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Uncertainty relation on Wigner–Yanase–Dyson skew information ☆

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ABSTRACT

We give a trace inequality related to the uncertainty relation of Wigner–Yanase–Dyson skew information. This inequality corresponds to a generalization of the uncertainty relation derived by S. Luo (2005) [8] for the quantum uncertainty quantity excluding the classical mixture.

1. Introduction

Wigner–Yanase skew information

$$\begin{aligned} I_\rho(H) &= \frac{1}{2} Tr\big[\big(i\big[\rho^{1/2}, H\big]\big)^2\big] \\ &= Tr\big[\rho H^2\big] - Tr\big[\rho^{1/2} H \rho^{1/2} H\big] \end{aligned}$$

was defined in [10]. This quantity can be considered as a kind of the degree for non-commutativity between a quantum state ρ and an observable H. Here we denote the commutator by $[X, Y] = XY - YX$. This quantity was generalized by Dyson

$$\begin{aligned} I_{\rho,\alpha}(H) &= \frac{1}{2} Tr\big[\big(i\big[\rho^\alpha, H\big]\big)\big(i\big[\rho^{1-\alpha}, H\big]\big)\big] \\ &= Tr\big[\rho H^2\big] - Tr\big[\rho^\alpha H \rho^{1-\alpha} H\big], \quad \alpha \in [0,1] \end{aligned}$$

which is known as the Wigner–Yanase–Dyson skew information. It is famous that the convexity of $I_{\rho,\alpha}(H)$ with respect to ρ was successfully proven by E.H. Lieb in [7]. From the physical point of view, an observable H is generally considered to be an unbounded operator, however in the present paper, unless otherwise stated, we consider $H \in B(\mathcal{H})$ represents the set of all bounded linear operators on the Hilbert space $\mathcal{H}$, as a mathematical interest. We also denote the set of all self-adjoint operators (observables) by $\mathcal{L}_h(\mathcal{H})$ and the set of all density operators (quantum states) by $\mathcal{S}(\mathcal{H})$ on the Hilbert space $\mathcal{H}$. The relation between the Wigner–Yanase skew information and the uncertainty relation was studied in [9]. Moreover the relation between the Wigner–Yanase–Dyson skew information and the uncertainty relation was studied in [5,11]. In our paper [11], we defined a generalized skew information and then derived a kind of an uncertainty relation. In Section 2, we discuss various properties of the Wigner–Yanase–Dyson skew information. Finally in Section 3, we give our main result and its proof.

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2. Trace inequalities of Wigner–Yanase–Dyson skew information

We review the relation between the Wigner–Yanase skew information and the uncertainty relation. In quantum mechanical system, the expectation value of an observable H in a quantum state ρ is expressed by $Tr[\rho H]$. It is natural that the variance for a quantum state ρ and an observable H is defined by $V_\rho(H) = Tr[\rho(H - Tr[\rho H]I)^2] = Tr[\rho H^2] - Tr[\rho H]^2$. It is famous that we have

$$V_\rho(A)V_\rho(B) \geqslant \frac{1}{4}\big|Tr\big[\rho[A,B]\big]\big|^2 \tag{2.1}$$

for a quantum state ρ and two observables A and B (see [4]). The further strong results was given by Robertson and Schrödinger

$$V_\rho(A)V_\rho(B) - \big|Cov_\rho(A,B)\big|^2 \geqslant \frac{1}{4}\big|Tr\big[\rho[A,B]\big]\big|^2,$$

where the covariance is defined by $Cov_\rho(A,B) = Tr[\rho(A - Tr[\rho A]I)(B - Tr[\rho B]I)]$. However, the uncertainty relation for the Wigner–Yanase skew information failed. (See [5,9,11].)

$$I_\rho(A)I_\rho(B) \geqslant \frac{1}{4}\big|Tr\big[\rho[A,B]\big]\big|^2.$$

Recently, S. Luo introduced the quantity $U_\rho(H)$ representing a quantum uncertainty excluding the classical mixture:

$$U_\rho(H) = \sqrt{V_\rho(H)^2 - \big(V_\rho(H) - I_\rho(H)\big)^2}, \tag{2.2}$$

then he derived the uncertainty relation on $U_\rho(H)$ in [8]:

$$U_\rho(A)U_\rho(B) \geqslant \frac{1}{4}\big|Tr\big[\rho[A,B]\big]\big|^2. \tag{2.3}$$

Note that we have the following relation

$$0 \leqslant I_\rho(H) \leqslant U_\rho(H) \leqslant V_\rho(H). \tag{2.4}$$

The inequality (2.3) is a refinement of the inequality (2.1) in the sense of (2.4). In this section, we study one-parameter extended inequality for the inequality (2.3).

Definition 2.1. For $0 \leqslant \alpha \leqslant 1$, a quantum state ρ and an observable H, we define the Wigner–Yanase–Dyson skew information

$$\begin{aligned} I_{\rho,\alpha}(H) &= \frac{1}{2}Tr\big[\big(i\big[\rho^\alpha, H_0\big]\big)\big(i\big[\rho^{1-\alpha}, H_0\big]\big)\big] \\ &= Tr\big[\rho H_0^2\big] - Tr\big[\rho^\alpha H_0 \rho^{1-\alpha} H_0\big] \end{aligned} \tag{2.5}$$

and we also define

$$\begin{aligned} J_{\rho,\alpha}(H) &= \frac{1}{2}Tr\big[\big\{\rho^\alpha, H_0\big\}\big\{\rho^{1-\alpha}, H_0\big\}\big] \\ &= Tr\big[\rho H_0^2\big] + Tr\big[\rho^\alpha H_0 \rho^{1-\alpha} H_0\big], \end{aligned} \tag{2.6}$$

where $H_0 = H - Tr[\rho H]I$ and we denote the anti-commutator by $\{X, Y\} = XY + YX$.

Note that we have

$$\frac{1}{2}Tr\big[\big(i\big[\rho^\alpha, H_0\big]\big)\big(i\big[\rho^{1-\alpha}, H_0\big]\big)\big] = \frac{1}{2}Tr\big[\big(i\big[\rho^\alpha, H\big]\big)\big(i\big[\rho^{1-\alpha}, H\big]\big)\big]$$

but we have

$$\frac{1}{2}Tr\big[\big\{\rho^\alpha, H_0\big\}\big\{\rho^{1-\alpha}, H_0\big\}\big] \neq \frac{1}{2}Tr\big[\big\{\rho^\alpha, H\big\}\big\{\rho^{1-\alpha}, H\big\}\big].$$

Then we have the following inequalities:

$$I_{\rho,\alpha}(H) \leqslant I_\rho(H) \leqslant J_\rho(H) \leqslant J_{\rho,\alpha}(H), \tag{2.7}$$

since we have $Tr[\rho^{1/2}H\rho^{1/2}H] \leqslant Tr[\rho^\alpha H \rho^{1-\alpha} H]$. (See [1,2] for example.) If we define

$$U_{\rho,\alpha}(H) = \sqrt{V_\rho(H)^2 - \big(V_\rho(H) - I_{\rho,\alpha}(H)\big)^2}, \tag{2.8}$$

as a direct generalization of Eq. (2.2), then we have

$$0 \leqslant I_{\rho,\alpha}(H) \leqslant U_{\rho,\alpha}(H) \leqslant U_{\rho}(H) \tag{2.9}$$

due to the first inequality of (2.7). We also have

$$U_{\rho,\alpha}(H) = \sqrt{I_{\rho,\alpha}(H) J_{\rho,\alpha}(H)}.$$

From the inequalities (2.4), (2.8), (2.9), our situation is that we have

$$0 \leqslant I_{\rho,\alpha}(H) \leqslant I_{\rho}(H) \leqslant U_{\rho}(H)$$

and

$$0 \leqslant I_{\rho,\alpha}(H) \leqslant U_{\rho,\alpha}(H) \leqslant U_{\rho}(H).$$

Our concern is to show an uncertainty relation with respect to $U_{\rho,\alpha}(H)$ as a direct generalization of the inequality (2.3) such that

$$U_{\rho,\alpha}(A) U_{\rho,\alpha}(B) \geqslant \frac{1}{4} \left| Tr\left[\rho[A,B]\right]\right|^2. \tag{2.10}$$

On the other hand, we introduced a generalized Wigner–Yanase skew information which is a generalization of the inequality (2.10), but different from the Wigner–Yanase–Dyson skew information defined in (2.5) and gave the following theorem in [3].

Theorem 2.1. *For $0 \leqslant \alpha \leqslant 1$, a quantum state ρ and an observable H, we define a generalized Wigner–Yanase skew information by*

$$K_{\rho,\alpha}(H) = \frac{1}{2} Tr\left[\left(i\left[\frac{\rho^{\alpha} + \rho^{1-\alpha}}{2}, H_0\right]\right)^2\right]$$

and we also define

$$L_{\rho,\alpha}(H) = \frac{1}{2} Tr\left[\left(i\left\{\frac{\rho^{\alpha} + \rho^{1-\alpha}}{2}, H_0\right\}\right)^2\right],$$

and

$$W_{\rho,\alpha}(H) = \sqrt{K_{\rho,\alpha}(H) L_{\rho,\alpha}(H)}.$$

Then for a quantum state ρ and observables A, B and $\alpha \in [0,1]$, we have

$$W_{\rho,\alpha}(A) W_{\rho,\alpha}(B) \geqslant \frac{1}{4} \left| Tr\left[\left(\frac{\rho^{\alpha} + \rho^{1-\alpha}}{2}\right)^2 [A,B]\right]\right|^2.$$

3. Main theorem

We give the main theorem as follows;

Theorem 3.1. *For a quantum state ρ and observables A, B and $0 \leqslant \alpha \leqslant 1$, we have*

$$U_{\rho,\alpha}(A) U_{\rho,\alpha}(B) \geqslant \alpha(1-\alpha) \left| Tr\left[\rho[A,B]\right]\right|^2. \tag{3.1}$$

We use the several lemmas to prove Theorem 3.1. By spectral decomposition, there exists an orthonormal basis $\{\phi_1, \phi_2, \ldots\}$ consisting of eigenvectors of ρ. Let $\lambda_1, \lambda_2, \ldots$ be the corresponding eigenvalues, where $\sum_{i=1}^{\infty} \lambda_i = 1$ and $\lambda_i \geqslant 0$. Thus, ρ has a spectral representation

$$\rho = \sum_{i=1}^{\infty} \lambda_i |\phi_i\rangle\langle\phi_i|. \tag{3.2}$$

Lemma 3.1.

$$I_{\rho,\alpha}(H) = \sum_{i<j} \left(\lambda_i + \lambda_j - \lambda_i^{\alpha}\lambda_j^{1-\alpha} - \lambda_i^{1-\alpha}\lambda_j^{\alpha}\right) \left|\langle\phi_i|H_0|\phi_j\rangle\right|^2.$$

Proof. By (3.2),

$$\rho H_0^2 = \sum_{i=1}^{\infty} \lambda_i |\phi_i\rangle\langle\phi_i| H_0^2.$$

Then

$$Tr\big[\rho H_0^2\big] = \sum_{i=1}^{\infty} \lambda_i \langle\phi_i| H_0^2 |\phi_i\rangle = \sum_{i=1}^{\infty} \lambda_i \big\| H_0|\phi_i\rangle \big\|^2. \tag{3.3}$$

Since

$$\rho^{\alpha} H_0 = \sum_{i=1}^{\infty} \lambda_i^{\alpha} |\phi_i\rangle\langle\phi_i| H_0$$

and

$$\rho^{1-\alpha} H_0 = \sum_{i=1}^{\infty} \lambda_i^{1-\alpha} |\phi_i\rangle\langle\phi_i| H_0,$$

we have

$$\rho^{\alpha} H_0 \rho^{1-\alpha} H_0 = \sum_{i,j=1}^{\infty} \lambda_i^{\alpha} \lambda_j^{1-\alpha} |\phi_i\rangle\langle\phi_i| H_0 |\phi_j\rangle\langle\phi_j| H_0.$$

Thus

$$\begin{aligned} Tr\big[\rho^{\alpha} H_0 \rho^{1-\alpha} H_0\big] &= \sum_{i,j=1}^{\infty} \lambda_i^{\alpha} \lambda_j^{1-\alpha} \langle\phi_i| H_0 |\phi_j\rangle\langle\phi_j| H_0 |\phi_i\rangle \\ &= \sum_{i,j=1}^{\infty} \lambda_i^{\alpha} \lambda_j^{1-\alpha} \big|\langle\phi_i| H_0 |\phi_j\rangle\big|^2. \end{aligned} \tag{3.4}$$

From (2.5), (3.3), (3.4),

$$\begin{aligned} I_{\rho,\alpha}(H) &= \sum_{i=1}^{\infty} \lambda_i \big\| H_0|\phi_i\rangle \big\|^2 - \sum_{i,j=1}^{\infty} \lambda_i^{\alpha} \lambda_j^{1-\alpha} \big|\langle\phi_i| H_0 |\phi_j\rangle\big|^2 \\ &= \sum_{i,j=1}^{\infty} \big(\lambda_i - \lambda_i^{\alpha} \lambda_j^{1-\alpha}\big) \big|\langle\phi_i| H_0 |\phi_j\rangle\big|^2 \\ &= \sum_{i<j} \big(\lambda_i + \lambda_j - \lambda_i^{\alpha} \lambda_j^{1-\alpha} - \lambda_i^{1-\alpha} \lambda_j^{\alpha}\big) \big|\langle\phi_i| H_0 |\phi_j\rangle\big|^2. \qquad \square \end{aligned}$$

Lemma 3.2.

$$J_{\rho,\alpha}(H) \geqslant \sum_{i<j} \big(\lambda_i + \lambda_j + \lambda_i^{\alpha} \lambda_j^{1-\alpha} + \lambda_i^{1-\alpha} \lambda_j^{\alpha}\big) \big|\langle\phi_i| H_0 |\phi_j\rangle\big|^2.$$

Proof. By (2.6), (3.3), (3.4), we have

$$\begin{aligned} J_{\rho,\alpha}(H) &= \sum_{i=1}^{\infty} \lambda_i \big\| H_0|\phi_i\rangle \big\|^2 + \sum_{i,j=1}^{\infty} \lambda_i^{\alpha} \lambda_j^{1-\alpha} \big|\langle\phi_i| H_0 |\phi_j\rangle\big|^2 \\ &= \sum_{i,j=1}^{\infty} \big(\lambda_i + \lambda_i^{\alpha} \lambda_j^{1-\alpha}\big) \big|\langle\phi_i| H_0 |\phi_j\rangle\big|^2 \\ &= 2\sum_{i=1}^{\infty} \lambda_i \big|\langle\phi_i| H_0 |\phi_i\rangle\big|^2 + \sum_{i\neq j} \big(\lambda_i + \lambda_i^{\alpha} \lambda_j^{1-\alpha}\big) \big|\langle\phi_i| H_0 |\phi_j\rangle\big|^2 \end{aligned}$$

$$= 2\sum_{i=1}^{\infty}\lambda_i\left|\langle\phi_i|H_0|\phi_i\rangle\right|^2 + \sum_{i<j}\left(\lambda_i+\lambda_j+\lambda_i^{\alpha}\lambda_j^{1-\alpha}+\lambda_i^{1-\alpha}\lambda_j^{\alpha}\right)\left|\langle\phi_i|H_0|\phi_j\rangle\right|^2$$
$$\geqslant \sum_{i<j}\left(\lambda_i+\lambda_j+\lambda_i^{\alpha}\lambda_j^{1-\alpha}+\lambda_i^{1-\alpha}\lambda_j^{\alpha}\right)\left|\langle\phi_i|H_0|\phi_j\rangle\right|^2. \qquad \square$$

Lemma 3.3. *For any $t > 0$ and $0 \leqslant \alpha \leqslant 1$, the following inequality holds;*

$$(1-2\alpha)^2(t-1)^2 - \left(t^{\alpha}-t^{1-\alpha}\right)^2 \geqslant 0. \tag{3.5}$$

Proof. If $\alpha = 0$ or $\frac{1}{2}$ or 1, then it is clear that (3.5) is satisfied. Now we put

$$F(t) = (1-2\alpha)^2(t-1)^2 - \left(t^{\alpha}-t^{1-\alpha}\right)^2.$$

We have

$$F'(t) = 2(1-2\alpha)^2 t - 2\alpha t^{2\alpha-1} - 2(1-\alpha)t^{1-2\alpha} + 8\alpha(1-\alpha).$$

And we also have

$$F''(t) = 2(1-2\alpha)^2 - 2\alpha(2\alpha-1)t^{2\alpha-2} - 2(1-\alpha)(1-2\alpha)t^{-2\alpha}$$

and

$$F'''(t) = 4\alpha(1-2\alpha)(1-\alpha)t^{-2\alpha-1} - 4\alpha(1-2\alpha)(1-\alpha)t^{2\alpha-3}$$
$$= 4\alpha(1-2\alpha)(1-\alpha)\left(\frac{1}{t^{1+2\alpha}} - \frac{1}{t^{3-2\alpha}}\right).$$

If $\frac{1}{2} < \alpha < 1$, then $1+2\alpha > 3-2\alpha$. Then it is easy to show that $F'''(t) < 0$ for $t < 1$ and $F'''(t) > 0$ for $t > 1$. On the other hand if $0 < \alpha < \frac{1}{2}$, then $1+2\alpha < 3-2\alpha$. Then it is easy to show that $F'''(t) < 0$ for $t < 1$ and $F'''(t) > 0$ for $t > 1$. Since $F''(1) = 0$, we can get $F''(t) > 0$. Since $F'(1) = 0$, we also have $F'(t) < 0$ for $t < 1$ and $F'(t) > 0$ for $t > 1$. Since $F(1) = 0$, we finally get $F(t) \geqslant 0$ for all $t > 0$. Therefore we have (3.5). $\square$

Proof of Theorem 3.1. We put $t = \frac{\lambda_i}{\lambda_j}$ in (3.5). Then we have

$$(1-2\alpha)^2\left(\frac{\lambda_i}{\lambda_j}-1\right)^2 - \left(\left(\frac{\lambda_i}{\lambda_j}\right)^{\alpha} - \left(\frac{\lambda_i}{\lambda_j}\right)^{1-\alpha}\right)^2 \geqslant 0.$$

And we get

$$(1-2\alpha)^2(\lambda_i-\lambda_j)^2 - \left(\lambda_i^{\alpha}\lambda_j^{1-\alpha} - \lambda_i^{1-\alpha}\lambda_j^{\alpha}\right)^2 \geqslant 0$$

and

$$(\lambda_i-\lambda_j)^2 - \left(\lambda_i^{\alpha}\lambda_j^{1-\alpha} - \lambda_i^{1-\alpha}\lambda_j^{\alpha}\right)^2 \geqslant 4\alpha(1-\alpha)(\lambda_i-\lambda_j)^2$$

and

$$(\lambda_i+\lambda_j)^2 - \left(\lambda_i^{\alpha}\lambda_j^{1-\alpha} + \lambda_i^{1-\alpha}\lambda_j^{\alpha}\right)^2 \geqslant 4\alpha(1-\alpha)(\lambda_i-\lambda_j)^2. \tag{3.6}$$

Since

$$Tr\left[\rho[A,B]\right] = Tr\left[\rho[A_0,B_0]\right]$$
$$= 2i\, Im\, Tr[\rho A_0 B_0]$$
$$= 2i\, Im \sum_{i<j}(\lambda_i-\lambda_j)\langle\phi_i|A_0|\phi_j\rangle\langle\phi_j|B_0|\phi_i\rangle$$
$$= 2i\sum_{i<j}(\lambda_i-\lambda_j)\, Im\langle\phi_i|A_0|\phi_j\rangle\langle\phi_j|B_0|\phi_i\rangle,$$

$$\left|Tr\left[\rho[A,B]\right]\right| = 2\left|\sum_{i<j}(\lambda_i-\lambda_j)\, Im\langle\phi_i|A_0|\phi_j\rangle\langle\phi_j|B_0|\phi_i\rangle\right|$$
$$\leqslant 2\sum_{i<j}|\lambda_i-\lambda_j|\left|Im\langle\phi_i|A_0|\phi_j\rangle\langle\phi_j|B_0|\phi_i\rangle\right|.$$

Then we have

$$\left|Tr\left[\rho[A,B]\right]\right|^2 \leqslant 4\left\{\sum_{i<j}|\lambda_i-\lambda_j|\left|Im\langle\phi_i|A_0|\phi_j\rangle\langle\phi_j|_0|\phi_i\rangle\right|\right\}^2.$$

By (3.6) and Schwarz inequality,

$$\begin{aligned}
\alpha(1-\alpha)\left|Tr\left[\rho[A,B]\right]\right|^2 &\leqslant 4\alpha(1-\alpha)\left\{\sum_{i<j}|\lambda_i-\lambda_j|\left|Im\langle\phi_i|A_0|\phi_j\rangle\langle\phi_j|B_0|\phi_i\rangle\right|\right\}^2 \\
&= \left\{\sum_{i<j}2\sqrt{\alpha(1-\alpha)}|\lambda_i-\lambda_j|\left|Im\langle\phi_i|A_0|\phi_j\rangle\langle\phi_j|B_0|\phi_i\rangle\right|\right\}^2 \\
&\leqslant \left\{\sum_{i<j}2\sqrt{\alpha(1-\alpha)}|\lambda_i-\lambda_j|\left|\langle\phi_i|A_0|\phi_j\rangle\right|\left|\langle\phi_j|B_0|\phi_i\rangle\right|\right\}^2 \\
&\leqslant \left\{\sum_{i<j}\left\{(\lambda_i+\lambda_j)^2-\left(\lambda_i^{\alpha}\lambda_j^{1-\alpha}+\lambda_i^{1-\alpha}\lambda_j^{\alpha}\right)^2\right\}^{1/2}\left|\langle\phi_i|A_0|\phi_j\rangle\right|\left|\langle\phi_j|B_0|\phi_i\rangle\right|\right\}^2 \\
&\leqslant \sum_{i<j}\left(\lambda_i+\lambda_j-\lambda_i^{\alpha}\lambda_j^{1-\alpha}-\lambda_i^{1-\alpha}\lambda_j^{\alpha}\right)\left|\langle\phi_i|A_0|\phi_j\rangle\right|^2 \\
&\quad\times\sum_{i<j}\left(\lambda_i+\lambda_j+\lambda_i^{\alpha}\lambda_j^{1-\alpha}+\lambda_i^{1-\alpha}\lambda_j^{\alpha}\right)\left|\langle\phi_i|B_0|\phi_j\rangle\right|^2.
\end{aligned}$$

Then we have

$$I_{\rho,\alpha}(A)J_{\rho,\alpha}(B) \geqslant \alpha(1-\alpha)\left|Tr\left[\rho[A,B]\right]\right|^2.$$

We also have

$$I_{\rho,\alpha}(B)J_{\rho,\alpha}(A) \geqslant \alpha(1-\alpha)\left|Tr\left[\rho[A,B]\right]\right|^2.$$

Hence we have the final result (3.1). □

Remark 3.1. We remark that (2.3) is derived by putting $\alpha = 1/2$ in (3.1). Then Theorem 3.1 is a generalization of the result of Luo [8].

Remark 3.2. We remark that Conjecture 2.3 in [3] does not hold in general. The conjecture is (2.10). A counterexample is given as follows. Let

$$\rho = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4}\end{pmatrix}, \qquad A = \begin{pmatrix} 0 & i \\ -i & 0\end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}, \qquad \alpha = \frac{1}{3}.$$

We have

$$I_{\rho,\alpha}(A)J_{\rho,\alpha}(B) = I_{\rho,\alpha}(B)I_{\rho,\alpha}(A) = 0.22457296\ldots$$

and $|Tr[\rho[A,B]]|^2 = 1$. These imply

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) = 0.22457296\ldots < \frac{1}{4}\left|Tr\left[\rho[A,B]\right]\right|^2 = 0.25.$$

On the other hand we have

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) > \alpha(1-\alpha)\left|Tr\left[\rho[A,B]\right]\right|^2 = 0.2222222\ldots.$$

We also give a counterexample for Conjecture 2.10 in [3]. The inequality

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) \geqslant \frac{1}{4}\left|Tr\left[\left(\frac{\rho^{\alpha}+\rho^{1-\alpha}}{2}\right)^2[A,B]\right]\right|^2$$

is not correct in general, because $LHS = 0.22457296\ldots$, $RHS = 0.23828105995\ldots$.

Remark 3.3. In the recent literature another generalization for inequality (2.3) has been proved in [6] as follows; for any ρ, A, B and $0 \leqslant \alpha \leqslant 1$

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) \geqslant \frac{1}{4}\left|Tr\left[\left(\rho - \rho^{|2\alpha-1|}\right)[A,B]\right]\right|^2.$$

However we gave the counter example for this inequality. Let

$$\rho = \begin{pmatrix} \frac{1}{64} & 0 & 0 \\ 0 & \frac{1}{16} & 0 \\ 0 & 0 & \frac{59}{64} \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \alpha = \frac{3}{4}.$$

Then we have

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) = 0.00170898\ldots,$$
$$\frac{1}{4}\left|Tr\left[\left(\rho - \rho^{|2\alpha-1|}\right)[A,B]\right]\right|^2 = 0.00610351\ldots.$$

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