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Zero product preserving maps on Banach algebras of Lipschitz functions ☆

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ABSTRACT

Let (K,d) be a non-empty, compact metric space and $\alpha \in]0,1[$. Let A be either $\operatorname{lip}_\alpha(K)$ or $\operatorname{Lip}_\alpha(K)$ and let B be a commutative unital Banach algebra. We show that every continuous linear map $T : A \to B$ with the property that $T(f)T(g) = 0$ whenever $f, g \in A$ are such that $fg = 0$ is of the form $T = w\Phi$ for some invertible element w in B and some continuous epimorphism $\Phi : A \to B$.

1. Introduction

Our motivation to study zero product preserving maps on Banach algebras of Lipschitz functions comes from two very recent papers by A. Jiménez-Vargas [10] and J. Araujo and L. Dubarbie [6]. In those papers, the authors have paid attention to the problem of describing the so-called *separating maps* (also considered under the name of *disjointness preserving maps*) between Banach algebras of Lipschitz functions and Banach spaces of Lipschitz functions, respectively. The typical examples of separating maps between function spaces are provided by the so-called *weighted composition operators* and the standard problem consists in determining whether these are the canonical examples.

The so-called *preserver problems* have became an area of lively interest in many parts of mathematics, including operator theory and Banach algebra theory. We refer the reader to [11] for a thorough account of the theory. One of the most popular preserver problems is that of the *zero product preserving maps*. This problem is concerned with the question of describing the operators $T : A \to B$ between Banach algebras A and B which preserve the zero product in the sense that

$$a, b \in A, \quad ab = 0 \quad \Rightarrow \quad T(a)T(b) = 0.$$

A typical example of operator T from the above definition is the one given by $T = W \circ \Phi$ where $\Phi : A \to B$ is a homomorphism and $W : B \to B$ is a centralizer i.e., $W(ab) = W(a)b = aW(b)$ $(a, b \in B)$. The basic task consists in determining whether every operator preserving the zero product is necessarily given by such a weighted homomorphism (see [3] and the references given therein). Of course, this problem is closely related to the problem of describing the separating maps in the context of function spaces.

In this paper we are concerned with zero product preserving maps from a Banach algebra of Lipschitz functions into any arbitrary commutative unital Banach algebra. In the last years the authors of this paper have developed a powerful tool for analysing a zero product preserving linear map $T : A \to B$ between Banach algebras A and B [1–5]. The method consists in giving rise to the continuous bilinear map $\varphi : A \times A \to B$ defined by $\varphi(a,b) = T(a)T(b)$ $(a, b \in A)$ which obviously satisfies

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the property that $\varphi(a,b)=0$ whenever $a,b\in A$ are such that $ab=0$. Very often the analysis of such a bilinear map may be reduced to the analysis of a bilinear map $\varphi: A(\mathbb{T})\times A(\mathbb{T})\to B$ with the property that $\varphi(f,g)=0$ whenever $f,g\in A(\mathbb{T})$ are such that $\operatorname{supp}(f)\cap\operatorname{supp}(g)=\emptyset$. As usual, $A(\mathbb{T})$ stands for the classical Fourier algebra on the circle group $\mathbb{T}$. It turned out in [3] that such a map necessarily satisfies $\varphi(\mathbf{z},\mathbf{1})=\varphi(\mathbf{1},\mathbf{z})$ and this is just the key fact for solving the problem of describing the zero product preserving maps for a large class of Banach algebras which includes C^*-algebras and group algebras. Nevertheless, in the case when we are concerned with the algebras of Lipschitz functions we are required to replace the Fourier algebra $A(\mathbb{T})$ by a weighted Fourier algebra $A_\alpha(\mathbb{T})$. Motivated by this fact, our starting point in this paper is to study the continuous bilinear maps φ: $A_\alpha(\mathbb{T})\times A_\alpha(\mathbb{T})\to X$ into some Banach space X with the property that

$$f,g\in A_\alpha(\mathbb{T}),\quad \operatorname{supp}(f)\cap\operatorname{supp}(g)=\emptyset \quad\Rightarrow\quad \varphi(f,g)=0.$$

We prove in Section 2 that such a map necessarily satisfies

$$\sum_{n=0}^{N}\binom{N}{n}(-1)^n\varphi\bigl(\mathbf{z}^{N-n},\mathbf{z}^n\bigr)=0.$$

This is applied in Section 3 for proving that in the case when we replace $A_\alpha(\mathbb{T})$ by any of the algebras $\operatorname{lip}_\alpha(K)$ and $\operatorname{Lip}_\alpha(K)$ of Lipschitz functions on a non-empty, compact metric space (K,d) and $\alpha\in\,]0,1[$, then

$$\varphi(f,g)+\varphi(g,f)=\varphi(fg,\mathbf{1})+\varphi(\mathbf{1},fg).$$

The above identity is the starting point in Section 4 for proving that every continuous surjective linear map $T:\operatorname{lip}_\alpha(K)\times[\operatorname{Lip}_\alpha(K)]\to B$ onto any commutative unital Banach algebra B which preserves the zero product is of the form $T=w\Phi$ for some invertible element w in B and some continuous epimorphism $\Phi:\operatorname{lip}_\alpha(K)[\operatorname{Lip}_\alpha(K)]\to B$.

All Banach spaces and Banach algebras which we consider throughout this paper are assumed to be complex.

2. Bilinear maps on weighted Fourier algebras

For $n\in\mathbb{N}$ and $\alpha\geqslant 0$, let $A_\alpha(\mathbb{T}^n)$ denote the *weighted Fourier algebra* consisting of all functions $f\in C(\mathbb{T}^n)$ such that

$$\|f\|=\sum_{k\in\mathbb{Z}^n}\bigl|\hat f(k)\bigr|\bigl(1+|k|\bigr)^\alpha<\infty.$$

Lemma 2.1. *Let $\alpha\geqslant 0$ and let $N\in\mathbb{N}$ be with $N>\alpha$. Let $f\in A_\alpha(\mathbb{T}^2)$ be the function defined by $f(z,w)=(z-w)^N$ $(z,w\in\mathbb{T})$. Then there exists a sequence (f_n) in $A_\alpha(\mathbb{T}^2)$ such that $\lim f_n=f$ and f_n vanishes on a neighbourhood of $\{(z,z)\colon z\in\mathbb{T}\}$ for each $n\in\mathbb{N}$.*

Proof. A result by C. Herz (see [9,13]) shows that if $g\in A_\alpha(\mathbb{T})$ is such that $g(z_0)=g'(z_0)=\cdots=g^{[\alpha]}(z_0)=0$ for some $z_0\in\mathbb{T}$ ($[\alpha]$ stands for the integer part of α), then there exists a sequence (g_n) in $A_\alpha(\mathbb{T})$ such that $\lim g_n=g$ and g_n vanishes on a neighbourhood of z_0 for each $n\in\mathbb{N}$. We apply this result to the function $g\in A_\alpha(\mathbb{T})$ defined by $g(z)=(z-1)^N$ $(z\in\mathbb{T})$ to obtain a sequence (g_n) in $A_\alpha(\mathbb{T})$ such that $\lim g_n=g$ and g_n vanishes on a neighbourhood of 1 for each $n\in\mathbb{N}$.

For every $n\in\mathbb{N}$ we consider the function

$$f_n:\mathbb{T}^2\to\mathbb{C},\quad f_n(z,w)=g_n\bigl(zw^{-1}\bigr)w^N\quad(z,w\in\mathbb{T}).$$

It is easily seen that the sequence (f_n) satisfies our requirements. $\square$

Theorem 2.2. *Let $\alpha\geqslant 0$ and let $\varphi: A_\alpha(\mathbb{T})\times A_\alpha(\mathbb{T})\to X$ be a continuous bilinear map into some Banach space X with the property that*

$$f,g\in A_\alpha(\mathbb{T}),\quad \operatorname{supp}(f)\cap\operatorname{supp}(g)=\emptyset \quad\Rightarrow\quad \varphi(f,g)=0. \tag{2.1}$$

Then

$$\sum_{n=0}^{N}\binom{N}{n}(-1)^n\varphi\bigl(\mathbf{z}^{N-n},\mathbf{z}^n\bigr)=0$$

for each $N>2\alpha$, where $\mathbf{z}$ stands for the function on $\mathbb{T}$ defined by $\mathbf{z}(z)=z$ $(z\in\mathbb{T})$.

Proof. The map φ gives rise to a continuous linear operator $\Phi: A_{2\alpha}(\mathbb{T}^2)\to X$ by defining

$$\Phi(f)=\sum_{j,k\in\mathbb{Z}}\hat f(j,k)\varphi\bigl(\mathbf{z}^j,\mathbf{z}^k\bigr)\quad\bigl(f\in A_{2\alpha}\bigl(\mathbb{T}^2\bigr)\bigr).$$

It is straightforward to check, if $f,g\in A_{2\alpha}(\mathbb{T})$, then $f\otimes g\in A_{2\alpha}(\mathbb{T}^2)$, where $f\otimes g$ is defined by $(f\otimes g)(z,w)=f(z)g(w)$ $(z,w\in\mathbb{T})$. Furthermore, on account of the continuity of φ, we have

$$\begin{aligned}\Phi(f\otimes g)&=\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\hat f(j)\hat g(k)\varphi\big(\mathbf{z}^j,\mathbf{z}^k\big)\\&=\varphi\bigg(\sum_{j\in\mathbb{Z}}\hat f(j)\mathbf{z}^j,\sum_{k\in\mathbb{Z}}\hat g(k)\mathbf{z}^k\bigg)=\varphi(f,g)\quad\big(f,g\in A_{2\alpha}(\mathbb{T})\big).\end{aligned}\tag{2.2}$$

We now claim that Φ has the following property

$$f\in A_{2\alpha}\big(\mathbb{T}^2\big),\quad \operatorname{supp}(f)\cap\big\{(z,z)\colon z\in\mathbb{T}\big\}=\emptyset\quad\Rightarrow\quad\Phi(f)=0.\tag{2.3}$$

Let $f\in A_{2\alpha}(\mathbb{T}^2)$ satisfy $\operatorname{supp}(f)\cap\{(z,z)\colon z\in\mathbb{T}\}=\emptyset$. We then pick $\delta>0$ such that

$$\delta\leqslant|z-u|+|w-u|\quad\big((z,w)\in\operatorname{supp}(f),\ u\in\mathbb{T}\big)\tag{2.4}$$

and let $m\in\mathbb{N}$ be such that $8\sin(\pi/m)<\delta$. We now consider the open covering of $\mathbb{T}$ given by

$$U_k=\left\{z\in\mathbb{T}\colon\left|z-\exp\left(\frac{2\pi k}{m}i\right)\right|<2\sin(\pi/m)\right\}\quad(k=1,\dots,m).$$

There exist functions $\omega_1,\dots,\omega_m\in C^\infty(\mathbb{T})$ with $\omega_1+\dots+\omega_m=\mathbf{1}$ and $\operatorname{supp}(\omega_k)\subset U_k$ for $k=1,\dots,m$. It is easily seen that

$$\operatorname{supp}(f)\cap(U_j\times U_k)=\emptyset$$

whenever U_j and U_k are such that $U_j\cap U_k\neq\emptyset$. Accordingly, we have

$$f=\sum_{p,q=1}^m f(\omega_p\otimes\omega_q)=\sum_{U_p\cap U_q=\emptyset}f(\omega_p\otimes\omega_q).$$

On the other hand, we have

$$f=\sum_{j,k\in\mathbb{Z}}\hat f(j,k)\mathbf{z}^j\otimes\mathbf{z}^k.$$

We thus get

$$f=\sum_{U_p\cap U_q=\emptyset}\ \sum_{j,k\in\mathbb{Z}}\hat f(j,k)\big(\mathbf{z}^j\omega_p\big)\otimes\big(\mathbf{z}^k\omega_q\big).$$

On account of the continuity of Φ we arrive at

$$\Phi(f)=\sum_{U_p\cap U_q=\emptyset}\ \sum_{j,k\in\mathbb{Z}}\hat f(j,k)\Phi\big(\big(\mathbf{z}^j\omega_p\big)\otimes\big(\mathbf{z}^k\omega_q\big)\big).$$

According to [8, 12.1.1], every smooth function $u\in C^\infty(\mathbb{T})$ satisfies

$$\lim_{|k|\to+\infty}k^n\hat u(k)=0$$

for each $n\in\mathbb{N}$, which clearly entails that $u\in A_\beta(\mathbb{T})$ for each $\beta\geqslant 0$. Accordingly, by the smoothness of the functions $\mathbf{z}^k$ ($k\in\mathbb{Z}$) and ω_p ($p=1,\dots,m$), we can apply (2.2) to get

$$\Phi(f)=\sum_{U_p\cap U_q=\emptyset}\ \sum_{j,k\in\mathbb{Z}}\hat f(j,k)\varphi\big(\mathbf{z}^j\omega_p,\mathbf{z}^k\omega_q\big)=0,$$

because of (2.1) and the fact that

$$\operatorname{supp}\big(\mathbf{z}^j\omega_p\big)\cap\operatorname{supp}\big(\mathbf{z}^k\omega_q\big)\subset\operatorname{supp}(\omega_p)\cap\operatorname{supp}(\omega_q)\subset U_p\cap U_q=\emptyset$$

for all the terms appearing in the preceding identity for $\Phi(f)$.

Let $N\in\mathbb{N}$ with $N>2\alpha$ and consider the function $f\in A_{2\alpha}(\mathbb{T}^2)$ defined by

$$f(z,w)=(z-w)^N\quad(z,w\in\mathbb{T}).$$

According to Lemma 2.1, there exists a sequence (f_n) in $A_{2\alpha}(\mathbb{T}^2)$ such that $\lim f_n=f$ and $\operatorname{supp}(f_n)\cap\{(z,z)\colon z\in\mathbb{T}\}=\emptyset$ for each $n\in\mathbb{N}$. On account of the continuity of Φ and (2.3), we have

$$\Phi(f)=\lim\Phi(f_n)=0.$$

Finally, it is a simple matter to check that

$$\Phi(f)=\sum_{n=0}^N\binom{N}{n}(-1)^n\varphi\big(\mathbf{z}^{N-n},\mathbf{z}^n\big),$$

which completes the proof. $\square$

Corollary 2.3. *Let A be a two-sided ideal of a unital Banach algebra B and let $\varphi : A \times A \to X$ be a continuous bilinear map into some Banach space X with the property that*

$$a, b \in A, \quad ab = 0 \quad \Rightarrow \quad \varphi(a, b) = 0. \tag{2.5}$$

If $u \in B$ is invertible and

$$\|u^k\| = O\big(|k|^\alpha\big) \quad \text{as } |k| \to \infty,$$

for some $\alpha \geqslant 0$, then

$$\sum_{n=0}^{N} \binom{N}{n} (-1)^n \varphi\big(au^{N-n}, u^n b\big) = 0 \quad (a, b \in A)$$

for each $N > 2\alpha$.

Proof. We begin by defining a continuous homomorphism

$$\Psi : A_\alpha(\mathbb{T}) \to B, \quad \Psi(f) = \sum_{k\in\mathbb{Z}} \hat{f}(k) u^k \quad \big(f \in A_\alpha(\mathbb{T})\big).$$

We now pick $a, b \in A$ and we define a continuous bilinear map

$$\psi : A_\alpha(\mathbb{T}) \times A_\alpha(\mathbb{T}) \to X, \quad \psi(f, g) = \varphi\big(a\Psi(f), \Psi(g)b\big) \quad \big(f, g \in A_\alpha(\mathbb{T})\big).$$

It is immediate to see that ψ satisfies the condition (2.1). We then apply Theorem 2.2 to get the desired conclusion. □

3. Bilinear maps on Lipschitz algebras

Let (K, d) be a non-empty, compact metric space, and take $\alpha \in]0, 1]$. Then $\mathrm{Lip}_\alpha(K)$ is the Banach algebra of complex-valued functions f on K such that

$$p_\alpha(f) = \sup\left\{ \frac{|f(x) - f(y)|}{d(x, y)^\alpha} \colon x, y \in K,\ x \neq y \right\} < \infty,$$

and $\mathrm{lip}_\alpha(K)$ is the closed subalgebra of $\mathrm{Lip}_\alpha(K)$ consisting of functions f that

$$\frac{|f(x) - f(y)|}{d(x, y)^\alpha} \to 0 \quad \text{as } d(x, y) \to 0.$$

The norm of a function $f \in \mathrm{Lip}_\alpha(K)$ is defined by $\|f\| = \|f\|_\infty + p_\alpha(f)$. We refer the reader to [7, §4.4] for the basic properties of these Banach algebras.

Corollary 3.1. *Let (K, d) be a non-empty, compact metric space, $\alpha \in]0, 1[$, and let $\varphi : \mathrm{lip}_\alpha(K) \times \mathrm{lip}_\alpha(K) \to X$ be a continuous bilinear map into some Banach space X with the property that*

$$f, g \in \mathrm{lip}_\alpha(K), \quad fg = 0 \quad \Rightarrow \quad \varphi(f, g) = 0.$$

Then

$$\varphi(f, g) + \varphi(g, f) = \varphi(fg, \mathbf{1}) + \varphi(\mathbf{1}, fg) \quad \big(f, g \in \mathrm{lip}_\alpha(K)\big).$$

Proof. Let $h : K \to \mathbb{R}$ be a Lipschitz function with Lipschitz constant L and consider the function $u = \exp(ih)$. Then $u \in \mathrm{lip}_\alpha(K)$ is clearly invertible and we claim that

$$\|u^k\| \leqslant 1 + 2^{1-\alpha/2} L^\alpha |k|^\alpha \quad (k \in \mathbb{Z}). \tag{3.1}$$

We can certainly assume that $k \geqslant 1$. Obviously $\|u^k\|_\infty = 1$ and, for $x \neq y$ in K, we have

$$\begin{aligned}
\frac{|u(x)^k - u(y)^k|}{d(x, y)^\alpha} &= \big|u(x)^k - u(y)^k\big|^{1-\alpha} \left(\frac{|u(x)^k - u(y)^k|}{d(x, y)} \right)^\alpha \\
&\leqslant 2^{1-\alpha} \left(\frac{|u(x)^k - u(y)^k|}{d(x, y)} \right)^\alpha = 2^{1-\alpha} \left| \sum_{j=0}^{k-1} u(x)^{k-1-j} u(y)^j \right|^\alpha \left(\frac{|u(x) - u(y)|}{d(x, y)} \right)^\alpha \\
&\leqslant 2^{1-\alpha} k^\alpha \left(\frac{\sqrt{2}|h(x) - h(y)|}{d(x, y)} \right)^\alpha \leqslant 2^{1-\alpha} k^\alpha 2^{\alpha/2} L^\alpha,
\end{aligned}$$

which establishes (3.1).

We now take a real-valued Lipschitz function h on K. On account of (3.1), we can apply Corollary 2.3 with $a = b = \mathbf{1}$, $u = \exp(ith)$ $(t \in \mathbb{R})$, and $N = 2$, which yields

$$\varphi\big(\exp(i2th), \mathbf{1}\big) - 2\varphi\big(\exp(ith), \exp(ith)\big) + \varphi\big(\mathbf{1}, \exp(i2th)\big) = 0 \quad (t \in \mathbb{R}). \tag{3.2}$$

By computing the second derivative (with respect to t) at the point $t = 0$ of both sides in (3.2) we arrive at

$$\varphi\big(h^2, \mathbf{1}\big) - 2\varphi(h, h) + \varphi\big(\mathbf{1}, h^2\big) = 0 \tag{3.3}$$

for each real-valued Lipschitz function h on K. The linearization of (3.3) now yields

$$\varphi(fg, \mathbf{1}) - \varphi(f, g) - \varphi(g, f) + \varphi(\mathbf{1}, fg) = 0 \tag{3.4}$$

for all real-valued Lipschitz functions f, g on K. Since identity (3.4) is (complex) linear in both f and g and the (complex) linear span of all real-valued Lipschitz functions on K is nothing but the space of complex-valued Lipschitz functions on K, it follows that identity (3.4) holds for all complex-valued Lipschitz functions f, g on K. On the other hand, the space of all complex-valued Lipschitz functions f, g on K is dense in $\mathrm{lip}_\alpha(K)$ [7, Corollary 4.4.28(i)] and therefore (3.4) holds for $f, g \in \mathrm{lip}_\alpha(K)$ and this proves our assertion in the corollary. □

Corollary 3.2. *Let (K, d) be a non-empty, compact metric space, $\alpha \in]0, 1[$, and let $\varphi : \mathrm{Lip}_\alpha(K) \times \mathrm{Lip}_\alpha(K) \to X$ be a continuous bilinear map into some Banach space X with the property that*

$$f, g \in \mathrm{Lip}_\alpha(K), \quad fg = 0 \quad \Rightarrow \quad \varphi(f, g) = 0.$$

Then

$$\varphi(f, g) + \varphi(g, f) = \varphi(fg, \mathbf{1}) + \varphi(\mathbf{1}, fg) \quad \big(f, g \in \mathrm{Lip}_\alpha(K)\big).$$

Proof. By applying Corollary 3.1 to the restriction of φ to $\mathrm{lip}_\alpha(K) \times \mathrm{lip}_\alpha(K)$ we obtain

$$\varphi(f, g) + \varphi(g, f) = \varphi(fg, \mathbf{1}) + \varphi(\mathbf{1}, fg) \quad \big(f, g \in \mathrm{lip}_\alpha(K)\big). \tag{3.5}$$

On the other hand, by [7, Theorem 4.4.34], the Banach algebra $\mathrm{lip}_\alpha(K)$ is Arens regular, and its bidual $\mathrm{lip}_\alpha(K)^{**}$ is isometrically isomorphic to $\mathrm{Lip}_\alpha(K)$. This entails that, for every $f \in \mathrm{Lip}_\alpha(K)$, the map $g \mapsto fg$ from $\mathrm{Lip}_\alpha(K)$ into itself is w^*-continuous. Let $f, g \in \mathrm{Lip}_\alpha(K)$. From the w^*-denseness of $\mathrm{lip}_\alpha(K)$ in $\mathrm{Lip}_\alpha(K)$, it follows that there are nets (f_i) and (g_j) in $\mathrm{lip}_\alpha(K)$ with $w^*\text{-}\lim f_i = f$ and $w^*\text{-}\lim g_j = g$, respectively. According to (3.5), we have

$$\varphi(f_i, g_j) + \varphi(g_j, f_i) = \varphi(f_i g_j, \mathbf{1}) + \varphi(\mathbf{1}, f_i g_j). \tag{3.6}$$

We now proceed to take the limits in (3.6). We first fix j and take the limit in i. It should be pointed out that $w^*\text{-}\lim f_i g_j = f g_j$ and, on the other hand, the maps $\varphi(\cdot, g_j)$, $\varphi(g_j, \cdot)$, $\varphi(\cdot, \mathbf{1})$, and $\varphi(\mathbf{1}, \cdot)$ from $\mathrm{Lip}_\alpha(K)$ into X are continuous and hence w^*–w-continuous. From this and (3.6), it may be concluded that

$$\varphi(f, g_j) + \varphi(g_j, f) = \varphi(f g_j, \mathbf{1}) + \varphi(\mathbf{1}, f g_j). \tag{3.7}$$

We now take the limit in j in (3.7). To this end we note that $fg_j \to fg$ and that the maps $\varphi(f, \cdot)$, $\varphi(\cdot, f)$, $\varphi(\cdot, \mathbf{1})$, and $\varphi(\mathbf{1}, \cdot)$ from $\mathrm{Lip}_\alpha(K)$ into X are w^*–w-continuous. We thus obtain that

$$\varphi(f, g) + \varphi(g, f) = \varphi(fg, \mathbf{1}) + \varphi(\mathbf{1}, fg)$$

and therefore that (3.5) holds for all functions $f, g \in \mathrm{Lip}_\alpha(K)$, as claimed. □

4. Zero product preserving maps on Lipschitz algebras

Theorem 4.1. *Let (K, d) be a non-empty, compact metric space, $\alpha \in]0, 1[$, and let B be a commutative unital Banach algebra. If $T : \mathrm{lip}_\alpha(K) \to B$ is a continuous surjective linear map with the property that*

$$f, g \in \mathrm{lip}_\alpha(K), \quad fg = 0 \quad \Rightarrow \quad T(f)T(g) = 0,$$

then there exist an invertible element w in B and a continuous epimorphism $\Phi : \mathrm{lip}_\alpha(K) \to B$ such that $T = w\Phi$.

Proof. Define a continuous bilinear map $\varphi : \mathrm{lip}_\alpha(K) \times \mathrm{lip}_\alpha(K) \to B$ by $\varphi(f, g) = T(f)T(g)$ $(f, g \in \mathrm{lip}_\alpha(K))$. Then φ satisfies the requirement in Corollary 3.1 and therefore $T(f)T(g) + T(g)T(f) = T(fg)T(\mathbf{1}) + T(\mathbf{1})T(fg)$ $(f, g \in \mathrm{lip}_\alpha(K))$. Since B is commutative, the preceding identity gives

$$T(fg)T(\mathbf{1}) = T(f)T(g) \quad \big(f, g \in \mathrm{lip}_\alpha(K)\big). \tag{4.1}$$

From (4.1), together with the surjectivity of T, we deduce immediately that $w = T(\mathbf{1})$ is invertible in B. Finally, from (4.1) we deduce at once that the map $\Phi = w^{-1}T$ is a homomorphism. □

Theorem 4.2. *Let (K,d) be a non-empty, compact metric space, $\alpha \in]0,1[$, and let B be a commutative unital Banach algebra. If $T:\mathrm{Lip}_\alpha(K) \to B$ is a continuous surjective linear map with the property that*

$$f, g \in \mathrm{Lip}_\alpha(K), \quad fg = 0 \quad \Rightarrow \quad T(f)T(g) = 0,$$

then there exist an invertible element w in B and a continuous epimorphism $\Phi:\mathrm{Lip}_\alpha(K) \to B$ such that $T = w\Phi$.

Proof. This follows by the same method as in Theorem 4.1, the only difference being in the application of Corollary 3.2 instead of Corollary 3.1. □

Remark 4.3. It is worth pointing out that both Corollary 3.2 and Theorem 4.2 may fail to be true for the Banach algebra $\mathrm{Lip}_1(K)$. The bilinear map

$$\varphi:\mathrm{Lip}_1\big([0,1]\big) \times \mathrm{Lip}_1\big([0,1]\big) \to L^\infty\big([0,1]\big), \quad \varphi(f,g) = f'g' \quad \big(f, g \in \mathrm{Lip}_1\big([0,1]\big)\big)$$

satisfies the condition required in Corollary 3.2. Indeed, let $f, g \in \mathrm{Lip}_1([0,1])$ such that $fg = 0$ and assume that $x \in [0,1]$ is such that $f'(x) \neq 0$. Then $x \in \mathrm{supp}(f)$, because otherwise f would vanish on a neighbourhood of x which would imply that $f'(x) = 0$. Consequently, there exists a sequence (x_n) in $[0,1]$ with $\lim x_n = x$ and $f(x_n) \neq 0$ for each $n \in \mathbb{N}$. Since $fg = 0$, it follows that $g(x_n) = 0$ for each $n \in \mathbb{N}$ and so $g(x) = 0$. Hence $g'(x) = 0$. We thus get $f'g' = 0$, as claimed. Nevertheless, it is easily seen that the property claimed in Corollary 3.2 fails to be true for φ. On the other hand, Theorem 4.2 fails for the linear map

$$T:\mathrm{Lip}_1\big([0,1]\big) \to L^\infty\big([0,1]\big), \quad T(f) = f' \quad \big(f \in \mathrm{Lip}_1\big([0,1]\big)\big).$$

Remark 4.4. It is well known that Lipschitz algebras are semisimple and that every homomorphism from a Banach algebra into a commutative semisimple Banach algebra is automatically continuous [7]. Accordingly, every homomorphism from a Banach algebra into a Lipschitz algebra is continuous. On the other hand, it is important to know that every homomorphism from a Lipschitz algebra $\mathrm{Lip}_1(K_1)$ into another $\mathrm{Lip}_1(K_2)$ is described in [12] as a composition operator. Here we show that every epimorphism $\Phi:\mathrm{Lip}_\alpha(K_1) \to \mathrm{Lip}_\alpha(K_2)$ with $\alpha \in]0,1]$ and every epimorphism $\Phi:\mathrm{lip}_\alpha(K_1) \to \mathrm{lip}_\alpha(K_2)$ with $\alpha \in]0,1[$ is a composition operator C_ϕ for a map $\phi:K_2 \to K_1$ that satisfies $md_2(x,y) \leqslant d_1(\phi(x),\phi(y)) \leqslant Md_2(x,y)$ $(x, y \in K_1)$ for some positive constants m and M. Indeed, for any non-empty, compact metric space (K,d) and $\alpha \in]0,1]$, we have $\mathrm{Lip}_\alpha(K) = \mathrm{Lip}(K, d^\alpha)$, where we are following the notation of [12] in the right side of the equality and d^α is the distance on K defined by $d^\alpha(x,y) = d(x,y)^\alpha$ $(x, y \in K)$. Then, according to [12, Theorem 5.1], we deduce that every epimorphism $\Phi:\mathrm{Lip}_\alpha(K_1) \to \mathrm{Lip}_\alpha(K_2)$ with $\alpha \in]0,1]$ is of the form $\Phi(f) = f \circ \phi$ $(f \in \mathrm{Lip}_\alpha(K_1))$ for a map $\phi:K_2 \to K_1$ that satisfies $md_2(x,y) \leqslant d_1(\phi(x),\phi(y)) \leqslant Md_2(x,y)$ $(x, y \in K_1)$ for some positive constants m and M. Moreover, if $\Phi:\mathrm{lip}_\alpha(K_1) \to \mathrm{lip}_\alpha(K_2)$ is an epimorphism, then we deduce from the Arens regularity of both $\mathrm{lip}_\alpha(K_1)$ and $\mathrm{lip}_\alpha(K_2)$ [7, Theorem 4.4.34] and the w^*-continuity of the bi-adjoint operator $\Phi^{**}:\mathrm{lip}_\alpha(K_1)^{**} \to \mathrm{lip}_\alpha(K_2)^{**}$ that Φ^{**} is an epimorphism. Since $\mathrm{lip}_\alpha(K_1)^{**}$ and $\mathrm{lip}_\alpha(K_2)^{**}$ are isometrically isomorphic to $\mathrm{Lip}_\alpha(K_1)$ and $\mathrm{Lip}_\alpha(K_2)$, respectively [7, Theorem 4.4.34], it follows that Φ^{**} is a composition operator C_ϕ for a map $\phi:K_2 \to K_1$ that satisfies $md_2(x,y) \leqslant d_1(\phi(x),\phi(y)) \leqslant Md_2(x,y)$ $(x, y \in K_1)$ for some positive constants m and M, and therefore that Φ is nothing but the restriction to $\mathrm{lip}_\alpha(K_1)$ of C_ϕ, as claimed.

Corollary 4.5. *Let (K_1,d_1) and (K_2,d_2) be non-empty, compact metric spaces and $\alpha \in]0,1[$.*

(1) *If $T:\mathrm{lip}_\alpha(K_1) \to \mathrm{lip}_\alpha(K_2)$ is a continuous surjective linear map with the property that $T(f)T(g) = 0$ whenever $f, g \in \mathrm{lip}_\alpha(K_1)$ are such that $fg = 0$, then there exist a nonvanishing function $w \in \mathrm{lip}_\alpha(K_2)$ and a map $\phi:K_2 \to K_1$ that satisfies $md_2(x,y) \leqslant d_1(\phi(x),\phi(y)) \leqslant Md_2(x,y)$ $(x, y \in K_1)$ for some positive constants m and M such that $T(f) = w(f \circ \phi)$ $(f \in \mathrm{lip}_\alpha(K_1))$.*
(2) *If $T:\mathrm{Lip}_\alpha(K_1) \to \mathrm{Lip}_\alpha(K_2)$ is a continuous bijective linear map with the property that $T(f)T(g) = 0$ whenever $f, g \in \mathrm{Lip}_\alpha(K_1)$ are such that $fg = 0$, then there exist a nonvanishing function $w \in \mathrm{Lip}_\alpha(K_2)$ and a map $\phi:K_2 \to K_1$ that satisfies $md_2(x,y) \leqslant d_1(\phi(x),\phi(y)) \leqslant Md_2(x,y)$ $(x, y \in K_1)$ for some positive constants m and M such that $T(f) = w(f \circ \phi)$ $(f \in \mathrm{Lip}_\alpha(K_1))$.*

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