J. Math. Anal. Appl. 375 (2011) 685–698

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Lifespan Theorem for simple constrained surface diffusion flows

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ARTICLE INFO

Article history:
Received 21 March 2010
Available online 1 October 2010
Submitted by Steven G. Krantz

Keywords:
Global differential geometry
Fourth order
Geometric analysis
Parabolic partial differential equations

ABSTRACT

We consider closed immersed hypersurfaces in $\mathbb{R}^3$ and $\mathbb{R}^4$ evolving by a special class of constrained surface diffusion flows. This class of constrained flows includes the classical surface diffusion flow. In this paper we present a Lifespan Theorem for these flows, which gives a positive lower bound on the time for which a smooth solution exists, and a small upper bound on the total curvature during this time. The hypothesis of the theorem is that the surface is not already singular in terms of concentration of curvature. This turns out to be a deep property of the initial manifold, as the lower bound on maximal time obtained depends precisely upon the concentration of curvature of the initial manifold in L^2 for M^2 immersed in $\mathbb{R}^3$ and additionally on the concentration in L^3 for M^3 immersed in $\mathbb{R}^4$. This is stronger than a previous result on a different class of constrained surface diffusion flows, as here we obtain an improved lower bound on maximal time, a better estimate during this period, and eliminate any assumption on the area of the evolving hypersurface.

1. Introduction

Let $f : M^n \times [0, T) \to \mathbb{R}^{n+1}$ be a family of compact immersed hypersurfaces $f(\cdot, t) = f_t : M^n \to f_t(M) = M_t \subset \mathbb{R}^{n+1}$ with associated Laplace–Beltrami operator Δ, unit normal vector field ν, and mean curvature function H. In this paper we study the constrained surface diffusion flows, where f evolves by

$$\frac{\partial}{\partial t} f = (\Delta H + h)\nu, \tag{1}$$

where $h : [0, T) \subset I \to \mathbb{R}$ is called the *constraint function*. The study of the fourth order degenerate parabolic quasilinear system of Eqs. (1) is motivated primarily by choice of constraint function. The trivial example of $h = 0$, classical surface diffusion flow, is instructive and for this paper our chief motivator.

Indeed, there does already exist a large body of work on the classical surface diffusion flow. First proposed by the physicist Mullins [31] in 1957, it was originally designed to model the formation of tiny thermal grooves in phase interfaces where the contribution due to evaporation–condensation was insignificant. Some time later, Davì, Gurtin, Cahn and Taylor [7,10] proposed many other physical models which give rise to the surface diffusion flow. These all exhibit a reduction of free surface energy and conservation of volume; an essential characteristic of surface diffusion flow. There are also other motivations for the study of surface diffusion flow. For example, two years later Cahn, Elliot and Novick-Cohen [6] proved that the surface diffusion flow is the singular limit of the Cahn–Hilliard equation with a concentration dependent mobility. Among other applications, this arises in the modeling of isothermal separation of compound materials.

Analysis of the surface diffusion flow began slowly, with the first works appearing in the early 80s. Baras, Duchon and Robert [3] showed the global existence of weak solutions for two-dimensional strip-like domains in 1984. Later, in 1997

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.09.043

Elliot and Garcke [11] analysed the surface diffusion flow of curves, and obtained local existence and regularity for C^4-initial curves, and global existence for small perturbations of circles. Significantly, Ito [19] showed in 1998 that convexity will not be preserved under the surface diffusion flow, even for smooth, rotationally symmetric, closed, compact, strictly convex initial hypersurfaces. In contrast with the case for second order flows such as mean curvature flow, this behaviour appears pathological. Escher, Mayer and Simonett [12] gave several numerical schemes for modeling surface diffusion flow, and have also given the only two known numerical examples [26] of the development of a singularity: a tubular spiral and thin-necked dumbbell. They also provide an example of an immersion which will self-intersect under the flow, a figure eight knot. In 2001, Simonett [34] used centre manifold techniques to show that for initial data $C^{2,\alpha}$-close to a sphere, both the surface diffusion and Willmore flows (Willmore flow in one codimension is $\partial_t f = \Delta H + \|A^o\|^2 H$, where $A^o = A - \frac{1}{n} gH$) exist for all time and converge asymptotically to a sphere.

There have been many important works on fourth order flows of a slightly different character, from Willmore flow of surfaces to Calabi flow, a fourth order flow of metrics. Significant contributions to the analysis of these flows by the authors Kuwert, Schätzle, Polden, Huisken, Mantegazza and Chruściel [8,21,22,25,32] are particularly relevant, as the methods employed there are similar to ours here. For the study of constrained flows, we mention the papers [1,2,4,5,9,15,14,16,24, 27–29,38,33,39], which contain a plethora of applications to motivate the study of non-trivial constraint functions h.

The issue of local well-posedness of (1) is delicate, although standard, and overcome with standard techniques as in [12], with the constraint function causing no additional difficulty. We make no effort to pose an optimal version, although the interested reader may enjoy [20] for recent progress in this direction.

Theorem 1.1 *(Short time existence). For any smooth initial immersion $f_0 : M^n \to \mathbb{R}^{n+1}$ and bounded constraint function $h : I \to \mathbb{R}$, with I an interval containing 0, there exists a unique nonextendable smooth solution $f : M \times [0,T) \to \mathbb{R}^{n+1}$ to (1) with $f(\cdot,0) = f_0$, where $0 < T \leqslant \infty$.*

The main issue then becomes global existence. While we do not treat this explicitly here, we do present a result with applications to singularity analysis, as can be seen in [36]. In our proof, we exploit the fact that for an n-dimensional immersion the integral

$$\int_M \|A\|^n \, d\mu$$

is scale invariant. The technique used by Struwe [35] is then relevant, although as with all higher order flows the major difficulty is in overcoming the lack of powerful techniques unique to the second order case. In particular, we are without the maximum principle, and this implies that the geometry of the surface could deteriorate, as in [19]. Drawing inspiration from Kuwert and Schätzle [22] in particular, we use local integral estimates to derive derivative curvature bounds under a local smallness of curvature assumption. In calculating these estimates it is crucial to only use inequalities which involve universal constants. Interpolation inequalities similar in nature to those used by Ladyzhenskaya, Solonnikov and Ural'tseva [23] and Hamilton [13], and the Sobolev inequality of Michael and Simon [30], are invaluable in this regard.

Following Hamilton [13], we denote polynomials in the iterated covariant derivatives of a tensor T by

$$P^i_j(T) = \sum_{k_1+\cdots+k_j=i} c_{k_1\cdots k_j} \nabla_{(k_1)} T * \cdots * \nabla_{(k_j)} T,$$

where $c_{k_1\cdots k_j} \in \mathbb{R}$ and $\nabla_{(k)} T$ is the k-th iterated covariant derivative of T; see Section 2 for more details. For a large class of constrained surface diffusion flows the following theorem applies.

Theorem 1.2 *(Lifespan Theorem). (See [38].) Suppose $n \in \{2,3\}$ and let $f : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a compact immersion with C^∞ initial data evolving by (1). Suppose that for some $j,k,l \in \mathbb{N}_0$ the constraint function $h : I \supset [0,T) \to \mathbb{R}$ obeys an estimate*

$$h \leqslant \int_M P^2_j(A) + P^1_k(A) + P^0_l(A) \, d\mu. \tag{2}$$

Then there are constants $\rho > 0$, $\epsilon_0 > 0$, and $c < \infty$ such that

$$\int_{f^{-1}(B_\rho(x))} \|A\|^m \, d\mu \bigg|_{t=0} = \epsilon(x) \leqslant \epsilon_0, \quad \text{for any } x \in \mathbb{R}^{n+1} \tag{3}$$

where $m = \max\{2k-2, 2j-k, l, n^2+n-2\}$; and there exists an absolute constant $C_{AB} \in (0,\infty)$ such that

$$\|M_t\| \leqslant C_{AB}, \quad \text{for } 0 \leqslant t \leqslant \frac{1}{c}\rho^4; \tag{4}$$

then the maximal time T of smooth existence for the flow (1) *with initial data $f_0 = f(\cdot, 0)$ satisfies*

$$T \geqslant \frac{1}{c}\rho^4, \tag{5}$$

and we have the estimate

$$\int_{f^{-1}(B_\rho(x))} \|A\|^n \, d\mu \leqslant c\epsilon(x), \quad \text{for } 0 \leqslant t \leqslant \frac{1}{c}\rho^4. \tag{6}$$

The result we present here is new for the surface diffusion flow, stronger than Theorem 1.2, and plays a key role in the analysis of the asymptotic behaviour of the flow. In particular the main theorem of this papers enables one to guarantee that under certain conditions finite time curvature singularities possess properties which combined with the results on blowups in [36] allows one to rule out their development entirely. The key improvements are that the assumption on the evolving surface area (4) is completely removed, and the concentration of curvature assumption (3) is in L^2 for two-dimensional manifolds and additionally in L^3 for three-dimensional manifolds.

The reason for these improvements is that we consider only constraint functions which fit into the following natural class. A constraint function $h : [0, T) \subset I \to \mathbb{R}$ which satisfies an estimate

$$\|h\|_{\infty, J} \leqslant c_h < \infty \tag{7}$$

on any closed interval $J \subset [0, T)$ with $c_h = c_h(J)$ is called *simple*. Note that this includes constraint functions which are unbounded on $\mathbb{R}$, change sign, and so on. The corresponding constrained surface diffusion flow where the constraint function is simple is called briefly a *simple constrained surface diffusion flow*. Our main result in this paper is the following.

Theorem 1.3. *Suppose $n \in \{2, 3\}$ and let $f : M^n \times [0, T) \to \mathbb{R}^{n+1}$ be a simple constrained surface diffusion flow. Then there are constants $\rho > 0$, $\epsilon_0 > 0$, and $c < \infty$ such that*

$$\int_{f^{-1}(B_\rho(x))} \|A\|^m \, d\mu \bigg|_{t=0} = \epsilon(x) \leqslant \epsilon_0, \quad \text{for } m = 2, n, \text{ any } x \in \mathbb{R}^{n+1}, \tag{8}$$

and h is simple on $[0, \frac{1}{c}\rho^4]$, then the maximal time T satisfies

$$T \geqslant \frac{1}{c}\rho^4, \tag{9}$$

and we have the estimate

$$\int_{f^{-1}(B_\rho(x))} \|A\|^n + \|A\|^2 \, d\mu \leqslant c\epsilon(x), \quad \text{for } 0 \leqslant t \leqslant \frac{1}{c}\rho^4. \tag{10}$$

There is no easy relationship between the geometrically motivated constraint functions considered in [38] and the simple constraint functions considered here. Despite the stronger statement Theorem 1.3, one may consider the class of simple constraint functions as being ‘larger’ than the class of constraint functions which satisfy the geometric growth condition (2). This is due to the following fact. In [38] we prove that every constraint function satisfying the growth condition (2) and giving rise to an area bound as in (4) is in fact bounded, given that the concentration of curvature in a high enough L^p norm is sufficiently small. In this sense one may regard those functions as satisfying (7) under the additional condition that (8) holds for later times in a higher L^p norm. Additionally, note that there are constraint functions such as $h(t) = e^t$, $h(t) = \sin t$, $h(t) = \frac{1}{1+t}$, $h(t) = -t$, which easily satisfy (7) but do not fit into the framework of [38]. These may be of interest to model expanding, breathing, stabilising and shrinking solutions. Thus we feel that, given the motivating example of classical surface diffusion flow, one must take both Theorem 1.2 and Theorem 1.3 into account to form a complete picture.

2. Notation and preliminary results

In this section we will collect various general formulae from differential geometry which we will need when performing the later analysis. We have as our principal object of study a smooth immersion $f : M^n \to \mathbb{R}^{n+1}$ of an n-dimensional orientable compact hypersurface M^n, and induced metric tensor with components

$$g_{ij} = \left(\frac{\partial}{\partial x_i} f \,\bigg|\, \frac{\partial}{\partial x_j} f \right),$$

so that the pair (M, g) is a Riemannian manifold. In the above equation $(\cdot|\cdot)$ denotes the regular Euclidean inner product, and $\frac{\partial}{\partial x_i}$ is the derivative in the direction of the i-th tangent vector. When convenient we frequently use the abbreviation $\partial_i = \frac{\partial}{\partial x_i}$.

The Riemannian metric induces an inner product structure on all tensors, which we define as the trace over pairs of indices with the metric:

$$\langle T^i_{jk}, S^i_{jk}\rangle = g_{is}g^{jr}g^{ku}T^i_{jk}S^s_{ru}, \qquad \|T\|^2 = \langle T, T\rangle,$$

where repeated indices are summed over from 1 to n. The mean curvature H is defined by

$$H = g^{ij}A_{ij} = A^i_i,$$

where the components A_{ij} of the second fundamental form A are given by

$$A_{ij} = -\left(\frac{\partial^2}{\partial x_i \partial x_j} f \,\middle|\, \nu\right) = \left(\frac{\partial}{\partial x_j} f \,\middle|\, \frac{\partial}{\partial x_i}\nu\right), \tag{11}$$

where ν is the outer unit normal vector field on M.

The Christoffel symbols of the induced connection are determined by the metric,

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}\left(\frac{\partial}{\partial x_i}g_{jl} + \frac{\partial}{\partial x_j}g_{il} - \frac{\partial}{\partial x_l}g_{ij}\right), \tag{12}$$

so that the covariant derivative on M of a vector X and of a covector Y is

$$\nabla_j X^i = \frac{\partial}{\partial x_j}X^i + \Gamma^i_{jk}X^k, \quad \text{and}$$
$$\nabla_j Y_i = \frac{\partial}{\partial x_j}Y_i - \Gamma^k_{ij}Y_k$$

respectively.

From the expression (11) and the smoothness of f we can see that the second fundamental form is symmetric; less obvious but equally important is the symmetry of the first covariant derivatives of A,

$$\nabla_i A_{jk} = \nabla_j A_{ik} = \nabla_k A_{ij},$$

commonly referred to as the Codazzi equation.

The fundamental relations between components of the Riemann curvature tensor, the Ricci tensor and scalar curvature are given by Gauss' equation

$$R_{ijkl} = A_{ik}A_{jl} - A_{il}A_{jk},$$

with contractions

$$g^{jl}R_{ijkl} = R_{ik} = HA_{ik} - A^j_i A_{jk}, \quad \text{and}$$
$$g^{ik}R_{ik} = R = H^2 - \|A\|^2.$$

We will need to interchange covariant derivatives; for vectors X and covectors Y we obtain

$$\nabla_{ij}X^h - \nabla_{ji}X^h = R^h_{ijk}X^k = (A_{lj}A_{ik} - A_{lk}A_{ij})g^{hl}X^k,$$
$$\nabla_{ij}Y_k - \nabla_{ji}Y_k = R_{ijkl}g^{lm}Y_m = (A_{lj}A_{ik} - A_{il}A_{jk})g^{lm}Y_m,$$

where $\nabla_{i_1\cdots i_n} = \nabla_{i_1}\cdots\nabla_{i_n}$. Further, we define $\nabla_{(n)}T$ to be the tensor with components $\nabla_{i_1\cdots i_n}T^{k_1\cdots}_{j_1\cdots}$. We also use for tensors T and S the notation $T * S$ to denote a new tensor formed by summations of contractions of pairs of indices from T and S by the metric g, with possible multiplication of each summation by a universal constant. The resultant tensor will have the same type as the other quantities in the equation it appears. Keeping these in mind we also denote polynomials in the iterated covariant derivatives of these terms by

$$P^i_j(T) = \sum_{k_1+\cdots+k_j=i} c_{k_1\cdots k_j}\nabla_{(k_1)}T * \cdots * \nabla_{(k_j)}T,$$

where the constant $c_{k_1\cdots k_j} \in \mathbb{R}$ is absolute. As is common for the $*$-notation, we slightly abuse this constant when certain subterms do not appear in our P-style terms. For example

$$\begin{aligned}\|\nabla A\|^2 &= \langle\nabla A, \nabla A\rangle \\ &= 1\cdot(\nabla_{(1)}A * \nabla_{(1)}A) + 0\cdot(A * \nabla_{(2)}A) \\ &= P^2_2(A).\end{aligned}$$

This will occur throughout the paper without further comment.

The Laplacian we will use is the Laplace–Beltrami operator on M^n, with the components of ΔT given by

$$\Delta T^i_{jk} = g^{pq}\nabla_{pq}T^i_{jk} = \nabla^p\nabla_p T^i_{jk}.$$

Using the Codazzi equation with the interchange of covariant derivative formula given above, we obtain Simons' identity:

$$\begin{aligned}\Delta A_{ij} &= \nabla_{ij}H + HA_{il}g^{lm}A_{mj} - \|A\|^2 A_{ij}\\ &= \nabla_{ij}H + HA^l_i A_{lj} - \|A\|^2 A_{ij},\end{aligned}$$

or in $*$-notation

$$\Delta A = \nabla_{(2)}H + A * A * A. \tag{13}$$

In the coming sections we will be concerned with calculating the evolution of the iterated covariant derivatives of curvature quantities. The following less precise interchange of covariant derivatives formula (derived from the fundamental equations above) will be useful to keep in mind:

$$\nabla_{ij}T = \nabla_{ji}T + P^0_2(A) * T.$$

In most of our integral estimates, we will be including a function $\gamma : M \to \mathbb{R}$ in the integrand. Eventually, this will be specialised to a smooth cutoff function between concentric geodesic balls on M. For now however let us only assume that $\gamma = \tilde\gamma \circ f$, where

$$0 \leqslant \tilde\gamma \leqslant 1, \quad \text{and} \quad \|\tilde\gamma\|_{C^2(\mathbb{R}^{n+1})} \leqslant c_{\tilde\gamma} < \infty.$$

Using the chain rule, this implies $D\gamma = (D\tilde\gamma \circ f)Df$ and then $D^2\gamma = (D^2\tilde\gamma \circ f)(Df, Df) + (D\tilde\gamma \circ f)D^2 f(\cdot,\cdot)$. Using the expression (12) for the Christoffel symbols to convert the computations above to covariant derivatives, and the Weingarten relations to convert the derivatives of ν to factors of the second fundamental form with the basis vectors $\partial_i f$, we obtain the estimates

$$\|\nabla\gamma\| \leqslant c_{\gamma 1}, \quad \text{and} \quad \|\nabla_{(2)}\gamma\| \leqslant c_{\gamma 2}\big(1 + \|A\|\big). \tag{14}$$

At times we will use the set $[\gamma > c] = \{p \in M\colon \gamma(p) > c\}$ or the set $[\gamma = c] = \{p \in M\colon \gamma(p) = c\}$ as the domain of integration.

3. Integral estimates

We now establish the fundamental integral estimates which allow us to exert control upon the curvature and derivatives of curvature by controlling the concentration of the curvature. Throughout this section we will need various Sobolev and interpolation inequalities. These are collected in Appendix A for the convenience of the reader.

We begin with the following lemma, whose proof is straightforward, see [18] for example.

Lemma 3.1. *For $f : M^n \times [0, T) \to \mathbb{R}^{n+1}$ evolving by $\partial_t f = F\nu$ the following equations hold*:

$$\begin{aligned}&\frac{\partial}{\partial t} g_{ij} = 2FA_{ij}, \qquad \frac{\partial}{\partial t} g^{ij} = -2FA^{ij}, \qquad \frac{\partial}{\partial t} d\mu = (HF)\,d\mu,\\ &\frac{\partial}{\partial t}\nu = -\nabla F, \qquad \frac{\partial}{\partial t} A_{ij} = -\nabla_{ij}F + FA^p_i A_{pj},\\ &\frac{\partial}{\partial t} H = -\Delta F - F\|A\|^2, \quad \text{and}\\ &\frac{\partial}{\partial t} A^o_{ij} = -S^o(\nabla_{(2)}F) + F\left(A^p_i A_{pj} + \frac{1}{n} g_{ij}|A|^2 - \frac{2}{n} HA_{ij}\right),\end{aligned}$$

where $S^o(T)$ denotes the tracefree part of a symmetric bilinear form T. If $F = \Delta H + h$ then the following evolution equation additionally holds:

$$\frac{\partial}{\partial t} A_{ij} = -\Delta^2 A_{ij} + \|A\|^2 A_{ij} + (\Delta H - H + h)A_{ik}A^k_j.$$

Lemma 3.2. *Let $f : M^n \times [0, T) \to \mathbb{R}^{n+1}$ be a constrained surface diffusion flow. Then the following equation holds*:

$$\frac{\partial}{\partial t}\nabla_{(k)}A = -\Delta^2\nabla_{(k)}A + hP^k_2(A) + P^{k+2}_3(A).$$

Corollary 3.3. *Let $f : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a constrained surface diffusion flow. Then the following equation holds*:

$$\frac{\partial}{\partial t}\|\nabla_{(k)}A\|^2 = -2\big\langle \nabla_{(k)}A, \nabla^p \Delta \nabla_p \nabla_{(k)}A\big\rangle + \big[hP_2^k(A) + P_3^{k+2}(A)\big] * \nabla_{(k)}A.$$

Integration by parts gives us our most basic localised integral estimate.

Corollary 3.4. *Let $f : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a constrained surface diffusion flow, and γ as in (14). Then for any $s \geqslant 0$,*

$$\begin{aligned}
&\frac{d}{dt}\int_M \|\nabla_{(k)}A\|^2\gamma^s\,d\mu + 2\int_M \|\nabla_{(k+2)}A\|^2\gamma^s\,d\mu \\
&\quad= \int_M \|\nabla_{(k)}A\|^2\big(\partial_t\gamma^s\big)\,d\mu + 2\int_M \big\langle\big(\nabla\gamma^s\big)(\nabla_{(k)}A), \Delta\nabla_{(k+1)}A\big\rangle\,d\mu - 2\int_M \big\langle\big(\nabla\gamma^s\big)(\nabla_{(k+1)}A), \nabla_{(k+2)}A\big\rangle\,d\mu \\
&\qquad+ \int_M \gamma^s\big[\big(P_3^{k+2}(A) + hP_2^k(A)\big) * \nabla_{(k)}A\big]\,d\mu.
\end{aligned}$$

Combining the above with standard integral estimates and interpolation inequalities as in [22] gives the following proposition.

Proposition 3.5. *Let $f : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a simple constrained surface diffusion flow with γ a cutoff function as in (14). Then for a fixed $\theta > 0$ and $s \geqslant 2k+4$,*

$$\begin{aligned}
&\frac{d}{dt}\int_M \|\nabla_{(k)}A\|^2\gamma^s\,d\mu + (2-\theta)\int_M \|\nabla_{(k+2)}A\|^2\gamma^s\,d\mu \\
&\quad\leqslant (c+ch)\int_M \|A\|^2\gamma^{s-4-2k}\,d\mu + ch\int_M \big(\nabla_{(k)}[A*A]*\nabla_{(k)}A\big)\gamma^s\,d\mu \\
&\qquad+ c\int_M \big(\big[P_3^{k+2}(A) + P_5^k(A)\big]*\nabla_{(k)}A\big)\gamma^s\,d\mu,
\end{aligned}$$

where c depends on $c_{\gamma 1}$, $c_{\gamma 2}$, s, k, $c_h([0,T))$, and θ.

We now use the above and specialised multiplicative Sobolev inequalities to demonstrate that small concentration of curvature along the flow allows one to control the L^2 norm of first and second derivatives of curvature.

Proposition 3.6. *Let $n \in \{2,3\}$. Suppose $f : M^n \times [0,T^*] \to \mathbb{R}^{n+1}$ is a simple constrained surface diffusion flow and γ a cutoff function as in (14). Then there is an ϵ_0 depending on $c_{\gamma 1}$, $c_{\gamma 2}$, and $c_h([0,T^*])$ such that if*

$$\epsilon = \sup_{[0,T^*]}\int_{[\gamma>0]} \|A\|^n\,d\mu \leqslant \epsilon_0 \tag{15}$$

then for any $t \in [0,T^*]$ we have

$$\begin{aligned}
&\int_{[\gamma=1]} \|A\|^2\,d\mu + \int_0^t\int_{[\gamma=1]} \big(\|\nabla_{(2)}A\|^2 + \|A\|^2\|\nabla A\|^2 + \|A\|^6\big)\,d\mu\,d\tau \\
&\quad\leqslant \big(1+(n-2)t\big)\int_{[\gamma>0]} \|A\|^2\,d\mu\bigg|_{t=0} + c\big(t+(n-2)e^t\big)\epsilon^{\frac{2}{n}},
\end{aligned} \tag{16}$$

where c depends on $c_{\gamma 1}$, $c_{\gamma 2}$, and $c_h([0,T^*])$.

Proof. The idea of the proof is to integrate Proposition 3.5, and then use the multiplicative Sobolev inequality Lemma A.1. This will introduce a multiplicative factor of $\|A\|_{n,[\gamma>0]}$ in front of several integrals, which we can then absorb on the left.

Setting $k=0$, $s=4$ and $\theta=\frac{1}{2}$ in Proposition 3.5 we have

$$\frac{d}{dt}\int_M \|A\|^2\gamma^4\,d\mu + \frac{3}{2}\int_M \|\nabla_{(2)}A\|^2\gamma^4\,d\mu$$
$$\leqslant (c+ch)\int_{[\gamma>0]} \|A\|^2\,d\mu + ch\int_M \big([A*A]*A\big)\gamma^4\,d\mu + c\int_M \big(\big[P_3^2(A)+P_5^0(A)\big]*A\big)\gamma^4\,d\mu. \tag{17}$$

First we estimate the P-style terms:

$$\int_M \big(\big[P_3^2(A)+P_5^0(A)\big]*A\big)\gamma^4\,d\mu$$
$$\leqslant c\int_M \big(\big[\|A\|^2\cdot\|\nabla_{(2)}A\| + \|\nabla A\|^2\cdot\|A\| + \|A\|^5\big]\|A\|\big)\gamma^4\,d\mu$$
$$\leqslant c\int_M \big[\|A\|^3\cdot\|\nabla_{(2)}A\| + \|\nabla A\|^2\cdot\|A\|^2 + \|A\|^6\big]\gamma^4\,d\mu$$
$$\leqslant \theta\int_M \|\nabla_{(2)}A\|^2\gamma^4\,d\mu + c_\theta\int_M \big(\|A\|^6 + \|\nabla A\|^2\|A\|^2\big)\gamma^4\,d\mu.$$

We use Lemma A.1 to estimate the second integral and obtain for $n=2$

$$\int_M \big(\big[P_3^2(A)+P_5^0(A)\big]*A\big)\gamma^4\,d\mu$$
$$\leqslant \theta\int_M \|\nabla_{(2)}A\|^2\gamma^4\,d\mu + c_\theta\int_{[\gamma>0]} \|A\|^2\,d\mu\int_M \big(\|\nabla_{(2)}A\|^2 + \|A\|^6\big)\gamma^4\,d\mu$$
$$+ c_\theta\bigg(\int_{[\gamma>0]} \|A\|^2\,d\mu\bigg)^2, \tag{18}$$

and for $n=3$

$$\int_M \big(\big[P_3^2(A)+P_5^0(A)\big]*A\big)\gamma^4\,d\mu$$
$$\leqslant \theta\int_M \|\nabla_{(2)}A\|^2\gamma^4\,d\mu + c_\theta\|A\|_{3,[\gamma>0]}^{\frac{3}{2}}\int_M \big(\|\nabla_{(2)}A\|^2 + \|A\|^6\big)\gamma^4\,d\mu$$
$$+ c_\theta(c_{\gamma 1})^3\big(\|A\|_{3,[\gamma>0]}^3 + \|A\|_{3,[\gamma>0]}^{\frac{9}{2}}\big). \tag{19}$$

We add the integrals $\int_M \|A\|^6\gamma^4\,d\mu$ and $\int_M \|\nabla A\|^2\|A\|^2\gamma^4\,d\mu$ to (17) and obtain

$$\frac{d}{dt}\int_M \|A\|^2\gamma^4\,d\mu + \frac{3}{2}\int_M \big(\|\nabla_{(2)}A\|^2 + \|A\|^2\|\nabla A\|^2 + \|A\|^6\big)\gamma^4\,d\mu$$
$$\leqslant (c+ch)\int_{[\gamma>0]} \|A\|^2\,d\mu + ch\int_M \big([A*A]*A\big)\gamma^4\,d\mu$$
$$+ c\int_M \big(\|A\|^2\|\nabla A\|^2 + \|A\|^6\big)\gamma^4\,d\mu + c\int_M \big(\big[P_3^2(A)+P_5^0(A)\big]*A\big)\gamma^4\,d\mu$$
$$\leqslant c\big(1+h^2\big)\int_{[\gamma>0]} \|A\|^2\,d\mu + c\int_M \big(\|A\|^3\|\nabla_{(2)}A\| + \|A\|^2\|\nabla A\|^2 + \|A\|^6\big)\gamma^4\,d\mu.$$

For $n=2$, we use the estimate (18) above and obtain

$$\frac{d}{dt}\int_M \|A\|^2\gamma^4\,d\mu + \frac{3}{2}\int_M \big(\|\nabla_{(2)}A\|^2 + \|A\|^2\|\nabla A\|^2 + \|A\|^6\big)\gamma^4\,d\mu$$
$$\leqslant c\big(1+h^2\big)\int_{[\gamma>0]} \|A\|^2\,d\mu + \theta\int_M \|\nabla_{(2)}A\|^2\gamma^4\,d\mu$$
$$+ c_\theta \int_{[\gamma>0]} \|A\|^2\,d\mu \int_M \big(\|\nabla_{(2)}A\|^2 + \|A\|^6\big)\gamma^4\,d\mu + c_\theta\bigg(\int_{[\gamma>0]} \|A\|^2\,d\mu\bigg)^2.$$

For $n=3$, we use instead (19) to obtain

$$\frac{d}{dt}\int_M \|A\|^2\gamma^4\,d\mu + \frac{3}{2}\int_M \big(\|\nabla_{(2)}A\|^2 + \|A\|^2\|\nabla A\|^2 + \|A\|^6\big)\gamma^4\,d\mu$$
$$\leqslant c\big(1+h^2\big)\int_{[\gamma>0]} \|A\|^2\,d\mu + \theta\int_M \|\nabla_{(2)}A\|^2\gamma^4\,d\mu$$
$$+ c_\theta \|A\|^{\frac{3}{2}}_{3,[\gamma>0]} \int_M \big(\|\nabla_{(2)}A\|^2 + \|A\|^6\big)\gamma^4\,d\mu$$
$$+ c_\theta (c_{\gamma 1})^3\big(\|A\|^3_{3,[\gamma>0]} + \|A\|^{\frac{9}{2}}_{3,[\gamma>0]}\big).$$

Absorbing, we obtain for $n=2$

$$\frac{d}{dt}\int_M \|A\|^2\gamma^4\,d\mu + (3-2\theta-2c_\theta\epsilon_0)\frac{1}{2}\int_M \big(\|\nabla_{(2)}A\|^2 + \|A\|^2\|\nabla A\|^2 + \|A\|^6\big)\gamma^4\,d\mu$$
$$\leqslant c_\theta\big(1+\epsilon_0+\|h\|^2_{\infty,[0,T^*]}\big)\epsilon$$
$$\leqslant c_\theta\epsilon,$$

and for $n=3$

$$\frac{d}{dt}\int_M \|A\|^2\gamma^4\,d\mu + (3-2\theta-2c_\theta\sqrt{\epsilon_0})\frac{1}{2}\int_M \big(\|\nabla_{(2)}A\|^2 + \|A\|^2\|\nabla A\|^2 + \|A\|^6\big)\gamma^4\,d\mu$$
$$\leqslant c_\theta\big(1+\|h\|^2_{\infty,[0,T^*]}\big)\int_{[\gamma>0]} \|A\|^2\,d\mu + c_\theta\big(\epsilon_0^{\frac{1}{3}} + \epsilon_0^{\frac{5}{6}}\big)\epsilon^{\frac{2}{3}}.$$

For θ, ϵ_0 small enough we have

$$\frac{d}{dt}\int_M \|A\|^2\gamma^4\,d\mu + \int_M \big(\|\nabla_{(2)}A\|^2 + \|A\|^2\|\nabla A\|^2 + \|A\|^6\big)\gamma^4\,d\mu$$
$$\leqslant c\epsilon^{\frac{2}{n}} + c(n-2)\int_{[\gamma>0]} \|A\|^2\,d\mu,$$

with c depending on ϵ_0, $c_h([0,t^*])$, $c_{\gamma 1}$, and $c_{\gamma 2}$. Integrating, we have for $n=2$

$$\int_{[\gamma=1]} \|A\|^2\gamma^4\,d\mu + \int_0^t \int_{[\gamma=1]} \big(\|\nabla_{(2)}A\|^2 + \|A\|^2\|\nabla A\|^2 + \|A\|^6\big)\,d\mu\,d\tau$$
$$\leqslant \int_{[\gamma>0]} \|A\|^2\,d\mu\bigg|_{t=0} + c\epsilon t,$$

where we used the fact $[\gamma=1]\subset[\gamma>0]$ and $0\leqslant\gamma\leqslant 1$. For $n=3$ we use a covering argument and Gronwall's inequality after integrating to obtain

$$\begin{aligned}
&\int_{[\gamma=1]} \|A\|^2\gamma^4\,d\mu + \int_0^t \int_{[\gamma=1]} \big(\|\nabla_{(2)}A\|^2 + \|A\|^2\|\nabla A\|^2 + \|A\|^6\big)\,d\mu\,d\tau \\
&\quad\leqslant \int_{[\gamma>0]} \|A\|^2\,d\mu\bigg|_{t=0} + c\epsilon^{\frac{2}{3}}t + c\int_0^t \bigg(\int_{[\gamma>0]} \|A\|^2\,d\mu\bigg|_{t=0} + c\epsilon^{\frac{2}{3}}\tau\bigg) e^{\int_\tau^t c\,d\nu}\,d\tau \\
&\quad= (1+ct)\int_{[\gamma>0]} \|A\|^2\,d\mu\bigg|_{t=0} + c\epsilon^{\frac{2}{3}}t + c\epsilon^{\frac{2}{3}}\int_0^t \tau e^{c(t-\tau)}\,d\tau \\
&\quad\leqslant (1+ct)\int_{[\gamma>0]} \|A\|^2\,d\mu\bigg|_{t=0} + c\big(t+e^t\big)\epsilon^{\frac{2}{3}}.
\end{aligned}$$

This finishes the proof. □

We now move on to obtaining estimates for the higher derivatives of curvature in L^∞. The first issue is in dealing with the P-style terms from Proposition 3.5. These are easily interpolated as in [22] with the extra terms involving the constraint function presenting little difficulty.

Proposition 3.7. *Suppose $f : M^n \times [0,T] \to \mathbb{R}^{n+1}$ is a constrained surface diffusion flow and γ a cutoff function as in* (14)*. Then, for $s \geqslant 2k+4$ the following estimate holds*:

$$\begin{aligned}
&\frac{d}{dt}\int_M \|\nabla_{(k)}A\|^2\gamma^s\,d\mu + \int_M \|\nabla_{(k+2)}A\|^2\gamma^s\,d\mu \\
&\quad\leqslant c\|A\|^4_{\infty,[\gamma>0]}\int_M \|\nabla_{(k)}A\|^2\gamma^s\,d\mu + c\|A\|^2_{2,[\gamma>0]}\big(1+\|A\|^4_{\infty,[\gamma>0]}\big) \\
&\qquad + ch\bigg(h^{\frac{1}{3}}\int_M \|\nabla_{(k)}A\|^2\gamma^s\,d\mu + \big(1+h^{\frac{1}{3}}\big)\|A\|^2_{2,[\gamma>0]}\bigg).
\end{aligned} \tag{20}$$

We now prove that controlling the concentration of curvature in a ball gives pointwise bounds on all derivatives of curvature in that ball.

Proposition 3.8. *Let $n \in \{2,3\}$. Suppose $f : M^n \times [0,T^*] \to \mathbb{R}^{n+1}$ is a simple constrained surface diffusion flow and γ is as in* (14)*. Then there is an ϵ_0 depending on the constants in* (14) *and $c_h([0,T^*])$ such that if*

$$\sup_{[0,T^*]}\int_{[\gamma>0]} \|A\|^n\,d\mu \leqslant \epsilon_0, \tag{21}$$

we can conclude

$$\|\nabla_{(k)}A\|^2_{\infty,[\gamma=1]} \leqslant c \tag{22}$$

where c depends on k, T^*, $c_{\gamma 1}$, $c_{\gamma 2}$, $c_h([0,T^*])$, and $\alpha_0(k+2)$. The latter is defined by

$$\alpha_0(k) = \sum_{j=0}^k \|\nabla_{(j)}A\|_{2,[\gamma>0]}\bigg|_{t=0}.$$

Proof. As before, the idea is to use our previous estimates and then integrate. The ϵ_0 which we will use is exactly the same as that in Proposition 3.6. We fix γ and consider cutoff functions $\gamma_{\sigma,\tau}$ which will allow us to combine our previous estimates. Define for $0 \leqslant \sigma < \tau \leqslant 1$ functions $\gamma_{\sigma,\tau} = \psi_{\sigma,\tau} \circ \gamma$ satisfying $\gamma_{\sigma,\tau} = 0$ for $\gamma \leqslant \sigma$ and $\gamma_{\sigma,\tau} = 1$ for $\gamma \geqslant \tau$. The function $\psi_{\sigma,\tau}$ is chosen such that $\gamma_{\sigma,\tau}$ satisfies (14), although with different constants. Acceptable choices are

$$c_{\gamma_{\sigma,\tau}1} = \|\nabla\psi_{\sigma,\tau}\|_\infty \cdot c_{\gamma 1}, \quad \text{and} \quad c_{\gamma_{\sigma,\tau}2} = \max\big\{c^2_{\gamma 1}\|\nabla_{(2)}\psi_{\sigma,\tau}\|_\infty, c_{\gamma 2}\|\nabla\psi_{\sigma,\tau}\|_\infty\big\}.$$

Using the cutoff function $\gamma_{0,\frac{1}{2}}$ instead of γ in Proposition 3.6 gives

$$\int_{[\gamma_{0,\frac{1}{2}}=1]} \|A\|^2\,d\mu + \int_0^{T^*}\int_{[\gamma_{0,\frac{1}{2}}=1]} \|\nabla_{(2)}A\|^2 + \|A\|^6\,d\mu\,d\tau \leqslant c\epsilon_0^{\frac{2}{n}}T^* + \|A\|^2_{2,[\gamma>0]}\Big|_{t=0}$$

which is for $n=2$

$$\int_{[\gamma\geqslant\frac{1}{2}]} \|A\|^2\,d\mu + \int_0^{T^*}\int_{[\gamma\geqslant\frac{1}{2}]} \|\nabla_{(2)}A\|^2 + \|A\|^6\,d\mu\,d\tau \leqslant c\big(1+T^*\big)\epsilon_0 \tag{23}$$

and for $n=3$

$$\int_{[\gamma\geqslant\frac{1}{2}]} \|A\|^2\,d\mu + \int_0^{T^*}\int_{[\gamma\geqslant\frac{1}{2}]} \|\nabla_{(2)}A\|^2 + \|A\|^6\,d\mu\,d\tau \leqslant c\big(1+T^*\big)\big(\delta+\epsilon_0^{\frac{2}{3}}\big),$$

where $\delta = \|A\|^2_{2,[\gamma>0]}|_{t=0}$. Note that we do not need any smallness of δ.

Recall the multiplicative Sobolev inequality Proposition A.2:

$$\|T\|^4_{\infty,[\gamma=1]} \leqslant c\|T\|^{4-n}_{2,[\gamma>0]}\big(\|\nabla_{(2)}T\|^n_{2,[\gamma>0]} + \|TA^2\|^n_{2,[\gamma>0]} + \|T\|^n_{2,[\gamma>0]}\big). \tag{A.2}$$

Using this with $\gamma_{\frac{1}{2},\frac{3}{4}}$ and (23) above we obtain for $n=2$

$$\begin{aligned}\int_0^T \|A\|^4_{\infty,[\gamma\geqslant\frac{3}{4}]}\,d\tau &\leqslant c\epsilon_0\big(c\epsilon_0\big(1+T^*\big)+\epsilon_0T^*\big)\\ &\leqslant c\epsilon_0.\end{aligned} \tag{24}$$

For $n=3$ we similarly obtain

$$\begin{aligned}\int_0^T \|A\|^4_{\infty,[\gamma\geqslant\frac{3}{4}]}\,d\tau &\leqslant \sqrt{c\big(1+T^*\big)\big(\delta+\epsilon_0^{\frac{2}{3}}\big)}\Big[c\big(1+T^*\big)\big(\delta+\epsilon_0^{\frac{2}{3}}\big)\Big]^{\frac{3}{2}}\\ &\leqslant c\big(\sqrt{\delta}+\epsilon_0^{\frac{1}{3}}\big),\end{aligned} \tag{25}$$

where c depends on $c_h([0,T^*])$, $c_{\gamma 1}$, $c_{\gamma 2}$, T^*, n, and ϵ_0.

We now use (20) with $\gamma_{\frac{3}{4},\frac{7}{8}}$. Factorising, we have

$$\begin{aligned}\frac{d}{dt}\int_M \|\nabla_{(k)}A\|^2\gamma^s_{\frac{3}{4},\frac{7}{8}}\,d\mu &\leqslant c\|A\|^4_{\infty,[\gamma_{\frac{3}{4},\frac{7}{8}}\geqslant 0]}\int_M \|\nabla_{(k)}A\|^2\gamma^s_{\frac{3}{4},\frac{7}{8}}\,d\mu\\ &\quad + c\|A\|^2_{2,[\gamma_{\frac{3}{4},\frac{7}{8}}\geqslant 0]}\big(1+h+\|A\|^4_{\infty,[\gamma_{\frac{3}{4},\frac{7}{8}}\geqslant 0]}\big)\\ &\quad + ch^{\frac{4}{3}}\bigg(\int_M \|\nabla_{(k)}A\|^2\gamma^s_{\frac{3}{4},\frac{7}{8}}\,d\mu + \|A\|^2_{2,[\gamma_{\frac{3}{4},\frac{7}{8}}\geqslant 0]}\bigg)\\ &\leqslant c\big(\|A\|^4_{\infty,[\gamma\geqslant\frac{3}{4}]}+h^{\frac{4}{3}}\big)\int_M \|\nabla_{(k)}A\|^2\gamma^s_{\frac{3}{4},\frac{7}{8}}\,d\mu\\ &\quad + c\|A\|^2_{2,[\gamma\geqslant\frac{3}{4}]}\big(1+\|A\|^4_{\infty,[\gamma\geqslant\frac{3}{4}]}+h+h^{\frac{4}{3}}\big).\end{aligned}$$

Integrating,

$$\int_M \|\nabla_{(k)}A\|^2\gamma^s_{\frac{3}{4},\frac{7}{8}}\,d\mu - \int_M \|\nabla_{(k)}A\|^2\gamma^s_{\frac{3}{4},\frac{7}{8}}\,d\mu\bigg|_{t=0}$$
$$\leqslant c\int_0^t \bigg[\big(\|A\|^4_{\infty,[\gamma\geqslant\frac{3}{4}]}+h^{\frac{4}{3}}\big)\int_M \|\nabla_{(k)}A\|^2\gamma^s_{\frac{3}{4},\frac{7}{8}}\,d\mu\bigg]\,d\tau$$
$$+\,c\int_0^t \big[\|A\|^2_{2,[\gamma\geqslant\frac{3}{4}]}\big(1+\|A\|^4_{\infty,[\gamma\geqslant\frac{3}{4}]}+h+h^{\frac{4}{3}}\big)\big]\,d\tau. \tag{26}$$

Now from our earlier calculation (24) we have

$$\int_0^t \big(\|A\|^4_{\infty,[\gamma\geqslant\frac{3}{4}]}+h^{\frac{4}{3}}\big)\,d\tau \leqslant c,$$

and, using our assumption (21)

$$c\int_0^t \big[\|A\|^2_{2,[\gamma\geqslant\frac{3}{4}]}\big(1+\|A\|^4_{\infty,[\gamma\geqslant\frac{3}{4}]}+h+h^{\frac{4}{3}}\big)\big]\,d\tau \leqslant c.$$

Also, we have

$$\int_M \|\nabla_{(k)}A\|^2\gamma^s_{\frac{3}{4},\frac{7}{8}}\,d\mu\bigg|_{t=0} \leqslant c\alpha_0(k),$$

where α_0 is as in the statement of the proposition. Therefore, Eq. (26) is of the form

$$\alpha(t)\leqslant\beta(t)+\int_c^t \lambda(\tau)\alpha(\tau)\,d\tau,$$

where

$$\alpha(t)=\int_M \|\nabla_{(k)}A\|^2\gamma^s_{\frac{3}{4},\frac{7}{8}}\,d\mu,$$

$$\beta(t)=\int_M \|\nabla_{(k)}A\|^2\gamma^s_{\frac{3}{4},\frac{7}{8}}\,d\mu\bigg|_{t=0}+c\int_0^t \big[\|A\|^2_{2,[\gamma\geqslant\frac{3}{4}]}\big(1+\|A\|^4_{\infty,[\gamma\geqslant\frac{3}{4}]}+h+h^{\frac{4}{3}}\big)\big]\,d\tau,$$

and

$$\lambda(t)=\|A\|^4_{\infty,[\gamma\geqslant\frac{3}{4}]}+h^{\frac{4}{3}}.$$

Noting that β and $\int\lambda\,d\tau$ are bounded by the constants shown above, we can invoke Gronwall's inequality and conclude

$$\int_{[\gamma\geqslant\frac{7}{8}]} \|\nabla_{(k)}A\|^2\,d\mu \leqslant \beta(t)+\int_0^t \beta(\tau)\lambda(\tau)e^{\int_\tau^t\lambda(\nu)\,d\nu}\,d\tau \leqslant c\big(k,\alpha_0(k)\big).$$

Trivially, we also have

$$\int_{[\gamma\geqslant\frac{7}{8}]} \|\nabla_{(k+2)}A\|^2\,d\mu \leqslant c\big(k+2,\alpha_0(k+2)\big).$$

Therefore using (A.2) with $\gamma_{\frac{7}{8},\frac{15}{16}}$, and taking into account the $n=3$ statement of Lemma A.1 we can bound $\|A\|_\infty$ on a smaller ball:

$$\|A\|^4_{\infty,[\gamma\geqslant\frac{15}{16}]}\leqslant c\big(0,\alpha_0(0)\big)^{\frac{4-n}{2}}\big(c\big(2,\alpha_0(2)\big)^{\frac{n}{2}}+c\big(0,\alpha_0(0)\big)^{\frac{n}{2}}\big)\leqslant c.$$

Finally, using (A.2) with $T=\nabla_{(k)}A$ and $\gamma=\gamma_{\frac{15}{16},1}$ we obtain

$$\begin{aligned}\|\nabla_{(k)}A\|^4_{\infty,[\gamma=1]} &\leqslant c\|\nabla_{(k)}A\|^{4-n}_{2,[\gamma>\frac{15}{16}]}\big(\|\nabla_{(k+2)}A\|^n_{2,[\gamma>\frac{15}{16}]}+\big(\|A\|^{2n}_{\infty,[\gamma>\frac{15}{16}]}+1\big)\|\nabla_{(k)}A\|^n_{2,[\gamma>\frac{15}{16}]}\big)\\ &\leqslant c\big(k,\alpha_0(k+2)\big).\end{aligned}$$

This completes the proof. □

4. Proof of the Lifespan Theorem

We begin by scaling $\tilde{f}(x,t)=\frac{1}{\rho}f(x,\rho^4 t)$. Note that $\|A\|^n_n$ is scale invariant, and so we may assume $\rho=1$. Note that h may scale in a non-invariant fashion but this introduces a single change in the constant c_h only, and certainly a scaled simple h (we only perform this rescaling once) remains simple. We make the definition

$$\eta(t)=\sup_{x\in\mathbb{R}^3}\int_{f^{-1}(B_1(x))}\|A\|^n\,d\mu. \tag{27}$$

By covering B_1 with several translated copies of $B_{\frac{1}{2}}$ there is a constant c_η such that

$$\eta(t)\leqslant c_\eta\sup_{x\in\mathbb{R}^3}\int_{f^{-1}(B_{\frac{1}{2}}(x))}\|A\|^n\,d\mu. \tag{28}$$

Note that $c_\eta=4^{n+1}$ is sufficient.

By short time existence we have that $f(M\times[0,t])$ is compact for $t<T$ and so the function $\eta:[0,T)\to\mathbb{R}$ is continuous. We now define

$$t_0^{(n)}=\sup\big\{0\leqslant t\leqslant\min(T,\lambda_n)\colon \eta(\tau)\leqslant 3c_\eta\sigma(n)\text{ for }0\leqslant\tau\leqslant t\big\}, \tag{29}$$

where

$$\sigma(n)=\begin{cases}\epsilon_0 & \text{for } n=2,\\ c_{P8}c^*(\delta+\epsilon_0^{2/3}) & \text{for } n=3\end{cases}$$

with $\delta=\sup_{x\in\mathbb{R}^4}\|A\|^2_{2,f^{-1}(B_1(x))}|_{t=0}$, λ_n a parameter to be specified later and

$$c^*=c_{P8}+c_0c_\eta e^{c_P 5/c_0 c_\eta}.$$

The constant c_{P8} is the maximum of 1 and the constant from Proposition 3.8, and c_0 is the maximum of all the constants on the right-hand side of Proposition 3.6. Note that the ϵ_0 on the right-hand side of the inequality is from Eq. (8). Unlike earlier in Proposition 3.8, we require δ small as described in the statement of Theorem 1.3.

The proof continues in three steps. First, we show that it must be the case that $t_0^{(n)}=\min(T,\lambda_n)$. Second, we show that if $t_0^{(n)}=\lambda_n$, then we can conclude Theorem 1.3. Finally, we prove by contradiction that if $T\neq\infty$, then $t_0^{(n)}\neq T$. We label these steps as

$$t_0^{(n)}=\min(T,\lambda_n), \tag{30}$$

$$t_0^{(n)}=\lambda_n \implies \text{Theorem 1.3}, \tag{31}$$

$$T\neq\infty \implies t_0^{(n)}\neq T. \tag{32}$$

The three statements (30), (31), (32) together imply Theorem 1.3. We expand the sketch of the argument given above as follows: first notice that by (30) $t_0^{(n)}=\lambda_n$ or $t_0^{(n)}=T$, and if $t_0^{(n)}=\lambda_n$ then by (31) we have Theorem 1.3. Also notice that if $t_0^{(n)}=\infty$ then $T=\infty$ and Theorem 1.3 follows from estimate (35) below (used to prove statement (31)). Therefore the only remaining case where Theorem 1.3 may fail to be true is when $t_0^{(n)}=T<\infty$. But this is impossible by statement (32), so we are finished.

We now give the proof of the first step, statement (30). From the assumption (8),

$$\eta(0)\leqslant\epsilon_0<\begin{cases}3c_\eta\epsilon_0, & \text{for } n=2,\\ 3c_{P8}c_\eta c^*(\delta+\epsilon_0^{2/3}), & \text{for } n=3,\end{cases}$$

and therefore (29) implies $t_0^{(n)}>0$. Assume for the sake of contradiction that $t_0^{(n)}<\min(T,\lambda_n)$. Then from the definition (29) of $t_0^{(n)}$ and the continuity of η we have

$$\eta\big(t_0^{(n)}\big)=\begin{cases}3c_\eta\epsilon_0, & \text{for } n=2,\\ 3c_{P8}c_\eta c^*(\delta+\epsilon_0^{2/3}), & \text{for } n=3,\end{cases} \tag{33}$$

so long as $\epsilon_0 \leqslant 1$ and $c_{P8} \geqslant 1$. Recall Proposition 3.6. We will now set γ to be a cutoff function as in (14) such that

$$\chi_{B_{\frac{1}{2}}(x)} \leqslant \tilde{\gamma} \leqslant \chi_{B_1(x)},$$

for some $x \in f(M,t)$. Choosing a small enough ϵ_0 (by varying ρ in (8)), definition (29) implies that the smallness condition (15) is satisfied on $[0, t_0^{(n)})$. Therefore we have satisfied all the requirements of Proposition 3.6, and so we conclude

$$\begin{aligned}
&\int_{f^{-1}(B_{\frac{1}{2}}(x))} \|A\|^2\,d\mu \\
&\quad\leqslant \big(1+(n-2)t\big) \int_{f^{-1}(B_1(x))} \|A\|^2\,d\mu\bigg|_{t=0} + c\big(t+(n-2)e^t\big)c_\eta \epsilon^{\frac{2}{n}} \\
&\quad\leqslant \begin{cases} 2\epsilon_0, & \text{for } n=2 \text{ and } \lambda_2 = \frac{1}{c_0 c_\eta}, \\ 2c_{P8}c^*(\delta+\epsilon_0^{2/3}), & \text{for } n=3 \text{ and } \lambda_3 = c_{P8}\frac{1}{c_0 c_\eta}, \end{cases}
\end{aligned} \tag{34}$$

for all $t \in [0, t^*]$, where $t^* < t_0^{(n)}$. Thus Eq. (34) above is true for all $t \in [0, t_0^{(n)})$. We combine this with (28) to conclude

$$\eta(t) \leqslant c_{P8}^{n-2} c_\eta \sup_{x\in\mathbb{R}^3} \int_{f^{-1}(B_{\frac{1}{2}}(x))} \|A\|^n\,d\mu \leqslant \begin{cases} 2c_\eta\epsilon_0, & \text{for } n=2, \\ 2c_{P8}c_\eta c^*(\delta+\epsilon_0^{2/3}), & \text{for } n=3, \end{cases} \tag{35}$$

where $0 \leqslant t < t_0^{(n)}$. Since η is continuous, we can let $t \to t_0^{(n)}$ and obtain a contradiction with (33). Therefore, with the choice of λ_n in Eq. (34), the assumption that $t_0^{(n)} < \min(T, \lambda_n)$ is incorrect. Thus we have shown (30). We have also proved the second step (31). Observe that if $t_0^{(n)} = \lambda_n$ then by the definition (29) of $t_0^{(n)}$,

$$T \geqslant \lambda_n,$$

which is (9). Also, (35) implies (10). That is, we have proved if $t_0^{(n)} = \lambda_n$, then the Lifespan Theorem holds, which is the second step (31). It only remains to prove Eq. (32).

We assume

$$t_0^{(n)} = T \neq \infty;$$

since if $T = \infty$ then (9) holds automatically and again (35) implies (10). Note also that we can safely assume $T < \lambda_n$, since otherwise we can apply step two to conclude the Lifespan Theorem.

Our strategy is to show that in this case the flow exists smoothly up to and including time T, allowing us to extend the flow, thus contradicting the finite maximality of T. Since h is simple, it presents no difficulty, and for finite T, h satisfies the requirements of short time existence. To show that the immersion $f(\cdot, T)$ satisfies the requirements of short time existence, we use Proposition 3.8 to obtain pointwise bounds for the higher derivatives of curvature everywhere on $f(\cdot, T)$ and follow a standard proof such as that found in [22] or [17]. Therefore we can extend the flow, contradicting the maximality of T. This establishes (32) and the theorem is proved.

Acknowledgments

This work forms part of the author's PhD thesis under Dr. James McCoy and Prof. Graham Williams, supported by an Australian Postgraduate Award and the University of Wollongong. He is grateful for their advice and for many helpful discussions on this topic.

Appendix A. Sobolev and interpolation inequalities

Here we state the multiplicative Sobolev and interpolation inequalities we have used in the paper. We have generalised the inequalities in [22] to the case of three intrinsic dimensions. Although the proofs are long and involved, they are straightforward and standard and so we have omitted them, referring the reader to the appendix in [22] or [37] instead.

Lemma A.1. *Let γ be as in (14). Then for an immersed surface $f : M^2 \to \mathbb{R}^3$ we have*

$$\begin{aligned}
&\int_M \|A\|^6\gamma^s\,d\mu + \int_M \|A\|^2\|\nabla A\|^2\gamma^s\,d\mu \\
&\quad\leqslant c\int_{[\gamma>0]} \|A\|^2\,d\mu \int_M \big(\|\nabla_{(2)}A\|^2 + \|A\|^6\big)\gamma^s\,d\mu + c(c_{\gamma 1})^4\bigg(\int_{[\gamma>0]} \|A\|^2\,d\mu\bigg)^2,
\end{aligned}$$

and for an immersion $f : M^3 \to \mathbb{R}^4$,

$$\int_M \|A\|^6 \gamma^s \, d\mu + \int_M \|A\|^2 \|\nabla A\|^2 \gamma^s \, d\mu$$
$$\leqslant \theta \int_M \|\nabla_{(2)} A\|^2 \gamma^s \, d\mu + c\|A\|^{\frac{3}{2}}_{3,[\gamma>0]} \int_M \big(\|\nabla_{(2)} A\|^2 + \|A\|^6\big)\gamma^s \, d\mu$$
$$+ c(c_{\gamma 1})^3 \big(\|A\|^3_{3,[\gamma>0]} + \|A\|^{\frac{9}{2}}_{3,[\gamma>0]}\big),$$

where $\theta \in (0, \infty)$ and c is an absolute constant depending on s and θ.

Proposition A.2. *Let $n \in \{2, 3\}$. Then for any tensor T and γ as in (14),*

$$\|T\|^4_{\infty,[\gamma=1]} \leqslant c\|T\|^{4-n}_{2,[\gamma>0]} \big(\|\nabla_{(2)} T\|^n_{2,[\gamma>0]} + \|TA^2\|^n_{2,[\gamma>0]} + \|T\|^n_{2,[\gamma>0]}\big),$$

where c depends on $c_{\gamma 1}$, and n. Assume $T = A$. Then there exists an ϵ_0 depending on $c_{\gamma 1}$, $c_{\gamma 2}$, and n such that if

$$\|A\|^n_{n,[\gamma>0]} \leqslant \epsilon_0$$

we have

$$\|A\|^{8n-12}_{\infty,[\gamma=1]} \leqslant c\epsilon_0 \big(\|\nabla_{(2)} A\|^{2n^2-3n}_{2,[\gamma>0]} + \epsilon_0\big),$$

with c depending on $c_{\gamma 1}$, $c_{\gamma 2}$, n, and ϵ_0.

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