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Jacobi polynomials and associated reproducing kernel Hilbert spaces

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ABSTRACT

This paper deals with a generating function of the Jacobi polynomials that satisfies the following properties (I) and (II). (I) The generating function is the kernel of an integral operator that is unitary. (II) The image of the unitary operator is a reproducing kernel Hilbert space of analytic functions and the reproducing kernel is given as a special value of the generating function above. A generating function that satisfies (I) is given in Watanabe (1998) [11]. The purpose of this paper is to give a generating function that satisfies (I) and (II). From a group theoretical point of view, a similar construction for zonal spherical functions is given in Watanabe (2006) [12].

1. Introduction

This paper deals with generating functions that define unitary operators. In our previous paper [11], we constructed a unitary operator given by an integral operator whose kernel is a generating function of the Jacobi polynomials. Problems of this kind on classical orthogonal polynomials were discussed first in Bargmann [1]. He dealt with the Hermite polynomials.

On the other hand, we gave in [12] a general construction of generating functions for zonal spherical functions which define unitary operators. We shall explain it. Denote by $\mathbf{N}_0$ the set of nonnegative integers, and B the unit open disk $|z| < 1$ in $\mathbf{C}$. Let G be a compact connected Lie group, K a closed subgroup of G, and (G, K) a Riemannian symmetric pair of rank ℓ such that G/K is simply connected. And denote by $L^2(G, K)$ the space of square integrable functions on G that are bi-invariant under K. Then it is known (cf. [8]) that the set of all the zonal spherical functions on (G, K) is parametrized by $\mathbf{N}_0^\ell$. (Note that the existence of parametrization by $\mathbf{N}_0^\ell$ is not unique.) Denote the set by $\{\varphi_m \mid m \in \mathbf{N}_0^\ell\}$. Further, for each $m \in \mathbf{N}_0^\ell$ we denote by d_m the degree of the representation corresponding to φ_m. Then, by [12, Theorem 1], there exists a generating function of the following form such that it is the kernel of an integral operator on $L^2(G, K)$ which is unitary:

$$\sum_{m\in\mathbf{N}_0^\ell} d_m \varphi_m(x) z^m, \quad x \in G,\ z \in B^\ell, \tag{1}$$

where $z^m = z_1^{m_1} \cdots z_\ell^{m_\ell}$ for $z = (z_1, \ldots, z_\ell)$ and $m = (m_1, \ldots, m_\ell)$. We denote by $\Phi(z, x)$ the generating function. Furthermore, by [12, Lemmas 4 and 6], the image of the unitary operator is a reproducing kernel Hilbert space of analytic functions on B^ℓ, and the reproducing kernel is given by

$$\sum_{m\in\mathbf{N}_0^\ell} d_m z^m \overline{w}^m, \quad z, w \in B^\ell, \tag{2}$$

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that is, $\Phi(z\overline{w}, e)$, where $z\overline{w} = (z_1\overline{w}_1, \dots, z_\ell\overline{w}_\ell)$ for $z = (z_1, \dots, z_\ell)$ and $w = (w_1, \dots, w_\ell)$, and e is the identity element of G. Notice the relation between the generating function and the reproducing kernel:

$$\text{generating function} = \Phi(z, x),$$

$$\text{reproducing kernel} = \Phi(z\overline{w}, e).$$

Let us return to Jacobi's case that we constructed in [11]. In this case, there is not a similar relation between the generating function and the reproducing kernel. Thus, we naturally turned our interest to a construction for Jacobi's case which is similar to the general construction described above. The purpose of this paper is to give a construction similar to the general one for Jacobi's case. That is, we shall show the following: for the Jacobi polynomials $P_m^{(\alpha,\beta)}(x)$, $m = 0, 1, 2, \dots,$ there exists a generating function $\Phi_{\alpha,\beta}(z, x)$ such that it is the kernel of an integral operator which is unitary. Furthermore, in the case of $\alpha \geqslant \beta \geqslant 0$, the reproducing kernel of the image of the unitary operator is given by $\Phi_{\alpha,\beta}(z\overline{w}, 1)$. In the case of $0 \leqslant \alpha < \beta$, it is given by $\Phi_{\alpha,\beta}(z\overline{w}, -1)$.

The paper is organized as follows. In Section 3 we shall construct a reproducing kernel Hilbert space associated with the Jacobi polynomials. In Sections 4 and 5 we shall construct a unitary operator associated with a generating function of the Jacobi polynomials in the case of $\alpha \geqslant \beta \geqslant 0$, and that in the case of $0 \leqslant \alpha < \beta$, respectively. In Section 6 we give some remarks to two special cases: one is for the case of $\alpha = \beta = \lambda - 1/2$ ($\lambda > 0$), and the other is for that of $\alpha = n - 1$ and $\beta = 0$.

2. Notation and preliminaries

2.1. General notation

We shall use the notation $\mathbf{N}_0, \mathbf{R}, \mathbf{C}$ for the set of nonnegative integers, the field of real numbers and the field of complex numbers, respectively. We denote by B the unit open disk in $\mathbf{C}$. For a subset $A \subset \mathbf{R}$ we denote by A^2 the direct product of A with itself. We shall use the notation $[a, b]$, $[a, b)$ for the interval $\{x \in \mathbf{R} \mid a \leqslant x \leqslant b\}$, and the interval $\{x \in \mathbf{R} \mid a \leqslant x < b\}$, respectively. For $\zeta \in \mathbf{C}$ let $\operatorname{Re}\zeta$ be the real part of ζ, and $\zeta \mapsto \overline{\zeta}$ the usual conjugation in $\mathbf{C}$. We denote the Gamma function by $\Gamma(x)$, the hypergeometric function by $F(a, b; c; x)$, and $\Gamma(a+m)/\Gamma(a)$ by $(a)_m$. The maximum value of a and b is denoted by $\max(a, b)$. A function is assumed to be complex-valued.

2.2. Jacobi polynomials

We shall describe the definition and some properties of the Jacobi polynomials. For references on the Jacobi polynomials, see [3,6,9].

For $\alpha, \beta \geqslant 0$ the Jacobi polynomials $P_m^{(\alpha,\beta)}(x)$, $m = 0, 1, 2, \dots,$ are defined by

$$P_m^{(\alpha,\beta)}(x) = \frac{(-1)^m}{2^m m!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^m}{dx^m}\big[(1-x)^{\alpha+m}(1+x)^{\beta+m}\big].$$

They have the following orthogonality relation:

$$\int_{-1}^{1} P_m^{(\alpha,\beta)}(x)P_{m'}^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta\,dx = \begin{cases} \frac{2^{\alpha+\beta+1}}{\alpha+\beta+2m+1}\frac{\Gamma(\alpha+m+1)\Gamma(\beta+m+1)}{m!\Gamma(\alpha+\beta+m+1)}, & m = m', \\ 0, & m \neq m'. \end{cases} \tag{3}$$

The Jacobi polynomials have the following generating function:

$$\sum_{m\in\mathbf{N}_0}\frac{(\alpha+\beta+2m+1)(\alpha+\beta+1)_m}{(\alpha+1)_m}P_m^{(\alpha,\beta)}(x)z^m$$
$$= \frac{(\alpha+\beta+1)(z+1)}{(1-z)^{\alpha+\beta+2}}F\left(\frac{\alpha+\beta+2}{2}, \frac{\alpha+\beta+3}{2}; \alpha+1; \frac{2z(x-1)}{(1-z)^2}\right), \quad -1 \leqslant x \leqslant 1,\ z \in B. \tag{4}$$

The maximum value of $|P_m^{(\alpha,\beta)}(x)|$ in $-1 \leqslant x \leqslant 1$ is $(q+1)_m/m!$, where $q = \max(\alpha, \beta)$. The following numbers $d_m^{(\alpha,\beta)}$, $m = 0, 1, 2, \dots,$ play important roles in this paper.

Definition 1. For $m \in \mathbf{N}_0$ define

$$d_m^{(\alpha,\beta)} = \frac{(\alpha+\beta+2m+1)(\beta+1)_m(\alpha+\beta+1)_m}{m!(\alpha+1)_m}.$$

3. Reproducing kernel Hilbert space

In this section we shall construct a reproducing kernel Hilbert space associated with the Jacobi polynomials. First of all, we shall define a measure on the interval $[0,1)$ associated with the sequence $\{(d_m^{(\alpha,\beta)})^{-1}\}_{m\in\mathbf{N}_0}$. Next, making use of the measure, we define a measure on B and construct a reproducing kernel Hilbert space of analytic functions on B.

3.1. Measure on $[0,1)$ associated with $\{(d_m^{(\alpha,\beta)})^{-1}\}_{m\in\mathbf{N}_0}$

Let $\alpha,\beta>0$. For $0<t<1$ define

$$\rho_{\alpha,\beta}(t)=t^{\frac{\alpha+\beta-1}{2}}\int_t^1 u^{-\frac{\alpha+\beta+1}{2}}(1-u)^{\beta-1}\,du\int_{\frac{t}{u}}^1 v^{-\frac{\beta-\alpha+1}{2}}(1-v)^{\beta-1}\,dv,$$

and set

$$\sigma_{\alpha,\beta}(t)=\frac{\beta\Gamma(\alpha+\beta+1)}{2\pi\Gamma(\beta)\Gamma(\alpha+1)}\rho_{\alpha,\beta}(t).$$

Lemma 1. *For $\alpha,\beta>0$ we have*

$$\int_B |z^m|^2\sigma_{\alpha,\beta}(|z|^2)\,dz=\frac{1}{d_m^{(\alpha,\beta)}},\quad m\in\mathbf{N}_0,$$

where dz is the Lebesgue measure on B induced from the identification $\mathbf{C}\cong\mathbf{R}^2$.

Proof. Exchanging orders of integrals, we obtain

$$\begin{aligned}\int_B |z^m|^2\rho_{\alpha,\beta}(|z|^2)\,dz&=\pi\int_0^1 t^m\rho_{\alpha,\beta}(t)\,dt=\pi\int_0^1 t^{m+\frac{\alpha+\beta-1}{2}}\,dt\int_t^1 u^{-\frac{\alpha+\beta+1}{2}}(1-u)^{\beta-1}\,du\int_{\frac{t}{u}}^1 v^{-\frac{\beta-\alpha+1}{2}}(1-v)^{\beta-1}\,dv\\&=\pi\iint_{0\leqslant u,v\leqslant 1}(1-u)^{\beta-1}v^{\alpha}(1-v)^{\beta-1}\left[(uv)^{-\frac{\alpha+\beta+1}{2}}\int_0^{uv}t^{m+\frac{\alpha+\beta-1}{2}}\,dt\right]du\,dv\\&=\frac{2\pi}{\alpha+\beta+2m+1}\iint_{0\leqslant u,v\leqslant 1}(1-u)^{\beta-1}v^{\alpha}(1-v)^{\beta-1}(uv)^m\,du\,dv\\&=\frac{2\pi}{\alpha+\beta+2m+1}\frac{m!\Gamma(\beta)}{\Gamma(\beta+m+1)}\frac{\Gamma(\alpha+m+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+m+1)},\end{aligned}$$

which implies our assertion. □

Lemma 1 is equivalent to the following

$$\int_{[0,1]} t^m\sigma_{\alpha,\beta}(t)\,dt=\frac{1}{\pi d_m^{(\alpha,\beta)}},\quad \alpha,\beta>0,\ m\in\mathbf{N}_0. \tag{5}$$

Combining this result with Theorem 1 in [5], for $\alpha,\beta>0$ we see that the sequence $\{(d_m^{(\alpha,\beta)})^{-1}\}_{m\in\mathbf{N}_0}$ is completely monotonic. On the other hand, for a fixed $m\in\mathbf{N}_0$ we can regard $d_m^{(\alpha,\beta)}$ as a continuous function with respect to $(\alpha,\beta)\in[0,\infty)^2$. Hence, for a fixed $\alpha>0$, by considering the limit $\lim_{\beta\to 0}d_m^{(\alpha,\beta)}=d_m^{(\alpha,0)}$, we obtain that the sequence $\{(d_m^{(\alpha,0)})^{-1}\}_{m\in\mathbf{N}_0}$ is also completely monotonic. Applying this result to Theorem 1 in [5], it is easy to see that there exists a measure $\tilde{\sigma}_{\alpha,0}$ on the interval $[0,1]$ such that

$$\int_{[0,1]} t^m\,d\tilde{\sigma}_{\alpha,0}(t)=\frac{1}{\pi d_m^{(\alpha,0)}},\quad m\in\mathbf{N}_0. \tag{6}$$

In the same way, we see that there exists a measure $\tilde{\sigma}_{0,\beta}$ on $[0,1]$ such that

$$\int_{[0,1]} t^m\,d\tilde{\sigma}_{0,\beta}(t)=\frac{1}{\pi d_m^{(0,\beta)}},\quad m\in\mathbf{N}_0. \tag{7}$$

Further, considering the limit $\lim_{\beta\to 0} d_m^{(0,\beta)} = d_m^{(0,0)}$, we see that there exists a measure $\tilde{\sigma}_{0,0}$ on $[0,1]$ such that

$$\int\limits_{[0,1]} t^m\, d\tilde{\sigma}_{0,0}(t) = \frac{1}{\pi d_m^{(0,0)}}, \quad m \in \mathbf{N}_0. \tag{8}$$

For $\alpha, \beta \geqslant 0$ define a measure $\tau_{\alpha,\beta}$ on $[0,1]$ by

$$d\tau_{\alpha,\beta}(t) = \begin{cases} \pi\,\sigma_{\alpha,\beta}(t)\,dt, & \alpha, \beta > 0, \\ \pi\, d\tilde{\sigma}_{\alpha,0}(t), & \alpha > 0,\ \beta = 0, \\ \pi\, d\tilde{\sigma}_{0,\beta}(t), & \alpha = 0,\ \beta > 0, \\ \pi\, d\tilde{\sigma}_{0,0}(t), & \alpha = \beta = 0. \end{cases}$$

Then we can summarize (5), (6), (7) and (8) as follows

$$\int\limits_{[0,1]} t^m\, d\tau_{\alpha,\beta}(t) = \frac{1}{d_m^{(\alpha,\beta)}}, \quad m \in \mathbf{N}_0. \tag{9}$$

By the definition of $d_m^{(\alpha,\beta)}$ we have $\lim_{m\to\infty} (d_m^{(\alpha,\beta)})^{-1} = 0$. Further, by Lebesgue's convergence theorem the left-hand side of (9) goes to $\tau_{\alpha,\beta}(\{1\})$ as $m \to \infty$. Thus, we can conclude that $\tau_{\alpha,\beta}(\{1\}) = 0$. Therefore, we can rewrite (9) as follows

$$\int\limits_{[0,1)} t^m\, d\tau_{\alpha,\beta}(t) = \frac{1}{d_m^{(\alpha,\beta)}}, \quad m \in \mathbf{N}_0. \tag{10}$$

Let us consider the measure $\tau_{\alpha,\beta}$ as a measure on $[0,1)$. Let $([0,1), \mathfrak{B})$ be the measurable space associated with the measure $\tau_{\alpha,\beta}$. And define the mapping T of $[0,1)$ onto itself by $T(t) = \sqrt{t}$ and set

$$\mathfrak{B}' = \bigl\{ E \subset [0,1) \bigm| T^{-1}(E) \in \mathfrak{B} \bigr\},$$
$$\tilde{\tau}_{\alpha,\beta}(E) = \tau_{\alpha,\beta}\bigl(T^{-1}(E)\bigr), \quad E \in \mathfrak{B}'.$$

Then $([0,1), \mathfrak{B}', \tilde{\tau}_{\alpha,\beta})$ is a measure space and satisfies

$$\int\limits_{[0,1)} h\bigl(T^{-1}(r)\bigr)\, d\tilde{\tau}_{\alpha,\beta}(r) = \int\limits_{[0,1)} h(t)\, d\tau_{\alpha,\beta}(t)$$

for any $\tau_{\alpha,\beta}$-integrable function h. In particular, we have

$$\int\limits_{[0,1)} r^{2m}\, d\tilde{\tau}_{\alpha,\beta}(r) = \int\limits_{[0,1)} t^m\, d\tau_{\alpha,\beta}(t), \quad m \in \mathbf{N}_0. \tag{11}$$

3.2. Measure on B and an associated reproducing kernel Hilbert space of analytic functions on B

In this subsection, first we define a measure on B related to the measure $\tilde{\tau}_{\alpha,\beta}$ on $[0,1)$ given in the preceding subsection. Next we define a reproducing kernel Hilbert space associated with the measure on B. The constructions in this step can be found in [2,4,7]. In what follows, we assume that $\alpha, \beta \geqslant 0$.

Set $U = \{e^{\sqrt{-1}\theta} \mid \theta \in \mathbf{R}\}$, and let τ_U be the normalized Haar measure on the compact group U. Further, we denote by $\nu_{\alpha,\beta}$ the product measure $\tilde{\tau}_{\alpha,\beta} \times \tau_U$ on the product set $[0,1) \times U$, and identify B with $[0,1) \times U$. Then a measure on B is induced from the identification. We shall also denote it by $\nu_{\alpha,\beta}$. If f is a $\nu_{\alpha,\beta}$-integrable function, the following holds

$$\int\limits_B f(z)\, d\nu_{\alpha,\beta}(z) = \int\limits_{[0,1)\times U} f(ru)\, d(\tilde{\tau}_{\alpha,\beta} \times \tau_U)(r,u). \tag{12}$$

Let $L^2(B, d\nu_{\alpha,\beta}(z))$ be the Hilbert space of $\nu_{\alpha,\beta}$-measurable functions f on B with

$$\|f\|_{\alpha,\beta} = \sqrt{\int\limits_B \bigl|f(z)\bigr|^2\, d\nu_{\alpha,\beta}(z)} < \infty.$$

The inner product is given by

$$\langle f, g\rangle_{\alpha,\beta} = \int_B f(z)\overline{g(z)}\, d\nu_{\alpha,\beta}(z). \tag{13}$$

Let $\mathcal{F}_{\alpha,\beta}$ be the space of analytic functions on B which belong to $L^2(B, d\nu_{\alpha,\beta}(z))$. The following proposition is a straightforward consequence of the results in [2,4,7] to which the interested reader is referred for details.

Proposition 1. *The space $\mathcal{F}_{\alpha,\beta}$ is a closed subspace of $L^2(B, d\nu_{\alpha,\beta}(z))$ with orthonormal basis given by $u_m^{(\alpha,\beta)}(z) = (d_m^{(\alpha,\beta)})^{1/2} z^m$, $m \in \mathbf{N}_0$. Moreover, $\mathcal{F}_{\alpha,\beta}$ is a reproducing kernel Hilbert space with kernel, for $z, w \in B$, given by*

$$\begin{aligned} g_w^{(\alpha,\beta)}(z) &= \sum_{m\in\mathbf{N}_0} u_m^{(\alpha,\beta)}(\overline{w}) u_m^{(\alpha,\beta)}(z) \\ &= \sum_{m\in\mathbf{N}_0} d_m^{(\alpha,\beta)} z^m \overline{w}^m. \end{aligned} \tag{14}$$

Proof. See [2,4,7]. $\square$

The reader should note the similarity of the above formula for the reproducing kernel to the series (2) and that the series

$$\sum_{m\in\mathbf{N}_0} d_m^{(\alpha,\beta)} \zeta^m \tag{15}$$

is absolutely convergent for $|\zeta| < 1$.

4. Unitary operator – the case of $\alpha \geqslant \beta \geqslant 0$

In this section we shall construct a unitary operator associated with a generating function of the Jacobi polynomials in the case of $\alpha \geqslant \beta \geqslant 0$. To do this, we divide this section into three subsections.

4.1. L^2-space associated with the Jacobi polynomials

As described in Section 2, the maximum value of $|P_m^{(\alpha,\beta)}(x)|$ in $-1 \leqslant x \leqslant 1$ is equal to $(\alpha+1)_m/m!$. On the other hand, it is known that $P_m^{(\alpha,\beta)}(1) = (\alpha+1)_m/m!$ (cf. [3]). Define

$$\varphi_m^{(\alpha,\beta)}(x) = \frac{P_m^{(\alpha,\beta)}(x)}{P_m^{(\alpha,\beta)}(1)}, \quad m \in \mathbf{N}_0.$$

Then we have

$$\left|\varphi_m^{(\alpha,\beta)}(x)\right| \leqslant \varphi_m^{(\alpha,\beta)}(1) = 1 \quad \text{for } -1 \leqslant x \leqslant 1. \tag{16}$$

Further, we can rewrite the orthogonality relation (3) as follows

$$\int_{-1}^{1} \varphi_m^{(\alpha,\beta)}(x)\varphi_{m'}^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta\, dx = \begin{cases} \frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)d_m^{(\beta,\alpha)}}, & m = m', \\ 0, & m \neq m'. \end{cases} \tag{17}$$

Set

$$d\mu_{\alpha,\beta}(x) = \frac{\Gamma(\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}(1-x)^\alpha(1+x)^\beta\, dx.$$

Then the orthogonality relation (17) results in

$$\int_{-1}^{1} \varphi_m^{(\alpha,\beta)}(x)\varphi_{m'}^{(\alpha,\beta)}(x)\,d\mu_{\alpha,\beta}(x) = \begin{cases} (d_m^{(\beta,\alpha)})^{-1}, & m = m', \\ 0, & m \neq m'. \end{cases} \tag{18}$$

Denote by $L^2((-1,1), d\mu_{\alpha,\beta}(x))$ the Hilbert space of Lebesgue measurable functions φ on the open interval $(-1,1)$ with

$$\|\varphi\|_{\alpha,\beta} = \sqrt{\int_{-1}^{1} |\varphi(x)|^2\,d\mu_{\alpha,\beta}(x)} < \infty.$$

The inner product is given by

$$(\varphi,\psi)_{\alpha,\beta} = \int_{-1}^{1} \varphi(x)\overline{\psi(x)}\,d\mu_{\alpha,\beta}(x), \quad \varphi,\psi \in L^2\big((-1,1), d\mu_{\alpha,\beta}(x)\big).$$

Set

$$\psi_m^{(\alpha,\beta)} = \sqrt{d_m^{(\beta,\alpha)}}\,\varphi_m^{(\alpha,\beta)}, \quad m \in \mathbf{N}_0.$$

Then, it follows from (18) that the system $\{\psi_m^{(\alpha,\beta)} \mid m \in \mathbf{N}_0\}$ is a complete orthonormal system of $L^2((-1,1), d\mu_{\alpha,\beta}(x))$.

4.2. Generating function similar to the form of (1)

Let us consider a generating function of the Jacobi polynomials which is written as a series similar to the form of (1). For $-1 \leqslant x \leqslant 1$ and $z \in B$ define

$$\Phi_{\alpha,\beta}(z,x) = \frac{(\alpha+\beta+1)(1-z)}{(1+z)^{\alpha+\beta+2}} F\left(\frac{\alpha+\beta+2}{2}, \frac{\alpha+\beta+3}{2}; \beta+1; \frac{2z(1+x)}{(1+z)^2}\right).$$

By (4) the function $\Phi_{\alpha,\beta}(z,x)$ is equal to

$$\sum_{m\in\mathbf{N}_0} \frac{(\alpha+\beta+2m+1)(\alpha+\beta+1)_m}{(\beta+1)_m} P_m^{(\beta,\alpha)}(-x)(-z)^m.$$

Considering the relation $P_m^{(\alpha,\beta)}(-x) = (-1)^m P_m^{(\beta,\alpha)}(x)$, we can rewrite the function $\Phi_{\alpha,\beta}(z,x)$ as follows

$$\begin{aligned} \Phi_{\alpha,\beta}(z,x) &= \sum_{m\in\mathbf{N}_0} \frac{(\alpha+\beta+2m+1)(\alpha+\beta+1)_m}{(\beta+1)_m} P_m^{(\alpha,\beta)}(x) z^m \\ &= \sum_{m\in\mathbf{N}_0} d_m^{(\beta,\alpha)} \frac{m!}{(\alpha+1)_m} P_m^{(\alpha,\beta)}(x) z^m \\ &= \sum_{m\in\mathbf{N}_0} d_m^{(\beta,\alpha)} \varphi_m^{(\alpha,\beta)}(x) z^m & (19) \\ &= \sum_{m\in\mathbf{N}_0} u_m^{(\beta,\alpha)}(z) \psi_m^{(\alpha,\beta)}(x). & (20) \end{aligned}$$

The function $\Phi_{\alpha,\beta}(z,x)$ is a generating function of the Jacobi polynomials $P_m^{(\alpha,\beta)}(x)$, $m = 0,1,2,\dots,$ and by (19) it is written as a series similar to the form of (1).

Remark 1. By (14), (16) and (19), we obtain the following relation

$$g_w^{(\beta,\alpha)}(z) = \Phi_{\alpha,\beta}(z\overline{w}, 1).$$

4.3. Unitary operator constructed by the generating function $\Phi_{\alpha,\beta}(z,x)$

A unitary operator on $L^2((-1,1), d\mu_{\alpha,\beta}(x))$ to $\mathcal{F}_{\beta,\alpha}$ is defined as follows. For $\varphi \in L^2((-1,1), d\mu_{\alpha,\beta}(x))$ set

$$(\Phi_{\alpha,\beta}\varphi)(z) = \int_{-1}^{1} \Phi_{\alpha,\beta}(z,x)\varphi(x)\,d\mu_{\alpha,\beta}(x), \quad z \in B. \tag{21}$$

Then $\Phi_{\alpha,\beta}\varphi \in \mathcal{F}_{\beta,\alpha}$ and $\Phi_{\alpha,\beta}$ is unitary.

Theorem 1. *The operator $\Phi_{\alpha,\beta}$ on $L^2((-1,1), d\mu_{\alpha,\beta}(x))$ to $\mathcal{F}_{\beta,\alpha}$ is unitary.*

Proof. Let φ be an element of $L^2((-1,1), d\mu_{\alpha,\beta}(x))$. First of all, we show $\Phi_{\alpha,\beta}\varphi \in \mathcal{F}_{\beta,\alpha}$. As we remarked in the series (15), $\sum_m d_m^{(\beta,\alpha)}|z|^m$ converges for $z \in B$. It follows from this fact and (16) that for a fixed $z \in B$ the series (19) converges uniformly on $[-1,1]$. Then by (20) we see

$$(\Phi_{\alpha,\beta}\varphi)(z) = \sum_{m\in\mathbf{N}_0} \left(\varphi, \psi_m^{(\alpha,\beta)}\right)_{\alpha,\beta} u_m^{(\beta,\alpha)}(z). \tag{22}$$

Since $\sum_m |(\varphi, \psi_m^{(\alpha,\beta)})_{\alpha,\beta}|^2 < \infty$ and $\sum_m |u_m^{(\beta,\alpha)}(z)|^2 < \infty$ $(z \in B)$, the right-hand side of (22) converges absolutely on B and converges with respect to the norm of $\mathcal{F}_{\beta,\alpha}$. These imply $\Phi_{\alpha,\beta}\varphi \in \mathcal{F}_{\beta,\alpha}$. Next we show that $\Phi_{\alpha,\beta}$ is unitary. Applying the Parseval formula to (22), we obtain

$$\|\Phi_{\alpha,\beta}\varphi\|_{\beta,\alpha}^2 = \sum_{m\in\mathbf{N}_0} \left|\left(\varphi, \psi_m^{(\alpha,\beta)}\right)_{\alpha,\beta}\right|^2 = \|\varphi\|_{\alpha,\beta}^2,$$

which means $\Phi_{\alpha,\beta}$ is an isometry. Further, let us take $\varphi = \psi_m^{(\alpha,\beta)}$ in (22). Then we obtain

$$\Phi_{\alpha,\beta}\psi_m^{(\alpha,\beta)} = u_m^{(\beta,\alpha)}, \quad m \in \mathbf{N}_0,$$

which means $\Phi_{\alpha,\beta}$ is surjective. This completes the proof. $\square$

Therefore, we obtain the desired result:

$$\text{generating function} = \Phi_{\alpha,\beta}(z,x),$$
$$\text{reproducing kernel} = \Phi_{\alpha,\beta}(z\overline{w}, 1).$$

5. Unitary operator – the case of $0 \leqslant \alpha < \beta$

In this section we shall construct a unitary operator associated with a generating function of the Jacobi polynomials in the case of $0 \leqslant \alpha < \beta$.

5.1. L^2-space associated with the Jacobi polynomials

As described in Section 2, the maximum value of $|P_m^{(\alpha,\beta)}(x)|$ in $-1 \leqslant x \leqslant 1$ is equal to $(\beta+1)_m/m!$. On the other hand, it is known that $P_m^{(\alpha,\beta)}(-1) = ((-1)^m(\beta+1)_m)/m!$ (cf. [3]). Define

$$\varphi_m^{(\alpha,\beta)}(x) = \frac{P_m^{(\alpha,\beta)}(x)}{P_m^{(\alpha,\beta)}(-1)}, \quad m \in \mathbf{N}_0.$$

Then we have

$$\left|\varphi_m^{(\alpha,\beta)}(x)\right| \leqslant \varphi_m^{(\alpha,\beta)}(-1) = 1 \quad \text{for } -1 \leqslant x \leqslant 1. \tag{23}$$

Further, we can rewrite the orthogonality relation (3) as follows

$$\int_{-1}^{1} \varphi_m^{(\alpha,\beta)}(x)\varphi_{m'}^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta\,dx = \begin{cases} \frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)d_m^{(\alpha,\beta)}}, & m = m', \\ 0, & m \neq m'. \end{cases} \tag{24}$$

Set

$$d\mu_{\alpha,\beta}(x)=\frac{\Gamma(\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}(1-x)^{\alpha}(1+x)^{\beta}\,dx.$$

Then the orthogonality relation (24) results in

$$\int_{-1}^{1}\varphi_m^{(\alpha,\beta)}(x)\varphi_{m'}^{(\alpha,\beta)}(x)\,d\mu_{\alpha,\beta}(x)=\begin{cases}(d_m^{(\alpha,\beta)})^{-1}, & m=m',\\ 0, & m\neq m'.\end{cases}\tag{25}$$

Denote by $L^2((-1,1),d\mu_{\alpha,\beta}(x))$ the Hilbert space of Lebesgue measurable functions φ on the open interval $(-1,1)$ with

$$\|\varphi\|_{\alpha,\beta}=\sqrt{\int_{-1}^{1}\left|\varphi(x)\right|^2 d\mu_{\alpha,\beta}(x)}<\infty.$$

The inner product is given by

$$(\varphi,\psi)_{\alpha,\beta}=\int_{-1}^{1}\varphi(x)\overline{\psi(x)}\,d\mu_{\alpha,\beta}(x),\quad \varphi,\psi\in L^2\big((-1,1),d\mu_{\alpha,\beta}(x)\big).$$

Set

$$\psi_m^{(\alpha,\beta)}=\sqrt{d_m^{(\alpha,\beta)}}\varphi_m^{(\alpha,\beta)},\quad m\in\mathbf{N}_0.$$

Then, it follows from (25) that the system $\{\psi_m^{(\alpha,\beta)}\mid m\in\mathbf{N}_0\}$ is a complete orthonormal system of $L^2((-1,1),d\mu_{\alpha,\beta}(x))$.

5.2. Generating function similar to the form of (1)

Let us consider a generating function of the Jacobi polynomials which is written as a series similar to the form of (1). For $-1\leqslant x\leqslant 1$ and $z\in B$ define

$$\Phi_{\alpha,\beta}(z,x)=\frac{(\alpha+\beta+1)(1-z)}{(1+z)^{\alpha+\beta+2}}F\left(\frac{\alpha+\beta+2}{2},\frac{\alpha+\beta+3}{2};\alpha+1;\frac{2z(1-x)}{(1+z)^2}\right).$$

By (4) we can rewrite the function $\Phi_{\alpha,\beta}(z,x)$ as follows

$$\begin{aligned}\Phi_{\alpha,\beta}(z,x)&=\sum_{m\in\mathbf{N}_0}\frac{(\alpha+\beta+2m+1)(\alpha+\beta+1)_m}{(\alpha+1)_m}P_m^{(\alpha,\beta)}(x)(-z)^m\\&=\sum_{m\in\mathbf{N}_0}d_m^{(\alpha,\beta)}\frac{m!}{(-1)^m(\beta+1)_m}P_m^{(\alpha,\beta)}(x)z^m\\&=\sum_{m\in\mathbf{N}_0}d_m^{(\alpha,\beta)}\varphi_m^{(\alpha,\beta)}(x)z^m &(26)\\&=\sum_{m\in\mathbf{N}_0}u_m^{(\alpha,\beta)}(z)\psi_m^{(\alpha,\beta)}(x). &(27)\end{aligned}$$

The function $\Phi_{\alpha,\beta}(z,x)$ is a generating function of the Jacobi polynomials $P_m^{(\alpha,\beta)}(x)$, $m=0,1,2,\ldots,$ and by (26) it is written as a series similar to the form of (1).

Remark 2. By (14), (23) and (26), we obtain the following relation

$$g_w^{(\alpha,\beta)}(z)=\Phi_{\alpha,\beta}(z\overline{w},-1).$$

5.3. Unitary operator constructed by the generating function $\Phi_{\alpha,\beta}(z,x)$

A unitary operator on $L^2((-1,1),d\mu_{\alpha,\beta}(x))$ to $\mathcal{F}_{\alpha,\beta}$ is defined as follows. For $\varphi \in L^2((-1,1),d\mu_{\alpha,\beta}(x))$ set

$$(\Phi_{\alpha,\beta}\varphi)(z) = \int_{-1}^{1} \Phi_{\alpha,\beta}(z,x)\varphi(x)\,d\mu_{\alpha,\beta}(x), \quad z \in B. \tag{28}$$

Then $\Phi_{\alpha,\beta}\varphi \in \mathcal{F}_{\alpha,\beta}$ and $\Phi_{\alpha,\beta}$ is unitary.

Theorem 2. *The operator $\Phi_{\alpha,\beta}$ on $L^2((-1,1),d\mu_{\alpha,\beta}(x))$ to $\mathcal{F}_{\alpha,\beta}$ is unitary.*

Proof. The definition (28) corresponds to (21). Similarly, the formulae (23), (26) and (27) correspond to (16), (19) and (20), respectively. Therefore, in the same way as in the proof of Theorem 1, we obtain the unitarity of $\Phi_{\alpha,\beta}$ defined by (28). □

Therefore, we obtain the desired result:

$$\text{generating function} = \Phi_{\alpha,\beta}(z,x),$$
$$\text{reproducing kernel} = \Phi_{\alpha,\beta}(z\overline{w},-1).$$

6. Some remarks to special cases

6.1. The case of $\alpha = \beta = \lambda - 1/2$

Let $\lambda > 0$. The Gegenbauer polynomials $C_m^\lambda(x)$, $m = 0,1,2,\ldots,$ are defined as the Jacobi polynomials with $\alpha = \beta = \lambda - 1/2$ (cf. [3]):

$$C_m^\lambda(x) = \frac{(2\lambda)_m}{(\lambda+1/2)_m} P_m^{(\lambda-1/2,\lambda-1/2)}(x).$$

Notice the formula (1) in [10]:

$$\int_B |z|^{2m}\rho_\lambda\big(|z|^2\big)\,dz = \frac{\pi m!}{(m+\lambda)\Gamma(m+2\lambda)}, \quad m \in \mathbf{N}_0,$$

where

$$\rho_\lambda(t) = \begin{cases} \frac{1}{\Gamma(2\lambda-1)} t^{\lambda-1}\int_t^1 s^{-\lambda}(1-s)^{2\lambda-2}\,ds, & \lambda > 1/2, \\ t^{\lambda-1}\big(\frac{\Gamma(1-\lambda)}{\Gamma(\lambda)} - \frac{1}{\Gamma(2\lambda-1)}\int_0^t s^{-\lambda}(1-s)^{2\lambda-2}\,ds\big), & 0 < \lambda \leqslant 1/2. \end{cases}$$

Combining this formula with Proposition 1, we see that the measure $d\nu_{\alpha,\beta}(z)$ for $\alpha = \beta = \lambda - 1/2$ is equal to

$$\frac{\Gamma(2\lambda)}{2\pi}\rho_\lambda\big(|z|^2\big)\,dz.$$

Therefore, the construction for $\alpha = \beta = \lambda - 1/2$ in this paper is equivalent to that in [10].

On the other hand, the generating function $\Phi_{\alpha,\beta}(z,x)$ for $\alpha = \beta = \lambda - 1/2$ is

$$\sum_{m\in\mathbf{N}_0} d_m^{(\lambda-1/2,\lambda-1/2)}\varphi_m^{(\lambda-1/2,\lambda-1/2)}(x)z^m,$$

where

$$\varphi_m^{(\lambda-1/2,\lambda-1/2)}(x) = \frac{P_m^{(\lambda-1/2,\lambda-1/2)}(x)}{P_m^{(\lambda-1/2,\lambda-1/2)}(1)} = \frac{m!}{(2\lambda)_m}C_m^\lambda(x),$$
$$d_m^{(\lambda-1/2,\lambda-1/2)} = \frac{2(\lambda+m)(2\lambda)_m}{m!}.$$

Thus, the generating function $\Phi_{\alpha,\beta}(z,x)$ for $\alpha = \beta = \lambda - 1/2$ can be rewritten as follows

$$2\sum_{m\in\mathbf{N}_0}(\lambda+m)C_m^\lambda(x)z^m = 2\lambda\frac{1-z^2}{(1-2zx+z^2)^{\lambda+1}}, \tag{29}$$

where we use the following formula (cf. [3]):

$$\sum_{m\in\mathbf{N}_0} C_m^\lambda(x)z^m = \left(1-2zx+z^2\right)^{-\lambda}.$$

Set $x=1$ and replace z by $z\overline{w}$ in (29). Considering Remark 1, we can obtain the explicit expression for the reproducing kernel $g_w^{(\lambda-1/2,\lambda-1/2)}$ of $\mathcal{F}_{\lambda-1/2,\lambda-1/2}$:

$$g_w^{(\lambda-1/2,\lambda-1/2)}(z) = 2\lambda\frac{1+z\overline{w}}{(1-z\overline{w})^{2\lambda+1}}.$$

The measure $d\nu_{0,0}(z)$ corresponding to the Legendre polynomials $P_m(x)$, $m\in\mathbf{N}_0$, which are the Gegenbauer polynomials with $\lambda=1/2$, is given by

$$d\nu_{0,0}(z) = \frac{1}{2\pi|z|}\,dz.$$

The reproducing kernel $g_w^{(0,0)}$ of the space $\mathcal{F}_{0,0}$ is given by

$$g_w^{(0,0)}(z) = \frac{1+z\overline{w}}{(1-z\overline{w})^2}.$$

6.2. The case of $\alpha=n-1$ and $\beta=0$

Let n be a positive integer. Consider the Jacobi polynomials $P_m^{(n-1,0)}(x)$, $m=0,1,2,\dots$. By (19), the associated generating function $\Phi_{n-1,0}(z,x)$ is equal to

$$\sum_{m\in\mathbf{N}_0} d_m^{(0,n-1)}\varphi_m^{(n-1,0)}(x)z^m,$$

where

$$\varphi_m^{(n-1,0)}(x) = \frac{P_m^{(n-1,0)}(x)}{P_m^{(n-1,0)}(1)}, \qquad d_m^{(0,n-1)} = \frac{(n+2m)((n)_m)^2}{(m!)^2}.$$

On the other hand, as is well known (cf. [8]), the functions $\varphi_m^{(n-1,0)}$, $m=0,1,2,\dots,$ are the zonal spherical functions on the n-dimensional complex projective space $SU(n+1)/S(U(1)\times U(n))$, where $U(n)$ is the unitary group of degree n and $SU(n+1)$ is the special unitary group of degree $n+1$. Set $G=SU(n+1)$ and $K=S(U(1)\times U(n))$. Then the pair (G,K) is a Riemannian symmetric pair of rank 1 such that G/K is simply connected. Thus, by Theorem 2 in [12], the pair (G,K) has a generating function associated with itself. By Example 4 in [12], the generating function is given as follows: let d_m denote the degree of the spherical representation corresponding to $\varphi_m^{(n-1,0)}$, which is given by

$$d_m = \frac{(n+2m)((n)_m)^2}{n(m!)^2},$$

that is,

$$nd_m = d_m^{(0,n-1)}.$$

Then the generating function associated with the pair (G,K) is given by

$$\sum_{m\in\mathbf{N}_0} d_m\varphi_m^{(n-1,0)}(x)z^m = \frac{1}{n}\sum_{m\in\mathbf{N}_0} d_m^{(0,n-1)}\varphi_m^{(n-1,0)}(x)z^m.$$

Therefore, the construction for $\alpha=n-1$ and $\beta=0$ in this paper is equivalent to that in [12].

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