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Acoustic limit for the Boltzmann equation in the whole space

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A R T I C L E   I N F O

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A B S T R A C T

This paper is devoted to the following rescaled Boltzmann equation in the acoustic time scaling in the whole space

$$\partial_t F^\epsilon + v \cdot \nabla_x F^\epsilon = \frac{1}{\epsilon} Q\left(F^\epsilon, F^\epsilon\right), \quad x \in \mathbf{R}^3,\ t > 0, \tag{0.1}$$

with prescribed initial data

$$F^\epsilon|_{t=0} = F^\epsilon(0, x, v), \quad x \in \mathbf{R}^3.$$

For a solution

$$F^\epsilon(t, x, v) = \mu + \sqrt{\mu}\epsilon f^\epsilon(t, x, v)$$

to the rescaled Boltzmann equation (0.1) in the whole space $\mathbf{R}^3$ for all $t \geqslant 0$ with initial data

$$F^\epsilon(0, x, v) = F_0^\epsilon(x, v) = \mu + \sqrt{\mu}\epsilon f^\epsilon(0, x, v), \quad x, v \in \mathbf{R}^3,$$

our main purpose is to justify the global-in-time uniform energy estimates of $f^\epsilon(t, x, v)$ in ϵ and prove that $f^\epsilon(t, x, v)$ converges strongly to $f(t, x, v)$ whose dynamic is governed by the acoustic system.

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1. Introduction and main results

In recent years the study on limiting process leading from the rescaled Boltzmann equation of the classical kinetic theory to the equations governing the macroscopic fluid dynamics has been observed by many mathematicians (see [1–4, 6,8,9,16,18] and the references therein). While there has been considerable success at the formal level, full mathematical justifications have proven elusive.

- The program initiated by Bardos, Golse, and Levermore [2] was to derive the fluid limits which include incompressible Stokes, Navier–Stokes, Euler equations, and acoustic system from the DiPerna–Lions renormalized solutions.
- The rigorous proofs of convergence of DiPerna–Lions' renormalized solutions to the Leray–Hopf's weak solutions of the incompressible Navier–Stokes–Fourier equations were given by F. Golse and L. Saint-Raymond [9].
- Higher order approximations with the unified energy method have been shown by Y. Guo [16] to give rise to a rigorous passage from the Boltzmann equations to the Navier–Stokes–Fourier systems beyond the Navier–Stokes approximations in the framework of classical solutions.
- In a periodic box, in [3], the acoustic limit was justified for Maxwell molecular collisions under some assumption on the amplitude of the fluctuations, and recently, J. Jang and N. Jiang [18] established the acoustic limit from the Boltzmann equation in the framework of classical solutions.

The acoustic system is the linearization of the compressible Euler system about a constant state. In the case where this constant state corresponds to density and temperature equal to 1 and velocity field equal to 0, the fluid fluctuations (ρ, u, θ) satisfy

$$\begin{aligned}
&\partial_t\rho + \nabla\cdot u = 0, \qquad \rho(x,0) = \rho^0(x),\\
&\partial_t u + \nabla\cdot(\rho+\theta) = 0, \qquad u(x,0) = u^0(x),\\
&\partial_t\theta + \frac{2}{3}\nabla\cdot u = 0, \qquad \theta(x,0) = \theta^0(x).
\end{aligned} \tag{1.1}$$

This is one of the simplest system of fluid dynamical equations imaginable, being essentially the wave equation. It may be derived directly from the Boltzmann equation as the formal limit of moment equations for an appropriately scaled family of Boltzmann solutions as the Knudsen number tends to zero.

However, the status for rigorously deriving the acoustic system from DiPerna–Lions solutions of Boltzmann equation in the whole space is still incomplete. In this paper, we try to establish the acoustic limit from the Boltzmann equation in the whole space in the framework of classical solutions. Working with classical solutions has several advantages than working with the DiPerna–Lions solutions. For example, the classical solutions automatically satisfy local conservation laws and have good regularities; the nonlinear interaction can be controlled by linear dissipation for small solutions.

In this paper, we consider the following rescaled Boltzmann and Landau equation for dynamics of dilute particles, the equation reads as:

$$\partial_t F^\epsilon + v\cdot\nabla_x F^\epsilon = \frac{1}{\epsilon} Q\left(F^\epsilon, F^\epsilon\right). \tag{1.2}$$

For the Boltzmann equation, the collision operator Q is given by the following form

$$Q(f,g) = \int_{\mathbf{R}^3}\int_{\mathbf{S}^2} |v-u|^\gamma f\left(u'\right) g\left(v'\right) B(\theta)\,du\,d\omega - \int_{\mathbf{R}^3}\int_{\mathbf{S}^2} |v-u|^\gamma f(u) g(v) B(\theta)\,du\,d\omega, \tag{1.3}$$

where $u' = u + ((v-u)\cdot\omega)\omega$, $v' = v - ((v-u)\cdot\omega)\omega$ denote the velocities of the particles after the collision, with u and v being their velocities before the collision, and

$$-N < \gamma \leqslant 1, \qquad B(\theta) \leqslant c|\cos\theta|, \qquad \cos\theta = \frac{(u-v)\cdot\omega}{|u-v|}. \tag{1.4}$$

Clearly, under a slightly generalized version of Grad's angular cutoff, this is satisfied by the hard potential case with $0 \leqslant \gamma \leqslant 1$ and by the soft potential case with $-3 < \gamma < 0$.

As to the Landau equation, the collision operator Q is defined as

$$Q(f,g) = \nabla\cdot\left\{\int_{\mathbf{R}^3} \phi(v-u)\left[f(u)\nabla_v g(v) - g(v)\nabla_u f(u)\right] du\right\}, \tag{1.5}$$

where

$$\phi_{ij}(v) = \left\{\delta_{ij} - \frac{v_i v_j}{|v|^2}\right\}|v|^{\gamma+2}, \quad \gamma \geqslant -3. \tag{1.6}$$

Let

$$F^\epsilon = \mu + \epsilon\sqrt{\mu} f^\epsilon$$

be the perturbation around the global Maxwellian

$$\mu = e^{-|v|^2}.$$

Eq. (1.2) is written in terms of the perturbation f^ϵ as follows:

$$\partial_t f^\epsilon + v\cdot\nabla_x f^\epsilon + \frac{1}{\epsilon} L f^\epsilon = \Gamma\big(f^\epsilon, f^\epsilon\big). \tag{1.7}$$

For the linearized Boltzmann collision operator L, we have (see [5,7,10,12,14])

$$Lg = \nu(v)g - Kg,$$

where the collision frequency $\nu(v)$ is

$$\nu(v) = \int_{\mathbf{R}^3} |v-u|^\gamma \mu(u) B(\theta)\, du\, d\omega = c\int_{\mathbf{R}^3} |v-u|^\gamma \mu(u)\, du, \tag{1.8}$$

for some constant $c > 0$; notice that $\nu(v)$ behaves like $(1+|v|)^\gamma$, and where the operator $K = K_2 - K_1$ is defined as in [14]

$$[K_1 g](v) = \int |u-v|^\gamma \mu^{\frac{1}{2}}(u)\mu^{\frac{1}{2}}(v) g(u) B(\theta)\, du\, d\omega, \tag{1.9}$$

$$[K_2 g](v) = \int |u-v|^\gamma \mu^{\frac{1}{2}}(u)\big\{\mu^{\frac{1}{2}}\big(u'\big) g\big(v'\big) + \mu^{\frac{1}{2}}\big(v'\big) g\big(u'\big)\big\} B(\theta)\, du\, d\omega. \tag{1.10}$$

Moreover, in [14] the (non-symmetric) bilinear form $\Gamma[g_1, g_2]$ is given by

$$\begin{aligned}\Gamma[g_1,g_2] &= \mu^{-\frac{1}{2}}(v) Q\big[\mu^{-\frac{1}{2}} g_1, \mu^{-\frac{1}{2}} g_2\big] = \Gamma_{gain}[g_1,g_2] - \Gamma_{loss}[g_1,g_2] \\ &= \int_{\mathbf{R}^3}\int_{\mathbf{S}^2} |u-v|^\gamma B(\theta)\mu^{\frac{1}{2}}(u) g_1\big(u'\big) g_2\big(v'\big)\, du\, d\omega - g_2(v)\int_{\mathbf{R}^3}\int_{\mathbf{S}^2} |u-v|^\gamma B(\theta)\mu^{\frac{1}{2}}(u) g_1(u)\, du\, d\omega.\end{aligned} \tag{1.11}$$

For the Landau equation, the linearized collision operator L is defined by $Lf = -Af - Kf$, where

$$Af = \frac{1}{\sqrt{\mu}} Q(\sqrt{\mu} f, \mu), \qquad Kf = \frac{1}{\sqrt{\mu}} Q(\mu, \sqrt{\mu} f),$$

and the bilinear collision operator $\Gamma(f,g)$ is given by $\Gamma(f,g) = \frac{1}{\sqrt{\mu}} Q(\sqrt{\mu} f, \sqrt{\mu} g)$.

We first recall that the operator $L \geqslant 0$, and for any fixed (t,x), the null space of L is generated by $[\sqrt{\mu}, v\sqrt{\mu}, |v|^2\sqrt{\mu}]$. For any function $f(t,x,v)$ we thus can decompose

$$f = \mathbf{P}f + (\mathbf{I}-\mathbf{P})f,$$

where $\mathbf{P}f$ (the hydrodynamic part) is the L^2_v projection on the null space for L for the given (t,x). We can further denote

$$\mathbf{P}f = \left\{\rho_f(t,x) + v\cdot u_f(t,x) + \left(\frac{|v|^2}{2} - \frac{3}{2}\right)\theta_f(t,x)\right\}\sqrt{\mu}.$$

For notational simplicity, we use $\langle\cdot,\cdot\rangle$ to denote the L^2_v inner product in $\mathbf{R}^3_v$, with its L^2 norm given by $|\cdot|$, moreover $(\cdot,\cdot)$ is the L^2 inner product either in $\mathbf{R}^3_x \times \mathbf{R}^3_v$ or in $\mathbf{R}^3_x$ with corresponding L^2 norm $\|\cdot\|$. We use the standard notation H^s to denote the Sobolev space. For the Boltzmann equation, we introduce a weight function of v as $w_1(v) = [1+|v|]^\gamma$, while we introduce another weight function of v as $w_1(v) = [1+|v|]^{\gamma+2}$ for the Landau equation. We always denote the two weight functions as $w = w(v)$.

For the Boltzmann equation, define the dissipation norm as

$$|g|^2_\nu = \int_{\mathbf{R}^3} g^2(v)\nu(v)\, dv, \qquad \|g\|^2_\nu = \int_{\mathbf{R}^3\times\mathbf{R}^3} g^2(x,v)\nu(v)\, dx\, dv.$$

For the Landau equation, on the other hand, let

$$\sigma_{ij} = \int_{\mathbf{R}^3} \phi^{ij}(v-u)\mu(u)\, du.$$

The natural norms are given by the σ-norm

$$|g|_\sigma^2 = \sum_{i,j=1}^{3} \int_{\mathbf{R}^3} \{\sigma_{ij}\partial^i g \partial^j g + \sigma_{ij} v^i v^j g^2\}\, dv,$$

$$\|g\|_\sigma^2 = \sum_{i,j=1}^{3} \int_{\mathbf{R}^3\times\mathbf{R}^3} \{\sigma_{ij}\partial^i g \partial^j g + \sigma_{ij} v^i v^j g^2\}\, dx\, dv.$$

We also use a unified notation for dissipation as $|g|_D$ and $\|g\|_D$ to denote either $|g|_\nu$ or $|g|_\sigma$, $\|g\|_\nu$ or $\|g\|_\sigma$, respectively.

In order to be consistent with the hydrodynamic equations, we define

$$\partial_\alpha^\beta = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}\partial_{v_1}^{\beta_1}\partial_{v_2}^{\beta_2}\partial_{v_3}^{\beta_3},$$

where $\alpha = [\alpha_1, \alpha_2, \alpha_3]$ is related to the space derivatives, while $\beta = [\beta_1, \beta_2, \beta_3]$ is related to the velocity derivatives.

For the sake of capturing the structure of the rescaled Boltzmann equation (1.2) and getting the uniform estimates on the remainder term, we introduce the instant energy functionals $\mathcal{E}_s$ and the dissipate rate $\mathcal{D}_s$ as following.

Definition 1.1 *(Instant energy).* For $s \geqslant 8$, and some constant $C > 0$, an instant energy functional $\mathcal{E}_s(g)$ satisfies:

(i) for the hard potential case with $\gamma \geqslant 0$ in (1.8)

$$\frac{1}{C}\mathcal{E}_s(g) \leqslant \sum_{|\alpha|\leqslant s+1} \|\partial_\alpha g\|^2 + \sum_{|\alpha|+|\beta|\leqslant s} \left\|\partial_\alpha^\beta g\right\|^2 \leqslant C\mathcal{E}_s(g), \tag{1.12}$$

(ii) for the soft potential case with $-3 < \gamma < 0$ in (1.8) and Landau equation with $\gamma \geqslant -3$ in (1.6)

$$\frac{1}{C}\mathcal{E}_s(g) \leqslant \sum_{|\alpha|\leqslant s+1} \|\partial_\alpha g\|^2 + \sum_{|\alpha|+|\beta|\leqslant s} \left\|w^{|\beta|}\partial_\alpha^\beta g\right\|^2 \leqslant C\mathcal{E}_s(g), \tag{1.13}$$

for all function $g(t, x, v)$.

Definition 1.2 *(Dissipation rate).* For $s \geqslant 8$, the dissipation rate $\mathcal{D}_s(g)$ is defined as:

(i) for the hard potential case with $\gamma \geqslant 0$ in (1.8)

$$\mathcal{D}_s(g) = \sum_{0<|\alpha|\leqslant s+1} \epsilon\|\partial_\alpha \mathbf{P}g\|^2 + \sum_{|\alpha|\leqslant s+1} \frac{1}{\epsilon}\left\|\partial_\alpha(\mathbf{I}-\mathbf{P})g\right\|_\nu^2 + \frac{1}{\epsilon}\sum_{|\alpha|+|\beta|\leqslant s} \left\|\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})g\right\|_\nu^2, \tag{1.14}$$

(ii) for the soft potential case with $-3 < \gamma < 0$ in (1.8) and Landau equation with $\gamma \geqslant -3$ in (1.6)

$$\mathcal{D}_s(g) = \sum_{0<|\alpha|\leqslant s+1} \epsilon\|\partial_\alpha \mathbf{P}g\|^2 + \sum_{|\alpha|\leqslant s+1} \frac{1}{\epsilon}\left\|\partial_\alpha(\mathbf{I}-\mathbf{P})g\right\|_D^2 + \frac{1}{\epsilon}\sum_{|\alpha|+|\beta|\leqslant s} \left\|w^{|\beta|}\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})g\right\|_D^2. \tag{1.15}$$

Remark 1.1.

(i) For the soft potential and Landau equation, $\mathcal{E}_s(g)$ and $\mathcal{D}_s(g)$ involve a weight function in v which depends on the number of velocity derivatives ∂^β, this is designed to control the velocity derivatives for the screaming terms $v\cdot\nabla_x$ by a weaker dissipation rate as proposed in [14].

(ii) Noticing that zeroth-order term of macroscopic part $\mathbf{P}g$ does not appear in (1.14) and (1.15), we cannot compare the instant energy with the dissipation rate, which brings much difficulties to study the large time behaviors of the solutions.

(iii) The dissipate rates in (1.14) and (1.15) in which the hydrodynamic part $\mathbf{P}g$ has ϵ scale reflect that we do not observe the dissipation rate in the limit, which is exactly the case of the acoustic system.

Lastly, the Einstein's convention is used for Greek letters from time to time. We use c or C to denote a constant independent of ϵ which may be different from line to line. The main results are stated as follows.

1.1. Main results

Theorem 1.1. *Set $s \geqslant 8$, let $0 < \epsilon \leqslant \frac{1}{4}$ be given, assume $f^\epsilon(0,x,v) = f_0^\epsilon(x,v)$, $F^\epsilon(0,x,v) = \mu + \epsilon\sqrt{\mu} f_0^\epsilon(x,v) \geqslant 0$. For any sufficiently small $M_0 > 0$, if there exists an instant energy functional $\mathcal{E}_s$ such that*

$$\mathcal{E}_s\big(f^\epsilon\big)(0) < M_0,$$

then (1.7) *admits a unique global solution $f^\epsilon(t,x,v)$ with $F^\epsilon(t,x,v) = \mu + \epsilon\sqrt{\mu} f^\epsilon(t,x,v) \geqslant 0$ and*

$$\frac{d}{dt}\mathcal{E}_s\big(f^\epsilon\big)(t) + \mathcal{D}_s\big(f^\epsilon\big)(t) \leqslant 0. \tag{1.16}$$

Furthermore

$$\sup_{0 \leqslant t \leqslant \infty} \mathcal{E}_s\big(f^\epsilon\big)(t) \leqslant \mathcal{E}_s\big(f^\epsilon\big)(0). \tag{1.17}$$

Remark 1.2. The global existence of solution $f^\epsilon(t,x,v)$ to (1.7) follows from the a priori global energy bound (1.17) by rather standard method (see [11,14,15]). In this article, we focus on proving the uniform bound.

Recently, Y. Guo studied the classical solution of the Boltzmann equation and Landau equation by a nonlinear energy method [11–15]. And we can see that this method is also useful to study the limiting process from the rescaled Boltzmann equation to the fluid dynamic equations [16,18]. However, both of their works [16,18] are focus on the periodic boundary problems, i.e. the spatial variables $x \in \mathbf{T}^3$. In this paper, we study the acoustic limits of the rescaled Boltzmann equation in the whole space. Our results generalize the classical results in [3] to the very soft potentials for the Boltzmann equation and also extend the classical results in [18] in the periodic box to the whole space for the Boltzmann equation and Landau equation in the Coulomb interaction. Although it has the same framework as [12,13,15,19], there are several major difficulties. Firstly, since $\gamma < 0$, it is impossible to directly control v-derivatives ∂_β of the linear streaming term $v_j \partial^j f$ in the terms of the weaker $\|\partial_\beta f\|_\nu$. Thus, like [11,14,16–18], we introduce additional w-weighted functions as in $\mathcal{E}_s(f)$ and $\mathcal{D}_s(f)$ which depend on the number of v-derivatives of f to overcome this first difficulty. Secondly, the key point of Guo's method is to prove that L is actually positive definite for any solution f with small amplitude to the fully Boltzmann equation, and this fact is really true in the case of periodic box [14], since the macroscopic part $\mathbf{P}f$ is bounded by its microscopic part $(\mathbf{I}-\mathbf{P})f$ due to Poincaré's inequality and the conservation of mass momentum and energy. In the current whole space, however, only derivatives of the macroscopic part $\mathbf{P}f$ not itself can be directly estimated from the macroscopic equations, see (4.1) in Lemma 4.1, hence the nature dissipation rate $\mathcal{D}_s(f)$ is weaker than that in the periodic box. Thirdly, we have to carefully estimate the nonlinear term $\Gamma(f,f)$ via the weaker $\mathcal{D}_s(f)$ defined in (1.15) which does not contain the L^2 norm of $\mathbf{P}f$. Our idea is to split

$$\Gamma(f,f) = \Gamma(\mathbf{P}f,\mathbf{P}f) + \Gamma\big(\{\mathbf{I}-\mathbf{P}\}f,\mathbf{P}f\big) + \Gamma\big(\mathbf{P}f,\{\mathbf{I}-\mathbf{P}\}f\big) + \Gamma\big(\{\mathbf{I}-\mathbf{P}\}f,\{\mathbf{I}-\mathbf{P}\}f\big),$$

and use control of L^6 norm of $(a_f(t,x), b_f(t,x), c_f(t,x))$ in $\mathbf{R}^3$.

The remaining parts of this paper is arranged as follows. In Section 2, we list some basic estimates on the collision operators with various kernels. We give a very brief formal derivation of the acoustic system in Section 3. Section 4 is devoted to the proof of Theorem 1.1.

2. Basic estimates

In this section, we sum up some basic estimates of collision operators for various kernels considered in this paper. The following is the positivity of L.

Lemma 2.1. *(See [16].) There exists a $\delta > 0$ such that*

$$\langle Lf, f\rangle \geqslant \delta\big|(\mathbf{I}-\mathbf{P})f\big|_D^2. \tag{2.1}$$

We now summarize some estimates for x, v derivatives of the collision operators L and Γ.

Lemma 2.2. *(See [16].) For the hard potential with $0 \leqslant \gamma \leqslant 1$, let $\beta > 0$. There exists $C_{|\beta|} > 0$ such that*

$$\big(\partial_\alpha^\beta Lg, \partial_\alpha^\beta g\big) \geqslant \frac{1}{2}\big\|\partial_\alpha^\beta g\big\|_\nu^2 - C_{|\beta|}\|\partial_\alpha g\|_\nu^2. \tag{2.2}$$

The next lemma is devoted to the estimates for nonlinear collision term $\Gamma(f,g)$.

Lemma 2.3. *For the hard potential with $0 \leqslant \gamma \leqslant 1$, let f, g, h be smooth functions, and assume that $|\alpha| + |\beta| \leqslant s$, $|\beta_1| + |\beta_2| \leqslant |\beta|$, then there exists $C > 0$ such that*

$$\begin{aligned}\langle \partial_\alpha^\beta \Gamma(f,g), \partial_\alpha^\beta h\rangle \leqslant{}& C \sum_{\substack{\beta_1+\beta_2\leqslant\beta\\ \alpha_1+\alpha_2=\alpha}} \left[\int_{\mathbf{R}^3} \nu\left|\partial_{\alpha_1}^{\beta_1} f\right|^2 dv\right]^{\frac{1}{2}} \times \left[\int_{\mathbf{R}^3} \left|\partial_{\alpha_2}^{\beta_2} g\right|^2 dv\right]^{\frac{1}{2}} \times \left[\int_{\mathbf{R}^3} \nu\left|\partial_\alpha^\beta h\right|^2 dv\right]^{\frac{1}{2}}\\ &+ C \sum_{\substack{\beta_1+\beta_2\leqslant\beta\\ \alpha_1+\alpha_2=\alpha}} \left[\int_{\mathbf{R}^3} \nu\left|\partial_{\alpha_2}^{\beta_2} g\right|^2 dv\right]^{\frac{1}{2}} \times \left[\int_{\mathbf{R}^3} \left|\partial_{\alpha_1}^{\beta_1} f\right|^2 dv\right]^{\frac{1}{2}} \times \left[\int_{\mathbf{R}^N} \nu\left|\partial_\alpha^\beta h\right|^2 dv\right]^{\frac{1}{2}},\end{aligned} \tag{2.3}$$

where the summation is over $|\alpha| + |\beta| \leqslant s$.

Furthermore, suppose $\alpha_1 + \alpha_2 = \alpha > 0$, then:

if $\alpha_1 \leqslant \frac{|\alpha|}{2}$ and $\alpha_1 + \beta_1 \leqslant \frac{s}{2}$, we obtain

$$\left(\partial_\alpha^\beta \Gamma(f,g), \partial_\alpha^\beta h\right) \leqslant C \sum_{\substack{|\alpha_1'|+|\beta_1|\leqslant s\\ |\alpha_2|\geqslant\frac{|\alpha|}{2}}} \left\{\left\|\partial_{\alpha_1'}^{\beta_1} f\right\| \cdot \left\|\partial_{\alpha_2}^{\beta_2} g\right\|_\nu + \left\|\partial_{\alpha_1'}^{\beta_1} f\right\|_\nu \cdot \left\|\partial_{\alpha_2}^{\beta_2} g\right\|\right\} \left\|\partial_\alpha^\beta h\right\|_\nu, \tag{2.4}$$

if $\alpha_1 \leqslant \frac{|\alpha|}{2}$ and $\alpha_1 + \beta_1 \geqslant \frac{s}{2}$, we get

$$\left(\partial_\alpha^\beta \Gamma(f,g), \partial_\alpha^\beta h\right) \leqslant C \sum_{\substack{|\alpha_2'|+|\beta_2|\leqslant s\\ |\alpha_2'|\geqslant\frac{|\alpha|}{2}}} \left\{\left\|\partial_{\alpha_1}^{\beta_1} f\right\| \cdot \left\|\partial_{\alpha_2'}^{\beta_2} g\right\|_\nu + \left\|\partial_{\alpha_1}^{\beta_1} f\right\|_\nu \cdot \left\|\partial_{\alpha_2'}^{\beta_2} g\right\|\right\} \left\|\partial_\alpha^\beta h\right\|_\nu, \tag{2.5}$$

if $\alpha_1 \geqslant \frac{|\alpha|}{2}$ then $\alpha_2 \leqslant \frac{|\alpha|}{2}$, we can get the similar results as above.

The proof is similar as those of Lemma 2.3 and Theorem 3.1 in [12], we omit the details here for brevity.

Remark 2.1. If $|\alpha| > 0$ we can always find that both α_i and α_i' $(i = 1, 2)$ are not zero, so that the right-hand sides of (2.4), (2.5) can be bounded by $C\{\mathcal{E}_s^{\frac{1}{2}}(f)\mathcal{D}_s^{\frac{1}{2}}(g)\mathcal{D}_s^{\frac{1}{2}}(h) + \mathcal{E}_s^{\frac{1}{2}}(g)\mathcal{D}_s^{\frac{1}{2}}(f)\mathcal{D}_s^{\frac{1}{2}}(h)\}$.

We now consider the case for a soft potential, $\gamma < 0$. The following lemma is a modified version of Theorem 3 in [14].

Lemma 2.4. *For the soft potential with $-3 < \gamma < 0$, we further assume that $l \geqslant 0$, $\beta_1 + \beta_2 \leqslant \beta$, $\alpha_1 + \alpha_2 = \alpha$, $|\alpha| + |\beta| \leqslant s$,*

if $\alpha_1 + \beta_1 \leqslant \frac{s}{2}$, then

$$\begin{aligned}\left(w^{2l}\partial_\alpha^\beta \Gamma(f,g), \partial_\alpha^\beta h\right) \leqslant{}& C \sum_{|\alpha_1'|+|\beta_1'|\leqslant s} \left\{\left\|w^{|\beta_1'|}\partial_{\alpha_1'}^{\beta_1'} f\right\|\right\} \left\|w^l \partial_{\alpha_2}^{\beta_2} g\right\|_\nu \left\|w^l \partial_\alpha^\beta h\right\|_\nu\\ &+ C \sum_{|\alpha_1'|+|\beta_1|\leqslant s} \left\{\left\|w^{|\beta_1|}\partial_{\alpha_1'}^{\beta_1} f\right\|\right\} \left\|w^{l-|\beta_1|} \partial_{\alpha_2}^{\beta_2} g\right\|_\nu \left\|w^l \partial_\alpha^\beta h\right\|_\nu,\end{aligned} \tag{2.6}$$

if $\alpha_1 + \beta_1 \geqslant \frac{s}{2}$, then

$$\begin{aligned}\left(w^{2l}\partial_\alpha^\beta \Gamma(f,g), \partial_\alpha^\beta h\right) \leqslant{}& C \sum_{|\alpha_2'|+|\beta_2'|\leqslant s} \left\{\left\|w^{|\beta_2'|}\partial_{\alpha_2'}^{\beta_2'} g\right\|\right\} \left\|w^l \partial_{\alpha_1}^{\beta_1} f\right\|_\nu \left\|w^l \partial_\alpha^\beta h\right\|_\nu\\ &+ C \sum_{|\alpha_2'|+|\beta_2|\leqslant s} \left\{\left\|w^{|\beta_2|}\partial_{\alpha_2'}^{\beta_2} g\right\|\right\} \left\|w^{l-|\beta_2|} \partial_{\alpha_1}^{\beta_1} f\right\|_\nu \left\|w^l \partial_\alpha^\beta h\right\|_\nu,\end{aligned} \tag{2.7}$$

where C is a positive constant.

Performing the same calculations as in the proof of Lemma 2.4, we have the following results.

Lemma 2.5. *For the inverse power law with $-3 < \gamma < 0$, for any $l \geqslant 0$, and $\beta_1 + \beta_2 \leqslant \beta$, $\alpha_1 + \alpha_2 = \alpha$, $|\alpha| + |\beta| \leqslant s$, let $\zeta(v)$ be a function that decays exponentially. If $f = a(x)\zeta(v)$, then*

$$\left(w^{2l}\partial_\alpha^\beta \Gamma(f, g), \partial_\alpha^\beta h\right) \leqslant C \int_{\mathbf{R}^3} \left|\partial_{\alpha_1} a(x)\right| \left|w^l \partial_{\alpha_2}^{\beta_2} g\right|_\nu \left|w^l \partial_\alpha^\beta h\right|_\nu \, dx, \tag{2.8}$$

if $g = a(x)\zeta(v)$, then

$$\left(w^{2l}\partial_\alpha^\beta \Gamma(f, g), \partial_\alpha^\beta h\right) \leqslant C \int_{\mathbf{R}^3} \left|\partial_{\alpha_2} a(x)\right| \left|w^l \partial_{\alpha_1}^{\beta_1} f\right|_\nu \left|w^l \partial_\alpha^\beta h\right|_\nu \, dx. \tag{2.9}$$

Now, we give some estimates on the nonlinear Landau operators [11].

Lemma 2.6. *For the Landau kernel, for $l \geqslant 0$, $|\alpha| + |\beta| \leqslant s$, there exists $C > 0$, such that*

$$\left\langle w^{2l}\partial_\alpha^\beta \Gamma(f, g), \partial_\alpha^\beta h\right\rangle \leqslant C \sum \left[\left|w^l \partial_{\alpha_1}^{\bar\beta} f\right| \left|w^l \partial_{\alpha-\alpha_1}^{\beta-\beta_1} g\right|_\sigma + \left|w^l \partial_{\alpha_1}^{\bar\beta} f\right|_\sigma \left|w^l \partial_{\alpha-\alpha_1}^{\beta-\beta_1} g\right|\right] \left|w^l \partial_\alpha^\beta h\right|_\sigma, \tag{2.10}$$

where the summation is over $|\alpha_1| + |\beta_1| \leqslant s$, $\alpha_1 \leqslant \alpha$, $\bar\beta \leqslant \beta_1 \leqslant \beta$.

Moreover, suppose $\alpha_1 + \alpha_2 = \alpha > 0$, then:

if $\alpha_1 \leqslant \frac{|\alpha|}{2}$ and $\alpha_1 + \bar\beta \leqslant \frac{s}{2}$, we obtain

$$\left(w^{2l}\partial_\alpha^\beta \Gamma(f, g), \partial_\alpha^\beta h\right) \leqslant C \sum_{\substack{|\alpha_1'| + |\bar\beta| \leqslant s \\ |\alpha_2| \geqslant \frac{|\alpha|}{2}}} \left\{\left\|w^l \partial_{\alpha_1'}^{\bar\beta} f\right\| \cdot \left\|w^l \partial_{\alpha_2}^{\beta-\beta_1} g\right\|_\sigma + \left\|w^l \partial_{\alpha_1'}^{\bar\beta} f\right\|_\sigma \cdot \left\|w^l \partial_{\alpha_2}^{\beta-\beta_1} g\right\|\right\} \left\|w^l \partial_\alpha^\beta h\right\|_\sigma, \tag{2.11}$$

if $\alpha_1 \leqslant \frac{|\alpha|}{2}$ and $\alpha_1 + \bar\beta \geqslant \frac{s}{2}$, we get

$$\left(w^{2l}\partial_\alpha^\beta \Gamma(f, g), \partial_\alpha^\beta h\right) \leqslant C \sum_{\substack{|\alpha_2'| + |\beta_2| \leqslant s \\ |\alpha_2'| \geqslant \frac{|\alpha|}{2}}} \left\{\left\|w^l \partial_{\alpha_1}^{\bar\beta} f\right\| \cdot \left\|w^l \partial_{\alpha_2'}^{\beta-\beta_1} g\right\|_\sigma + \left\|w^l \partial_{\alpha_1}^{\bar\beta} f\right\|_\sigma \cdot \left\|w^l \partial_{\alpha_2'}^{\beta-\beta_1} g\right\|\right\} \left\|w^l \partial_\alpha^\beta h\right\|_\sigma, \tag{2.12}$$

if $\alpha_1 \geqslant \frac{|\alpha|}{2}$ then $\alpha_2 \leqslant \frac{|\alpha|}{2}$, we can get the similar results as above.

The estimates for the linear operator L with soft potential and Landau kernels can be summarized as the following unified form.

Lemma 2.7. *(See [11,14].) For the soft potential with $-3 < \gamma < 0$ and Landau kernel with $\gamma \geqslant -3$, let $\beta > 0$, $l \geqslant 0$. There exists $C_{|\beta|} > 0$ such that*

$$\left(w^{2l}\partial_\alpha^\beta Lg, \partial_\alpha^\beta g\right) \geqslant \frac{1}{2}\left\|w^l \partial_\alpha^\beta g\right\|_D^2 - C_{|\beta|}\|\partial_\alpha g\|_D^2. \tag{2.13}$$

For use later in the uniform spatial energy estimates, the following lemma is needed.

Lemma 2.8. *Let $\zeta(v)$ be a smooth function that decays exponentially, assume further that*

$$\partial_\alpha \Gamma(f, g) = \sum_{\alpha_1 + \alpha_2 = \alpha} \Gamma(\partial_{\alpha_1} f, \partial_{\alpha_2} g),$$

where $|\alpha| \leqslant s$. Then there is a $C_\zeta > 0$ such that

$$\left\|\int \Gamma(\partial_{\alpha_1} f, \partial_{\alpha_2} g)\zeta \, dv\right\| \leqslant C_\zeta \mathcal{E}_s^{\frac{1}{2}}(f)\|\partial_{\alpha_2} g\|_D, \quad \text{if } |\alpha_1| \leqslant \frac{|\alpha|}{2}, \tag{2.14}$$

$$\left\|\int \Gamma(\partial_{\alpha_1} f, \partial_{\alpha_2} g)\zeta \, dv\right\| \leqslant C_\zeta \mathcal{E}_s^{\frac{1}{2}}(g)\|\partial_{\alpha_1} f\|_D, \quad \text{if } |\alpha_1| \geqslant \frac{|\alpha|}{2}, \tag{2.15}$$

and

$$\langle L\partial_\alpha g, \zeta\rangle \leqslant C_\zeta |\partial_\alpha g|_D. \tag{2.16}$$

Eqs. (2.14) and (2.15) are direct consequences of Lemmas 2.3, 2.4, and 2.6, while (2.16) can be obtained by the straightforward calculations.

3. Derivation of the acoustic system

In this section, we derive the acoustic system as the hydrodynamic limit of the solutions f^ϵ to the rescaled Boltzmann equation (1.7).

Since we have the uniform energy bound in ϵ by Theorem 1.1, there exists the unique limit f of f^ϵ in ϵ and we remark that due to higher order energy bound, all the limits in the below are strongly convergent. First, by letting $\epsilon \to 0$ in (1.7), one finds that $Lf = 0$. Therefore f can be written as follows:

$$f = \left\{\rho + v \cdot u + \left(\frac{|v|^2}{2} - \frac{3}{2}\right)\right\}\sqrt{\mu},$$

for ρ, u, θ being functions of t, x. In order to determine the dynamics of ρ, u, θ, by multiplying $[\sqrt{\mu}, v\sqrt{\mu}, |v|^2\sqrt{\mu}]$ with (1.7), we get

$$\left\langle \partial_t f^\epsilon + v \cdot \nabla_x f^\epsilon, \left\{1, v, \left(\frac{|v|^2}{3} - 1\right)\right\}\sqrt{\mu}\right\rangle = 0,$$

and take limit $\epsilon \to 0$ to get

$$\left\langle \partial_t f + v \cdot \nabla_x f, \left\{1, v, \left(\frac{|v|^2}{3} - 1\right)\right\}\sqrt{\mu}\right\rangle = 0.$$

Since $f = \mathbf{P}f$, this is equivalent to

$$\begin{aligned}
&\partial_t \rho + \nabla \cdot u = 0,\\
&\partial_t u + \nabla \cdot (\rho + \theta) = 0,\\
&\partial_t \theta + \frac{2}{3}\nabla \cdot u = 0.
\end{aligned} \tag{3.1}$$

Thus we have shown the following proposition on the mathematical derivation of the acoustic system from the Boltzmann equation.

Proposition 3.1. *Assume that $F^\epsilon = \mu + \epsilon\sqrt{\mu} f^\epsilon$ solves the rescaled Boltzmann equation* (1.2) *where f^ϵ is obtained from Theorem* 1.1. *Then there exists the hydrodynamic limit f of f^ϵ such that $f = \mathbf{P}f$, and furthermore its macroscopic variables ρ, u, θ solve the acoustic system* (3.1).

The acoustic system is a linear system and it is globally well-posed in the Sobolev space.

Lemma 3.1. *The acoustic system* (3.1) *is well-posed in $H^s(\mathbf{R}^3)$ ($s > \frac{3}{2}$). Moreover, we obtain the following estimate*:

$$\frac{d}{dt}\left\{\|\rho\|_{H^s}^2 + \|u\|_{H^s}^2 + \frac{3}{2}\|\theta\|_{H^s}^2\right\} = 0. \tag{3.2}$$

Proof. The existence of solutions can be easily obtained, for instance, by solving the ordinary differential equation after taking Fourier transform in $x \in \mathbf{R}^3$. $s > \frac{3}{2}$, the energy estimates give rise to the conservation of energy (3.2). □

4. Proof of Theorem 1.1

This section is devoted to proving Theorem 1.1. Firstly, we establish spatial energy estimate for f^ϵ, a solution to (1.7):

$$\partial_t f^\epsilon + v \cdot \nabla_x f^\epsilon + \frac{1}{\epsilon} L f^\epsilon = \Gamma\big(f^\epsilon, f^\epsilon\big).$$

For the convenience, we rewrite the fluid part $\mathbf{P}f^\epsilon$ as follows:

$$\mathbf{P}f^\epsilon = \big\{a_\epsilon(t,x) + b_\epsilon(t,x) \cdot v + c_\epsilon |v|^2\big\}\sqrt{\mu}.$$

Our goal is to estimate $a_\epsilon(t,x), b_\epsilon(t,x), c_\epsilon(t,x)$ in terms of $(\mathbf{I} - \mathbf{P})f^\epsilon$.

Lemma 4.1. *Assume f^ϵ is a solution to Eq.* (1.7), *then there exists a constant $C_1 > 0$ such that*

$$\sum_{0<|\alpha|\leqslant s+1} \epsilon\big\|\partial_\alpha \mathbf{P}f^\epsilon\big\|^2 \leqslant \epsilon\frac{dG(t)}{dt} + \frac{C_1}{\epsilon}\sum_{|\alpha|\leqslant s+1}\big\|\partial_\alpha(\mathbf{I} - \mathbf{P})f^\epsilon\big\|_D^2 + \epsilon\sum_{|\alpha|\leqslant s}\big\|\partial_\alpha \Gamma\big(f^\epsilon, f^\epsilon\big)_\perp\big\|^2, \tag{4.1}$$

for ϵ sufficiently small. Here $G(t)$ is defined as

$$-\sum_{|\alpha|\leqslant s}\int_{\mathbf{R}^3}\big\{\langle(\mathbf{I}-\mathbf{P})\partial_\alpha f^\epsilon,\zeta_c\rangle\cdot\nabla_x\partial_\alpha c_\epsilon-\langle(\mathbf{I}-\mathbf{P})\partial_\alpha f^\epsilon,\zeta_{ij}\rangle\cdot\partial_j\partial_\alpha b_\epsilon\big\}\,dx$$
$$-\sum_{|\alpha|\leqslant s}\int_{\mathbf{R}^3}\big\{\langle(\mathbf{I}-\mathbf{P})\partial_\alpha f^\epsilon,\zeta_a\rangle\cdot\nabla_x\partial_\alpha a_\epsilon-\partial_\alpha b_\epsilon\cdot\nabla_x\partial_\alpha a_\epsilon\big\}\,dx, \tag{4.2}$$

where $\zeta_c(v)$, $\zeta_{ij}(v)$, and $\zeta_a(v)$ are some fixed linear combination of the basis

$$\big[\sqrt{\mu},\ v_i\sqrt{\mu},\ v_iv_j\sqrt{\mu},\ v_i|v|^2\sqrt{\mu}\big],$$

$1\leqslant i,j\leqslant 3$, and $\partial_\alpha\Gamma(f^\epsilon,f^\epsilon)_\perp$ is the L^2_v projection of $\partial_\alpha\Gamma(f^\epsilon,f^\epsilon)$ onto the subspace generated by the same basis.

The proof of Lemma 4.1 is similar to the one of Lemma 6.1 in [16] and Lemma 5.1 in [18]. For the clear presentation of this article, we omit the details here.

Now we proceed to deduce the basic energy estimate. For results in this direction, we have

Lemma 4.2. *Assume that f^ϵ is a solution to Eq. (1.7), then there exists a constant $C_1>0$ such that the following energy estimate is valid*:

$$\frac{d}{dt}\Big\{C_1\sum_{|\alpha|\leqslant s+1}\|\partial_\alpha f^\epsilon\|^2-\epsilon\delta G(t)\Big\}+\delta\Big\{\frac{1}{\epsilon}\sum_{|\alpha|\leqslant s+1}\|\partial_\alpha(\mathbf{I}-\mathbf{P})f^\epsilon\|_D^2+\sum_{0<|\alpha|\leqslant s+1}\epsilon\|\partial_\alpha\mathbf{P}f^\epsilon\|^2\Big\}$$
$$\leqslant 2C_1\sum_{|\alpha|\leqslant s+1}\big(\partial_\alpha\Gamma(f^\epsilon,f^\epsilon),\partial_\alpha f^\epsilon\big)+\epsilon\delta\sum_{|\alpha|\leqslant s}\|\partial_\alpha\Gamma(f^\epsilon,f^\epsilon)_\perp\|^2$$
$$\leqslant C\big\{\mathcal{E}_s^{\frac12}(f^\epsilon)+\mathcal{E}_s(f^\epsilon)\big\}\mathcal{D}_s(f^\epsilon). \tag{4.3}$$

Proof. We take ∂_α of (1.7) then make the inner product with $\partial_\alpha f^\epsilon$ to get

$$\frac12\frac{d}{dt}\|\partial_\alpha f^\epsilon\|^2+\frac{\delta}{\epsilon}\big\{\|\partial_\alpha(\mathbf{I}-\mathbf{P})f^\epsilon\|_D^2\big\}\leqslant\big(\partial_\alpha\Gamma(f^\epsilon,f^\epsilon),\partial_\alpha f^\epsilon\big). \tag{4.4}$$

Returning now to Lemma 4.1, we find

$$\frac{\delta}{\epsilon}\sum_{|\alpha|\leqslant s+1}\|\partial_\alpha(\mathbf{I}-\mathbf{P})f^\epsilon\|_D^2\geqslant\frac{1}{C_1}\Big\{\delta\sum_{0<|\alpha|\leqslant s+1}\|\partial_\alpha\mathbf{P}f^\epsilon\|^2-\epsilon\delta G'(t)-\epsilon\delta\sum_{|\alpha|\leqslant s}\|\partial_\alpha\Gamma(f^\epsilon,f^\epsilon)_\perp^\epsilon\|\Big\}. \tag{4.5}$$

Inserting (4.5) into (4.4) and adjusting constants, we deduce the first inequality in (4.3). Furthermore, if $|\alpha|>0$, by Lemmas 2.3, 2.4, 2.6 and 2.8, it is easy to derive that

$$\|\partial_\alpha\Gamma(f^\epsilon,f^\epsilon)_\perp\|^2\leqslant C\mathcal{E}_s(f^\epsilon)\mathcal{D}_s(f^\epsilon),$$

and

$$\big(\partial_\alpha\Gamma(f^\epsilon,f^\epsilon),\partial_\alpha f^\epsilon\big)=\big(\partial_\alpha\Gamma(f^\epsilon,f^\epsilon),(\mathbf{I}-\mathbf{P})\partial_\alpha f^\epsilon\big)\leqslant C\mathcal{E}_s^{\frac12}(f^\epsilon)\mathcal{D}_s(f^\epsilon).$$

If $\alpha=0$, we split $f^\epsilon=\mathbf{P}f^\epsilon+\{\mathbf{I}-\mathbf{P}\}f^\epsilon$, so that $\|\Gamma(f^\epsilon,f^\epsilon)_\perp\|$ is decomposed into

$$\|\Gamma(\mathbf{P}f^\epsilon,\mathbf{P}f^\epsilon)_\perp\|+\|\Gamma(\mathbf{P}f^\epsilon,\{\mathbf{I}-\mathbf{P}\}f^\epsilon)_\perp\|+\|\Gamma(\{\mathbf{I}-\mathbf{P}\}f^\epsilon,\mathbf{P}f^\epsilon)_\perp\|+\|\Gamma(\{\mathbf{I}-\mathbf{P}\}f^\epsilon,\{\mathbf{I}-\mathbf{P}\}f^\epsilon)_\perp\|.$$

Applying Lemma 2.8, we can see that the last three terms are bounded by

$$C\mathcal{E}_s^{\frac12}(f^\epsilon)\mathcal{D}_s^{\frac12}(f^\epsilon).$$

On the other hand, we plug $\mathbf{P}f^\epsilon=\{a_\epsilon(t,x)+b_\epsilon(t,x)\cdot v+c_\epsilon(t,x)|v|^2\}\sqrt{\mu}$ into the first term to get

$$\|\Gamma(\mathbf{P}f^\epsilon,\mathbf{P}f^\epsilon)_\perp\|\leqslant C\|a_\epsilon^2+|b_\epsilon|^2+c_\epsilon^2\|.$$

Since $\frac16+\frac13=\frac12$, it is therefore bounded by the generalized Hölder inequality

$$\|a_\epsilon^2+b_\epsilon^2+c_\epsilon^2\|\leqslant C\big\{\|a_\epsilon(t,x)\|_{L^6}+\|b_\epsilon(t,x)\|_{L^6}+\|c_\epsilon(t,x)\|_{L^6}\big\}$$
$$\times\big\{\|a_\epsilon(t,x)\|_{L^3}+\|b_\epsilon(t,x)\|_{L^3}+\|c_\epsilon(t,x)\|_{L^3}\big\}. \tag{4.6}$$

The first factor is dominated by Sobolev's inequality in $\mathbf{R}^3$

$$C\{\|\nabla_x a_\epsilon(t,x)\| + \|\nabla_x b_\epsilon(t,x)\| + \|\nabla_x c_\epsilon(t,x)\|\} \leqslant C\mathcal{D}_s^{\frac{1}{2}}(f^\epsilon).$$

The second factor is bounded by an interpolation as

$$C\{\|a_\epsilon(t,x)\|_{H^s} + \|b_\epsilon(t,x)\|_{H^s} + \|c_\epsilon(t,x)\|_{H^s}\} \leqslant C\mathcal{E}_s^{\frac{1}{2}}(f^\epsilon).$$

Similarly, $(\Gamma(f^\epsilon, f^\epsilon), \{\mathbf{I}-\mathbf{P}\}f^\epsilon)$ can be decomposed into

$$\begin{aligned}&\big(\Gamma\big(\mathbf{P}f^\epsilon, \mathbf{P}f^\epsilon\big), \{\mathbf{I}-\mathbf{P}\}f^\epsilon\big) + \big(\Gamma\big(\mathbf{P}f^\epsilon, \{\mathbf{I}-\mathbf{P}\}f^\epsilon\big), \{\mathbf{I}-\mathbf{P}\}f^\epsilon\big)\\ &\quad + \big(\Gamma\big(\{\mathbf{I}-\mathbf{P}\}f^\epsilon, \mathbf{P}f^\epsilon\big), \{\mathbf{I}-\mathbf{P}\}f^\epsilon\big) + \big(\Gamma\big(\{\mathbf{I}-\mathbf{P}\}f^\epsilon, \{\mathbf{I}-\mathbf{P}\}f^\epsilon\big), \{\mathbf{I}-\mathbf{P}\}f^\epsilon\big).\end{aligned}$$

We deduce from Lemmas 2.3, 2.4 and 2.6 that the last three terms are dominated by

$$C\mathcal{E}_s^{\frac{1}{2}}(f^\epsilon)\mathcal{D}_s(f^\epsilon).$$

For the first term, we plug $\mathbf{P}f^\epsilon = \{a_\epsilon(t,x) + b_\epsilon(t,x)\cdot v + c_\epsilon(t,x)|v|^2\}\sqrt{\mu}$ into the expression and apply Lemmas 2.3, 2.4 and 2.6 to get

$$\begin{aligned}\big(\Gamma\big(\mathbf{P}f^\epsilon, \mathbf{P}f^\epsilon\big), \{\mathbf{I}-\mathbf{P}\}f^\epsilon\big) &\leqslant C\int_{\mathbf{R}^3}\big(a_\epsilon^2 + |b_\epsilon|^2 + c_\epsilon^2\big)\big|\{\mathbf{I}-\mathbf{P}\}f^\epsilon\big|_D\,dx\\ &\leqslant C\big\|a_\epsilon^2 + |b_\epsilon|^2 + c_\epsilon^2\big\|\big\|\{\mathbf{I}-\mathbf{P}\}f^\epsilon\big\|_D\\ &\leqslant C\mathcal{E}_s^{\frac{1}{2}}(f^\epsilon)\mathcal{D}_s(f^\epsilon),\end{aligned}\tag{4.7}$$

where in the last inequality we have used (4.6) again. We thus conclude the proof of Lemma 4.2. □

Now we turn to prove Theorem 1.1. We already established a pure spatial energy estimate for all collision kernels in Lemma 4.2. To estimate the velocity derivatives as well as velocity weights, however, we will separate two different cases.

4.1. Proof of hard potential case for Theorem 1.1

We first notice that for the hydrodynamic part $\mathbf{P}f^\epsilon$

$$\|\partial_\alpha^\beta\mathbf{P}f^\epsilon\| \leqslant c\|\partial_\alpha\mathbf{P}f^\epsilon\|,$$

which has been estimated in Lemma 4.2. It suffices to estimate the remaining microscopic part

$$\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f^\epsilon$$

for $|\alpha|+|\beta| \leqslant s$. We take ∂_α^β of Eq. (1.7) and sum over $|\alpha|+|\beta|\leqslant s$, $|\beta|\geqslant 1$ to get

$$\begin{aligned}&\partial_t\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f^\epsilon + v\cdot\nabla_x\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f^\epsilon + \frac{1}{\epsilon}\partial_\alpha^\beta L(\mathbf{I}-\mathbf{P})f^\epsilon + \big\{\partial_t\partial_\alpha^\beta\mathbf{P}f^\epsilon + v\cdot\nabla_x\partial_\alpha^\beta\mathbf{P}f^\epsilon + C_\beta^{\beta_1}\partial^{\beta_1}v\cdot\nabla_x\partial_\alpha^{\beta-\beta_1}f^\epsilon\big\}\\ &\quad = \partial_\alpha^\beta\Gamma\big(f^\epsilon, f^\epsilon\big),\end{aligned}$$

where $|\beta_1| = 1$. Taking the inner product with $\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f^\epsilon$, we get

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{1}{2}\big\|\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f^\epsilon\big\|^2\right\} + \frac{1}{\epsilon}\big(\partial_\alpha^\beta L(\mathbf{I}-\mathbf{P})f^\epsilon, \partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f^\epsilon\big)\\ &\qquad + \big(\partial_t\partial_\alpha^\beta\mathbf{P}f^\epsilon + v\cdot\nabla_x\partial_\alpha^\beta\mathbf{P}f^\epsilon + C_\beta^{\beta_1}\partial^{\beta_1}v\cdot\nabla_x\partial_\alpha^{\beta-\beta_1}f^\epsilon, \partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f^\epsilon\big)\\ &\quad \leqslant \big(\partial_\alpha^\beta\Gamma\big(f^\epsilon, f^\epsilon\big), \partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f^\epsilon\big).\end{aligned}\tag{4.8}$$

From the estimates on L in Lemma 2.2, we know

$$\frac{1}{\epsilon}\big(\partial_\alpha^\beta L(\mathbf{I}-\mathbf{P})f^\epsilon, \partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f^\epsilon\big) \geqslant \frac{1}{2\epsilon}\big\|\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f^\epsilon\big\|_\nu^2 - \frac{C}{\epsilon}\big\|\partial_\alpha(\mathbf{I}-\mathbf{P})f^\epsilon\big\|_\nu^2.$$

We now turn to the second line in (4.8); according to Section 6 in [16] we calculate

$$\sum_{|\alpha|+|\beta|\leqslant s}\left\|\partial_t\partial_\alpha^\beta \mathbf{P}f^\epsilon\right\| \leqslant \sum_{|\alpha|\leqslant s}\left\{\|\partial_t\partial_\alpha a_\epsilon\|+\|\partial_t\partial_\alpha b_\epsilon\|+\|\partial_t\partial_\alpha c_\epsilon\|\right\}$$
$$\leqslant C\sum_{0<|\alpha|\leqslant s+1}\left\|\partial_\alpha(\mathbf{I}-\mathbf{P})f^\epsilon\right\|.$$

For the second term, we have

$$\sum_{|\alpha|+|\beta|\leqslant s}\left\|v\cdot\nabla_x\partial_\alpha^\beta \mathbf{P}f^\epsilon\right\|\leqslant c\sum_{|\alpha|\leqslant s}\left\|\nabla_x\partial_\alpha \mathbf{P}f^\epsilon\right\|\leqslant c\sum_{0<|\alpha|\leqslant s+1}\left\|\partial_\alpha \mathbf{P}f^\epsilon\right\|.$$

Hence Cauchy inequality with ϵ yields the estimate

$$\left(\partial_t\partial_\alpha^\beta \mathbf{P}f^\epsilon+v\cdot\nabla_x\partial_\alpha^\beta \mathbf{P}f^\epsilon,\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f^\epsilon\right)\leqslant\frac{1}{8\epsilon}\sum_{|\alpha|\leqslant s}\left\|\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f^\epsilon\right\|_\nu^2+C\epsilon\sum_{0<|\alpha|\leqslant s+1}\left\|\partial_\alpha f^\epsilon\right\|.$$

The last term in the same line is bounded by

$$\begin{aligned}
&\left|C_\beta^{\beta_1}\left(\partial^{\beta_1}v\cdot\nabla_x\partial_\alpha^{\beta-\beta_1}f^\epsilon,\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f^\epsilon\right)\right|\\
&\quad\leqslant C\left|\left(\partial^{\beta_1}v\cdot\nabla_x\partial_\alpha^{\beta-\beta_1}(\mathbf{I}-\mathbf{P})f^\epsilon,\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f^\epsilon\right)\right|+C\left|\left(\partial^{\beta_1}v\cdot\nabla_x\partial_\alpha^{\beta-\beta_1}\mathbf{P}f^\epsilon,\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f^\epsilon\right)\right|\\
&\quad\leqslant\left\|\nabla_x\partial_\alpha^{\beta-\beta_1}(\mathbf{I}-\mathbf{P})f^\epsilon\right\|_\nu^2+\frac{1}{8\epsilon}\left\|\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f^\epsilon\right\|_\nu^2+C\epsilon\left\|\partial_\alpha\nabla\mathbf{P}f^\epsilon\right\|^2\\
&\quad\leqslant\epsilon\mathcal{D}_s\left(f^\epsilon\right)+\frac{1}{8\epsilon}\left\|\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f^\epsilon\right\|_\nu^2+C\epsilon\left\|\partial_\alpha\nabla\mathbf{P}f^\epsilon\right\|^2.
\end{aligned}$$

Now we turn to the nonlinear term in (4.8), our goal is to show

$$\left(\partial_\alpha^\beta\Gamma\left(f^\epsilon,f^\epsilon\right),\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f^\epsilon\right)\leqslant\mathcal{E}_s^{1/2}\left(f^\epsilon\right)\mathcal{D}_s\left(f^\epsilon\right).\tag{4.9}$$

To verify (4.9), we still separate two cases. If $\alpha=0$, we split $f^\epsilon=\mathbf{P}f^\epsilon+(\mathbf{I}-\mathbf{P})f^\epsilon$ to further decompose $(\partial^\beta\Gamma(f^\epsilon,f^\epsilon),\partial^\beta(\mathbf{I}-\mathbf{P})f^\epsilon)$ into

$$\begin{aligned}
&\left(\partial^\beta\Gamma\left(\mathbf{P}f^\epsilon,\mathbf{P}f^\epsilon\right),\partial^\beta(\mathbf{I}-\mathbf{P})f^\epsilon\right)+\left(\partial^\beta\Gamma\left((\mathbf{I}-\mathbf{P})f^\epsilon,\mathbf{P}f^\epsilon\right),\partial^\beta(\mathbf{I}-\mathbf{P})f^\epsilon\right)\\
&\quad+\left(\partial^\beta\Gamma\left(f^\epsilon,(\mathbf{I}-\mathbf{P})f^\epsilon\right),\partial^\beta(\mathbf{I}-\mathbf{P})f^\epsilon\right).
\end{aligned}$$

By Lemma 2.3, we can easily get that the second term and the third term of the right-hand side is bounded by $\mathcal{E}_s^{1/2}(f^\epsilon)\mathcal{D}_s(f^\epsilon)$.

Furthermore, by plugging $\mathbf{P}f^\epsilon=\{a_\epsilon+b_\epsilon\cdot v+c_\epsilon|v|^2\}\sqrt{\mu}$ into the first term, we obtain

$$\left(\partial^\beta\Gamma\left(\mathbf{P}f^\epsilon,\mathbf{P}f^\epsilon\right),\partial^\beta(\mathbf{I}-\mathbf{P})f^\epsilon\right)\leqslant C\left\|a_\epsilon^2+b_\epsilon^2+c_\epsilon^2\right\|\cdot\left\|\partial^\beta(\mathbf{I}-\mathbf{P})f^\epsilon\right\|_\nu.$$

The second factor is obviously dominated by $\mathcal{D}_s^{1/2}(f^\epsilon)$, employing (4.6) to the first factor, we thus conclude (4.9).

As to the general case $|\alpha|\geqslant 1$, let $\alpha'>0$; utilizing Lemma 2.3 we simplify to discover

$$\begin{aligned}
\left(\partial_\alpha^\beta\Gamma\left(f^\epsilon,f^\epsilon\right),\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f^\epsilon\right)&\leqslant C\mathcal{E}_s^{1/2}\left(f^\epsilon\right)\sum_{\alpha'>0,\,\beta_1\leqslant\beta}\left\|\partial_{\alpha'}^{\beta_1}f^\epsilon\right\|_\nu\times\left\{\left\|\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f^\epsilon\right\|_\nu\right\}\\
&\leqslant C\mathcal{E}_s^{1/2}\left(f^\epsilon\right)\mathcal{D}_s\left(f^\epsilon\right).
\end{aligned}$$

Absorbing a total of $\frac{1}{4\epsilon}\|\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f^\epsilon\|_\nu^2$ from the right-hand side and collecting terms, we have

$$\begin{aligned}
&\sum_{|\alpha|+|\beta|\leqslant s}\left(\frac{d}{dt}\left\{\frac{1}{2}\left\|\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f^\epsilon\right\|^2\right\}+\frac{1}{4\epsilon}\left\|\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f^\epsilon\right\|_\nu^2\right)\\
&\quad\leqslant C\epsilon\sum_{0<|\alpha|\leqslant s+1}\left\|\partial_\alpha f^\epsilon\right\|+C\left\{\mathcal{E}_s^{1/2}\left(f^\epsilon\right)+\mathcal{E}_s\left(f^\epsilon\right)+\epsilon\right\}\mathcal{D}_s\left(f^\epsilon\right).
\end{aligned}$$

Multiplying above a factor of 4 and adding a large multiple K of (4.3), we get

$$\begin{aligned}
&\frac{d}{dt}\left\{K\left\{C_1\sum_{|\alpha|\leqslant s}\left\|\partial_\alpha f^\epsilon\right\|^2-\epsilon\delta G(t)\right\}+2\sum_{|\alpha|+|\beta|\leqslant s}\left\|\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f^\epsilon\right\|^2\right\}+\mathcal{D}_s\left(f^\epsilon\right)\\
&\quad\leqslant C_1\left\{\mathcal{E}_s^{1/2}\left(f^\epsilon\right)+\mathcal{E}_s\left(f^\epsilon\right)+\epsilon^2\right\}\mathcal{D}_s\left(f^\epsilon\right).
\end{aligned}$$

Remember that

$$\left\|\partial_\alpha^\beta \mathbf{P} f^\epsilon\right\|^2 \leqslant C\left\|\partial_\alpha f^\epsilon\right\|^2,$$

and by (4.2)

$$G(t) \leqslant C \sum_{|\alpha|\leqslant s}\left\{\left\|\partial_\alpha \mathbf{P} f^\epsilon\right\|\right\}\left\{\left\|(\mathbf{I}-\mathbf{P})\partial_\alpha f^\epsilon\right\|+\left\|\partial_\alpha \mathbf{P} f^\epsilon\right\|\right\},$$

we thus redefine an instant energy by

$$\mathcal{E}_s\left(f^\epsilon\right)=K\left\{C_1 \sum_{|\alpha|\leqslant s+1}\left\|\partial_\alpha f^\epsilon\right\|^2-\epsilon\delta G(t)\right\}+2\sum_{|\alpha|+|\beta|\leqslant s}\left\|\partial_\alpha^\beta(\mathbf{I}-\mathbf{P}) f^\epsilon\right\|^2, \tag{4.10}$$

for ϵ sufficiently small, which yields

$$\frac{d}{dt}\mathcal{E}_s\left(f^\epsilon\right)+\mathcal{D}_s\left(f^\epsilon\right) \leqslant C\left\{\mathcal{E}_s^{1/2}\left(f^\epsilon\right)+\mathcal{E}_s\left(f^\epsilon\right)+\epsilon^2\right\}\mathcal{D}_s\left(f^\epsilon\right). \tag{4.11}$$

Suppose there is M_1 such that

$$M_1^{\frac{1}{2}}+M_1+\epsilon^2=\frac{1}{2C},$$

then we set $M=\min\{M_1, M_0\}$, and choose initial data so that $\mathcal{E}_s(f^\epsilon)(0) \leqslant \frac{M}{2} < M_0$.

Continuing, we choose $T>0$ so that

$$T=\sup\left\{t \mid \mathcal{E}_s\left(f^\epsilon\right)(t) \leqslant M\right\}>0,$$

since $\mathcal{E}_s(f^\epsilon)(t)$ is continuous.

For $0 \leqslant t \leqslant T$, from (4.11) we get

$$\mathcal{E}_s\left(f^\epsilon\right)(t)+\frac{1}{2}\int_0^t \mathcal{D}_s\left(f^\epsilon\right) \leqslant \mathcal{E}_s\left(f^\epsilon\right)(0) \leqslant \frac{M}{2}<M,$$

thus $T=\infty$. We therefore complete the proof of Theorem 1.1 in the case of hard potential.

4.2. Proof of soft potential and Landau cases for Theorem 1.1

In this subsection, we follow the same procedure as in the hard potential case to establish (1.16) for both soft potentials and Landau kernels. We take ∂_α^β of Eq. (1.7) and sum over $|\alpha|+|\beta| \leqslant s$, $|\beta| \geqslant 1$ to get

$$\begin{aligned}
&\partial_t\partial_\alpha^\beta(\mathbf{I}-\mathbf{P}) f^\epsilon+\frac{1}{\epsilon} v\cdot\nabla_x\partial_\alpha^\beta(\mathbf{I}-\mathbf{P}) f^\epsilon+\frac{1}{\epsilon^2}\partial_\alpha^\beta L(\mathbf{I}-\mathbf{P}) f^\epsilon+\left\{\partial_t\partial_\alpha^\beta \mathbf{P} f^\epsilon+\frac{1}{\epsilon} v\cdot\nabla_x\partial_\alpha^\beta \mathbf{P} f^\epsilon+\frac{1}{\epsilon} C_\beta^{\beta_1}\partial^{\beta_1} v\cdot\nabla_x\partial_\alpha^{\beta-\beta_1} f^\epsilon\right\}\\
&\quad=\frac{1}{\epsilon}\partial_\alpha^\beta \Gamma\left(f^\epsilon, f^\epsilon\right),
\end{aligned}$$

where $|\beta_1|=1$. Taking the inner product with $w^{2|\beta|}\partial_\alpha^\beta(\mathbf{I}-\mathbf{P}) f^\epsilon$, we get

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{2}\left\|w^{|\beta|}\partial_\alpha^\beta(\mathbf{I}-\mathbf{P}) f^\epsilon\right\|^2\right\}+\frac{1}{\epsilon}\left(w^{2|\beta|}\partial_\alpha^\beta L(\mathbf{I}-\mathbf{P}) f^\epsilon, \partial_\alpha^\beta(\mathbf{I}-\mathbf{P}) f^\epsilon\right)\\
&\qquad+\left(\partial_t\partial_\alpha^\beta \mathbf{P} f^\epsilon+v\cdot\nabla_x\partial_\alpha^\beta \mathbf{P} f^\epsilon+C_\beta^{\beta_1}\partial^{\beta_1} v\cdot\nabla_x\partial_\alpha^{\beta-\beta_1} f^\epsilon, w^{2|\beta|}\partial_\alpha^\beta(\mathbf{I}-\mathbf{P}) f^\epsilon\right)\\
&\quad\leqslant\left(w^{2|\beta|}\partial_\alpha^\beta \Gamma\left(f^\epsilon, f^\epsilon\right), \partial_\alpha^\beta(\mathbf{I}-\mathbf{P}) f^\epsilon\right).
\end{aligned} \tag{4.12}$$

By the estimates on L in Lemma 2.7, we know

$$\frac{1}{\epsilon}\left(w^{2|\beta|}\partial_\alpha^\beta L(\mathbf{I}-\mathbf{P}) f^\epsilon, \partial_\alpha^\beta(\mathbf{I}-\mathbf{P}) f^\epsilon\right) \geqslant \frac{1}{2\epsilon}\left\|w^{|\beta|}\partial_\alpha^\beta(\mathbf{I}-\mathbf{P}) f^\epsilon\right\|_D^2-\frac{C}{\epsilon}\left\|\partial_\alpha(\mathbf{I}-\mathbf{P}) f^\epsilon\right\|_D^2.$$

From the local conservation laws as used in Section 6 in [16], we have

$$\sum_{|\alpha|+|\beta|\leqslant s}\left\|w^{|\beta|}\partial_t\partial_\alpha^\beta \mathbf{P} f^\epsilon\right\| \leqslant C\sum_{0<|\alpha|\leqslant s+1}\left\|\partial_\alpha(\mathbf{I}-\mathbf{P}) f^\epsilon\right\|_D.$$

We also have

$$\sum_{|\alpha|+|\beta|\leqslant s}\left\|w^{2|\beta|}v\cdot\nabla_x\partial_\alpha^\beta\mathbf{P}f^\epsilon\right\|\leqslant c\sum_{|\alpha|\leqslant s}\left\|\nabla_x\partial_\alpha\mathbf{P}f^\epsilon\right\|\leqslant c\sum_{0<|\alpha|\leqslant s+1}\left\|\partial_\alpha\mathbf{P}f^\epsilon\right\|.$$

Note that $\|\cdot\|_D$ is equivalent to $\|\cdot\|_\nu$ for the soft potential, and for the Landau equation, it is equivalent to $\|\cdot\|_\sigma$, then utilizing Cauchy inequality with ϵ to the first two inner product in the second line of (4.12), we get the upper bound

$$\frac{1}{8\epsilon}\sum_{|\alpha|\leqslant s}\left\|w^{|\beta|}\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f^\epsilon\right\|_D^2+C\epsilon\sum_{0<|\alpha|\leqslant s+1}\left\|\partial_\alpha f^\epsilon\right\|_D^2.$$

The weight function $w^{|\beta|}$ is so designed to treat the last term in the second line of (4.12)

$$\begin{aligned}
&\left|C_\beta^{\beta_1}\left(\frac{1}{\epsilon}\partial^{\beta_1}v\cdot\nabla_x\partial_\alpha^{\beta-\beta_1}f^\epsilon,\, w^{2|\beta|}\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f^\epsilon\right)\right| \\
&\quad\leqslant C\left|\left(\frac{1}{\epsilon}\partial^{\beta_1}v\cdot\nabla_x\partial_\alpha^{\beta-\beta_1}(\mathbf{I}-\mathbf{P})f^\epsilon,\, w^{2|\beta|}\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f^\epsilon\right)\right|+C\left|\left(\frac{1}{\epsilon}\partial^{\beta_1}v\cdot\nabla_x\partial_\alpha^{\beta-\beta_1}\mathbf{P}f^\epsilon,\, w^{2|\beta|}\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f^\epsilon\right)\right| \\
&\quad\leqslant\left\|w^{|\beta|-1}\nabla_x\partial_\alpha^{\beta-\beta_1}(\mathbf{I}-\mathbf{P})f^\epsilon\right\|_D^2+\frac{1}{8\epsilon}\left\|w^{|\beta|}\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f^\epsilon\right\|_D^2+C\epsilon\left\|\partial_\alpha\nabla\mathbf{P}f^\epsilon\right\|^2 \\
&\quad\leqslant\epsilon\mathcal{D}_s\big(f^\epsilon\big)+\frac{1}{8\epsilon}\left\|w^l\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f^\epsilon\right\|_D^2+C\epsilon\left\|\partial_\alpha\nabla\mathbf{P}f^\epsilon\right\|^2.
\end{aligned}$$

Finally, for the nonlinear term in (4.12), by using the same argument as used to deduce (4.9) we can show that

$$\big(w^{2|\beta|}\partial_\alpha^\beta\Gamma\big(f^\epsilon,f^\epsilon\big),\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f^\epsilon\big)\leqslant C\mathcal{E}_s^{1/2}\big(f^\epsilon\big)\mathcal{D}_s\big(f^\epsilon\big).$$

The rest of the proof is similar to the hard potential case, the nonlinear estimate (1.16) can be deduced by letting

$$\mathcal{E}_s\big(f^\epsilon\big)=K\left\{C_1\sum_{|\alpha|\leqslant s+1}\left\|\partial_\alpha f^\epsilon\right\|^2-\epsilon\delta G(t)\right\}+2\sum_{|\alpha|+|\beta|\leqslant s}\left\|w^{|\beta|}\partial_\alpha^\beta(\mathbf{I}-\mathbf{P})f^\epsilon\right\|^2.$$

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