



# Boundedness of the gradient for a doubly nonlinear parabolic equation

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## ABSTRACT

We prove the local boundedness of the gradient for positive solutions to a doubly nonlinear parabolic equation in the case when the standard Lebesgue measure has been replaced by a doubling measure which supports a weak Poincaré inequality.

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## 1. Introduction

We study the higher regularity for positive solutions of the doubly nonlinear equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \frac{\partial(u^{p-1})}{\partial t}, \quad p \geq 2. \quad (1.1)$$

More precisely, we give a clear and transparent proof for the boundedness of the gradient for a solution of this equation. As a consequence the solution is spatially locally Lipschitz continuous. This is the first step towards showing the  $C^{1,\alpha}$ -regularity result.

It is noteworthy that if  $u$  is a solution for this equation, also  $\lambda u$  is a solution for  $\lambda \in \mathbb{R}_+$ . However, we do not use this scaling property in the argument. Observe also that when  $p = 2$  we have the standard heat equation.

Hölder continuity for this kind of doubly nonlinear equations has been studied e.g. by Porzio and Vespri [21,18] and DiBenedetto [3] as well as Ivanov [11]. Recently, DiBenedetto, Gianazza, Surnachev and Vespri have also found new methods for proving Hölder regularity [8,9]. See also [17,4,20].

Particularly for this equation references for regularity results seem to be difficult to find. Harnack's inequality was studied by Trudinger already in the 1960s [19] but for this equation it does not seem to directly imply even Hölder continuity. The main problem comes from the power-type nonlinearity on the right-hand side of (1.1). In this paper our emphasis is on higher regularity questions. For the Hölder continuity argument we refer to [14].

For the evolution  $p$ -Laplace equation, in which the time derivative in the right-hand side of (1.1) is replaced by  $u_t$ , our result was proved by DiBenedetto and Friedman in [5]. See also [6,7]. Our proof is based on a similar argument. However, to emphasize the general principle behind our reasoning, we replace the Lebesgue measure by a more general Borel measure. More precisely, we only assume the measure to be doubling and support a Poincaré inequality. This is made possible by a simplification of the original argument. In particular, we shorten the proof by replacing the traditional De Giorgi type argument by a Moser iteration scheme.

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Recently, Lewis and Nyström proved a boundary Harnack principle for the time-independent  $p$ -Laplace equation [15]. See also [16]. Weighted higher regularity results are crucial in their argument. Since Eq. (1.1) admits scale and location invariant Harnack type interior estimates, see [12], it would be interesting to know whether this equation would allow parabolic generalization to boundary Harnack principles, as well. This is one of our main motivations to study this problem, particularly in the weighted case.

Our proof also applies for more general equations of type

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) = \frac{\partial(u^{m-1})}{\partial t}, \quad p, m \geq 2. \quad (1.2)$$

For the sake of completeness we formulate the proof in this more general case. Observe that if  $m = 2$  we give a simplified argument for the original result by DiBenedetto and Friedman in the weighted setting.

## 2. Preliminaries

Let  $\mu$  be a Borel measure and  $\Omega$  an open set in  $\mathbb{R}^d$ . The Sobolev space  $H^{1,p}(\Omega)$  is defined to be the completion of  $C^\infty(\Omega)$  with respect to the Sobolev norm

$$\|u\|_{1,p,\Omega} = \left( \int_{\Omega} |u|^p + |\nabla u|^p d\mu \right)^{1/p}.$$

A function  $u$  belongs to the local Sobolev space  $H_{loc}^{1,p}(\Omega)$  if it belongs to  $H^{1,p}(\Omega')$  for every  $\Omega' \Subset \Omega$ . Moreover, the Sobolev space with zero boundary values is defined as the completion of  $C_0^\infty(\Omega)$  with respect to the Sobolev norm. For more properties of Sobolev spaces, see e.g. [10] or [1].

The parabolic Sobolev space  $L^p(t_1, t_2; H^{1,p}(\Omega))$  is the space of functions  $u(x, t)$  such that for almost every  $t$ ,  $t_1 < t < t_2$ , the function  $u(\cdot, t)$  belongs to  $H^{1,p}(\Omega)$  and

$$\int_{t_1}^{t_2} \int_{\Omega} (|u(x, t)|^p + |\nabla u(x, t)|^p) d\mu(x) dt < \infty.$$

The definition of the space  $L_{loc}^p(t_1, t_2; H_{loc}^{1,p}(\Omega))$  is clear. We will denote the product measure by  $dv := d\mu dt$ .

**Definition 2.1.** A function  $u \in L_{loc}^p(t_1, t_2; H_{loc}^{1,p}(\Omega)) \cap L_{loc}^\infty(\Omega \times (t_1, t_2))$  is a weak solution of Eq. (1.2) in  $\Omega \times (t_1, t_2)$  if it satisfies the integral equality

$$\int_{t_1}^{t_2} \int_{\Omega} \left( |\nabla u|^{p-2} \nabla u \cdot \nabla \phi - u^{m-1} \frac{\partial \phi}{\partial t} \right) dv = 0$$

for all  $\phi \in C_0^\infty(\Omega \times (t_1, t_2))$ .

Let  $t_1 < \tau_1 < \tau_2 < t_2$ . If the test function  $\phi$  vanishes only on the lateral boundary  $\partial\Omega \times (\tau_1, \tau_2)$  the boundary terms have to be included as well. In this case the condition becomes

$$\int_{\tau_1}^{\tau_2} \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \phi dv + \left[ \int_{\Omega} u^{m-1} \phi d\mu \right]_{t=\tau_1}^{\tau_2} - \int_{\tau_1}^{\tau_2} \int_{\Omega} u^{m-1} \frac{\partial \phi}{\partial t} dv = 0. \quad (2.2)$$

The measure  $\mu$  is doubling if there is a universal constant  $D_0 \geq 1$  such that

$$\mu(B(x, 2r)) \leq D_0 \mu(B(x, r))$$

for all  $B(x, 2r) \subset \Omega$ . Here  $B(x, r)$  denotes the standard open ball in  $\mathbb{R}^d$

$$B(x, r) = \{y \in \mathbb{R}^d: |y - x| < r\}.$$

The dimension related to the measure is  $d_\mu := \log_2 D_0$ . In the case of Lebesgue measure this is  $d_\mu = d$ . Moreover, the measure is said to support a weak  $(1, p)$ -Poincaré inequality if there exist constants  $P_0 > 0$  and  $\sigma \geq 1$  such that

$$\int_{B(x,r)} |v - v_{B(x,r)}| d\mu \leq P_0 r \left( \int_{B(x,\sigma r)} |\nabla v|^p d\mu \right)^{1/p}, \quad (2.3)$$

for every  $v \in H^{1,p}(\Omega)$  and  $B(\sigma x, r) \subset \Omega$ . Here we use the notation

$$v_{B(x,r)} = \int_{B(x,r)} v \, d\mu = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} v \, d\mu.$$

The word weak refers to the constant  $\sigma \geq 1$ . If the inequality (2.3) is true for  $\sigma = 1$  we say that the measure supports a  $(1, p)$ -Poincaré inequality.

It is known that the weak  $(1, p)$ -Poincaré inequality and the doubling condition imply a Sobolev embedding.

**Theorem 2.4.** Suppose that  $v \in H_0^{1,p}(B(x, r))$ . Then

$$\left( \int_{B(x,r)} |v|^\kappa \, d\mu \right)^{1/\kappa} \leq Cr \left( \int_{B(x,r)} |\nabla v|^p \, d\mu \right)^{1/p}$$

where

$$\kappa = \begin{cases} \frac{d_\mu p}{d_\mu - p}, & \text{for } 1 < p < d_\mu, \\ 2p, & \text{otherwise.} \end{cases}$$

**Proof.** See for example [13].  $\square$

We recall the Barenblatt solution [2] of Eq. (1.1):

$$\mathcal{B}_p(x, t) = Ct^{-\frac{n}{p(p-1)}} \exp\left(-\frac{p-1}{p} \left(\frac{|x|^p}{pt}\right)^{\frac{1}{p-1}}\right), \quad 1 < p < \infty,$$

where  $C > 0$ . Observe that  $\mathcal{B}_p(x, t) > 0$  for every  $x \in \mathbb{R}^d$  and  $t > 0$ . This indicates the “infinite propagation speed” similar to the heat equation. Since this Barenblatt solution is in certain sense the fundamental solution for Eq. (1.1) we would expect similar behaviour also in general. In particular, our assumption of  $u > 0$  is related to the properties of this Barenblatt solution.

Our main theorem is the following regularity result.

**Theorem 2.5.** Let  $u \in L^p(t_1, t_2; H_{loc}^{1,p}(\Omega))$  be a positive and continuous weak solution of Eq. (1.2). Then

$$u \in L^p(t_1, t_2; H_{loc}^{1,\infty}(\Omega)).$$

In particular,  $u$  is locally Lipschitz continuous in the space direction.

**Remark 2.6.** In the case  $m = p$  we get Eq. (1.1), but the theorem also includes other fundamental examples like the porous medium equation which, after a suitable substitution, corresponds to the case  $p = 2$  or the evolution  $p$ -Laplace equation, when  $m = 2$ . In the latter case the assumption  $u > 0$  can be reduced to  $u \geq 0$  and we will get a generalization of the original result by DiBenedetto and Friedman [5]. This is achieved by a simplification of their argument which relies only on the Moser’s iteration scheme.

### 3. $L^\infty$ bound for the gradient

#### 3.1. A Caccioppoli inequality

We will fix a point  $(x_0, t_0)$  and assume  $u(x_0, t_0) > 0$ . Furthermore, we restrict our study to a small enough neighborhood  $U \times (t_1, t_2)$  of  $(x_0, t_0)$  so that

$$\frac{1}{2}u(x_0, t_0) \leq u(x, t) \leq \frac{3}{2}u(x_0, t_0) \tag{3.1}$$

for all  $(x, t) \in U \times (t_1, t_2)$ . This is possible since  $u$  is continuous [14].

We start by differentiating Eq. (1.2), like in [4,5], with respect to  $x_i$ . This gives

$$\frac{\partial}{\partial t}((m-1)u^{m-2}u_{x_i}) - \nabla \cdot \left( |\nabla u|^{p-2} \nabla u_{x_i} + \frac{\partial}{\partial x_i} (|\nabla u|^{p-2}) \nabla u \right) = 0.$$

In the weak formulation, similarly as in (2.2), this is

$$\begin{aligned}
& (m-1) \left[ \int_U u^{m-2} u_{x_i} \phi \, d\mu \right]_{t=\tau_1}^{\tau_2} - (m-1) \int_{\tau_1}^{\tau_2} \int_U u^{m-2} u_{x_i} \frac{\partial \phi}{\partial t} \, dv \\
& + \int_{\tau_1}^{\tau_2} \int_U \left( |\nabla u|^{p-2} \nabla u_{x_i} + \frac{\partial}{\partial x_i} (|\nabla u|^{p-2}) \nabla u \right) \cdot \nabla \phi \, dv = 0.
\end{aligned} \tag{3.2}$$

Our argument is based on freezing the factor  $u^{m-2}$  at the point  $(x_0, t_0)$  after which we can intrinsically scale it to the geometry. For this scaling to be possible we need the information that  $u > 0$  in a neighborhood of the point. If  $m = 2$  the factor disappears and this scaling is not needed. As a consequence, in this case  $u$  can be assumed to be merely non-negative instead of positive. We will start the argument with the following Caccioppoli inequality.

**Lemma 3.3.** *Let  $u > 0$  be such a continuous weak solution of Eq. (1.2) in  $U \times (t_1, t_2)$  that (3.1) holds. Denote  $v := |\nabla u|^2$ . Then for every  $\alpha^* > 0$  there exists a constant  $C = C(m, p, \alpha^*)$  such that for all  $\alpha \in \{0\} \cup [\alpha^*, \infty[$  we have*

$$\begin{aligned}
& \operatorname{ess\,sup}_{t_1 < t < t_2} \int_U v^{\alpha+1} \varphi^2 \, d\mu + \frac{C}{u(x_0, t_0)^{m-2}} \int_{t_1}^{t_2} \int_U |\nabla (v^{\frac{p+2\alpha}{4}})|^2 \varphi^2 \, dv \\
& \leq \frac{C}{u(x_0, t_0)^{m-2}} \int_{t_1}^{t_2} \int_U v^{\frac{p+2\alpha}{2}} |\nabla \varphi|^2 \, dv + C \int_{t_1}^{t_2} \int_U v^{\alpha+1} \varphi \left| \frac{\partial \varphi}{\partial t} \right| \, dv
\end{aligned}$$

for every  $\varphi \in C_0^\infty(U \times (t_1, t_2))$ .

**Proof.** Let  $t_1 < \tau_1 < \tau_2 < t_2$  and choose  $\phi = u_{x_i} |\nabla u|^{2\alpha} \varphi^2$ , where  $\varphi \in C_0^\infty(U \times (t_1, t_2))$ , in (3.2) to get

$$\begin{aligned}
& -(m-1) \int_{\tau_1}^{\tau_2} \int_U u^{m-2} u_{x_i} \frac{\partial u_{x_i}}{\partial t} |\nabla u|^{2\alpha} \varphi^2 + u^{m-2} u_{x_i}^2 \frac{\partial}{\partial t} (|\nabla u|^{2\alpha}) \varphi^2 \, dv - 2(m-1) \int_{\tau_1}^{\tau_2} \int_U u^{m-2} u_{x_i}^2 |\nabla u|^{2\alpha} \varphi \frac{\partial \varphi}{\partial t} \, dv \\
& + (m-1) \left[ \int_U u^{m-2} u_{x_i}^2 |\nabla u|^{2\alpha} \varphi^2 \, d\mu \right]_{t=\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} \int_U \left( |\nabla u|^{p-2} \nabla u_{x_i} + \frac{\partial}{\partial x_i} (|\nabla u|^{p-2}) \nabla u \right) \cdot \nabla (u_{x_i} |\nabla u|^{2\alpha} \varphi) \, dv = 0.
\end{aligned} \tag{3.4}$$

After summing over  $i$  the first term can be estimated as follows

$$\begin{aligned}
& \sum_{i=1}^d \int_{\tau_1}^{\tau_2} \int_U -(m-1) u^{m-2} u_{x_i} \frac{\partial u_{x_i}}{\partial t} |\nabla u|^{2\alpha} \varphi^2 \, dv = - \int_{\tau_1}^{\tau_2} \int_U \frac{m-1}{2} u^{m-2} \frac{\partial}{\partial t} (|\nabla u|^2) |\nabla u|^{2\alpha} \varphi^2 \, dv \\
& = - \int_{\tau_1}^{\tau_2} \int_U \frac{m-1}{2(\alpha+1)} u^{m-2} \frac{\partial}{\partial t} (|\nabla u|^{2\alpha+2}) \varphi^2 \, dv \\
& \geq - \frac{C_1(m-1)}{2(\alpha+1)} u(x_0, t_0)^{m-2} \int_{\tau_1}^{\tau_2} \int_U \frac{\partial}{\partial t} (|\nabla u|^{2\alpha+2}) \varphi^2 \, dv \\
& = - \frac{C_1(m-1)}{2(\alpha+1)} u(x_0, t_0)^{m-2} \left[ \int_U |\nabla u|^{2\alpha+2} \varphi^2 \, d\mu \right]_{\tau_1}^{\tau_2} \\
& \quad + \frac{C_1(m-1)}{\alpha+1} u(x_0, t_0)^{m-2} \int_{\tau_1}^{\tau_2} \int_U |\nabla u|^{2\alpha+2} \varphi \frac{\partial \varphi}{\partial t} \, dv,
\end{aligned} \tag{3.5}$$

where either  $C_1 = (1/2)^{m-2}$  or  $C_1 = (3/2)^{m-2}$  depending on whether the term on the left-hand side is negative or positive.

For the second term in (3.4) we will have

$$\begin{aligned} \sum_{i=1}^d \int_{\tau_1}^{\tau_2} \int_U -(m-1)u^{m-2}u_{x_i}^2 \frac{\partial}{\partial t}(|\nabla u|^{2\alpha})\varphi^2 dv &\geq -C_1(m-1)u(x_0, t_0)^{m-2} \int_{\tau_1}^{\tau_2} \int_U |\nabla u|^2 \frac{\partial}{\partial t}(|\nabla u|^{2\alpha})\varphi^2 dv \\ &= -\frac{C_1\alpha}{\alpha+1}(m-1)u(x_0, t_0)^{m-2} \int_{\tau_1}^{\tau_2} \int_U \frac{\partial}{\partial t}(|\nabla u|^{2\alpha+2})\varphi^2 dv \\ &= -\frac{C_1\alpha}{\alpha+1}(m-1)u(x_0, t_0)^{m-2} \left[ \int_U |\nabla u|^{2\alpha+2}\varphi^2 d\mu \right]_{\tau_1}^{\tau_2} \\ &\quad + \frac{2C_1\alpha}{\alpha+1}(m-1)u(x_0, t_0)^{m-2} \int_{\tau_1}^{\tau_2} \int_U |\nabla u|^{2\alpha+2}\varphi \frac{\partial \varphi}{\partial t} dv. \end{aligned}$$

Here we used the fact that  $\partial_t(|\nabla u|^{2\alpha})$  and  $\partial_t(|\nabla u|^{2\alpha+2})$  have the same sign so that the constant  $C_1$  will be the same as in (3.5).

The third and fourth terms of (3.4) are already in the required form so the first four terms in (3.4) can be estimated as

$$\begin{aligned} &-(m-1) \sum_{i=1}^d \int_{\tau_1}^{\tau_2} \int_U u^{m-2}u_{x_i} \frac{\partial u_{x_i}}{\partial t} |\nabla u|^{2\alpha}\varphi^2 + u^{m-2}u_{x_i}^2 \frac{\partial}{\partial t}(|\nabla u|^{2\alpha})\varphi^2 dv \\ &\quad - 2(m-1) \sum_{i=1}^d \int_{\tau_1}^{\tau_2} \int_U u^{m-2}u_{x_i}^2 |\nabla u|^{2\alpha}\varphi \frac{\partial \varphi}{\partial t} dv + (m-1) \sum_{i=1}^d \left[ \int_U u^{m-2}u_{x_i}^2 |\nabla u|^{2\alpha}\varphi^2 d\mu \right]_{t=\tau_1}^{\tau_2} \\ &\geq -\frac{C_1(m-1)}{2(\alpha+1)}u(x_0, t_0)^{m-2} \left[ \int_U |\nabla u|^{2\alpha+2}\varphi^2 d\mu \right]_{\tau_1}^{\tau_2} + \frac{C_1(m-1)}{\alpha+1}u(x_0, t_0)^{m-2} \int_{\tau_1}^{\tau_2} \int_U |\nabla u|^{2\alpha+2}\varphi \frac{\partial \varphi}{\partial t} dv \\ &\quad - \frac{C_1\alpha}{\alpha+1}(m-1)u(x_0, t_0)^{m-2} \left[ \int_U |\nabla u|^{2\alpha+2}\varphi^2 d\mu \right]_{\tau_1}^{\tau_2} + \frac{2C_1\alpha}{\alpha+1}(m-1)u(x_0, t_0)^{m-2} \int_{\tau_1}^{\tau_2} \int_U |\nabla u|^{2\alpha+2}\varphi \frac{\partial \varphi}{\partial t} dv \\ &\quad - 2C(m-1)u(x_0, t_0)^{m-2} \int_{\tau_1}^{\tau_2} \int_U |\nabla u|^{2\alpha+2}\varphi \frac{\partial \varphi}{\partial t} dv + \frac{1}{2^{m-2}}(m-1)u(x_0, t_0)^{m-2} \left[ \int_U |\nabla u|^{2\alpha+2}\varphi^2 d\mu \right]_{t=\tau_1}^{\tau_2}. \end{aligned}$$

Simplifying and discarding terms gives

$$\begin{aligned} &-(m-1) \sum_{i=1}^d \int_{\tau_1}^{\tau_2} \int_U u^{m-2}u_{x_i} \frac{\partial u_{x_i}}{\partial t} |\nabla u|^{2\alpha}\varphi^2 + u^{m-2}u_{x_i}^2 \frac{\partial}{\partial t}(|\nabla u|^{2\alpha})\varphi^2 dv \\ &\quad - 2(m-1) \sum_{i=1}^d \int_{\tau_1}^{\tau_2} \int_U u^{m-2}u_{x_i}^2 |\nabla u|^{2\alpha}\varphi \frac{\partial \varphi}{\partial t} dv + (m-1) \sum_{i=1}^d \left[ \int_U u^{m-2}u_{x_i}^2 |\nabla u|^{2\alpha}\varphi^2 d\mu \right]_{t=\tau_1}^{\tau_2} \\ &\geq -C_1(m-1) \frac{2\alpha+1}{2(\alpha+1)}u(x_0, t_0)^{m-2} \left[ \int_U |\nabla u|^{2\alpha+2}\varphi^2 d\mu \right]_{\tau_1}^{\tau_2} - C(m-1)u(x_0, t_0)^{m-2} \int_{\tau_1}^{\tau_2} \int_U |\nabla u|^{2\alpha+2}\varphi \left| \frac{\partial \varphi}{\partial t} \right| dv. \end{aligned}$$

Next we need to estimate the elliptic terms in (3.4). For the first one summing over  $i$  yields

$$\begin{aligned} &\sum_{i=1}^d \int_{\tau_1}^{\tau_2} \int_U |\nabla u|^{p-2} \nabla u_{x_i} \cdot \nabla (u_{x_i} |\nabla u|^{2\alpha}\varphi^2) dv \\ &= \sum_{i=1}^d \int_{\tau_1}^{\tau_2} \int_U |\nabla u|^{p-2} \nabla u_{x_i} \cdot (\nabla u_{x_i} |\nabla u|^{2\alpha}\varphi^2 + \alpha u_{x_i} |\nabla u|^{2\alpha-2} \nabla(|\nabla u|^2)\varphi^2 + 2u_{x_i} |\nabla u|^{2\alpha}\varphi \nabla \varphi) dv \end{aligned}$$

$$= \int_{\tau_1}^{\tau_2} \int_U |\nabla u|^{p-2+2\alpha} \sum_{i=1}^d |\nabla u_{x_i}|^2 \varphi^2 + \frac{\alpha}{2} |\nabla u|^{p+2\alpha-4} |\nabla(|\nabla u|^2)|^2 \varphi^2 \\ + |\nabla u|^{p+2\alpha-2} \nabla(|\nabla u|^2) \cdot \varphi \nabla \varphi \, dv.$$

And for the last term in (3.4) we have

$$\sum_{i=1}^d \int_{\tau_1}^{\tau_2} \int_U \left( \frac{\partial}{\partial x_i} (|\nabla u|^{p-2}) \nabla u \right) \cdot (\nabla u_{x_i} |\nabla u|^{2\alpha} \varphi^2 + \alpha u_{x_i} |\nabla u|^{2\alpha-2} \nabla(|\nabla u|^2) \varphi^2 + 2u_{x_i} |\nabla u|^{2\alpha} \varphi \nabla \varphi) \, dv \\ = \sum_{i=1}^d \int_{\tau_1}^{\tau_2} \int_U \frac{1}{2} \frac{\partial}{\partial x_i} (|\nabla u|^{p-2}) \frac{\partial}{\partial x_i} (|\nabla u|^2) |\nabla u|^{2\alpha} \varphi^2 + \alpha(p-2) |\nabla u|^{p-2\alpha-6} \frac{\partial}{\partial x_i} (|\nabla u|^2) u_{x_i} \nabla u \cdot \nabla(|\nabla u|^2) \varphi^2 \\ + 2(p-2) |\nabla u|^{p+2\alpha-4} \frac{\partial}{\partial x_i} (|\nabla u|^2) u_{x_i} \nabla u \cdot \varphi \nabla \varphi \, dv \\ = \int_{\tau_1}^{\tau_2} \int_U \frac{p-2}{2} |\nabla u|^{p+2\alpha-4} |\nabla(|\nabla u|^2)|^2 \varphi^2 + \alpha(p-2) |\nabla u|^{p+2\alpha-6} (\nabla u \cdot \nabla(|\nabla u|^2))^2 \varphi^2 \\ + (p-2) |\nabla u|^{p+2\alpha-4} (\nabla u \cdot \nabla(|\nabla u|^2)) \varphi \nabla u \cdot \nabla \varphi \, dv.$$

For simplicity we will use the notation  $v := |\nabla u|^2$ . Now combining all above estimates yields

$$0 \geq -C_1(m-1) \frac{2\alpha+1}{2(\alpha+1)} u(x_0, t_0)^{m-2} \left[ \int_U v^{\alpha+1} \varphi^2 \, d\mu \right]_{\tau_1}^{\tau_2} - C(m-1) u(x_0, t_0)^{m-2} \int_{\tau_1}^{\tau_2} \int_U v^{\alpha+1} \varphi \left| \frac{\partial \varphi}{\partial t} \right| \, dv \\ + \int_{\tau_1}^{\tau_2} \int_U v^{\frac{p-2}{2}+\alpha} \sum_{i=1}^d |\nabla u_{x_i}|^2 \varphi^2 + \frac{\alpha}{2} v^{\frac{p+2\alpha-4}{2}} |\nabla v|^2 \varphi^2 + v^{\frac{p+2\alpha-2}{2}} \nabla v \cdot \varphi \nabla \varphi \, dv \\ + \int_{\tau_1}^{\tau_2} \int_U \frac{p-2}{2} v^{\frac{p+2\alpha-4}{2}} |\nabla v|^2 \varphi^2 + \alpha(p-2) v^{\frac{p+2\alpha-6}{2}} (\nabla u \cdot \nabla v)^2 \varphi^2 + (p-2) v^{\frac{p+2\alpha-4}{2}} (\nabla u \cdot \nabla v) \varphi \nabla u \cdot \nabla \varphi \, dv.$$

Moving terms and trivial estimates gives

$$-C_1(m-1) \frac{2\alpha+1}{2(\alpha+1)} u(x_0, t_0)^{m-2} \left[ \int_U v^{\alpha+1} \varphi^2 \, d\mu \right]_{\tau_1}^{\tau_2} + \frac{\alpha+p-2}{2} \int_{\tau_1}^{\tau_2} \int_U v^{\frac{p+2\alpha-4}{2}} |\nabla v|^2 \varphi^2 \, dv \\ + \int_{\tau_1}^{\tau_2} \int_U v^{\frac{p-2}{2}+\alpha} \sum_{i=1}^d |\nabla u_{x_i}|^2 \varphi^2 \, dv + \alpha(p-2) \int_{\tau_1}^{\tau_2} \int_U v^{\frac{p+2\alpha-6}{2}} (\nabla u \cdot \nabla v)^2 \varphi^2 \, dv \\ \leq \int_{\tau_1}^{\tau_2} \int_U v^{\frac{p+2\alpha-2}{2}} |\nabla v \cdot \varphi \nabla \varphi| \, dv - (p-2) \int_{\tau_1}^{\tau_2} \int_U v^{\frac{p+2\alpha-4}{2}} (\nabla u \cdot \nabla v) \varphi \nabla u \cdot \nabla \varphi \, dv \\ + C(m-1) u(x_0, t_0)^{m-2} \int_{\tau_1}^{\tau_2} \int_U v^{\alpha+1} \varphi \left| \frac{\partial \varphi}{\partial t} \right| \, dv. \quad (3.6)$$

By Young's inequality the first term on the right-hand side can be estimated as

$$\int_{\tau_1}^{\tau_2} \int_U v^{\frac{p+2\alpha-2}{2}} |\nabla v \cdot \varphi \nabla \varphi| \, dv \leq \frac{\alpha+p-2}{4} \int_{\tau_1}^{\tau_2} \int_U v^{\frac{p+2\alpha-4}{2}} |\nabla v|^2 \varphi^2 \, dv + C(p, \alpha) \int_{\tau_1}^{\tau_2} \int_U v^{\frac{p+2\alpha}{2}} |\nabla \varphi|^2 \, dv.$$

Here the first term of the right-hand side can be absorbed to the second term on the left-hand side of (3.6). Note that the constant  $C(p, \alpha)$  behaves well both when  $\alpha \rightarrow 0$  and  $\alpha \rightarrow \infty$ .

For the second term on the right-hand side of (3.6) we have to consider the cases  $\alpha = 0$  and  $\alpha > 0$  separately. If  $\alpha = 0$  by Young's inequality we have

$$\begin{aligned} -(p-2) \int_{\tau_1}^{\tau_2} \int_U v^{\frac{p-4}{2}} (\nabla u \cdot \nabla v) \varphi \nabla u \cdot \nabla \varphi \, dv &\leq (p-2) \int_{\tau_1}^{\tau_2} \int_U v^{\frac{p-6}{2}} |\nabla u \cdot \nabla v| v^{3/2} \varphi |\nabla \varphi| \, dv \\ &\leq (p-2) \int_{\tau_1}^{\tau_2} \int_U \frac{1}{8} v^{\frac{p-4}{2}} |\nabla v|^2 \varphi^2 + C v^{\frac{p-8}{2}} |\nabla u|^2 v^3 |\nabla \varphi|^2 \, dv \\ &= (p-2) \int_{\tau_1}^{\tau_2} \int_U \frac{1}{8} v^{\frac{p-4}{2}} |\nabla v|^2 \varphi^2 + C v^{\frac{p}{2}} |\nabla \varphi|^2 \, dv. \end{aligned}$$

Again the first term can be absorbed to the second term on the left-hand side of (3.6). For  $\alpha > 0$  we estimate

$$\begin{aligned} -(p-2) \int_{\tau_1}^{\tau_2} \int_U v^{\frac{p+2\alpha-4}{2}} (\nabla u \cdot \nabla v) \varphi \nabla u \cdot \nabla \varphi \, dv &\leq (p-2) \int_{\tau_1}^{\tau_2} \int_U v^{\frac{p+2\alpha-6}{2}} |\nabla u \cdot \nabla v| v^{3/2} \varphi |\nabla \varphi| \, dv \\ &\leq (p-2) \int_{\tau_1}^{\tau_2} \int_U \frac{\alpha}{2} v^{\frac{p+2\alpha-6}{2}} (\nabla u \cdot \nabla v)^2 \varphi^2 + C(\alpha) v^{\frac{p+2\alpha-6}{2}} v^3 |\nabla \varphi|^2 \, dv \\ &= (p-2) \int_{\tau_1}^{\tau_2} \int_U \frac{\alpha}{2} v^{\frac{p+2\alpha-6}{2}} (\nabla u \cdot \nabla v)^2 \varphi^2 + C(\alpha) v^{\frac{p+2\alpha}{2}} |\nabla \varphi|^2 \, dv. \end{aligned}$$

In this case the first term on the right-hand side will be absorbed to the last term on the left-hand side of (3.6). It is also noteworthy that when  $\alpha$  gets bigger the constant  $C(\alpha)$  in the above inequality will get smaller. In particular, when  $\alpha$  is bounded below, say  $\alpha \geq \alpha^* > 0$ , we can choose the constant to be dependent only on  $\alpha^*$ .

With the above estimates inequality (3.6) takes the form

$$\begin{aligned} -C(m-1) \frac{2\alpha+1}{2(\alpha+1)} u(x_0, t_0)^{m-2} \left[ \int_U v^{\alpha+1} \varphi^2 \, d\mu \right]_{\tau_1}^{\tau_2} &+ \frac{\alpha+p-2}{8} \int_{\tau_1}^{\tau_2} \int_U v^{\frac{p+2\alpha-4}{2}} |\nabla v|^2 \varphi^2 \, dv \\ &+ \int_{\tau_1}^{\tau_2} \int_U v^{\frac{p-2}{2}+\alpha} \sum_{i=1}^d |\nabla u_{x_i}|^2 \varphi^2 \, dv + \frac{\alpha(p-2)}{2} \int_{\tau_1}^{\tau_2} \int_U v^{\frac{p+2\alpha-6}{2}} (\nabla u \cdot \nabla v)^2 \varphi^2 \, dv \\ &\leq C(\alpha^*) \int_{\tau_1}^{\tau_2} \int_U v^{\frac{p+2\alpha}{2}} |\nabla \varphi|^2 \, dv + C(m-1) u(x_0, t_0)^{m-2} \int_{\tau_1}^{\tau_2} \int_U v^{\alpha+1} \varphi \left| \frac{\partial \varphi}{\partial t} \right| \, dv. \end{aligned}$$

Choose now  $\tau_1 = \tau$  such that

$$\int_U v^{\alpha+1} \varphi^2(x, \tau) \, d\mu \geq \frac{1}{2} \operatorname{ess\,sup}_{t_1 < t < t_2} \int_U v^{\alpha+1} \varphi^2 \, d\mu$$

and let  $\tau_2 \rightarrow t_2$ . Finally, discarding terms and writing

$$v^{\frac{p+2\alpha-4}{2}} |\nabla v|^2 = \left| \nabla \left( v^{\frac{p+2\alpha}{4}} \right) \right|^2$$

yields

$$\begin{aligned} \operatorname{ess\,sup}_{t_1 < t < t_2} \int_U u(x_0, t_0)^{m-2} v^{\alpha+1} \varphi^2 \, d\mu &+ \int_{\tau}^{\tau_2} \int_U \left| \nabla \left( v^{\frac{p+2\alpha}{4}} \right) \right|^2 \varphi^2 \, dv \\ &\leq C \int_{\tau_1}^{\tau_2} \int_U v^{\frac{p+2\alpha}{2}} |\nabla \varphi|^2 \, dv + C \int_{\tau_1}^{\tau_2} \int_U u^{m-2} v^{\alpha+1} \varphi \left| \frac{\partial \varphi}{\partial t} \right| \, dv \end{aligned}$$

where  $C = C(m, p, \alpha^*)$ . Now divide both sides by  $u(x_0, t_0)^{p-2}$  to obtain

$$\begin{aligned} & \operatorname{ess\,sup}_{t_1 < t < t_2} \int_U v^{\alpha+1} \varphi^2 d\mu + \frac{C}{u(x_0, t_0)^{m-2}} \int_{\tau}^{t_2} \int_U |\nabla(v^{\frac{p+2\alpha}{4}})|^2 \varphi^2 dv \\ & \leq \frac{C}{u(x_0, t_0)^{m-2}} \int_{t_1}^{t_2} \int_U v^{\frac{p+2\alpha}{2}} |\nabla \varphi|^2 dv + C \int_{t_1}^{t_2} \int_U v^{\alpha+1} \varphi \left| \frac{\partial \varphi}{\partial t} \right| dv. \end{aligned}$$

By a similar argument the lower and upper integration limits in the second term of the left-hand side can be replaced by  $t_1$  and  $\tau$ , respectively. The lemma follows.  $\square$

### 3.2. Moser's iteration

Next we will prove the desired result of boundedness of the gradient of  $u$ . This will be done by Moser's iteration. Let

$$r_j = \frac{r}{2} + \frac{r}{2^j}$$

and denote

$$Q_j := B_j \times T_j = B(x, r_j) \times (t - u(x_0, t_0)^{m-2} r_j^2, t)$$

and

$$Q_\infty := \lim_{j \rightarrow \infty} Q_j.$$

Choose also  $r > 0$  such that  $Q_0 \subset U \times (t_1, t_2)$ . Now we are ready to state the theorem.

**Theorem 3.7.** *Let  $u > 0$  be a continuous weak solution of Eq. (1.2) in  $U \times (t_1, t_2)$ . Then there exists a constant  $C = C(m, p, D_0, P_0) > 0$  such that*

$$\operatorname{ess\,sup}_{Q_\infty} |\nabla u|^2 \leq C \left( \int_{Q_0} |\nabla u|^p dv + 1 \right).$$

**Proof.** The proof is based on the Moser iteration scheme. We will start by choosing cut-off functions  $\varphi_j \in C_0^\infty(Q_j)$  such that

$$0 \leq \varphi_j \leq 1, \quad \varphi_j = 1 \quad \text{in } Q_{j+1}$$

and

$$|\nabla \varphi_j| \leq \frac{2^j C}{r} \quad \text{and} \quad \left| \frac{\partial \varphi_j}{\partial t} \right| \leq \frac{2^{2j} C}{u(x_0, t_0)^{m-2} r^2}.$$

As before, denote also

$$v = |\nabla u|^2.$$

By Hölder's inequality in the spatial integral and by estimating the first factor by essential supremum we get

$$\int_{Q_{j+1}} v^{\frac{p+2\alpha}{2} + (\alpha+1)\frac{\kappa-2}{\kappa}} dv \leq \frac{|T_j| \mu(B_j)}{|T_{j+1}| \mu(B_{j+1})} \left( \operatorname{ess\,sup}_{T_j} \int_{B_j} v^{\alpha+1} \varphi_j^2 d\mu \right)^{\frac{\kappa-2}{\kappa}} \int_{T_j} \left( \int_{B_j} (v^{\frac{p+2\alpha}{4}} \varphi_j)^\kappa d\mu \right)^{2/\kappa} dt.$$

By the doubling property, the measure factor on the right-hand side is bounded. Using the Sobolev embedding and Lemma 3.3 yields

$$\begin{aligned} \int_{Q_{j+1}} v^{\frac{p+2\alpha}{2} + (\alpha+1)\frac{\kappa-2}{\kappa}} dv & \leq C \left( \operatorname{ess\,sup}_{T_j} \int_{B_j} v^{\alpha+1} \varphi_j^2 d\mu \right)^{\frac{\kappa-2}{\kappa}} r^2 \int_{Q_j} |\nabla(v^{\frac{p+2\alpha}{4}} \varphi_j)|^2 dv \\ & \leq C \left( \operatorname{ess\,sup}_{T_j} \int_{B_j} v^{\alpha+1} \varphi_j^2 d\mu + r^2 \int_{Q_j} |\nabla(v^{\frac{p+2\alpha}{4}})|^2 \varphi_j^2 + v^{\frac{p+2\alpha}{2}} |\nabla \varphi_j|^2 dv \right)^{2-2/\kappa} \end{aligned}$$



$$\begin{aligned}
&\leq C \left( r^2 \int_{Q_j} v^{\frac{p+2\alpha}{2}} |\nabla \varphi_j|^2 dv + C \int_{T_j} \int_{B_j} v^{\alpha+1} \varphi_j \left| \frac{\partial \varphi_j}{\partial t} \right| dv \right)^{2-2/\kappa} \\
&\leq C 2^{2j(2-2/\kappa)} \left( \int_{Q_j} v^{\frac{p+2\alpha}{2}} dv + \int_{Q_j} v^{\alpha+1} dv \right)^{2-2/\kappa} \\
&\leq C 2^{2j(2-2/\kappa)} \left( \int_{Q_j} v^{\frac{p+2\alpha}{2}} dv + \int_{Q_j} \max(v, 1)^{\frac{p+2\alpha}{2}} dv \right)^{2-2/\kappa} \\
&\leq C 2^{2j(2-2/\kappa)} \left( \int_{Q_j} v^{\frac{p+2\alpha}{2}} dv + 1 \right)^{2-2/\kappa}.
\end{aligned}$$

Now we are ready to start Moser's iteration. Let  $\gamma = 2 - 2/\kappa$  and  $\alpha_0 = 0$ . For  $j \geq 1$  define recursively

$$\beta_{j+1} := \frac{p + 2\alpha_{j+1}}{2} = \frac{p + 2\alpha_j}{2} + (\alpha_j + 1) \frac{\kappa - 2}{\kappa}.$$

Solving for  $\alpha_j$  gives  $\alpha_j = \gamma^j - 1$ . Moreover, denote

$$\psi_j := \int_{Q_j} v^{\beta_j} dv.$$

With these notations we have

$$\psi_{j+1} \leq C 4^{j\gamma} (\psi_j + 1)^\gamma$$

and hence by iteration

$$\begin{aligned}
\psi_{j+1} &\leq C 4^{j\gamma} (\psi_j + 1)^\gamma \leq C 4^{j\gamma} (C 4^{(j-1)\gamma} (\psi_{j-1} + 1)^\gamma + 1)^\gamma \\
&\leq C^{1+\gamma} 4^{j\gamma + (j-1)\gamma^2} 2^\gamma (\psi_{j-1} + 1)^{\gamma^2} \\
&\vdots \\
&\leq C^{\gamma^*} 4^{C_{\text{prod}}} 2^{\gamma^* - 1} (\psi_0 + 1)^{\gamma^{j+1}}
\end{aligned}$$

where

$$\gamma^* = 1 + \gamma + \dots + \gamma^j = \frac{\gamma^{j+1} - 1}{\gamma - 1}$$

and

$$C_{\text{prod}} = \sum_{k=0}^j \gamma^{k+1} (j - k).$$

Altogether we have

$$\psi_{j+1}^{1/\beta_{j+1}} \leq 4^{C_{\text{prod}}/\beta_{j+1}} C^{\gamma^*/\beta_{j+1}} 2^{(\gamma^*-1)/\beta_{j+1}} (\psi_0 + 1)^{\gamma^{j+1}/\beta_{j+1}}. \quad (3.8)$$

Since  $p \geq 2$ , we get

$$\begin{aligned}
\frac{C_{\text{prod}}}{\beta_{j+1}} &= \frac{2 \sum_{k=0}^j \gamma^{k+1} (j - k)}{p + 2(\gamma^{j+1} - 1)} \leq \sum_{k=1}^{\infty} k \gamma^{-k} < \infty, \\
\frac{\gamma^* - 1}{\beta_{j+1}} &\leq \frac{\gamma^*}{\beta_{j+1}} = \frac{2(\gamma^{j+1} - 1)}{(\gamma - 1)(p + 2(\gamma^{j+1} - 1))} \leq \frac{1}{(\gamma - 1)}
\end{aligned}$$

and

$$\frac{\gamma^{j+1}}{\beta_{j+1}} = \frac{2\gamma^{j+1}}{p + 2(\gamma^{j+1} - 1)} \rightarrow 1$$

as  $j \rightarrow \infty$ . Hence, by letting  $j \rightarrow \infty$  in (3.8) we obtain

$$\operatorname{ess\,sup}_{Q_\infty} |\nabla u|^2 \leq C \left( \int_{Q_0} |\nabla u|^p \, dv + 1 \right),$$

as required.  $\square$

Theorem 2.5 follows.

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