



# The dynamics of spreading and immune strategies of sexually transmitted diseases on scale-free network

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## ABSTRACT

We examine epidemic threshold and dynamics for sexually transmitted diseases (STDs) spread using a multiple susceptible–infected–removed–susceptible ODE model on scale-free networks. We derive the threshold for the epidemic to be zero in infinite scale-free network. For a hard cut off scale-free network, we also prove the stability of disease-free equilibrium and the persistence of STDs infection. The effects of two immunization schemes, including proportional scheme and targeted vaccination, are studied and compared. We find that targeted strategy compare favorably to a proportional scheme in terms of effectiveness. Theory and simulations both prove that an appropriate condom using has prominent effect to control STDs spread on scale-free networks.

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## 1. Introduction

Recently, a large number of statistical properties have been found to be common in the topology of real-world social, biological, and technological networks [1,2]. In particular many real world networks show the small-world phenomenon, related to a very small average path length between nodes [3,4]. More strikingly, in some cases this property is associated to a scale-free connectivity distribution, which nature is associated to a large heterogeneity in the connectivity properties of the system. In scale-free networks, the number of contacts or connections of a node with other nodes in the system, the degree (or connectivity)  $k$ , follows a power-law distribution,  $P(k) \sim k^{-2-\gamma}$ , with  $0 < \gamma \leq 1$ . Recent studies have shown the importance of the scale-free topology on the dynamics and function of the system under study [1,2]. For instance, scale-free networks are very robust to random failures but at the same time extremely fragile to targeted attacks of the highly connected nodes [5,6].

The knowledge of the mechanisms involved in disease spread and the relation between the network structure and the dynamical patterns of the spread process has improved in the last several years [7]. For instance, a very important example of scale-free networks is found in the web of human sexual contacts [8]. Data from national sex surveys [8,9] provide quantitative information on the number of sexual partners, i.e., the degree  $k$ , of an individual. The respondents are asked to provide information on sexual attitudes such as the number of sex partners they have had in the last 12 months or in their entire life. It turns out that the number of heterosexual partners reported from different populations is well described by power-law scale-free distributions [10,11].

Since diseases can spread through scale-free networks, such as the web of human sexual contacts, the study of epidemics and disease dynamics on scale-free networks is a relevant theoretical issue. Many mathematical models on this topic have been studied [12–17]. Reference [17] states that, in infinite scale-free networks, epidemic processes do not possess an

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epidemic threshold below which diseases cannot produce a major epidemic outbreak or the onset of an endemic state. The absence of an intrinsic epidemic threshold has been found in both the susceptible–infected–susceptible (SIS) model [17] and the susceptible–infected–removed (SIR) model [18] in infinite scale-free networks. Papers above normally only prove the existence of epidemic equilibrium, but from a mathematical aspect, such SIS models on a bounded network can be viewed as multiple SIS models [19,20]. In this way, the stabilities of equilibria also can be proved [20].

In this paper, we build an  $SI_1I_2RS$  model to analyze the spread of STDs in a scale-free network in such as China scenario. Our model relies on the following rough description of individuals in the population. Namely, each node of the graph represents an individual and each link is a connection along which the STDs can spread. Here we consider a population with two types of infected individuals:

1. one proportion of infected individuals who potentially have a small infection rate  $\beta_1$  since they use such as condoms to protect their partners (for most STDs, although a condom can reduce the chances of the transmission of these virus or bacterium if it covers the affected areas, it is not entirely effective. A condom may not cover all of the sores or rashes in the affected areas, and direct skin contact may give rise to transmission [21]);
2. the other proportion who do not have any protection (we say they have high-risk sexual behaviors) and potentially have a large infection rate  $\beta_2$ .

We suppose each susceptible (healthy) node is infected with rate  $\beta_1$  or  $\beta_2$  if it is connected to one or more infected nodes. Infected nodes are cured with rate  $\eta$  and recovered nodes again become susceptible with rate  $\gamma$ .

We find that epidemic processes of our model do not possess an epidemic threshold in infinite scale-free network, which is like the result in model SIS and SIR [17,18]. Since realistic systems are actually made up by a finite number of individuals, this finite population introduces a maximum connectivity  $\kappa_c$ , depending on the size of the network [1]. In this paper, we also discuss the stability of equilibria and the permanent of infection for our system on a bounded hard-cutoff scale-free network, which does not possess any node with connectivity  $k$  larger than  $\kappa_c$  [1]. Since a finite network has the effect of restoring a boundary in the connectivity fluctuations, in this way it produces an effective non-zero threshold.

When a vaccination for a disease exists, immunizing certain individuals against being infected by a disease as a pre-emptive strategy may be the most efficient way to prevent loss of time and funds due to the disease. Obviously, immunization of the entire population will eradicate the disease entirely, but this is not always possible, or may involve high costs and effort. Therefore, the choice of which individuals to immunize is an important step in the immunization process, and may increase the efficiency of the immunization strategy. So in this paper, we also discuss the effect of different vaccination strategies. We mainly pay attention to two kinds of immunity strategies: proportional immunization and targeted immunization. We get that, the second strategy has an overwhelming advantage compare with the first one.

Through theory and simulations, we prove the benefit that divides the infected individuals into two subgroup according to condom using or not. Condom using rate and immunization rate are the only two parameters that can be controlled. For a given immunization rate, we get the suitable condom using rate to control STDs spread on networks.

This paper is organized as follows. In Section 2, we develop the mathematical model. In Section 3.1, we give the threshold for the STDs spread on scale-free network. In Section 3.2, we analyze the stability of disease-free equilibrium and the persistence of STDs infection. In Section 4, we discuss and compare the immunization strategies. In Section 5, we illustrate the results with numerical simulations. Finally, in Section 6, we discuss the implications of the results.

## 2. The model

We consider a population with two types of infected individuals where one part has the small infection rate  $\beta_1$  and the other part has the large one  $\beta_2$ . Let  $s_k$ ,  $i_{1k}$ ,  $i_{2k}$  and  $r_k$  represent the relative densities of nodes of degree  $k$ . They also denote the densities of the susceptible, the infectious with low infectivity, the infectious with high infectivity and the recovered respectively. Then we have the following dynamics model.

$$\begin{cases} \frac{ds_k(t)}{dt} = \gamma r_k(t) - ks_k(t)\Theta(t), \\ \frac{di_{1k}(t)}{dt} = (1-p)ks_k(t)\Theta(t) - \eta i_{1k}(t), \\ \frac{di_{2k}(t)}{dt} = pks_k(t)\Theta(t) - \eta i_{2k}(t), \\ \frac{dr_k(t)}{dt} = \eta(i_{1k}(t) + i_{2k}(t)) - \gamma r_k(t), \end{cases} \quad (1)$$

where  $\Theta(t)$  is defined as

$$\Theta(t) = \frac{1}{\langle k \rangle} \sum_{k=0}^{\infty} \psi(k) P(k) [\beta_1 i_{1k}(t) + \beta_2 i_{2k}(t)]. \quad (2)$$

In this paper, we suppose that the connectivity of nodes on this network is uncorrelated. In the case of an uncorrelated random network, the probability that a link points to a node of connectivity  $s$  is independent of the connectivity  $k$  of the node from which the link is emanating.

The meanings for each parameter or item of system (1) are:

- Parameter  $\gamma$  represents the rate of immunization-lost for recovered individuals. Recovered individuals become susceptible after time span  $\frac{1}{\gamma}$ .
- In the first equation,  $ks_k(t)\Theta(t)$  represents the lost of susceptible individuals because of infection, which is proportional to the connectivity  $k$ , the densities of healthy nodes  $s_k$ , infected nodes  $i_{1k}$  and  $i_{2k}$ . Factors  $\psi(k)P(k)/\langle k \rangle$  in  $\Theta(t)$  represents the expectation that any given link emanating from a node of connectivity  $k$  points to an infected node. Parameters  $\beta_1$  and  $\beta_2$  in  $\Theta(t)$  are the STDs transmission rates for each sexual behavior of subgroups  $i_{1k}$  and  $i_{2k}$  respectively.
- In the second and third equations, parameter  $p$  is the non-usage of condom. Suppose that after being infected by STDs, there are  $p$  proportion individuals still keep their high risk behavior, but the other  $1 - p$  proportion infected individuals begin to control their behavior, such as using condom to protect their partners.
- Parameter  $\eta$  represents the recovery rate of infected individuals, i.e., infected individuals recovery from STDs after time span  $\frac{1}{\eta}$ .

All parameters in system (1) are positive. And according to their biological meanings, we have  $\beta_2 > \beta_1 > 0$ ,  $\eta > 0$ ,  $\gamma > 0$  and  $p \in [0, 1]$ .

Now we give some explanations to symbols in  $\Theta(t)$ .  $\langle k \rangle$  is the average degree of the network, i.e., the average number of edges that a node has. It can also be understood as the first moment of degree  $k$ :  $\langle k \rangle = \sum_k kP(k)$ . Here  $P(k)$  is the degree distribution, i.e. the probability that a randomly chosen node within the network has degree  $k$ . Function  $\psi(k)$  denotes the infectivity of a node with degree  $k$ . In [17], authors suppose that the infectivity  $\psi(k)$  of each node (each node's potential infection-activity) with degree  $k$  is  $\psi(k) = \alpha k$ , where  $\alpha$  is a positive constant,  $0 < \alpha \leq 1$ . Then they get the epidemic threshold  $\lambda_c = 0$  for sufficiently large networks. In [22], authors suppose the infectivity  $\psi(k)$  of a node with degree  $k$  is a constant  $A$ , which means every node has the same infectivity, no matter its degree. In this case,  $\lambda_c = \frac{1}{A}$  is a positive threshold which is independent of the topology. But for STDs spread, different kind of nodes, such as a sex workers and a normal woman, they of course have different numbers of sexual contacts in one time step. For this reason, we think that  $\psi(k) = \alpha k$  is much more suitable than a constant  $A$  one for each node of degree  $k$ .

Since the probability that a node of connectivity  $k$  is connected to an isolated node is zero, so we only consider the situation that  $k \geq 1$  in our paper. So system (1), combined with (2) and the initial conditions  $i_{1k}(0) = i_{1k}^0$ ,  $i_{2k}(0) = i_{2k}^0$ ,  $r_k(0) = r_k^0$  and  $s_k(0) = 1 - i_{1k}^0 - i_{2k}^0 - r_k^0$ , completely define the  $SI_1I_2RS$  model on an uncorrelated network with degree distribution  $P(k)$ .

### 3. Some results

#### 3.1. The threshold $R_0$ on infinite scale-free network

In this subsection, we discuss the existence of the epidemic equilibrium solution of system (1). We have the following Theorem 1.

**Theorem 1.** Define

$$R_0 = \Pi \frac{\langle k^2 \rangle}{\langle k \rangle}, \quad (3)$$

where  $\Pi = \frac{\alpha[p\beta_2 + (1-p)\beta_1]}{\eta}$ . There always exists a disease-free equilibrium solution  $E^0 = \{(1, 0, 0, 0)\}$  for system (1). When  $R_0 > 1$ , one and only one epidemic equilibrium solution of system (1) exists.

**Proof.** Since  $s_k$ ,  $i_{1k}$ ,  $i_{2k}$  and  $r_k$  represent the relative densities of nodes of degree  $k$  and we do not consider the death rate of each node, so these variables obey that

$$s_k(t) + i_{1k}(t) + i_{2k}(t) + r_k(t) = 1. \quad (4)$$

To get the epidemic solution  $(s_k(\infty), i_{1k}(\infty), i_{2k}(\infty), r_k(\infty))$ , we need to impose the right side of system (1) to be equal to zero. Then any equilibrium solution  $(s_k(\infty), i_{1k}(\infty), i_{2k}(\infty), r_k(\infty))$  should satisfy

$$\begin{aligned} i_{1k}(\infty) &= \frac{k\gamma(1-p)\Theta(\infty)}{\gamma\eta + k(\gamma + \eta)\Theta(\infty)}, & i_{2k}(\infty) &= \frac{p}{1-p}i_{1k}(\infty), \\ r_k(\infty) &= \frac{\eta}{\gamma}(i_{1k}(\infty) + i_{2k}(\infty)), & s_k(\infty) &= 1 - i_{1k}(\infty) - i_{2k}(\infty) - r_k(\infty). \end{aligned} \quad (5)$$

For simplicity, we omit the symbol  $\infty$  in the following. Substitute  $i_{1k}$  and  $i_{2k}$  of (5) in (2), we obtain a self-consistency equation as follows:

$$\Theta = \frac{\beta_1(1-p) + \beta_2 p}{\langle k \rangle} \sum_{k=1}^{\infty} k^2 \cdot \frac{\gamma \Theta}{\gamma \eta + k(\gamma + \eta) \Theta} \cdot P(k) = \frac{\eta \Pi}{\langle k \rangle} \left\langle k^2 \cdot \frac{\gamma \Theta}{\gamma \eta + k(\gamma + \eta) \Theta} \right\rangle \equiv f(\Theta). \quad (6)$$

Obviously,  $\Theta \equiv 0$  is a solution of (6), i.e.,  $f(0) = 0$ . So if a non-trivial solution exists, it should satisfy

$$g(\Theta) \equiv 1 - \frac{\eta \Pi}{\langle k \rangle} \sum_{k=1}^{\infty} k^2 \cdot \frac{\gamma}{\gamma \eta + k(\gamma + \eta) \Theta} \cdot P(k) = 1 - \frac{\eta \Pi}{\langle k \rangle} \left\langle k^2 \cdot \frac{\gamma}{\gamma \eta + k(\gamma + \eta) \Theta} \right\rangle = 0. \quad (7)$$

Taking into account that

$$\frac{dg(\Theta)}{d\Theta} > 0, \quad \lim_{\Theta \rightarrow +\infty} g(\Theta) = 1,$$

which means,  $g(\Theta)$  is a monotone increasing function and it tends to a positive value when  $\Theta$  tends to positive infinity. So a non-trivial solution exists if and only if

$$g(0) < 0 \quad \text{i.e.,} \quad \Pi \frac{\langle k^2 \rangle}{\langle k \rangle} > 1,$$

which yields the critical epidemic threshold  $R_0$  given in Eq. (3). So when  $R_0 > 1$ , one and only one epidemic equilibrium solution of system (1) exists. This finishes the proof.  $\square$

Clearly, for an infinite scale-free networks (in which situation  $\langle k^2 \rangle \rightarrow \infty$ ), the epidemic processes of our model do not possess an epidemic threshold, below which diseases cannot produce a major epidemic outbreak, like the results of standard SIS model and SIR model [17,18].

### 3.2. The stability of disease-free equilibrium and the persistence of STDs on a hard cutoff scale-free network

Real systems are actually made up by a finite number of individuals. This finite population introduces a maximum connectivity  $k_c$ . In this section, we will discuss the stability of equilibria for a hard cutoff scale-free network [1].

First we recall a theorem by Lajmanovich and York [19] that will be useful as a lemma in the following.

**Lemma 1** (Lajmanovich and York). Consider the system

$$\frac{dy}{dt} = Ay + N(y), \quad (8)$$

where  $A$  is an  $n \times n$  matrix and  $N(y)$  is continuously differentiable in a region  $D \subset R^n$ . Assume

1. the compact convex set  $C \subset D$  is positively invariant with respect to the system (8), and  $0 \in C$ ;
2.  $\lim_{y \rightarrow 0} \|N(y)\|/\|y\| = 0$ ;
3. there exist  $r > 0$  and a (real) eigenvector  $\omega$  of  $A^T$  such that  $(\omega \cdot y) \geq r\|y\|$  for all  $y \in C$ ;
4.  $\omega \cdot N(y) < 0$  for all  $y \in C$ ;
5.  $y = 0$  is the largest positively invariant set [for (8)] contained in  $H = \{y \in C \mid (\omega \cdot N(y)) = 0\}$ . Then either  $y = 0$  is globally asymptotically stable in  $C$ , or for any  $y_0 \in C - \{0\}$  the solution  $\phi(t, y_0)$  of (8) satisfies  $\liminf_{t \rightarrow \infty} \|\phi(t, y_0)\| \geq m$ , where  $m > 0$ , independent of  $y_0$ . Moreover, there exists a constant solution of (8),  $y = k$ ,  $k \in C - \{0\}$ .

About the stability of the disease-free equilibrium we have the following Theorem 2.

**Theorem 2.** Define  $R_0$  as (3). Then

1. when  $R_0 < 1$ , the disease-free equilibrium solution is globally asymptotically stable;
2. when  $R_0 > 1$ , one and only one epidemic equilibrium solution of system (1) exists. And, system (1) is permanent of infection, i.e., there exists an  $\varepsilon > 0$ , such that

$$\liminf_{t \rightarrow \infty} \{i_{1k}(t), i_{2k}(t)\}_{k=1}^{K_c} > \varepsilon,$$

for any solution of (1) with  $s_k(0) > 0$ ,  $i_{1k}(0) > 0$  (or  $i_{2k} > 0$  or both hold) and  $r_k \geq 0$ .

**Proof.** From (4) we know that system (1) can be rewritten as

$$\begin{cases} \frac{di_{1k}(t)}{dt} = (1-p)k[1-i_{1k}(t)-i_{2k}(t)-r_k(t)]\Theta(t) - \eta i_{1k}(t), \\ \frac{di_{2k}(t)}{dt} = pk[1-i_{1k}(t)-i_{2k}(t)-r_k(t)]\Theta(t) - \eta i_{2k}(t), \\ \frac{dr_k(t)}{dt} = \eta[i_{1k}(t)+i_{2k}(t)] - \gamma r_k(t). \end{cases} \quad (9)$$

Define  $C = \{(s_k, i_{1k}, i_{2k}, r_k)\}_{k=1}^{\kappa_c}$ ,  $s_k \geq 0$ ,  $i_{1k} \geq 0$ ,  $i_{2k} \geq 0$ ,  $r_k \geq 0$ ,  $s_k + i_{1k} + i_{2k} + r_k = 1$ ,  $k = 1, 2, \dots, \kappa_c$ . In the following, we discuss the dynamic of (1) in compact convex set  $C$ .

For simplicity, let

$$\frac{sP(s)}{\langle k \rangle} \equiv q_s. \quad (10)$$

So  $q_k$  is a function of degree  $k$ ,  $k = 1, 2, \dots, \kappa_c$ .

The Jacobian matrix of the disease-free equilibrium of system (9) which is a  $3\kappa_c \times 3\kappa_c$  matrix can be written as follows:

$$J = \begin{bmatrix} A & \cdots & B \\ \vdots & \ddots & \vdots \\ C & \cdots & D \end{bmatrix},$$

where

$$\begin{aligned} A &= \begin{bmatrix} (1-p)\beta_1\alpha q_1 - \eta & (1-p)\beta_2\alpha q_1 & 0 \\ p\beta_1\alpha q_1 & p\beta_2\alpha q_1 - \eta & 0 \\ \eta & \eta & -\gamma \end{bmatrix}, \\ B &= \begin{bmatrix} (1-p)\beta_1\alpha q_{\kappa_c} & (1-p)\beta_2\alpha q_{\kappa_c} & 0 \\ p\beta_1\alpha q_{\kappa_c} & p\beta_2\alpha q_{\kappa_c} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ C &= \begin{bmatrix} \kappa_c(1-p)\beta_1\alpha q_1 & \kappa_c(1-p)\beta_2\alpha q_1 & 0 \\ \kappa_c p\beta_1\alpha q_1 & \kappa_c p\beta_2\alpha q_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \end{aligned}$$

and

$$D = \begin{bmatrix} \kappa_c(1-p)\beta_1\alpha q_{\kappa_c} - \eta & \kappa_c(1-p)\beta_2\alpha q_{\kappa_c} & 0 \\ \kappa_c p\beta_1\alpha q_{\kappa_c} & \kappa_c p\beta_2\alpha q_{\kappa_c} - \eta & 0 \\ \eta & \eta & -\gamma \end{bmatrix}.$$

Using *mathematical induction method* we can calculate that, the polynomial equation of the disease-free equilibrium is

$$(\lambda + \gamma)^{\kappa_c} (\lambda + \eta)^{2\kappa_c - 1} \cdot \left[ (\lambda + \eta) - \alpha[(1-p)\beta_1 + p\beta_2] \frac{\langle k^2 \rangle}{\langle k \rangle} \right] = 0. \quad (11)$$

So, there exists a unique positive eigenvalue  $\lambda$  of  $J$  if and only if  $R_0 > 1$ , under which, the unique epidemic equilibrium exists. Otherwise all real-valued eigenvalues of  $J$  are negative. According to the Perron–Frobenius theorem, this implies that the maximal of the real parts of all eigenvalues of  $J$  is positive if and only if  $R_0 > 1$ . From Lemma 1 we finish the proof of Theorem 2.  $\square$

Our results show that, a sexual network with finite variance will have an epidemic threshold for positive transmissibility, for which situation, immunization strategies are quite necessary to reduce the threshold.

#### 4. Immunization strategies

Vaccination is very helpful in controlling vaccine preventable disease [13,23,24]. In this section we discuss system (1) on a scale-free network with two immunization schemes: the proportional immunization and the targeted immunization.

#### 4.1. Proportional immunization

Let  $\delta$  be the immunization rate,  $0 < \delta < 1$ , then system (1) becomes

$$\begin{cases} \frac{ds_k(t)}{dt} = \gamma r_k(t) - k(1 - \delta)s_k(t)\Theta(t), \\ \frac{di_{1k}(t)}{dt} = k(1 - p)(1 - \delta)s_k(t)\Theta(t) - \eta i_{1k}(t), \\ \frac{di_{2k}(t)}{dt} = pk(1 - \delta)s_k(t)\Theta(t) - \eta i_{2k}(t), \\ \frac{dr_k(t)}{dt} = \eta[i_{1k}(t) + i_{2k}(t)] - \gamma r_k(t). \end{cases}$$

Similar to system (1) we get that

$$\begin{aligned} i_{1k}(\infty) &= \frac{k\gamma(1 - \delta)(1 - p)\Theta(\infty)}{\gamma\eta + k(1 - \delta)(\gamma + \eta)\Theta(\infty)}, & i_{2k}(\infty) &= \frac{p}{1 - p}i_{1k}(\infty), \\ r_k(\infty) &= \frac{\eta}{\gamma}(i_{1k}(\infty) + i_{2k}(\infty)), & s_k(\infty) &= 1 - i_{1k}(\infty) - i_{2k}(\infty) - r_k(\infty), \end{aligned}$$

and

$$\Theta = \frac{(1 - \delta)\eta\Pi}{\langle k \rangle} \left\langle k^2 \cdot \frac{\gamma\Theta}{\gamma\eta + k(1 - \delta)(\gamma + \eta)\Theta} \right\rangle \equiv \bar{f}(\Theta).$$

By similar arguments to those in Section 3, the epidemic threshold is determined by the following inequality:

$$\left. \frac{d\bar{f}(\Theta)}{d\Theta} \right|_{\Theta=0} = (1 - \delta)\Pi \cdot \frac{\langle k^2 \rangle}{\langle k \rangle} = (1 - \delta)R_0 > 1. \quad (12)$$

From (12) we can get the following result.

**Theorem 3.** Suppose  $R_0 > 1$ . Define

$$\hat{R}_0 = (1 - \delta)R_0, \quad \delta_c = 1 - \frac{1}{\Pi} \cdot \frac{\langle k \rangle}{\langle k^2 \rangle}.$$

1. When  $\delta > \delta_c$  (we have  $\hat{R}_0 < 1$ ), then the epidemic cannot spread in the network.
2. Otherwise,
  - (a) when  $\delta = 0$ , that is, no immunization were done, then  $\hat{R}_0 = R_0 > 1$ ;
  - (b) when  $0 < \delta < \delta_c$ , i.e.,  $1 < \hat{R}_0 < R_0$ . This means the immunization strategy is effective, but not so effective to control the spread of STDs in network.

#### 4.2. Targeted immunization

Due to the heterogeneous nature of scale-free network: it is robust to random attacks, but fragile to selective attacks. Now we discuss another strategy, which is named targeted immunization scheme. Suppose all nodes with connectivity  $k \geq k^*$  will be immunized, here  $k^*$  is an upper threshold. So the immunization rate  $\delta_k$  can be defined as

$$\delta_k = \begin{cases} 1, & k \geq k^*, \\ 0, & k < k^*. \end{cases}$$

Then the epidemic dynamic model is

$$\begin{cases} \frac{ds_k(t)}{dt} = \gamma r_k(t) - k(1 - \delta_k)s_k(t)\Theta(t), \\ \frac{di_{1k}(t)}{dt} = k(1 - p)(1 - \delta_k)s_k(t)\Theta(t) - \eta i_{1k}(t), \\ \frac{di_{2k}(t)}{dt} = pk(1 - \delta_k)s_k(t)\Theta(t) - \eta i_{2k}(t), \\ \frac{dr_k(t)}{dt} = \eta[i_{1k}(t) + i_{2k}(t)] - \gamma r_k(t), \end{cases}$$

which leads to

$$\Theta = \frac{\eta \Pi}{\langle k \rangle} \left\langle k^2 \cdot \frac{(1 - \delta_k) \gamma \Theta}{\gamma \eta + k(1 - \delta_k)(\gamma + \eta) \Theta} \right\rangle \equiv \hat{f}(\Theta).$$

The epidemic threshold is determined by the following inequality:

$$\left. \frac{d\hat{f}(\Theta)}{d\Theta} \right|_{\Theta=0} = \Pi \cdot \frac{\langle k^2 \rangle - \langle \delta_k k^2 \rangle}{\langle k \rangle} > 1.$$

Define  $\bar{\delta}$  to be the average immunization rate, that is,  $\bar{\delta} = \sum_k \delta_k P(k)$ . Since

$$\langle \delta_k k^2 \rangle = \langle \delta_k \rangle \cdot \langle k^2 \rangle + \text{Cov}(\delta_k, k^2) = \bar{\delta} \cdot \langle k^2 \rangle + \langle (\delta_k - \bar{\delta}) \cdot (k^2 - \langle k^2 \rangle) \rangle = \bar{\delta} \cdot \langle k^2 \rangle + \langle (\delta_k - \bar{\delta}) \cdot (k^2 - \langle k^2 \rangle) \rangle.$$

For appropriate  $k$ ,  $\delta_k - \bar{\delta}$  and  $k^2 - \langle k^2 \rangle$  have the same signs, except for some  $k$ 's where  $\delta_k = \bar{\delta}$  and/or  $k^2 = \langle k^2 \rangle$ . So  $\text{Cov}(\delta_k, k^2) > 0$ . Then we have the following Theorem 4.

**Theorem 4.** Define

$$\check{R}_0 = \Pi \cdot \frac{\langle k^2 \rangle - \langle \delta_k k^2 \rangle}{\langle k \rangle}.$$

Then

1. when  $\check{R}_0 < 1$ , STDs can be controlled by the targeted immunization. Otherwise, epidemic still exists.
2.  $\check{R}_0 < R_0$ , which means the targeted immunization is effective.
3.  $\check{R}_0 < \frac{1-\bar{\delta}}{1-\delta} R_0$ . If  $0 < \bar{\delta} = \delta < 1$ , then  $\check{R}_0 < \hat{R}_0$ . This means, the targeted immunization strategy is more efficient than the proportional strategy for the same average immunization rate.

Since the non-usage of condom  $p$  is the only controllable parameter in system (1), so as the end of this section, we would like to discuss the effective condom using rate to control STDs spread on scale-free network for a given immunization rate (i.e.,  $\delta$  or  $k^*$  are given).

**Theorem 5.** Suppose  $R_0 > 1$ . Define

$$\begin{aligned} \tilde{p} &= \frac{\eta \langle k \rangle - \alpha \beta_1 \langle k^2 \rangle}{\alpha (\beta_2 - \beta_1) \langle k^2 \rangle}, \\ \hat{p} &= \frac{\eta \langle k \rangle - \alpha \beta_1 (1 - \delta) \langle k^2 \rangle}{\alpha (1 - \delta) (\beta_2 - \beta_1) \langle k^2 \rangle}, \\ \check{p} &= \frac{\eta \langle k \rangle - \alpha \beta_1 [\langle k^2 \rangle - \langle \delta_k k^2 \rangle]}{\alpha (\beta_2 - \beta_1) [\langle k^2 \rangle - \langle \delta_k k^2 \rangle]}. \end{aligned}$$

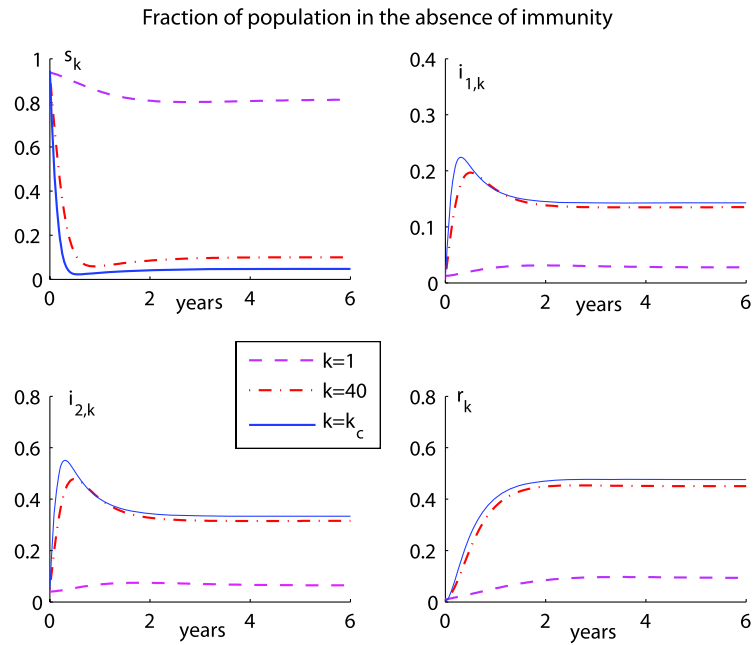
1. When  $p < \tilde{p}$ , STDs can be controlled in the network without immunization.
2. When  $p < \hat{p}$ , STDs can be controlled in the network under proportional immunization with immunization rate  $\delta$ .
3. When  $p < \check{p}$ , STDs can be controlled in the network under targeted immunization with immunization rate  $\delta_k$ .

## 5. Simulations

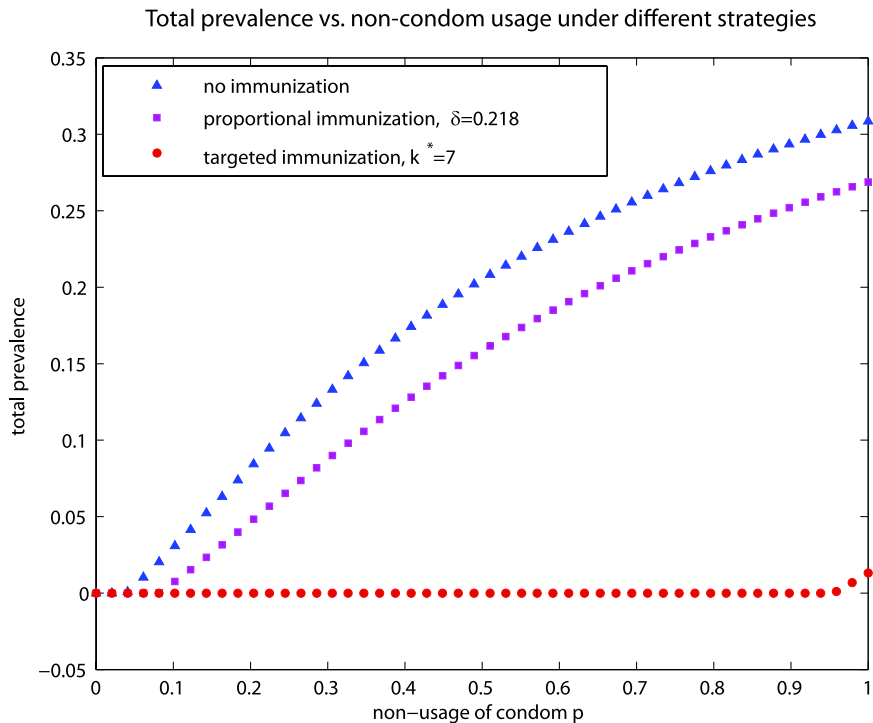
In this section we present the results of numerical simulations investigating the effectiveness of the immunization schemes. We use the preferential attachment algorithm of Barabási and Albert to generate a network with theoretical scale-free exponent 3. Parameters that are used in the simulations are listed as follows:  $\beta_1 = 0.05$ ,  $\beta_2 = 0.5$ ,  $\eta = 1$ ,  $\gamma = 1$ ,  $p = 0.7$  and  $\alpha = 1$ . Under the parameters above, the basic reproduction number  $R_0 = 5.3921 > 1$ , which implies that the disease will persist without immunization. According to Theorem 3,  $\delta_c = 0.8145$ . So under proportional immunization, the immunization rate should be as high as 0.8145 to control disease.

In the following, we give simulations from two angles:

1. The dynamics of nodes with different degree  $k$ . We investigate the dynamics of all fractions of the population as a function of time under different immunization policies (Fig. 1).
2. The total prevalence that changes as a function of parameters under different immunization policies. We mainly discuss the effect of the controllable parameter  $p$ , which is the non-usage of condom (Figs. 2 and 3).



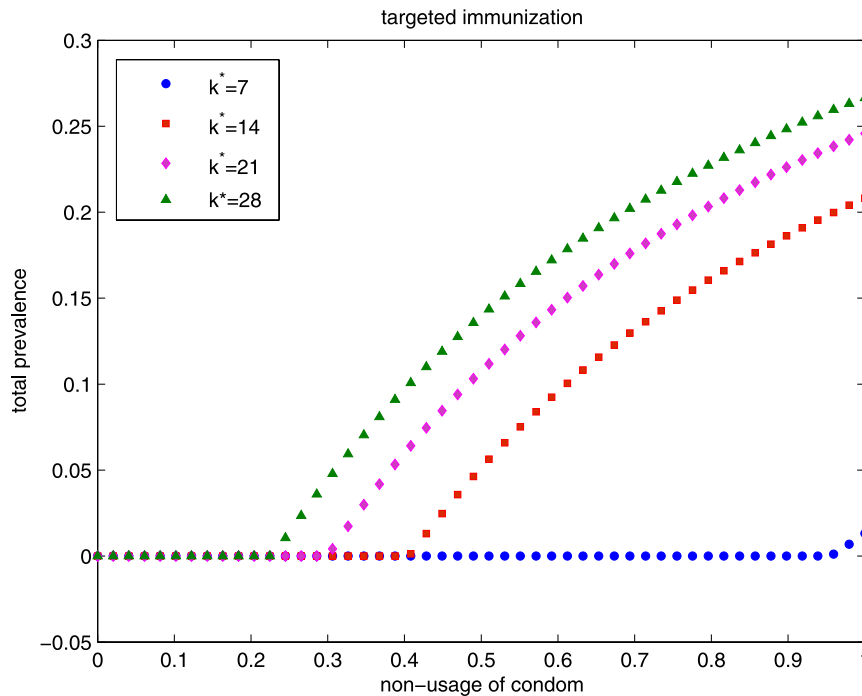
**Fig. 1.** Dynamics of nodes with degree  $k=1$ ,  $k=40$  and degree  $k=\kappa_c$ : fraction of the population as a function of time in the absence of immunity. The solid curve is of  $k=\kappa_c=89$ . The dash curve is of  $k=1$  and the dash-dot curve is of  $k=40$ . Obviously, the outcomes of  $k=40$  and  $k=\kappa_c$  have not big difference. The infection rate of  $i_{2k}$  is much higher than that of  $i_{1k}$ .



**Fig. 2.** The total prevalence changes with  $p$  for different immunization tactics. To control STDs spread on network, the non-usage of condom is 3.93%, 8.12% and 95.78% respectively for no immunization, proportional immunization of  $\delta=0.218$  and targeted immunization of  $k^*=7$  (according to Theorem 5).

Fig. 1 shows the dynamics of nodes change as a function of time for nodes with degree  $k=1$ ,  $k=40$  and  $k=\kappa_c$  respectively in the absence of immunity. Obviously, the prevalence rate of  $i_{2k}$  is much higher than that of  $i_{1k}$ . Interestingly, the outcomes of  $k=40$  and  $k=\kappa_c$  have not big difference.





**Fig. 3.** The total prevalence changes with  $p$  for different  $k^*$ . When nodes above degree 7 (include 7) are vaccinate (i.e.,  $k^* = 7$ ), more than 95% individual do not need to use condoms anymore. But there is not very big difference in effects to vaccinate nodes that degrees are bigger than 14 (i.e.,  $k^* \geq 14$ ).

What is the suitable condom using rate if the fund to vaccinate people is limit, in other words,  $\delta$  or  $k^*$  are limit? Suppose the average immunization rates for proportional immunization and targeted immunization are the same, for example,  $\delta = \bar{\delta} = 0.218$  (which is corresponding to  $k^* = 7$ ). Then we can get a comparison in Fig. 2 for the total prevalence as a function of non-usage of condom  $p$ . Fig. 2 shows that, as far as condom using rate, targeted immunization has the absolute advantage to control STDs spread compare with proportional immunization. Every point of all curves is obtained under enough long time running our programs which can guarantee that the system can infinitely near equilibrium.

Finally, we discuss the relation between the total prevalence and the suitable non-usage of condom  $\check{p}$  for different  $k^*$  in targeted immunization. We suppose  $k^* = 7, 14, 21$  and  $28$  respectively. According to Theorem 5, the corresponding  $\check{p} = 0.9578, 0.4067, 0.2998$  and  $0.2286$  respectively. Simulation results can be found in Fig. 3. Immunize these nodes that degrees are bigger than 7 (i.e.,  $k^* = 7$ ) can get very good result. But if the degrees of nodes that are immunized are bigger than 14 (i.e.,  $k^* \geq 14$ ), the final outcomes seem no big difference.

## 6. Discussion

Recent research into the properties of human sexual-contact networks has suggested that the degree distribution of the contact graph exhibits power-law scaling. One notable property of this power-law scaling is that the epidemic threshold for the population disappears when the scaling exponent  $\rho$  is in the range  $2 < \rho \leq 3$  for SIS and SIR models. Our multiple SIRS model also shows this character.

According to the actual meaning of node degree of STDs, infectivity between nodes in our model is modeled as a linear function of the node degree rather than a constant  $A$  since a commercial sex worker of course has much more sex partners in one unit time (such as one year) than a normal woman. So we think the constant infectivity  $A$  is not suitable at least for STDs spread model.

From a mathematical aspect, such  $SI_1I_2RS$ -type models on networks can be viewed as multi-type SIRS models [19,20] if the networks possess the bounded degree property. Ours simulations have proved the necessary to build a multi-type SIRS model for the STDs spread, which can be embodied by the non-usage of condom  $p$ . In fact,  $p$  is a controllable parameter compare with  $\beta_1, \beta_2, \eta$  and  $\gamma$ . For this reason, it has remarkable effect to give some more discussion on  $p$ . Especially under immunization schemes, we gave some useful discuss for the suitable usage of condom under the given limit fund of immunization.

The effects of two immunization schemes including proportional scheme and targeted vaccination are studied and compared in our paper. Both theory and simulations show that, the targeted immunization has prominent strengths compare with proportional immunization. The targeted immunization scheme maybe is difficult for some disease spread such as

SARS, but it is not so difficult for STDs spread since it is much easier to find these nodes with high degree  $k$ . We suggest to use target immunization scheme to decrease the spread of STDs, maybe also the spread of HIV/AIDS.

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