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An answer to a conjecture on Bernstein operators

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ABSTRACT

We give an affirmative answer to a conjecture of G.T. Tachev concerning the moments of the Bernstein operators.

1. Introduction

Let $f : [0,1] \to \mathbb{R}$. For any $n \in \mathbb{N}$ the Bernstein operator is defined by

$$B_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \quad x \in [0,1],$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$.

The classical Voronovskaja's theorem on the asymptotic behaviour of Bernstein polynomials can be stated as follows.

Theorem 1. *(See Voronovskaja [11].) If f is bounded on $[0,1]$ and, for some $x \in [0,1]$, f is differentiable in a neighbourhood of x and possesses a second derivative $f''(x)$, then*

$$\lim_{n\to\infty} n\big(B_n(f;x) - f(x)\big) = \frac{x(1-x)}{2} f''(x).$$

If $f \in C^2[0,1]$, the convergence is uniform.

S.N. Bernstein gave the following generalization of Voronovskaja's Theorem 1.

Theorem 2. *(See Bernstein [1].) If $q \in \mathbb{N}$ is even and $f \in C^q[0,1]$, then,*

$$\lim_{n\to\infty} n^{q/2}\left(B_n(f;x) - f(x) - \sum_{r=1}^{q} B_n\big((e_1 - x)^r; x\big)\frac{f^{(r)}(x)}{r!}\right) = 0,$$

uniformly on $[0,1]$.

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Throughout the paper, we denote by e_i, $i = 1, 2, \dots$, the monomial functions $e_i(t) = t^i$, $t \in [0,1]$. Mamedov obtained the following generalization of the previous Bernstein theorem.

Theorem 3. *(See Mamedov [5].) If $q \in \mathbb{N}$ is even, $f \in C^q[0,1]$, and $L_n : C[0,1] \to C[0,1]$ is a sequence of positive linear operators such that*

$$L_n e_0 = e_0,$$
$$\lim_{n\to\infty} \frac{L_n((e_1 - x)^{q+2j}; x)}{L_n((e_1 - x)^q; x)} = 0$$

for some $x \in [0,1]$ and for at least one integer $j > 0$, then

$$\lim_{n\to\infty} \frac{1}{L_n((e_1 - x)^q; x)} \left(L_n(f; x) - f(x) - \sum_{r=1}^{q} L_n\big((e_1 - x)^r; x\big) \frac{f^{(r)}(x)}{r!} \right) = 0.$$

Let $WC^q[0,1]$, $q \geqslant 0$, denote the set of all functions in $R^{[0,1]}$ possessing a q-th piecewise continuous derivative. A complete asymptotic expansion in quantitative form was given by Sikkema and van der Meer (1979).

Theorem 4. *(See Sikkema and van der Meer [8].) Let $(L_n)_{n\in\mathbb{N}}$ be a sequence of positive linear operators $L_n : WC^q[0,1] \to C[0,1]$ satisfying $L_n(e_0) = 1$. Then for all $f \in WC^q[0,1]$, $x \in [0,1]$, $n \in \mathbb{N}$ and $\delta > 0$, one has*

$$\left| L_n(f; x) - f(x) - \sum_{r=1}^{q} L_n\big((e_1 - x)^r; x\big) \frac{f^{(r)}(x)}{r!} \right| \leqslant C_{n,q}(x, \delta)\,\omega\big(f^{(q)}; \delta\big),$$

where:

$$C_{n,q}(x,\delta) = \delta^q L_n\left(s_{q,\mu}\left(\frac{e_1 - x}{\delta} \right); x \right),$$
$$\mu = \begin{cases} \frac{1}{2}, & \text{if } L_n((e_1 - x)^q; x) \geqslant 0, \\ -\frac{1}{2}, & \text{if } L_n((e_1 - x)^q; x) < 0, \end{cases}$$
$$s_{q,n}(u) = \frac{1}{q!}\left(\frac{1}{2}|u|^q + \mu u^q \right) + \frac{1}{(q+1)!}\big(b_{q+1}\big(|u|\big) - b_{q+1}\big(|u| - \lfloor |u| \rfloor\big)\big).$$

Here b_{q+1} is the Bernoulli polynomial of degree $q+1$, $\lfloor\ \rfloor$ denotes the integer part function and $\omega(g; \delta)$ denotes the classical first order modulus of continuity of $g \in C[0,1]$. Moreover the functions $C_{n,q}(x,\delta)$ are the best possible.

In 2007, H. Gonska proved the following result.

Theorem 5. *(See H. Gonska [3].) Let $q \in \mathbb{N}$, $f \in C^q[0,1]$ and $L : C[0,1] \to C[0,1]$ be a positive linear operator. Then*

$$\left| L(f; x) - \sum_{r=0}^{q} L\big((e_1 - x)^r; x\big) \frac{f^{(r)}(x)}{r!} \right|$$
$$\leqslant \frac{L(|e_1 - x|^q; x)}{q!} \tilde{\omega}\left(f^{(q)}; \frac{1}{q+1} \frac{L(|e_1 - x|^{q+1}; x)}{L(|e_1 - x|^q; x)} \right),$$

where the least concave majorant $\tilde{\omega}(f; \delta)$ is given by

$$\tilde{\omega}(f; \delta) = \begin{cases} \sup_{0 \leqslant x \leqslant \delta < y \leqslant 1} \frac{(\delta - x)\omega(f; y) + (y - \delta)\omega(f; x)}{y - x}, & 0 < \delta \leqslant 1, \\ \omega(f; 1), & \delta > 1. \end{cases}$$

By using the K-functionals technique, G.T. Tachev proved the following theorem.

Theorem 6. *(See G.T. Tachev [9].) If $q \in \mathbb{N}$ is odd and $f \in C^q[0,1]$, then*

$$\lim_{n\to\infty} n^{q/2} \left(B_n(f; x) - f(x) - \sum_{r=1}^{q} B_n\big((e_1 - x)^r; x\big) \frac{f^{(r)}(x)}{r!} \right) = 0,$$

uniformly on $[0,1]$.

In order to extend the Bernstein result (Theorem 2) to all $q \in \mathbb{N}$ by using Gonska's Theorem 5, it would be crucial to know the behaviour of the sequence

$$\frac{B_n((e_1 - x)^{q+1}; x)}{B_n(|e_1 - x|^q; x)}$$

as $n \to \infty$.

It is the context presented above that an answer to the following conjecture is of foremost importance.

Conjecture 7. *(See G.T. Tachev [9, p. 1183].) For $q \in \mathbb{N}$ and any $x \in (0, 1)$, the following statement holds true*:

$$\lim_{n\to\infty} \frac{B_n((e_1 - x)^{q+1}; x)}{B_n(|e_1 - x|^q; x)} = 0. \tag{1.1}$$

For $q = 0, 1, 2, 3$, Eq. (1.1) is easily verified. It remains to check it for all $q \geqslant 4$.

In Section 3, we will give an affirmative answer to Tachev's conjecture.

Moreover we provide stronger related inequalities that are crucial to extend Bernstein's Theorem 2.

2. Auxiliary results

Let $x \in [0, 1]$, $\alpha > 0$ and let us use the notation $X := x(1 - x)$, $x \in [0, 1]$. The following result was proved in 2008 by S. Telyakovskii.

Theorem 8. *(See S. Telyakovskii [10, Lemma 1].) The inequality*

$$B_n\big(|e_1 - x|^\alpha; x\big) \leqslant C(\alpha)\left(\frac{X}{n}\right)^{\alpha/2}, \tag{2.1}$$

where $C(\alpha)$ depends only on α, holds true for $nX \geqslant 1$.

Let $I = [a, b]$, Π be the linear space of all real polynomials defined on I and let S be a linear subspace of $C(I)$ such that $\Pi \subset S$. The divided difference of a function $f \in \mathbb{R}^I$ on the distinct knots $x_1, \ldots, x_n \in I$, is defined by

$$[x_1, \ldots, x_n; f] = \sum_{k=1}^{n} \frac{f(x_k)}{\ell'(x_k)},$$

where $\ell(x) = (x - x_1)\ldots(x - x_n)$. Recall the following definition of Tiberiu Popoviciu.

Definition 9. (See T. Popoviciu [6].) Let n be an integer, $n \geqslant -1$. A linear functional $A : S \to \mathbb{R}$ is said to be P_n-*simple* if the following conditions are satisfied:

(i) $A(e_{n+1}) \neq 0$;
(ii) For any $f \in S$ there exist distinct points $\xi_1, \ldots, \xi_{n+2} \in I$, depending on f, such that

$$A(f) = A(e_{n+1})[\xi_1, \ldots, \xi_{n+2}; f].$$

We will use the following result of T. Popoviciu.

Theorem 10. *(See T. Popoviciu [7].) Let $A : S \to \mathbb{R}$ be a bounded P_m-simple linear functional $(m \geqslant 0)$. Then, the functional $\bar{A} : C^{m+1}(I) \to \mathbb{R}$ defined by*

$$\bar{A}(f) = A(f) - A(e_{m+1})\frac{f^{(m+1)}(c)}{(m+1)!},$$

where

$$c = \frac{A(e_{m+2})}{(m+2)A(e_{m+1})},$$

is P_{m+2}-simple.

We give below a representation formula that will be essential in our proof.

Lemma 11. *Let $x \in [0,1]$ and $n \geqslant 1$. For any $g \in C^2[0,1]$, there exists distinct points $c_1, \ldots, c_5 \in [0,1]$, depending on g, such that the following representation formula for the Bernstein operator is satisfied*:

$$B_n(g;x) = g(x) + \frac{X}{2n} g''\left(x + \frac{1-2x}{3n}\right) + \frac{(9n-10)X^2 + X}{3n^3}[c_1, \ldots, c_5; g]. \tag{2.2}$$

Proof. The functional $A : C[0,1] \to \mathbb{R}$, $A(f) = B_n(f;x) - f(x)$, is P_1-simple. By Popoviciu's Theorem 10, the functional $\bar{A} : C^2[0,1] \to \mathbb{R}$,

$$\bar{A}(g) = B_n(g;x) - g(x) - \frac{X}{2n} g''(c),$$

where $c = \frac{1}{3}\frac{B_n(e_3;x)-x^3}{B_n(e_2;x)-x^2} = x + \frac{1-2x}{3n}$, is P_3-simple. It follows that for all $g \in C^2[0,1]$ there exist distinct points $c_1, \ldots, c_5 \in [0,1]$ such that

$$\bar{A}(g) = \bar{A}(e_4)[c_1, \ldots, c_5; g],$$

i.e.,

$$B_n(g;x) = g(x) + \frac{X}{2n} g''(c) + \left(B_n(e_4;x) - x^4 - 6\frac{X}{n}c^2\right)[c_1, \ldots, c_5; g]. \tag{2.3}$$

By using the equalities

$$\begin{aligned}
B_n(e_2;x) - x^2 &= \frac{X}{n}, \\
B_n(e_3;x) - x^3 &= \frac{X((3n-2)x+1)}{n^2}, \\
B_n(e_4;x) - x^4 &= \frac{X((6n^2-11n+6)x^2 + (7n-6)x + 1)}{n^3}, \\
B_n(e_4;x) - x^4 - 6\frac{X}{n}c^2 &= \frac{X((9n-10)X+1)}{3n^3},
\end{aligned}$$

in (2.3), the proof is concluded. □

We can prove the following lemma.

Lemma 12. *Let $x \in [0,1]$ be such that $nX \leqslant 1$. Then, for any $q \in \mathbb{N}^*$, there exists a constant $C_1(q)$ independent of n such that*:

$$\left|B_n\left((e_1 - x)^{q+1}; x\right)\right| \leqslant C_1(q)\frac{X}{n^q}. \tag{2.4}$$

Proof. It is known that (see, e.g., [2, Chapter 10, Theorem 1.1]) for any $s \in \mathbb{N}^*$, we have

$$B_n\left((e_1 - x)^{2s}; x\right) = \frac{1}{n^{2s}} \sum_{j=1}^{s} a_{j,s}(X) n^j X^j, \tag{2.5}$$

and

$$B_n\left((e_1 - x)^{2s+1}; x\right) = \frac{1-2x}{n^{2s+1}} \sum_{j=1}^{s} b_{j,s}(X)(nX)^j, \tag{2.6}$$

where $a_{j,s}$ and $b_{j,s}$ are polynomials of degree $s - j$ independent of n.

For $q = 2s + 1$, from (2.5), we get

$$B_n\left((e_1 - x)^{q+1}; x\right) \leqslant \frac{X}{n^q} \sum_{j=1}^{s} \|a_{j,s}\|. \tag{2.7}$$

From (2.6), for $nX \leqslant 1$, we obtain

$$\left|B_n\left((e_1 - x)^{2s+1}; x\right)\right| \leqslant \frac{X}{n^{2s}} \sum_{j=1}^{s} \|b_{j,s}\|. \tag{2.8}$$

From (2.8) with $q = 2s$, and (2.7) we obtain (2.4). □

3. An answer to Conjecture 7

The following theorem gives a positive answer to Conjecture 7.

Theorem 13. *For any $q \geqslant 4$ and $x \in (0,1)$ there exists a constant $K(q)$ such that*

$$\frac{|B_n((e_1-x)^{q+1};x)|}{B_n(|e_1-x|^q;x)} \leqslant \frac{K(q)}{\sqrt{n}}, \quad n \geqslant 5. \tag{3.1}$$

Proof. Since the function $t \mapsto t^{q/2}$, $t \in [0,1]$, is convex for $q \geqslant 4$, by Jessen's inequality [4], we obtain

$$B_n\big(|e_1-x|^q;x\big) = B_n\big(\big((e_1-x)^2\big)^{q/2};x\big) \geqslant \big(B_n\big((e_1-x)^2;x\big)\big)^{q/2} = \left(\frac{X}{n}\right)^{q/2}, \tag{3.2}$$

for all $x \in [0,1]$.

Now, from Telyakovskii's inequality (2.1) and (3.2) we get

$$\frac{|B_n((e_1-x)^{q+1};x)|}{B_n(|e_1-x|^q;x)} \leqslant C(q+1)\left(\frac{X}{n}\right)^{1/2}, \tag{3.3}$$

for any $x \in (0,1)$ such that $nX > 1$.

To prove an inequality similar to (3.3) in the case $nX \leqslant 1$ we will use our Lemma 11.

By taking $g = |e_1-x|^q$ in (2.2) and using the fact that the divided difference $[c_1,\dots,c_5;\ |e_1-x|^q]$ is positive for $q \geqslant 4$, we obtain

$$B_n\big(|e_1-x|^q;x\big) \geqslant \frac{X}{2n}q(q-1)\left|\frac{1-2x}{3n}\right|^{q-2},$$

or

$$B_n\big(|e_1-x|^q;x\big) \geqslant \frac{X}{2\cdot 3^{q-2}n^{q-1}}q(q-1)|1-2x|^{q-2}. \tag{3.4}$$

Since $nX \leqslant 1$, then $x \in \big(0, \frac{n-\sqrt{n^2-4n}}{2n}\big] \cup \big[\frac{n+\sqrt{n^2-4n}}{2n}, 1\big)$. So, from (3.4), for $n \geqslant 5$, we obtain

$$B_n\big(|e_1-x|^q;x\big) \geqslant \frac{X}{2(3\sqrt{5})^{q-2}n^{q-1}}q(q-1). \tag{3.5}$$

From (2.4) and (3.5), we get

$$\frac{|B_n((e_1-x)^{q+1};x)|}{B_n(|e_1-x|^q;x)} \leqslant \frac{C_1(q)2(3\sqrt{5})^{q-2}}{q(q-1)}\frac{1}{n}. \tag{3.6}$$

Thus, Eqs. (3.3) and (3.6) conclude the proof. □

4. Improved inequalities related to the conjecture

The following theorem enriches the answer given to the conjecture.

Theorem 14. *For any $q \geqslant 4$ and $x \in (0,1)$ there exists a constant $A(q)$ such that*

$$\frac{B_n(|e_1-x|^{q+1};x)}{B_n(|e_1-x|^q;x)} \leqslant \frac{A(q)}{\sqrt{n}}, \quad n \geqslant 5. \tag{4.1}$$

Proof. If $q = 2s+1$, inequality (4.1) follows from (3.1). Let $q = 2s$. By using the Cauchy–Schwarz inequality, we obtain

$$\frac{B_n(|e_1-x|^{2s+1};x)}{B_n(|e_1-x|^{2s};x)} \leqslant \sqrt{\frac{B_n((e_1-x)^{2s+2};x)}{B_n((e_1-x)^{2s};x)}}. \tag{4.2}$$

If $nX \geqslant 1$, by Theorem 8, there exists a constant $C(s)$ such that

$$B_n\big((e_1-x)^{2s+2};x\big) \leqslant C(s)\frac{X^{s+1}}{n^{s+1}}. \tag{4.3}$$

By Jessen's inequality [4], we obtain

$$B_n\big((e_1-x)^{2s};x\big)=B_n\big(\big((e_1-x)^2\big)^s;x\big)\geqslant\left(\frac{X}{n}\right)^s. \tag{4.4}$$

From (4.2), (4.3) and (4.4) we obtain

$$\frac{B_n(|e_1-x|^{2s+1};x)}{B_n(|e_1-x|^{2s};x)}\leqslant\sqrt{C(s)}\frac{\sqrt{X}}{\sqrt{n}},\quad nX\geqslant 1. \tag{4.5}$$

If $nX<1$, from Lemma 12, we get

$$\big|B_n\big((e_1-x)^{2s+2};x\big)\big|\leqslant C_1(2s+1)\frac{X}{n^{2s+1}},\quad nX<1. \tag{4.6}$$

From (3.5), we obtain

$$B_n\big(|e_1-x|^{2s};x\big)\geqslant\frac{X}{2(3\sqrt{5})^{2s-2}n^{2s-1}}2s(2s-1). \tag{4.7}$$

From (4.2), (4.6) and (4.7), we obtain

$$\frac{B_n(|e_1-x|^{2s+1};x)}{B_n(|e_1-x|^{2s};x)}\leqslant\sqrt{\frac{(3\sqrt{5})^{2s-2}C_1(2s+1)}{s(2s-1)}}\cdot\frac{1}{n},\quad nX<1,$$

and the proof is concluded. □

In what follows we provide a converse of inequality (4.1).

Theorem 15. *For any $q\in\mathbb{N}$, $q\geqslant 1$, we have*

$$\left\|\frac{B_n(|e_1-x|^{q+1};x)}{B_n(|e_1-x|^{q};x)}\right\|\geqslant\frac{M(q)}{\sqrt{n}},$$

where $M(q)$ is a positive constant independent of n and x.

Proof. We have

$$\left\|\frac{B_n(|e_1-x|^{q+1};x)}{B_n(|e_1-x|^{q};x)}\right\|\geqslant\max_{nX\geqslant 1}\frac{B_n(|e_1-x|^{q+1};x)}{B_n(|e_1-x|^{q};x)}. \tag{4.8}$$

By using Jessen's inequality, we have

$$B_n\big(|e_1-x|^{q+1};x\big)\geqslant\left(\frac{X}{n}\right)^{\frac{q+1}{2}},$$

and Telyakovskii's inequality (2.1) yields

$$B_n\big(|e_1-x|^{q};x\big)\leqslant C(q)\left(\frac{X}{n}\right)^{q/2},\quad n>1/X,$$

hence,

$$\frac{B_n(|e_1-x|^{q+1};x)}{B_n(|e_1-x|^{q};x)}\geqslant\frac{1}{C(q)}\frac{\sqrt{X}}{\sqrt{n}},\quad n>1/X. \tag{4.9}$$

From (4.8) and (4.9) we get

$$\left\|\frac{B_n(|e_1-x|^{q+1};x)}{B_n(|e_1-x|^{q};x)}\right\|\geqslant\frac{1}{2C(q)}\frac{1}{\sqrt{n}}.\qquad\square$$

Remark 16. From Theorems 14 and 15 we deduce the following estimation

$$\left\|\frac{B_n(|e_1-x|^{q+1};x)}{B_n(|e_1-x|^{q};x)}\right\|=O\left(\frac{1}{\sqrt{n}}\right),$$

which is essential to extend results from [1,3,9].

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