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# On fractional maximal function and fractional integrals associated with the Dunkl operator on the real line

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ABSTRACT

In this paper we obtain necessary and sufficient conditions on the parameters for the boundedness of the Dunkl-type fractional maximal operator $M_\beta$, and the Dunkl-type fractional integral operator $I_\beta$ from the spaces $L_{p,\alpha}(\mathbb{R})$ to the spaces $L_{q,\alpha}(\mathbb{R})$, $1 < p < q < \infty$, and from the spaces $L_{1,\alpha}(\mathbb{R})$ to the weak spaces $WL_{q,\alpha}(\mathbb{R})$, $1 < q < \infty$. In the case $p = \frac{2\alpha+2}{\beta}$, we prove that the operator $M_\beta$ is bounded from the space $L_{p,\alpha}(\mathbb{R})$ to the space $L_{\infty,\alpha}(\mathbb{R})$, and the Dunkl-type modified fractional integral operator $\widetilde{I}_\beta$ is bounded from the space $L_{p,\alpha}(\mathbb{R})$ to the Dunkl-type BMO space $BMO_\alpha(\mathbb{R})$. By this results we get boundedness of the operators $M_\beta$ and $I_\beta$ from the Dunkl-type Besov spaces $B^s_{p\theta,\alpha}(\mathbb{R})$ to the spaces $B^s_{q\theta,\alpha}(\mathbb{R})$, $1 < p < q < \infty$, $1/p - 1/q = \beta/(2\alpha+2)$, $1 \leqslant \theta \leqslant \infty$ and $0 < s < 1$.


## 1. Introduction

The Hardy–Littlewood maximal function, fractional maximal function and fractional integrals are important technical tools in harmonic analysis, theory of functions and partial differential equations. On the real line, the Dunkl operators are differential-difference operators associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. In the works [1,6,8,13] the maximal operator associated with the Dunkl operator on $\mathbb{R}$ were studied. In this work, we study the fractional maximal function (Dunkl-type fractional maximal function) and the fractional integral (Dunkl-type fractional integral) associated with the Dunkl operator on $\mathbb{R}$. We obtain the necessary and sufficient conditions for the boundedness of the Dunkl-type fractional maximal operator, and the Dunkl-type fractional integral operator from the spaces $L_{p,\alpha}(\mathbb{R})$ to the spaces $L_{q,\alpha}(\mathbb{R})$, $1 < p < q < \infty$, and from the spaces $L_{1,\alpha}(\mathbb{R})$ to the weak spaces $WL_{q,\alpha}(\mathbb{R})$, $1 < q < \infty$.

The paper is organized as follows. In Section 2, we give our main results on the boundedness of the Dunkl-type fractional maximal operator and Dunkl-type fractional integral operator. In Section 3, we present some definitions and auxiliary results. The main result of the paper is the inequality of Hardy–Littlewood–Sobolev type for the Dunkl-type fractional integral, established in Section 4. We prove that the Dunkl-type fractional maximal operator $M_\beta$ and Dunkl-type fractional integral operator $I_\beta$ are bounded from the spaces $L_{p,\alpha}(\mathbb{R})$ to the spaces $L_{q,\alpha}(\mathbb{R})$ and from the spaces $L_{1,\alpha}(\mathbb{R})$ to the weak spaces $WL_{q,\alpha}(\mathbb{R})$. We show that the conditions on the parameters ensuring the boundedness cannot be weakened. In the limiting case $p = \frac{2\alpha+2}{\beta}$ we also prove that the operator $M_\beta$ is bounded from the space $L_{p,\alpha}(\mathbb{R})$ to the space $L_{\infty,\alpha}(\mathbb{R})$,

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and the Dunkl-type modified fractional integral operator $\widetilde{I}_\beta$ is bounded from the space $L_{p,\alpha}(\mathbb{R})$ to the Dunkl-type BMO space $BMO_\alpha(\mathbb{R})$. As applications of these results, we prove that the operators $M_\beta$ and $I_\beta$ are bounded from the Dunkl-type Besov spaces $B^s_{p\theta,\alpha}(\mathbb{R})$ to $B^s_{q\theta,\alpha}(\mathbb{R})$ for $1<p<q<\infty$, $1/p-1/q=\beta/(2\alpha+2)$, $1\leqslant\theta\leqslant\infty$ and $0<s<1$.

## 2. Main results

Let $\alpha>-1/2$ be a fixed number and $\mu_\alpha$ be the weighted Lebesgue measure on $\mathbb{R}$, given by

$$d\mu_\alpha(x) := \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^{-1}|x|^{2\alpha+1}\,dx.$$

For every $1\leqslant p\leqslant\infty$, we denote by $L_{p,\alpha}(\mathbb{R})=L_p(\mathbb{R},d\mu_\alpha)$ the spaces of complex-valued functions $f$, measurable on $\mathbb{R}$ such that

$$\|f\|_{p,\alpha}\equiv\|f\|_{L_{p,\alpha}}=\left(\int_{\mathbb{R}}\left|f(x)\right|^p d\mu_\alpha(x)\right)^{1/p}<\infty\quad\text{if } p\in[1,\infty),$$

and

$$\|f\|_{\infty,\alpha}\equiv\|f\|_{L_\infty}=\operatorname*{ess\,sup}_{x\in\mathbb{R}}\left|f(x)\right|\quad\text{if } p=\infty.$$

For $1\leqslant p<\infty$ we denote by $WL_{p,\alpha}(\mathbb{R})$, the weak $L_{p,\alpha}(\mathbb{R})$ spaces defined as the set of locally integrable functions $f$ with the finite norm

$$\|f\|_{WL_{p,\alpha}}=\sup_{r>0} r\left(\mu_\alpha\left\{x\in\mathbb{R}:\ \left|f(x)\right|>r\right\}\right)^{1/p}.$$

Note that

$$L_{p,\alpha}\subset WL_{p,\alpha}\quad\text{and}\quad\|f\|_{WL_{p,\alpha}}\leqslant\|f\|_{p,\alpha}\quad\text{for all } f\in L_{p,\alpha}(\mathbb{R}).$$

Let $B(x,t)=\{y\in\mathbb{R}:\ |y|\in\,]\max\{0,|x|-t\},|x|+t[\}$ and $B_t\equiv B(0,t)=\,]-t,t[$, $t>0$. Then

$$\mu_\alpha B_t=b_\alpha t^{2\alpha+2},$$

where $b_\alpha=[2^{\alpha+1}(\alpha+1)\Gamma(\alpha+1)]^{-1}$.

We denote by $BMO_\alpha(\mathbb{R})$ (Dunkl-type BMO space) the set of locally integrable functions $f$ with finite norm (see [5])

$$\|f\|_{*,\alpha}=\sup_{r>0,\,x\in\mathbb{R}}\frac{1}{\mu_\alpha B_r}\int_{B_r}\left|\tau_x f(y)-f_{B_r}(x)\right|d\mu_\alpha(y)<\infty,$$

where

$$f_{B_r}(x)=\frac{1}{\mu_\alpha B_r}\int_{B_r}\tau_x f(y)\,d\mu_\alpha(y).$$

For all $x,y,z\in\mathbb{R}$, we put

$$W_\alpha(x,y,z)=(1-\sigma_{x,y,z}+\sigma_{z,x,y}+\sigma_{z,y,x})\Delta_\alpha(x,y,z)$$

where

$$\sigma_{x,y,z}=\begin{cases}\frac{x^2+y^2-z^2}{2xy}, & \text{if } x,y\in\mathbb{R}\setminus\{0\},\\ 0, & \text{otherwise}\end{cases}$$

and $\Delta_\alpha$ is the Bessel kernel given by

$$\Delta_\alpha(x,y,z)=\begin{cases}d_\alpha\frac{([(|x|+|y|)^2-z^2][z^2-(|x|-|y|)^2])^{\alpha-1/2}}{|xyz|^{2\alpha}}, & \text{if } |z|\in A_{x,y},\\ 0, & \text{otherwise,}\end{cases}$$

where $d_\alpha=(\Gamma(\alpha+1))^2/(2^{\alpha-1}\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2}))$ and $A_{x,y}=[||x|-|y||,|x|+|y|]$.

**Properties 1.** *(See Rösler [14].) The signed kernel $W_\alpha$ is even with respect to all variables and satisfies the following properties*

$$W_\alpha(x,y,z)=W_\alpha(y,x,z)=W_\alpha(-x,z,y),$$

$$W_\alpha(x,y,z)=W_\alpha(-z,y,-x)=W_\alpha(-x,-y,-z)$$

*and*

$$\int_{\mathbb{R}}\left|W_\alpha(x,y,z)\right|d\mu_\alpha(z)\leqslant 4.$$

In the sequel we consider the signed measure $\nu_{x,y}$, on $\mathbb{R}$, given by

$$\nu_{x,y} = \begin{cases} W_\alpha(x,y,z)\,d\mu_\alpha(z), & \text{if } x,y \in \mathbb{R}\setminus\{0\},\\ d\delta_x(z), & \text{if } y=0,\\ d\delta_y(z), & \text{if } x=0. \end{cases}$$

**Definition 1.** For $x, y \in \mathbb{R}$ and $f$ a continuous function on $\mathbb{R}$, we put

$$\tau_x f(y) = \int_{\mathbb{R}} f(z)\,d\nu_{x,y}(z).$$

The operators $\tau_x$, $x \in \mathbb{R}$, are called Dunkl translation operators on $\mathbb{R}$ and it can be expressed in the following form (see [14])

$$\tau_x f(y) = c_\alpha \int_0^\pi f_e\big((x,y)_\theta\big) h_1(x,y,\theta)(\sin\theta)^{2\alpha}\,d\theta + c_\alpha \int_0^\pi f_o\big((x,y)_\theta\big) h_2(x,y,\theta)(\sin\theta)^{2\alpha}\,d\theta,$$

where $(x,y)_\theta = \sqrt{x^2+y^2-2|xy|\cos\theta}$, $f = f_e + f_o$, $f_o$ and $f_e$ being respectively the odd and the even parts of $f$, with

$$c_\alpha \equiv \left(\int_0^\pi (\sin\theta)^{2\alpha}\,d\theta\right)^{-1} = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)},$$

$$h_1(x,y,\theta) = 1 - sgn(xy)\cos\theta$$

and

$$h_2(x,y,\theta) = \begin{cases} \frac{(x+y)[1-sgn(xy)\cos\theta]}{(x,y)_\theta}, & \text{if } xy \neq 0,\\ 0, & \text{if } xy = 0. \end{cases}$$

Using the change of variable $z = (x,y)_\theta$, we have also (see [3])

$$\tau_x f(y) = c_\alpha \int_0^\pi \left\{ f\big((x,y)_\theta\big) + f\big(-(x,y)_\theta\big) + \frac{x+y}{(x,y)_\theta}\big[f\big((x,y)_\theta\big) - f\big(-(x,y)_\theta\big)\big]\right\}(1-\cos\theta)(\sin\theta)^{2\alpha}\,d\theta.$$

Now we define the Dunkl-type fractional maximal function by

$$M_\beta f(x) = \sup_{r>0} (\mu_\alpha B_r)^{\frac{\beta}{2\alpha+2}-1} \int_{B_r} \tau_x |f|(y)\,d\mu_\alpha(y), \quad 0 \leqslant \beta < 2\alpha+2,$$

the Dunkl-type fractional integral by

$$I_\beta f(x) = \int_{\mathbb{R}} \tau_x |y|^{\beta-2\alpha-2} f(y)\,d\mu_\alpha(y), \quad 0 < \beta < 2\alpha+2,$$

and the Dunkl-type modified fractional integral by

$$\widetilde{I}_\beta f(x) = \int_{\mathbb{R}} \big(\tau_x |y|^{\beta-2\alpha-2} - |y|^{\beta-2\alpha-2}\chi_{\complement B(0,1)}\big) f(y)\,d\mu_\alpha(y), \quad 0 < \beta < 2\alpha+2.$$

If $\beta = 0$, then $M \equiv M_0$ is the Hardy–Littlewood maximal operator associated with the Dunkl operator (see [1,6,8,13]).

**Theorem 1.** *(See [1,8,13].)*

(1) *If $f \in L_{1,\alpha}(\mathbb{R})$, then for every $\beta > 0$*

$$\mu_\alpha\big\{x \in \mathbb{R}\colon\ Mf(x) > \beta\big\} \leqslant \frac{C_1}{\beta}\int_{\mathbb{R}} \big|f(x)\big|\,d\mu_\alpha(x),$$

*where $C_1 > 0$ is independent of $f$.*

(2) *If $f \in L_{p,\alpha}(\mathbb{R})$, $1 < p \leqslant \infty$, then $Mf \in L_{p,\alpha}(\mathbb{R})$ and*

$$\|Mf\|_{p,\alpha} \leqslant C_2 \|f\|_{p,\alpha},$$

*where $C_2 > 0$ is independent of $f$.*

**Corollary 1.** *If $f \in L^{loc}_{1,\alpha}(\mathbb{R})$, then*

$$\lim_{r\to 0} \frac{1}{\mu_\alpha B_r} \int_{B_r} \left|\tau_x f(y) - f(x)\right| d\mu_\alpha(y) = 0$$

*for a.e. $x \in \mathbb{R}$.*

**Corollary 2.** *If $f \in L^{loc}_{1,\alpha}(\mathbb{R})$, then*

$$\lim_{r\to 0} \frac{1}{\mu_\alpha B_r} \int_{B_r} \tau_x f(y)\, d\mu_\alpha(y) = f(x)$$

*for a.e. $x \in \mathbb{R}$.*

The following theorem is our main result in which we obtain the necessary and sufficient conditions for the Dunkl-type fractional integral operator $I_\beta$ to be bounded from the spaces $L_{p,\alpha}(\mathbb{R})$ to $L_{q,\alpha}(\mathbb{R})$, $1 < p < q < \infty$, and from the spaces $L_{1,\alpha}(\mathbb{R})$ to the weak spaces $WL_{q,\alpha}(\mathbb{R})$, $1 < q < \infty$.

**Theorem 2.** *Let $0 < \beta < 2\alpha + 2$ and $1 \leqslant p \leqslant \frac{2\alpha+2}{\beta}$.*

(1) *If $p = 1$, then the condition $1 - \frac{1}{q} = \frac{\beta}{2\alpha+2}$ is necessary and sufficient for the boundedness of $I_\beta$ from $L_{1,\alpha}(\mathbb{R})$ to $WL_{q,\alpha}(\mathbb{R})$.*
(2) *If $1 < p < \frac{2\alpha+2}{\beta}$, then the condition $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{2\alpha+2}$ is necessary and sufficient for the boundedness of $I_\beta$ from $L_{p,\alpha}(\mathbb{R})$ to $L_{q,\alpha}(\mathbb{R})$.*
(3) *If $p = \frac{2\alpha+2}{\beta}$, then the operator $\widetilde{I}_\beta$ is bounded from $L_{p,\alpha}(\mathbb{R})$ to $BMO_\alpha(\mathbb{R})$.*

Recall that, for $0 < \beta < 2\alpha + 2$, the following inequality hold

$$M_\beta f(x) \leqslant b_\alpha^{1-\frac{\beta}{2\alpha+2}} I_\beta\big(|f|\big)(x).$$

Hence the boundedness of the Dunkl-type fractional integral operator $I_\beta$ implies the boundedness of the Dunkl-type fractional maximal operator $M_\beta$.

**Corollary 3.** *Let $0 < \beta < 2\alpha + 2$ and $1 \leqslant p \leqslant \frac{2\alpha+2}{\beta}$.*

(1) *If $1 < p < \frac{2\alpha+2}{\beta}$, then the condition $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{2\alpha+2}$ is necessary and sufficient for the boundedness of $M_\beta$ from $L_{p,\alpha}(\mathbb{R})$ to $L_{q,\alpha}(\mathbb{R})$.*
(2) *If $p = 1$, then the condition $1 - \frac{1}{q} = \frac{\beta}{2\alpha+2}$ is necessary and sufficient for the boundedness of $M_\beta$ from $L_{1,\alpha}(\mathbb{R})$ to $WL_{q,\alpha}(\mathbb{R})$.*
(3) *If $p = \frac{2\alpha+2}{\beta}$, then $M_\beta$ is bounded from $L_{p,\alpha}(\mathbb{R})$ to $L_\infty(\mathbb{R})$.*

For $1 \leqslant p, \theta \leqslant \infty$ and $0 < s < 1$, the Besov space for the Dunkl operators on $\mathbb{R}$ (Besov–Dunkl space) $B^s_{p\theta,\alpha}(\mathbb{R})$ consists of all functions $f$ in $L_{p,\alpha}(\mathbb{R})$ so that

$$\|f\|_{B^s_{p\theta,\alpha}} = \|f\|_{p,\alpha} + \left(\int_{\mathbb{R}} \frac{\|\tau_x f(\cdot) - f(\cdot)\|^\theta_{p,\alpha}}{|x|^{2\alpha+2+s\theta}}\, d\mu_\alpha(x)\right)^{1/\theta} < \infty. \tag{1}$$

Besov spaces in the setting of the Dunkl operators studied by C. Abdelkefi and M. Sifi [2,3], R. Bouguila, M.N. Lazhari and M. Assal [4], L. Kamoun [10] and Y.Y. Mammadov [9]. In the following theorem we prove the boundedness of the Dunkl-type fractional integral operator $I_\beta$ in the Dunkl-type Besov spaces.

**Theorem 3.** *For $1 < p < q < \infty$, $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{2\alpha+2}$, $1 \leqslant \theta \leqslant \infty$ and $0 < s < 1$ the Dunkl-type fractional integral operator $I_\beta$ is bounded from $B^s_{p\theta,\alpha}(\mathbb{R})$ to $B^s_{q\theta,\alpha}(\mathbb{R})$. More precisely, there is a constant $C > 0$ such that*

$$\|I_\beta f\|_{B^s_{q\theta,\alpha}} \leqslant C\|f\|_{B^s_{p\theta,\alpha}}$$

*hold for all $f \in B^s_{p\theta,\alpha}(\mathbb{R})$.*

**Corollary 4.** *For $1 < p \leqslant q < \infty$, $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{2\alpha+2}$, $1 \leqslant \theta \leqslant \infty$ and $0 < s < 1$ the Dunkl-type fractional maximal operator $M_\beta$ is bounded from $B^s_{p\theta,\alpha}(\mathbb{R})$ to $B^s_{q\theta,\alpha}(\mathbb{R})$. More precisely, there is a constant $C > 0$ such that*

$$\|M_\beta f\|_{B^s_{q\theta,\alpha}} \leqslant C\|f\|_{B^s_{p\theta,\alpha}}$$

*hold for all $f \in B^s_{p\theta,\alpha}(\mathbb{R})$.*

**Remark 1.** Note that Corollary 4 in the case $\beta = 0$ was proved in [9].

## 3. Preliminaries

For a real parameter $\alpha \geqslant -1/2$, we consider the Dunkl operator, associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$:

$$\Lambda_\alpha(f)(x) = \frac{d}{dx} f(x) + \frac{2\alpha+1}{x}\left(\frac{f(x)-f(-x)}{2}\right). \tag{2}$$

Note that $\Lambda_{-1/2} = d/dx$.

For $\alpha \geqslant -1/2$ and $\lambda \in \mathbb{C}$, the initial value problem:

$$\Lambda_\alpha(f)(x) = \lambda f(x), \quad f(0) = 1, \quad x \in \mathbb{R},$$

has a unique solution $E_\alpha(\lambda x)$ called Dunkl kernel [7,11,15] and given by

$$E_\alpha(\lambda x) = j_\alpha(i\lambda x) + \frac{\lambda x}{2(\alpha+1)} j_{\alpha+1}(i\lambda x), \quad x \in \mathbb{R},$$

where $j_\alpha$ is the normalized Bessel function of the first kind and order $\alpha$ [16], defined by

$$j_\alpha(z) = 2^\alpha \Gamma(\alpha+1)\frac{J_\alpha(z)}{z^\alpha} = \Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n (z/2)^{2n}}{n!\,\Gamma(n+\alpha+1)}, \quad z \in \mathbb{C}.$$

We can write for $x \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ (see Rösler [14, p. 295])

$$E_\alpha(-i\lambda x) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)}\int_{-1}^{1}\left(1-t^2\right)^{\alpha-1/2}(1-t)e^{i\lambda x t}\,dt.$$

Note that $E_{-1/2}(\lambda x) = e^{\lambda x}$.

The Dunkl transform $\mathcal{F}_\alpha$ of a function $f \in L_{1,\alpha}(\mathbb{R})$, is given by

$$\mathcal{F}_\alpha f(\lambda) := \int_{\mathbb{R}} E_\alpha(-i\lambda x) f(x)\,d\mu_\alpha(x), \quad \lambda \in \mathbb{R}.$$

Here the integral makes sense since $|E_\alpha(ix)| \leqslant 1$ for every $x \in \mathbb{R}$ [14, p. 295].

Note that $\mathcal{F}_{-1/2}$ agrees with the classical Fourier transform $\mathcal{F}$, given by

$$\mathcal{F} f(\lambda) := (2\pi)^{-1/2}\int_{\mathbb{R}} e^{-i\lambda x} f(x)\,dx, \quad \lambda \in \mathbb{R}.$$

**Proposition 1.** *(See Soltani [12].)*

(i) *If $f$ is an even positive continuous function, then $\tau_x f$ is positive.*

(ii) *For all $x \in \mathbb{R}$ the operator $\tau_x$ extends to $L_{p,\alpha}(\mathbb{R})$, $p \geqslant 1$, and we have for $f \in L_{p,\alpha}(\mathbb{R})$,*

$$\|\tau_x f\|_{p,\alpha} \leqslant 4\|f\|_{p,\alpha}. \tag{3}$$

(iii) *For all $x, \lambda \in \mathbb{R}$ and $f \in L_{1,\alpha}(\mathbb{R})$, we have*

$$\mathcal{F}_\alpha(\tau_x f)(\lambda) = E_\alpha(i\lambda x)\mathcal{F}_\alpha f(\lambda).$$

Let $f$ and $g$ be two continuous functions on $\mathbb{R}$ with compact support. We define the generalized convolution $*_\alpha$ of $f$ and $g$ by

$$f *_\alpha g(x) := \int_{\mathbb{R}} \tau_x f(-y) g(y)\,d\mu_\alpha(y), \quad x \in \mathbb{R}.$$

The generalized convolution $*_\alpha$ is associative and commutative [14]. Note that $*_{-1/2}$ agrees with the standard convolution $*$.

**Proposition 2.** *(See Soltani [12].)*

(i) *If $f$ is an even positive function and $g$ a positive function with compact support, then $f *_\alpha g$ is positive.*

(ii) *Assume that $p, q, r \in [1, +\infty[$ satisfying $1/p + 1/q = 1 + 1/r$ (the Young condition). Then the map $(f, g) \mapsto f *_\alpha g$, defined on $\mathcal{E}_c \times \mathcal{E}_c$, extends to a continuous map from $L_{p,\alpha}(\mathbb{R}) \times L_{q,\alpha}(\mathbb{R})$ to $L_{r,\alpha}(\mathbb{R})$, and we have*

$$\|f *_\alpha g\|_{r,\alpha} \leqslant 4\|f\|_{p,\alpha}\|g\|_{q,\alpha}.$$

(iii) *For all* $f \in L_{1,\alpha}(\mathbb{R})$ *and* $g \in L_{2,\alpha}(\mathbb{R})$, *we have*

$$\mathcal{F}_\alpha(f *_\alpha g) = (\mathcal{F}_\alpha f)(\mathcal{F}_\alpha g).$$

We need the following lemma.

**Lemma 1.** *Let* $0 < \beta < 2\alpha + 2$. *Then for* $2|x| \leqslant |y|$ *the following inequality is valid*

$$\left|\tau_y |x|^{\beta-2\alpha-2} - |y|^{\beta-2\alpha-2}\right| \leqslant 2^{2\alpha+4-\beta} |y|^{\beta-2\alpha-3} |x|. \tag{4}$$

**Proof.** We will show that

$$\left|\tau_y |x|^{\beta-2\alpha-2} - |y|^{\beta-2\alpha-2}\right| \leqslant 2C_\alpha \int_0^\pi \left|\left|(x,y)_\theta\right|^{\beta-2\alpha-2} - |y|^{\beta-2\alpha-2}\right| \sin^{2\alpha}\theta \, d\theta.$$

First estimate

$$\left|\left|(x,y)_\theta\right|^{\beta-2\alpha-2} - |y|^{\beta-2\alpha-2}\right|.$$

By the mean value theorem we get

$$\left|\left|(x,y)_\theta\right|^{\beta-2\alpha-2} - |y|^{\beta-2\alpha-2}\right| \leqslant \left|\left|(x,y)_\theta\right| - |y|\right| \cdot \xi^{\beta-2\alpha-3},$$

where $\min\{|(x,y)_\theta|, |y|\} \leqslant \xi \leqslant \max\{|(x,y)_\theta|, |y|\}$.

Note that we have

$$\left|(x,y)_\theta\right| \leqslant |x| + |y| \leqslant \frac{3}{2}|y|, \qquad \left|(x,y)_\theta\right| \geqslant |x-y| \geqslant |y| - |x| \geqslant \frac{1}{2}|y|$$

and

$$\left|(x,y)_\theta\right| - |y| \leqslant |x|, \qquad |y| - \left|(x,y)_\theta\right| \leqslant |y| - |x-y| \leqslant |x|.$$

Hence we get $\frac{1}{2}|y| \leqslant |(x,y)_\theta| \leqslant \frac{3}{2}|y|$ and $||(x,y)_\theta| - |y|| \leqslant |x|$.

Thus we obtain (4). □

## 4. Hardy–Littlewood–Sobolev theorem for the Dunkl-type fractional integral

It is easy to show that if $p \geqslant \frac{2\alpha+2}{\beta}$, then $I_\beta$ is not defined for some functions $f \in L_{p,\alpha}(\mathbb{R})$.

**Example 1.** Let $x \in \mathbb{R}$, $0 < \beta < 2\alpha + 2$, $f(x) = \frac{1}{|x|^\beta \ln|x|} \chi_{\complement B_2}(x)$, where $\complement B_r = \mathbb{R} \setminus B_r$, $r > 0$ and $\chi_E$ is the characteristic function of the set $E$. For $p = \frac{2\alpha+2}{\beta}$, we have $f \in L_{p,\alpha}(\mathbb{R})$ and $I_\beta f(x) = +\infty$.

**Example 2.** Let $x \in \mathbb{R}$, $0 < \beta < 2\alpha + 2$, $f(x) = |x|^{-\beta} \chi_{\complement B_2}(x)$. For $p > \frac{2\alpha+2}{\beta}$, we have $f \in L_{p,\alpha}(\mathbb{R})$ and $I_\beta f(x) = +\infty$.

For the Dunkl-type fractional integral the following analogue of Hardy–Littlewood–Sobolev theorem is valid.

**Theorem 4.** *Let* $0 < \beta < 2\alpha + 2$, $f \in L_{p,\alpha}(\mathbb{R})$ *and* $1 \leqslant p \leqslant \frac{2\alpha+2}{\beta}$.

(1) *If* $1 < p < \frac{2\alpha+2}{\beta}$ *and* $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{2\alpha+2}$, *then* $I_\beta f \in L_{q,\alpha}(\mathbb{R})$ *and*

$$\|I_\beta f\|_{q,\alpha} \leqslant C_{pq} \|f\|_{p,\alpha}, \tag{5}$$

*where* $C_{pq} = 2(C_4)^{1-p/q}(C_2 C_3)^{p/q}$, $C_3 = 2^{3\alpha+3}(\alpha+1)\Gamma(\alpha+1)/(2^\beta - 1)$, $C_4 = 4(2^{\alpha+1}(\alpha+1)\Gamma(\alpha+1)q/p')^{1/p'}$.

(2) *If* $p = 1$ *and* $1 - \frac{1}{q} = \frac{\beta}{2\alpha+2}$, *then* $I_\beta f \in WL_{q,\alpha}(\mathbb{R})$ *and*

$$\|I_\beta f\|_{WL_{q,\alpha}} \leqslant C_{1q} \|f\|_{1,\alpha}, \tag{6}$$

*where* $C_{1q} = 2(C_1 C_3)^{1/q}$.

(3) *If* $p = \frac{2\alpha+2}{\beta}$, *then* $\widetilde{I}_\beta f \in BMO_\alpha(\mathbb{R})$ *and*

$$\|\widetilde{I}_\beta f\|_{*,\alpha} \leqslant C \|f\|_{p,\alpha}, \tag{7}$$

*where* $C$ *is independent of* $f$.

*Moreover, if the integral $I_\beta f$ exists almost everywhere, then $I_\beta \in BMO_\alpha(\mathbb{R})$ and the following inequality is valid*

$$\|I_\beta f\|_{*,\alpha} \leqslant C\|f\|_{p,\alpha},$$

*where $C > 0$ is independent of $f$.*

**Proof.** (1) Let $f \in L_{p,\alpha}(\mathbb{R})$, $1 < p < \frac{2\alpha+2}{\beta}$. Then we write

$$I_\beta f(x) = \left(\int\limits_{B_r} + \int\limits_{\complement_{B_r}}\right)\tau_x f(y)|y|^{\beta-2\alpha-2}\,d\mu_\alpha(y) = A(x) + B(x).$$

By taking sum with respect to all integer $k > 0$, we get

$$\begin{aligned}
\bigl|A(x)\bigr| &\leqslant \int\limits_{B_r} \tau_x\bigl|f(y)\bigr||y|^{\beta-2\alpha-2}\,d\mu_\alpha(y) \\
&= \sum_{k=1}^{\infty} \int\limits_{B_{2^{-k+1}r}\setminus B_{2^{-k}r}} \tau_x\bigl|f(y)\bigr||y|^{\beta-2\alpha-2}\,d\mu_\alpha(y) \\
&\leqslant \sum_{k=1}^{\infty}\bigl(2^{-k}r\bigr)^{\beta-2\alpha-2} \int\limits_{B_{2^{-k+1}r}\setminus B_{2^{-k}r}} \tau_x\bigl|f(y)\bigr|\,d\mu_\alpha(y) \\
&\leqslant b_\alpha r^\beta Mf(x)\sum_{k=1}^{\infty}\bigl(2^{-k}\bigr)^{\beta-2\alpha-2}\bigl(2^{-k+1}\bigr)^{2\alpha+2} \\
&= 4b_\alpha r^\beta Mf(x)\sum_{k=1}^{\infty}2^{-k\beta} = C_3 r^\beta Mf(x).
\end{aligned}$$

Therefore it follows that

$$\bigl|A(x)\bigr| \leqslant C_3 r^\beta Mf(x). \tag{8}$$

By Hölder's inequality and the inequality (3) we have

$$\begin{aligned}
\bigl|B(x)\bigr| &\leqslant \bigl\|\tau_x|f|\bigr\|_{p,\alpha}\left(\int\limits_{\complement_{B_r}} |y|^{(\beta-2\alpha-2)p'}\,d\mu_\alpha(y)\right)^{1/p'} \\
&\leqslant 4\|f\|_{p,\alpha}\left(\int\limits_{\complement_{B_r}} |y|^{(\beta-2\alpha-2)p'}\,d\mu_\alpha(y)\right)^{1/p'} = C_4 r^{-(2\alpha+2)/q}\|f\|_{p,\alpha}.
\end{aligned}$$

Consequently, we get

$$\bigl|B(x)\bigr| \leqslant C_4 r^{-(2\alpha+2)/q}\|f\|_{p,\alpha}. \tag{9}$$

Thus, from the inequalities (8) and (9), we have

$$\bigl|I_\beta f(x)\bigr| \leqslant C_3 r^\beta Mf(x) + C_4 r^{-(2\alpha+2)/q}\|f\|_{p,\alpha}.$$

The minimum value of the right-hand side is attained at

$$r = \bigl[\bigl(C_3 Mf(x)\bigr)^{-1}C_4\|f\|_{p,\alpha}\bigr]^{p/(2\alpha+2)},$$

and hence

$$\bigl|I_\beta f(x)\bigr| \leqslant 2\bigl(C_3 Mf(x)\bigr)^{p/q}\bigl(C_4\|f\|_{p,\alpha}\bigr)^{1-p/q}.$$

By Theorem 1, we have

$$\begin{aligned}
\int\limits_{\mathbb{R}} \bigl|I_\beta f(y)\bigr|^q\,d\mu_\alpha(y) &\leqslant 2^q\bigl(C_4\|f\|_{p,\alpha}\bigr)^{q-p}\int\limits_{\mathbb{R}}\bigl(C_3 Mf(y)\bigr)^p\,d\mu_\alpha(y) \\
&\leqslant 2^q(C_4)^{q-p}(C_2C_3)^p\|f\|_{p,\alpha}^q.
\end{aligned}$$

Then we get

$$\|I_\beta f\|_{q,\alpha} \leqslant 2(C_4)^{1-p/q}(C_2C_3)^{p/q}\|f\|_{p,\alpha}.$$

(2) Let $f \in L_{1,\alpha}(\mathbb{R})$. We have

$$\mu_\alpha\{x\in\mathbb{R}\colon |I_\beta f(x)| > 2\lambda\} \leqslant \mu_\alpha\{x\in\mathbb{R}\colon |A(x)| > \lambda\} + \mu_\alpha\{x\in\mathbb{R}\colon |B(x)| > \lambda\}.$$

Taking into account the inequality (8) and applying Theorem 1 we have

$$\begin{aligned}\lambda\mu_\alpha\{x\in\mathbb{R}\colon |A(x)| > \lambda\} &\leqslant \lambda \int_{\{x\in\mathbb{R}\colon C_3 r^\beta Mf(x)>\lambda\}} d\mu_\alpha(x)\\ &= \lambda\mu_\alpha\left\{x\in\mathbb{R}\colon Mf(x) > \frac{\lambda}{C_3 r^\beta}\right\}\\ &\leqslant C_1C_3 r^\beta\|f\|_{1,\alpha}\end{aligned}$$

and

$$\begin{aligned}|B(x)| &\leqslant \int_{\complement_{B_r}} \tau_x|f(y)||y|^{\beta-2\alpha-2}\,d\mu_\alpha(y)\\ &\leqslant r^{\beta-2\alpha-2}\int_{\complement_{B_r}} \tau_x|f(y)|\,d\mu_\alpha(y)\\ &\leqslant 4r^{-\frac{2\alpha+2}{q}}\int_{\mathbb{R}} |f(x)|\,d\mu_\alpha(x) = 4r^{-\frac{2\alpha+2}{q}}\|f\|_{1,\alpha}.\end{aligned}$$

If $4r^{-\frac{2\alpha+2}{q}}\|f\|_{1,\alpha} = \lambda$, then $|B(x)| \leqslant \lambda$, and hence

$$\mu_\alpha\{x\in\mathbb{R}\colon |B(x)| > \lambda\} = 0.$$

Then we get

$$\begin{aligned}\mu_\alpha\{x\in\mathbb{R}\colon |I_\beta f(x)| > 2\lambda\} &\leqslant \mu_\alpha\{x\in\mathbb{R}\colon |A(x)| > \lambda\} + \mu_\alpha\{x\in\mathbb{R}\colon |B(x)| > \lambda\}\\ &\leqslant \frac{C_1C_3}{\lambda} r^\beta\|f\|_{1,\alpha} = C_1C_3 r^{\beta+\frac{2\alpha+2}{q}}\\ &= C_1C_3 r^{2\alpha+2} = C_1C_3\lambda^{-q}\|f\|_{1,\alpha}^q = \frac{C_1C_3}{\lambda^q}\|f\|_{1,\alpha}^q\end{aligned}$$

and hence

$$\|I_\beta f\|_{WL_{q,\alpha}} \leqslant 2(C_1C_3)^{1/q}\|f\|_{1,\alpha}.$$

(3) Let $f \in L_{p,\alpha}(\mathbb{R})$ and $p = \frac{2\alpha+2}{\beta}$. For given $t > 0$, put

$$f_1(x) = f(x)\chi_{B_{2t}}(x) \quad\text{and}\quad f_2(x) = f(x) - f_1(x). \tag{10}$$

Then

$$\widetilde{I}_\beta f(x) = \widetilde{I}_\beta f_1(x) + \widetilde{I}_\beta f_2(x) = F_1(x) + F_2(x),$$

where

$$F_1(x) = \int_{B_{2t}} \left(\tau_y|x|^{\beta-2\alpha-2} - |y|^{\beta-2\alpha-2}\chi_{\complement_{B_1}}(y)\right) f(y)\,d\mu_\alpha(y),$$

$$F_2(x) = \int_{\complement_{B_{2t}}} \left(\tau_y|x|^{\beta-2\alpha-2} - |y|^{\beta-2\alpha-2}\chi_{\complement_{B_1}}(y)\right) f(y)\,d\mu_\alpha(y).$$

Note that the function $f_1$ has compact (bounded) support and thus

$$a_1 = -\int_{B_{2t}\setminus B_{\min\{1,2t\}}} |y|^{\beta-2\alpha-2} f(y)\,d\mu_\alpha(y)$$

is finite.

Note also that

$$F_1(x) - a_1 = \int\limits_{B_{2t}} \tau_y|x|^{\beta-2\alpha-2} f(y)\,d\mu_\alpha(y) - \int\limits_{B_{2t}\setminus B_{\min\{1,2t\}}} |y|^{\beta-2\alpha-2} f(y)\,d\mu_\alpha(y) + \int\limits_{B_{2t}\setminus B_{\min\{1,2t\}}} |y|^{\beta-2\alpha-2} f(y)\,d\mu_\alpha(y)$$
$$= \int\limits_{\mathbb{R}} \tau_y|x|^{\beta-2\alpha-2} f_1(y)\,d\mu_\alpha(y) = I_\beta f_1(x).$$

Therefore

$$\left|F_1(x) - a_1\right| \leqslant \int\limits_{\mathbb{R}} |y|^{\beta-2\alpha-2}\left|\tau_y f_1(x)\right| d\mu_\alpha(y) = \int\limits_{B(x,2t)} |y|^{\beta-2\alpha-2}\left|\tau_y f(x)\right| d\mu_\alpha(y).$$

Further, for $x \in B_t,\ y \in B(x, 2t)$ we have

$$|y| \leqslant |x| + 2t < 3t.$$

Consequently

$$\left|F_1(x) - a_1\right| \leqslant \int\limits_{B_{3t}} |y|^{\beta-2\alpha-2}\tau_y\left|f(x)\right| d\mu_\alpha(y), \tag{11}$$

if $x \in B_t$.

By Theorem 1, and inequalities (3), (8) and (11) for $\beta p = 2\alpha + 2$

$$(\mu_\alpha B_t)^{-1} \int\limits_{B_t} \left|\tau_z F_1(x) - a_1\right| d\mu_\alpha(z) \leqslant b_\alpha^{-1} t^{-2\alpha-2} \int\limits_{B_t} \left( \int\limits_{B_{3t}} |y|^{\beta-2\alpha-2}\tau_y\tau_z\left|f(x)\right| d\mu_\alpha(y)\right) d\mu_\alpha(z)$$
$$\leqslant C_3 b_\alpha^{-1}(3t)^\beta t^{-2\alpha-2} \cdot \int\limits_{B_t} M(\tau_z f)(x)\,d\mu_\alpha(z)$$
$$\leqslant C_3 b_\alpha^{-1/p} t^{\beta-2\alpha-2} \cdot t^{(2\alpha+2)/p'} \left( \int\limits_{B_t} \big(M(\tau_z f)(x)\big)^p\,d\mu_\alpha(z)\right)^{1/p}$$
$$\leqslant C_3 b_\alpha^{-1/p} \left\|M(\tau_z f)\right\|_{p,\alpha} \leqslant C_2 C_3 b_\alpha^{-1/p} \|\tau_z f\|_{p,\alpha}$$
$$\leqslant 4C_2 C_3 b_\alpha^{-1/p} \|f\|_{p,\alpha}.$$

Therefore

$$\left|F_1(x) - a_1\right| \leqslant 4C_2 C_3 b_\alpha^{-1/p} \|f\|_{p,\alpha}. \tag{12}$$

Denote

$$a_2 = \int\limits_{B_{\max\{1,2t\}}\setminus B_{2t}} |y|^{\beta-2\alpha-2} f(y)\,d\mu_\alpha(y).$$

Now we estimate $|F_2(x) - a_2|$ for $x \in B_t$:

$$\left|F_2(x) - a_2\right| \leqslant \int\limits_{\complement_{B_{2t}}} \left|f(y)\right|\left|\tau_y|x|^{\beta-2\alpha-2} - |y|^{\beta-2\alpha-2}\right| d\mu_\alpha(y).$$

Applying Lemma 1 and Hölder inequality we obtain instead of we have

$$\begin{aligned}\left|F_2(x)-a_2\right| &\leqslant 2^{2\alpha+4-\beta}|x| \int_{\complement_{B_{2t}}} \left|f(y)\right||y|^{\beta-2\alpha-3}\,d\mu_\alpha(y) \\ &\leqslant 2^{2\alpha+4-\beta}|x|\left(\int_{\complement_{B_{2t}}} \left|f(y)\right|^p d\mu_\alpha(y)\right)^{1/p}\left(\int_{\complement_{B_{2t}}} |y|^{(\beta-2\alpha-3)p'}\,d\mu_\alpha(y)\right)^{1/p'} \\ &\leqslant C|x|t^{\beta-1-\frac{2\alpha+2}{p}}\|f\|_{p,\alpha} \leqslant C|x|t^{-1}\|f\|_{p,\alpha} \leqslant C\|f\|_{p,\alpha}.\end{aligned}$$

Note that if $|x| \leqslant t$, $|z| \leqslant 2t$, then $\tau_z|x| \leqslant |x|+|z| \leqslant 3t$. Thus for $\beta p = 2\alpha+2$ we obtain

$$\left|\tau_z F_2(x)-a_2\right| \leqslant \tau_z\left|F_2(x)-a_2\right| \leqslant C\|f\|_{p,\alpha}. \tag{13}$$

Denote

$$a_f = a_1 + a_2 = \int_{B_{\max\{1,2t\}}} |y|^{\beta-2\alpha-2} f(y)\,d\mu_\alpha(y).$$

Finally, from (12) and (13) we have

$$\sup_{x,t}(\mu_\alpha B_t)^{-1}\int_{B_t}\left|\tau_y \widetilde{I}_\beta f(x) - a_f\right| d\mu_\alpha(y) \leqslant C\|f\|_{p,\alpha}.$$

Thus

$$\|\widetilde{I}_\beta f\|_{*,\alpha} \leqslant 2\sup_{x,t}(\mu_\alpha B_t)^{-1}\int_{B_t}\left|\tau_y \widetilde{I}_\beta f(x) - a_f\right| d\mu_\alpha(y) \leqslant C\|f\|_{p,\alpha}.$$

Therefore the proof of Theorem 4 is completed. □

**Proof of Theorem 2.** Sufficiency part of the proof follows from Theorem 4.

*Necessity.* (1) Let $1 < p < \frac{2\alpha+2}{\alpha}$, $f \in L_{p,\alpha}(\mathbb{R})$ and assume that the inequality

$$\|I_\beta f\|_{q,\alpha} \leqslant C\|f\|_{p,\alpha} \tag{14}$$

holds, where $C$ depends only on $p$, $q$ and $\alpha$.

Define $f_r(x) := f(rx)$, then

$$\|f_r\|_{p,\alpha} = r^{-\frac{2\alpha+2}{p}}\|f\|_{p,\alpha}$$

and

$$\|I_\beta f_r\|_{q,\alpha} = r^{-\beta-\frac{2\alpha+2}{q}}\|I_\beta f\|_{q,\alpha}.$$

By the inequality (14)

$$\|I_\beta f\|_{q,\alpha} \leqslant C r^{\beta+\frac{2\alpha+2}{q}-\frac{2\alpha+2}{p}}\|f\|_{p,\alpha}.$$

If $\frac{1}{p} > \frac{1}{q} + \frac{\beta}{2\alpha+2}$, then for all $f \in L_{p,\alpha}(\mathbb{R})$ we have $\|I_\beta f\|_{q,\alpha} = 0$ as $r \to 0$, which is impossible. Similarly, if $\frac{1}{p} < \frac{1}{q} + \frac{\beta}{2\alpha+2}$, then for all $f \in L_{p,\alpha}(\mathbb{R})$ we obtain $\|I_\beta f\|_{q,\alpha} = 0$ as $r \to \infty$, which is also impossible.

Therefore we get $\frac{1}{p} = \frac{1}{q} + \frac{\beta}{2\alpha+2}$.

*Necessity.* Let $I_\beta$ be bounded from $L_{1,\alpha}(\mathbb{R})$ to $WL_{q,\alpha}(\mathbb{R})$. We have

$$\|I_\beta f_r\|_{WL_{q,\alpha}} = r^{-\beta-\frac{2\alpha+2}{q}}\|I_\beta f\|_{WL_{q,\alpha}}.$$

By the boundedness of $I_\beta$ from $L_{1,\alpha}(\mathbb{R})$ to $WL_{q,\alpha}(\mathbb{R})$ it follows

$$\begin{aligned}\|I_\beta f\|_{WL_{q,\alpha}} &= r^{\beta+\frac{2\alpha+2}{q}}\|I_\beta f_r\|_{WL_{q,\alpha}} \\ &\leqslant C r^{\beta+\frac{2\alpha+2}{q}}\|f_r\|_{1,\alpha} = C r^{\beta+\frac{2\alpha+2}{q}-(2\alpha+2)}\|f\|_{1,\alpha},\end{aligned}$$

where $C$ depends only on $q$ and $\alpha$.

If $1 < \frac{1}{q} + \frac{\beta}{2\alpha+2}$, then for all $f \in L_{1,\alpha}(\mathbb{R})$ we have $\|I_\beta f\|_{WL_{q,\alpha}} = 0$ as $r \to 0$. Similarly, if $1 > \frac{1}{q} + \frac{\beta}{2\alpha+2}$, then for all $f \in L_{1,\alpha}(\mathbb{R})$ we obtain $\|I_\beta f\|_{WL_{q,\alpha}} = 0$ as $r \to \infty$.

Hence we get $1 = \frac{1}{q} + \frac{\beta}{2\alpha+2}$. Thus the proof of Theorem 2 is completed. □

**Proof of Corollary 3.** Sufficiency part of the proof follows from Theorem 4 and the inequality

$$M_\beta f(x) \leqslant b_\alpha^{1-\frac{\beta}{2\alpha+2}} I_\beta\big(|f|\big)(x), \quad 0 < \beta < 2\alpha + 2.$$

*Necessity.* (1) Let $M_\beta$ be bounded from $L_{p,\alpha}(\mathbb{R})$ to $L_{q,\alpha}(\mathbb{R})$ for $1 < p < \frac{\beta}{2\alpha+2}$, $1 < p < q < \infty$. Then we have

$$M_\beta f_r(x) = r^{-\beta} M_\beta f(rx),$$
$$\|M_\beta f_r\|_{q,\alpha} = r^{-\beta-\frac{2\alpha+2}{q}} \|M_\beta f\|_{q,\alpha}$$

and by the same argument as in Theorem 2 we obtain $\frac{1}{p} = \frac{1}{q} + \frac{\beta}{2\alpha+2}$.

(2) Let $M_\beta$ be bounded from $L_{1,\alpha}(\mathbb{R})$ to $WL_{q,\alpha}(\mathbb{R})$. Then we have

$$\|M_\beta f_r\|_{WL_{q,\alpha}} = r^{-\beta-\frac{2\alpha+2}{q}} \|M_\beta f\|_{WL_{q,\alpha}}$$

and the inequality

$$\|M_\beta f\|_{q,\alpha} \leqslant C\|f\|_{p,\alpha}.$$

Hence, the equality $1 = \frac{1}{q} + \frac{\beta}{2\alpha+2}$ follows easily.

(3) Let $p = \frac{2\alpha+2}{\beta}$, $f \in L_{p,\alpha}(\mathbb{R})$, then applying Hölders inequality together with inequality (3) we obtain

$$\begin{aligned}(\mu_\alpha B_r)^{\frac{\beta}{2\alpha+2}-1} \int\limits_{B_r} \tau_x\big|f(y)\big|\,d\mu_\alpha(y) &\leqslant (\mu_\alpha B_r)^{\frac{\beta}{2\alpha+2}-1+\frac{1}{p'}} \Bigg(\int\limits_{B_r} \big(\tau_x\big|f(y)\big|\big)^p\,d\mu_\alpha(y)\Bigg)^{1/p}\\ &= \|\tau_x f\|_{p,\alpha} \leqslant 4\|f\|_{p,\alpha}.\end{aligned}$$

Thus the proof of Corollary 3 is completed. □

**Proof of Theorem 3.** For $x \in \mathbb{R}$, let $\tau_x$ be the generalized translation by $x$. By definition of the Dunkl-type Besov spaces it suffices to show that

$$\|\tau_x I_\beta f - I_\beta f\|_{p,\alpha} \leqslant C\|\tau_x f - f\|_{p,\alpha}.$$

It is easy to see that $\tau_x$ commutes with $I_\beta$, i.e. $\tau_x I_\beta f = I_\beta(\tau_x f)$. Hence we have

$$|\tau_x I_\beta f - I_\beta f| = \big|I_\beta(\tau_x f) - I_\beta f\big| \leqslant I_\beta\big(|\tau_x f - f|\big).$$

Taking $L_{q,\alpha}(\mathbb{R})$ norm on both ends of the above inequality, by the boundedness of $I_\beta$ from $L_{p,\alpha}(\mathbb{R})$ to $L_{q,\alpha}(\mathbb{R})$, we obtain the desired result. Theorem 3 is proved. □

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