behavior at large time. We resume problem (1.12) and prove

Theorem 3.1. *Let*

$$m \in C^1(\overline{\Omega}), \quad f \in L^\infty(\mathbf{R}; V'), \quad f(t) = f(t + T), \quad \text{a.e. } t \in \mathbf{R}.$$

Then problem (1.12) has a solution $u \in L^2_{\text{loc}}(\mathbf{R}; V)$ satisfying

$$\begin{aligned} mu &\in C(\mathbf{R}; L^2(\Omega)) \cap W^{1,2}(\mathbf{R}; V'), \\ u(x, t) &\leq u_s, \quad \text{a.e. } (x, t) \in \Omega \times \mathbf{R}, \\ \zeta &\in L^2_{\text{loc}}(\mathbf{R}; V), \quad \text{where } \zeta(x, t) \in \beta^*(u(x, t)), \quad \text{a.e. } (x, t) \in \Omega \times \mathbf{R}. \end{aligned}$$

Proof. We make the transformation $t' = t - T$ and denote $\tilde{u}(x, t') = u(x, t' + T)$ with $t' \in [0, T]$. Using now the periodicity of the function f we find again problem (1.13) which has a periodic solution $\tilde{u}(t')$ belonging to $C([0, T]; L^2(\Omega))$, such that $m(\tilde{u}(0) - \tilde{u}(T)) = 0$. Coming back to the variable t we obtain that (1.12) has a continuous periodic solution on $[T, 2T]$ such that $m(u(T) - u(2T)) = 0$ and the procedure is continued on each time period. \square

Eventually, we shall describe the longtime behavior of the solution to a Cauchy problem

$$\begin{aligned} \frac{\partial(m(x)u)}{\partial t} - \Delta\beta^*(u) &\ni f \quad \text{in } \Omega \times \mathbf{R}, \\ u(x, t) &= 0 \quad \text{on } \Gamma \times \mathbf{R}, \\ m(x)u(x, 0) &= v_0 \quad \text{in } \Omega, \end{aligned} \quad (3.1)$$

where $v_0 \in L^2(\Omega)$ is such that $u_0 := \frac{v_0}{m} \in L^2(\Omega)$, $\frac{v_0}{m}(x) \leq u_s$ a.e. $x \in \Omega$.

By Theorem 1.2, this problem has a unique solution u (obtained as an extension of the solution to (1.22) to $t \in \mathbf{R}$).

Corollary 3.2. *Let*

$$m \in C^1(\overline{\Omega}), \quad f \in L^\infty(\mathbf{R}; V'), \quad f(t) = f(t + T), \quad \text{a.e. } t \in \mathbf{R}.$$

Then, the solution u to the Cauchy problem (3.1) satisfies

$$\lim_{t \rightarrow \infty} \|\sqrt{m}(u(t) - \omega(t))\|_{V'} = 0, \quad (3.2)$$

where ω is the unique periodic solution to (1.12). Moreover,

$$\lim_{t \rightarrow \infty} \|u(t) - \omega(t)\|_{(H^1(\Omega_\delta))'} = 0 \quad (3.3)$$

with Ω_δ defined in (2.57).

Proof. Let ω be the unique solution to (1.12) given by Theorem 3.1 and u the unique solution corresponding to the initial datum v_0 and the same f . We multiply the difference of Eqs. (3.1) and (1.12) by $(u(t) - \omega(t))$ scalarly in V' and get

$$\frac{1}{2} \frac{d}{dt} \|\sqrt{m}(u(t) - \omega(t))\|_{V'}^2 + \int_{\Omega} \nabla(\zeta(t) - \zeta_{\omega}(t)) \cdot \nabla \psi \, dx \leq 0,$$

where $\zeta \in \beta^*(u)$ a.e. on $\Omega \times \mathbf{R}$ and $\zeta_m \in \beta^*(\omega)$ a.e. on $\Omega \times \mathbf{R}$, and ψ is the solution to $-\Delta \psi = u(t) - \omega(t)$, $\psi = 0$ on Γ . Therefore

$$\frac{d}{dt} \|\sqrt{m}(u(t) - \omega(t))\|_{V'}^2 + \rho \|u(t) - \omega(t)\|^2 \leq 0.$$

We have that $\|u(t)\|^2 \geq \|\sqrt{m}u(t)\|^2 \geq \|\sqrt{m}u(t)\|_{V'}^2$, so that

$$\frac{d}{dt} \|\sqrt{m}(u(t) - \omega(t))\|_{V'}^2 + \rho \|\sqrt{m}(u(t) - \omega(t))\|_{V'}^2 \leq 0.$$

From here we obtain

$$\|\sqrt{m}(u(t) - \omega(t))\|_{V'}^2 \leq \exp(-\rho t) \|\sqrt{m}(u_0 - \omega(0))\|_{V'}^2,$$

which implies (3.2).

In particular we shall derive the solution longtime behavior on $\Omega_{\delta} \times \mathbf{R}$ with Ω_{δ} defined in (2.57). We compute

$$\|\sqrt{m}(u(t) - \omega(t))\|_{V'}^2 = \langle \sqrt{m}(u(t) - \omega(t)), \phi(t) \rangle_{V', V} = \|\phi(t)\|_V^2,$$

where $\phi(t) \in V$ is the solution to the boundary value problem

$$-\Delta \phi(t) = \sqrt{m}(u(t) - \omega(t)) \quad \text{in } \Omega, \quad \phi(t) = 0 \quad \text{on } \Gamma, \quad \text{a.e. } t \in \mathbf{R}.$$

Hence $\phi(t) \rightarrow 0$ as $t \rightarrow \infty$ in $H_0^1(\Omega)$ implying that $\phi(t) \rightarrow 0$ in $H^1(\Omega_{\delta})$ for any $\delta > 0$. Thus

$$u(t) - \omega(t) = -\frac{1}{\sqrt{m}} \Delta \phi(t) \rightarrow 0 \quad \text{in } (H^1(\Omega_{\delta}))' \text{ as } t \rightarrow \infty, \text{ for any } \delta > 0,$$

which is equivalent to (3.3). \square

Theorem 3.3. *Let*

$$m \in C^1(\bar{Q}), \quad f \in L^{\infty}(\mathbf{R}; V'), \quad f(t) = f(t + T), \quad \text{a.e. } t \in \mathbf{R}.$$

Then (1.3) has a unique solution.

Proof. We have just proved that (1.12) has a unique periodic solution, i.e., $m(x)u(x, t)$ is periodic. Then, $\frac{d}{dt}(mu)$ is periodic too and by Eq. (1.3) it follows that $\Delta \zeta(x, t)$ is periodic, where $\zeta \in \beta^*(u(x, t))$ a.e. $(x, t) \in Q$. The inverse of the operator $-\Delta : V \rightarrow V'$ is periodic (because it is single valued and continuous) hence $\beta^*(u(x, t)) = \beta^*(u(x, t + T))$. Now, applying the inverse of β^* which is single valued we get that $u(x, t) = u(x, t + T)$ as claimed. \square

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