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Singular limit of a competition–diffusion system with large interspecific interaction

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ABSTRACT

We consider a competition–diffusion system for two competing species; the density of the first species satisfies a parabolic equation together with an inhomogeneous Dirichlet boundary condition whereas the second one either satisfies a parabolic equation with a homogeneous Neumann boundary condition, or an ordinary differential equation. Under the situation where the two species spatially segregate as the interspecific competition rate becomes large, we show that the resulting limit problem turns out to be a free boundary problem. We focus on the singular limit of the interspecific reaction term, which involves a measure located on the free boundary.

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1. Introduction

The understanding of the interaction of biological species arising in ecological systems has recently developed as a central problem in population ecology. In particular, problems of coexistence and exclusion of competing species have been theoretically investigated using models based on partial and ordinary differential equations. Among many models proposed so far, reaction–diffusion equation models are used to study the spatial segregation of competing species which move by diffusion. Consider a competing system which consists of n species living in a habitat $\Omega \subset \mathbb{R}^N$ $(N \geqslant 1)$. We denote by $(x,t) \mapsto u_i(x,t)$ $(i = 1, 2, \dots, n)$ their population densities at position $x \in \Omega$ and time $t \geqslant 0$. The evolution of u_i is described by

$$\partial_t u_i = d_i \Delta u_i + \left(r_i - a_i u_i - \sum_{j=1}^{n} b_{ij} u_j \right) u_i, \quad i = 1, 2, \dots, n,$$

where d_i is the diffusion rate, r_i the intrinsic growth rate, a_i the intraspecific competition rate, that is the competition between members of the *same* species u_i, and b_{ij} the interspecific competition rate, that is the competition between members of the *different* species u_i and u_j. All the rates are positive constants.

In this paper, we restrict ourselves to the case of two competing species, which reads as

$$\begin{cases} \partial_t u_1 = d_1 \Delta u_1 + r_1 u_1 (1 - u_1 - k u_2), \\ \partial_t u_2 = d_2 \Delta u_2 + r_2 u_2 (1 - u_2 - \alpha k u_1), \end{cases}$$

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where k and α are positive constants. We assume that k is the only parameter which is large and that all the other parameters are of order $\mathcal{O}(1)$. The coefficient $\alpha > 0$ is the competition ratio between the two species u_1 and u_2. If $\alpha > 1$, then u_1 has a competitive advantage over u_2, while if $\alpha < 1$, the situation is reversed. Nonlinear terms in the system are of Lotka–Volterra type composed of two different contributions: the intraspecific one, namely $u_i(1-u_i)$, is a growth term of Fisher-type, whereas the interspecific contribution $-ku_iu_j$ is a consumption term modelling the competition between the species.

We take k as a free parameter and keep the other parameters d_1, d_2, r_1, r_2 and α fixed. For values of k which are neither large nor small, it is shown that u_1 and u_2 exhibit spatial segregation with a rather wide zone of overlap. When the value of k increases, the zone of overlap becomes narrower. Thus, taking the limit $k \to \infty$, one can expect that u_1 and u_2 have disjoint supports (habitats) with only one common curve, which separates the habitats of the two competing species. The purpose of this paper is to derive the limiting system as $k \to \infty$, which is called the *spatial segregation limit*, to describe the time evolution of the supports of u_1 and u_2. As it will be proved below, the limiting system can be described by a *free boundary problem* which is a two-phase Stefan-like problem (with zero latent heat) with reaction terms.

In this paper, we consider the reaction–diffusion problem for (u, v):

$$(\mathcal{P}^k)\quad \begin{cases} \partial_t u = d_1 \Delta u + f(u) - kF(u, v), & \text{in } \Omega \times (0, T], \\ \partial_t v = d_2 \Delta v + g(v) - \alpha k F(u, v), & \text{in } \Omega \times (0, T], \\ u = \bar{u}, \quad d_2 \partial_n v = 0, & \text{on } \partial\Omega \times (0, T], \\ u(\cdot, 0) = u_0, \quad v(\cdot, 0) = v_0, & \text{on } \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. The functions f and g are the intraspecific growth functions, whereas $F(u, v)$ is the interspecific competition term; the diffusion coefficients d_1 and d_2 are such that $d_1 > 0$ and $d_2 \geqslant 0$, so that the population V can be mobile or immobile and α is a positive constant. The parameter k is the interspecific competition rate (k^{-1} can be also seen as a characteristic time of the interspecific competition process). We assume that the following hypotheses hold:

Assumption 1 *(Interaction of two species).*

- F is a Lipschitz continuous function: there exists $\gamma > 0$ such that

$$\left|F(u_1, v_1) - F(u_2, v_2)\right| \leqslant \gamma\left(|u_1 - u_2| + |v_1 - v_2|\right),$$

  for all $(u_1, v_1) \in [0, 1]^2$ and $(u_2, v_2) \in [0, 1]^2$,
- $F(0, s) = F(\tilde{s}, 0) = 0$ for $s \in [0, 1]$, $\tilde{s} \in [0, 1]$,
- $F(u, v) > 0$ for $(u, v) \in (0, 1] \times (0, 1]$,
- F is nondecreasing in u and v.

Assumption 2 *(Source terms for a single species).*

(i) f and g are continuously differentiable on $[0, +\infty)$ such that $f(0) = g(0) = 0$;
(ii) $f(s) < 0$, $g(s) < 0$ for all $s > 1$.

Assumption 3 *(Initial and boundary conditions).*

- u_0, v_0 and $\bar{u}$ are functions with values in $[0, 1]$,
- $\bar{u} \in C^{2,1}(\overline{\Omega} \times \mathbb{R}^+)$ and $\bar{u} > 0$ on $\partial\Omega \times \mathbb{R}^+$,
- $u_0 = \bar{u}(\cdot, 0)$.

In the sequel, we will always assume that Assumptions 1, 2 and 3 hold.

Numerous studies have been carried out for competition models of Lotka–Volterra type in the case of two competing species, see e.g. [28,33]. Let us also mention recent results of Squassina [34,35] who investigated from both theoretical and numerical viewpoints the long term behaviour for a class of competition–diffusion systems of Lotka–Volterra type for two competing species in the case of different interspecific reaction terms. Interestingly, a wide range of recent theoretical and numerical works has focused on different aspects of such systems: existence and uniqueness of classical solution for related free boundary problems [25], pattern formation [16], stability of competitive system with impulses [23,24], existence of traveling waves [36], control [1], analysis of coexistence for competing species by minimization approach [3], analysis of trajectories in configuration space of the population fractions for multi-species competing systems [11]. Other studies have focused on the fast-reaction limit: under suitable assumptions, as the reaction rate tends to infinity, competition–diffusion systems usually exhibit a limiting configuration with segregated habitats. We refer the reader to [4,5,7,10,12,13,22,27–29,31–33] and in particular to [4,6,31] for models involving Dirichlet boundary data. In particular, [37,9] exhibit some uniqueness and convergence results for a multi-species competing system and its singular limit, and an interior measure estimate of the free boundary for the singular limit; nevertheless, this is restricted to a uniform diffusion process with respect to the species. We refer to [10,15,22,30] for systems involving zero-flux boundary conditions.

Problem $(\mathcal{P}^k)$ with $d_1 > 0$ and $d_2 > 0$ has been studied in [8] in the case of homogeneous Neumann boundary conditions and by [4] in the case of inhomogeneous boundary conditions. Further we refer to [14,19–21] for studies of the singular limit of systems where a parabolic equation is coupled to an ordinary differential equation. In this paper we only suppose that $d_2 \geqslant 0$ so that Problem $(\mathcal{P}^k)$ contains both classes of systems. About the singular limit of the term $kF(u^k, v^k)$ in a one-dimensional context where a parabolic equation is coupled to an ordinary differential equation, we refer to [17,18]. Our aim is to show that the two competing species segregate more and more as k becomes large, and to describe the singular limit of the interspecific reaction term.

This paper is organized as follows:

- In Section 2, we prove that $(\mathcal{P}^k)$ admits a unique solution. The well-posedness of the PDE/PDE system is a straightforward application of a well-known result by Lunardi and the well-posedness of the PDE/ODE system is obtained as a limit case of the initial system.
- In Section 3, we focus on the fast reaction limit of $(\mathcal{P}^k)$ corresponding to an asymptotic study with respect to increasing values of k. We rigorously prove that the limit problem is a (well-posed) free boundary problem so that the two biological populations become disjoint.
- In Section 4, we consider again the limit problem: under some regularity assumption on the free boundary, we provide a strong formulation of the fast reaction limit and show that the support of the interspecific source term converges to a measure located at the free boundary. This is the main result of this article.

2. Existence and uniqueness results for the reaction–diffusion system

We first prove the well-posedness of the initial value problem. We have to apply different methods for the PDE/PDE system and the PDE/ODE system, due to the loss of regularity brought by the vanishing diffusion. In the first step (Section 2.1), we easily prove the well-posedness of the PDE/PDE system and then, in the second step (Section 2.2), we prove the well-posedness of the PDE/ODE system by passing to the limit in the diffusion parameter. Interestingly, this convergence analysis will be crucial also for the asymptotic study $k \to +\infty$ (see Section 3) as the estimates that are proven in this section are uniform not only with respect to the diffusion parameter d_2 but also with respect to the reaction rate k.

2.1. Well-posedness of the PDE/PDE system

Theorem 1. *If $d_2 > 0$ and $k > 0$, Problem $(\mathcal{P}^k)$ admits a unique classical solution*[1]

$$\left(u^k, v^k\right) \in C^{2,1}\left(\overline{\Omega} \times (0, T]\right) \cap C\left(\overline{\Omega} \times [0, T]\right).$$

Moreover,

$$0 \leqslant u^k, v^k \leqslant 1.$$

Proof. Define $U := u^k - \bar{u}$ and $V := v^k$. We can now apply Proposition 7.3.2, p. 277, in [26], to the corresponding problem for U and V with homogeneous boundary conditions to deduce that Problem $(\mathcal{P}^k)$ has a unique classical solution. Bounds are obtained as follows: we define

$$\mathcal{L}_1\left(u^k\right) := \partial_t u^k - d_1 \Delta u^k - f\left(u^k\right) + kF\left(u^k, v^k\right),$$
$$\mathcal{L}_2\left(v^k\right) := \partial_t v^k - d_2 \Delta v^k - g\left(v^k\right) + \alpha kF\left(u^k, v^k\right).$$

Since $\mathcal{L}_i(0) = 0$ and $\mathcal{L}_i(1) \geqslant 0$ for $i = 1, 2$, the assertion $0 \leqslant u^k, v^k \leqslant 1$ follows from the maximum principle; this completes the proof. $\square$

Note, using simple integrations, the following (classical) equalities which will be useful in the sequel. Let $T > 0$ be arbitrary; the function pair (u^k, v^k) is such that

$$\iint_{Q_T} u^k \partial_t \psi + \iint_{Q_T} \left\{ d_1 u^k \Delta \psi + \left(f\left(u^k\right) - kF\left(u^k, v^k\right)\right)\psi \right\} = -\int_{\Omega} u_0 \psi(\cdot, 0) + \int_0^T \int_{\partial\Omega} \bar{u}\, \partial_n \psi, \tag{1}$$

[1] By a classical solution of Problem $(\mathcal{P}^k)$ we mean a pair (u, v) such that $u, v \in C^{2,1}(\overline{\Omega} \times (0, T]) \cap C(\overline{\Omega} \times [0, T])$ and satisfies pointwise the partial differential equations as well as the boundary and initial conditions in Problem $(\mathcal{P}^k)$.

$$\iint\limits_{Q_T} v^k \partial_t \psi + \iint\limits_{Q_T} \{d_2 v^k \Delta\psi + \big(g\big(v^k\big) - \alpha k F\big(u^k, v^k\big)\big)\psi\} = -\int\limits_{\Omega} v_0 \psi(\cdot, 0) + \int\limits_0^T \int\limits_{\partial\Omega} d_2 v^k \partial_n \psi, \tag{2}$$

for all $\psi \in \mathcal{F}_T := \{\psi \in C^{2,1}(\overline{Q}_T),\ \psi(\cdot, T) = 0 \text{ on } \Omega \text{ and } \psi = 0 \text{ on } \partial\Omega \times [0, T]\}$.

2.2. Well-posedness of the PDE/ODE system

Lemma 1 *(Interspecific source term: estimates). For $d_2 > 0$ and $k > 0$, there exists a positive constant c_0 which does not depend on k and d_2 such that*

$$k \iint\limits_{Q_T} F\big(u^k, v^k\big) \leqslant c_0. \tag{3}$$

Proof. Integrating the equation for v^k over $Q_T := \Omega \times (0, T)$ yields

$$\begin{aligned} k \iint\limits_{Q_T} F\big(u^k, v^k\big) &= \alpha^{-1}\Bigg(\int\limits_0^T \int\limits_{\partial\Omega} d_2 \partial_n v^k + \iint\limits_{Q_T} g\big(v^k\big) - \int\limits_{\Omega} v^k(\cdot, T) + \int\limits_{\Omega} v_0\Bigg) \\ &\leqslant \alpha^{-1} \operatorname{meas}(\Omega)\big(2 + T\|g\|_{L^\infty(0,1)}\big) \end{aligned}$$

which implies the result. $\square$

Proposition 2. *For $d_2 > 0$ and $k > 0$, there exist positive constants c_1 and c_2 which do not depend on k and d_2 such that*

$$d_1 \iint\limits_{Q_T} \big|\nabla u^k\big|^2 \leqslant c_1, \tag{4}$$

$$d_2 \iint\limits_{Q_T} \big|\nabla v^k\big|^2 \leqslant c_2. \tag{5}$$

Proof. We proceed as follows:

- *Estimate for u^k.* We use the new unknown $U^k = u^k - \bar{u}$, so that the equation for u^k becomes

  $$\partial_t U^k = d_1 \Delta U^k + f\big(u^k\big) - kF\big(u^k, v^k\big) - \partial_t \bar{u} + d_1 \Delta \bar{u}.$$

  We multiply this equation by U^k and integrate over Ω. This yields the inequality

  $$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{\Omega} \big|U^k\big|^2 + d_1 \int\limits_{\Omega} \big|\nabla U^k\big|^2 + k \int\limits_{\Omega} F\big(u^k, v^k\big) U^k \leqslant 2 \operatorname{meas}(\Omega)\|f\|_{L^\infty(0,1)} + 2\int\limits_{\Omega} |\partial_t \bar{u}| + d_1 |\Delta \bar{u}|,$$

  which we integrate over $(0, T)$ to obtain (note that $U^k(\cdot, 0) = 0$)

  $$d_1 \iint\limits_{Q_T} \big|\nabla U^k\big|^2 + k \iint\limits_{Q_T} F\big(u^k, v^k\big) U^k \leqslant 2 \operatorname{meas}(\Omega) T \|f\|_{L^\infty(0,1)} - \frac{1}{2}\int\limits_{\Omega} \big|U^k\big|^2(\cdot, T) + 2\int\limits_{Q_T} |\partial_t \bar{u}| + d_1 |\Delta \bar{u}|.$$

  By Lemma 1, since F is nonnegative, $|U^k|$ is bounded by 2, we have

  $$d_1 \iint\limits_{Q_T} \big|\nabla U^k\big|^2 \leqslant 2\Bigg(c_0 + T \operatorname{meas}(\Omega)\|f\|_{L^\infty(0,1)} + \int\limits_{Q_T} |\partial_t \bar{u}| + d_1 |\Delta \bar{u}|\Bigg).$$

  Finally, we get

  $$\begin{aligned} d_1 \iint\limits_{Q_T} \big|\nabla u^k\big|^2 &= d_1 \iint\limits_{Q_T} \big|\nabla U^k + \nabla \bar{u}\big|^2 \\ &\leqslant 2d_1\Bigg(\iint\limits_{Q_T} \big|\nabla U^k\big|^2 + \iint\limits_{Q_T} |\nabla \bar{u}|^2\Bigg), \end{aligned}$$

which yields the estimate for u^k with

$$c_1 := 2c_0 + 4T\,\mathrm{meas}(\Omega)\|f\|_{L^\infty(0,1)} + 4\int_{Q_T} \big(|\partial_t \bar{u}| + d_1|\Delta\bar{u}|\big) + 2d_1 \iint_{Q_T} |\nabla\bar{u}|^2.$$

- *Estimate for v^k.* We multiply the equation for v^k

$$\partial_t v^k = d_2\Delta v^k + g\big(v^k\big) - \alpha k F\big(u^k, v^k\big)$$

by v^k and integrate over Ω. This yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \big|v^k\big|^2 + d_2\int_\Omega \big|\nabla v^k\big|^2 + \alpha k\int_\Omega F\big(u^k, v^k\big)v^k \leqslant \mathrm{meas}(\Omega)\|g\|_{L^\infty(0,1)}.$$

We integrate the result over $(0,T)$ and obtain

$$d_2\iint_{Q_T}\big|\nabla v^k\big|^2 + \alpha k\iint_{Q_T} F\big(u^k, v^k\big)v^k \leqslant \mathrm{meas}(\Omega)T\|g\|_{L^\infty(0,1)} + \frac{1}{2}\int_\Omega\big(\big|v^k\big|^2(\cdot,0) - \big|v^k\big|^2(\cdot,T)\big).$$

Since F is nonnegative and v^k, v_0 are functions with values in $[0,1]$, we get

$$d_2\iint_{Q_T}\big|\nabla v^k\big|^2 \leqslant \mathrm{meas}(\Omega)\big(T\|g\|_{L^\infty(0,1)} + 1\big),$$

which completes the proof with

$$c_2 := \mathrm{meas}(\Omega)\big(T\|g\|_{L^\infty(0,1)} + 1\big). \qquad \square$$

Next we state further uniform estimates with respect to d_2 and k. They will be essential for the convergence proof not only as d_2 tends to 0 but also as k tends to $+\infty$. Proposition 3 below is the key ingredient which will permit to apply the Riesz–Fréchet–Kolmogoroff theorem.

Proposition 3. *Assume that $d_2 > 0$ and $k > 0$. For $r > 0$ sufficiently small, say $r \in (0,\hat{r})$, we define*

$$\Omega_r = \big\{x\in\Omega,\ B(x,2r)\subset\Omega\big\}, \qquad \Omega_r' = \bigcup_{x\in\Omega_r} B(x,r),$$

where $B(x,r)$ denotes the ball in $\mathbb{R}^N$ with centre x and radius r. We also define, for any $F\in L^\infty(Q_T)$:

$$\forall\xi\in\overline{B(0,r)},\ \forall(x,t)\in\Omega_r'\times(0,T),\quad \mathcal{S}_\xi F(x,t) := F(x+\xi,t),$$
$$\forall\tau\in(0,T),\ \forall(x,t)\in\Omega\times(0,T),\quad \mathcal{T}_\tau F(x,t) := F(x,t+\tau).$$

For each $r\in(0,\hat{r})$, the following properties hold:

(i) *There exists a positive function G which does not depend on k and d_2, such that $G(\xi)\to 0$ as $\xi\to 0$ and*

$$\int_0^T\int_{\Omega_r}\big|\mathcal{S}_\xi u^k - u^k\big|^2 \leqslant c_1|\xi|^2, \qquad \int_0^T\int_{\Omega_r}\big|\mathcal{S}_\xi v^k - v^k\big| \leqslant G(\xi),$$

for all $\xi\in\overline{B(0,r)}$.

(ii) *There exist positive constants c_3 and c_4 which do not depend on k and d_2 such that,*

$$\int_0^{T-\tau}\int_{\Omega_r}\big|\mathcal{T}_\tau u^k - u^k\big|^2 \leqslant c_3\tau, \qquad \int_0^{T-\tau}\int_{\Omega_r}\big|\mathcal{T}_\tau v^k - v^k\big| \leqslant c_4\tau,$$

for all $\tau\in(0,T)$.

(iii) *For each $\varepsilon > 0$, there exists $\omega \Subset Q_T$ which does not depend on k and d_2 such that $\|u^k\|_{L^2(Q_T\setminus\omega)} < \varepsilon$, $\|v^k\|_{L^1(Q_T\setminus\omega)} < \varepsilon$.*

Proof. The proof of the left-hand inequality in (i) is based upon the fact that the sequence $\{\nabla u^k\}$ is bounded in $L^2(\Omega \times (0,T))$, whereas a key idea of the proof of the right-hand inequality in (i) is that if we would consider Problem $(\mathcal{P}^k)$ with $f = g = 0$, then the quantity

$$\int\limits_{\Omega} \left(\alpha\left|u_1^k(x,t) - u_2^k(x,t)\right| + \left|v_1^k(x,t) - v_2^k(x,t)\right|\right) \mathrm{d}x,$$

where (u_1^k, v_1^k) and (u_2^k, v_2^k) are two solution pairs, would decrease in time. The inequalities in (ii) follow from substituting the corresponding differential equations for $\{u^k\}$ and $\{v^k\}$ with the use of a suitable cut-off function. The inequalities in (iii) are a straightforward consequence of the uniform L^∞-boundedness of the solution.

∘ *Proof of* (i). This is a consequence of Proposition 2. We have

$$\begin{aligned}
d_1 \int\limits_0^T \int\limits_{\Omega_r} \left|\mathcal{S}_\xi u^k - u^k\right|^2 &= d_1 \int\limits_0^T \int\limits_{\Omega_r} \left|u^k(x+\xi,t) - u^k(x,t)\right|^2 \mathrm{d}x\,\mathrm{d}t \\
&= d_1 \int\limits_0^T \int\limits_{\Omega_r} \left| \int\limits_0^1 \nabla u^k(x+\theta\xi,t)\cdot\xi\,\mathrm{d}\theta\right|^2 \mathrm{d}x\,\mathrm{d}t \\
&\leqslant d_1|\xi|^2 \int\limits_0^1 \int\limits_0^T \int\limits_{\Omega_r} \left|\nabla u^k(x+\theta\xi,t)\right|^2 \mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\theta \\
&\leqslant d_1|\xi|^2 \int\limits_0^T \int\limits_{\Omega_r'} \left|\nabla u^k(x,t)\right|^2 \mathrm{d}x\,\mathrm{d}t \\
&\leqslant d_1|\xi|^2 \iint\limits_{Q_T} \left|\nabla u^k(x,t)\right|^2 \mathrm{d}x\,\mathrm{d}t \\
&\leqslant c_1|\xi|^2.
\end{aligned}$$

In the same way, we prove that

$$d_2 \int\limits_0^T \int\limits_{\Omega_r} \left|\mathcal{S}_\xi v^k - v^k\right|^2 \leqslant c_2|\xi|^2.$$

Next we focus on $|\mathcal{S}_\xi v^k - v^k|$, also allowing that $d_2 = 0$. Since $\Omega_r \subset \Omega_r' \subset \Omega$, we first construct a function $\psi \in C_0^\infty(\Omega_r')$, which only depends on Ω and r, such that

$$0 \leqslant \psi \leqslant 1 \quad \text{on } \Omega_r' \quad \text{and} \quad \psi = 1 \quad \text{on } \Omega_r,$$

with $|\nabla\psi|, |\Delta\psi| \leqslant C(r)$. To that purpose we set for $x \in \mathbb{R}^N$

$$\varrho(x) = \begin{cases} \varrho_0 \exp(-\frac{1}{1-|x|^2}), & \text{if } |x| < 1, \\ 0, & \text{otherwise,} \end{cases}$$

where the constant ϱ_0 is chosen such that $\int_{\mathbb{R}^N} \varrho = 1$,

$$\Omega_r'' = \bigcup_{x\in\Omega_r} B\left(x, \frac{r}{4}\right)$$

and

$$\psi(x) = \frac{4}{r} \int\limits_{\Omega_r''} \varrho\left(\frac{4(y-x)}{r}\right) \mathrm{d}y, \quad \text{for all } x \in \Omega_r'.$$

Take a smooth function $m : \mathbb{R} \to \mathbb{R}$ with

$$m \geqslant 0, \qquad m(0) = 0, \qquad m(r) = |r| - \frac{1}{2}, \quad |r| > 1,$$

and define for $\alpha > 0$ approximations of the modulus function

$$m_\alpha(r) = \alpha m\left(\frac{r}{\alpha}\right).$$

Now subtracting the equations for u^k and for $\mathcal{S}_\xi u^k$ (and also for v^k and for $\mathcal{S}_\xi v^k$) yields

$$\partial_t\left(u^k - \mathcal{S}_\xi u^k\right) - d_1\Delta\left(u^k - \mathcal{S}_\xi u^k\right) - \left(f\left(u^k\right) - f\left(\mathcal{S}_\xi u^k\right)\right) = -k\left(F\left(u^k, v^k\right) - F\left(\mathcal{S}_\xi u^k, \mathcal{S}_\xi v^k\right)\right), \tag{6}$$

and

$$\partial_t\left(v^k - \mathcal{S}_\xi v^k\right) - d_2\Delta\left(v^k - \mathcal{S}_\xi v^k\right) - \left(g\left(v^k\right) - g\left(\mathcal{S}_\xi v^k\right)\right) = -\alpha k\left(F\left(u^k, v^k\right) - F\left(\mathcal{S}_\xi u^k, \mathcal{S}_\xi v^k\right)\right). \tag{7}$$

For an arbitrary fixed $t_0 \in (0, T)$, we multiply Eq. (6) by $\psi m'_\alpha(u^k - \mathcal{S}_\xi u^k)$ and integrate this equation to obtain, after partial integration,

$$\begin{aligned}
&\int_0^{t_0}\int_{\Omega'_r} \psi\,\partial_t\left\{m_\alpha\left(u^k - \mathcal{S}_\xi u^k\right)\right\} \\
&\quad = -d_1\int_0^{t_0}\int_{\Omega'_r} \psi m''_\alpha\left(u^k - \mathcal{S}_\xi u^k\right)\left|\nabla\left(u^k - \mathcal{S}_\xi u^k\right)\right|^2 - d_1\int_0^{t_0}\int_{\Omega'_r} \nabla\psi m'_\alpha\left(u^k - \mathcal{S}_\xi u^k\right)\nabla\left(u^k - \mathcal{S}_\xi u^k\right) \\
&\qquad + \int_0^{t_0}\int_{\Omega'_r} \psi m'_\alpha\left(u^k - \mathcal{S}_\xi u^k\right)\left(f\left(u^k\right) - f\left(\mathcal{S}_\xi u^k\right)\right) - \int_0^{t_0}\int_{\Omega'_r} k\psi m'_\alpha\left(u^k - \mathcal{S}_\xi u^k\right)\left(F\left(u^k, v^k\right) - F\left(\mathcal{S}_\xi u^k, \mathcal{S}_\xi v^k\right)\right).
\end{aligned}$$

Evaluating the left-hand side, using $m''_\alpha \geqslant 0$ and integrating by parts again yield

$$\begin{aligned}
&\int_{\Omega'_r} \psi m_\alpha\left(u^k - \mathcal{S}_\xi u^k\right)(\cdot, t_0) - \int_{\Omega'_r} \psi m_\alpha\left(u^k - \mathcal{S}_\xi u^k\right)(\cdot, 0) \\
&\quad \leqslant d_1\int_0^{t_0}\int_{\Omega'_r} \Delta\psi m_\alpha\left(u^k - \mathcal{S}_\xi u^k\right) + \int_0^{t_0}\int_{\Omega'_r} \psi m'_\alpha\left(u^k - \mathcal{S}_\xi u^k\right)\left(f\left(u^k\right) - f\left(\mathcal{S}_\xi u^k\right)\right) \\
&\qquad - \int_0^{t_0}\int_{\Omega'_r} k\psi m'_\alpha\left(u^k - \mathcal{S}_\xi u^k\right)\left(F\left(u^k, v^k\right) - F\left(\mathcal{S}_\xi u^k, \mathcal{S}_\xi v^k\right)\right).
\end{aligned}$$

Now we let $\alpha \to 0$ and observe that $m_\alpha(r) \to |r|$ and $m'_\alpha(r) \to \operatorname{sign}(r)$. The dominated convergence theorem allows us to pass to the limit $\alpha \to 0$ in the last inequality to obtain

$$\begin{aligned}
&\int_{\Omega'_r} \psi\left|u^k - \mathcal{S}_\xi u^k\right|(\cdot, t_0) - \int_{\Omega'_r} \psi\left|u^k - \mathcal{S}_\xi u^k\right|(\cdot, 0) \\
&\quad \leqslant d_1\int_0^{t_0}\int_{\Omega'_r} \Delta\psi\left|u^k - \mathcal{S}_\xi u^k\right| + \mathcal{L}_f\int_0^{t_0}\int_{\Omega'_r} \psi\left|u^k - \mathcal{S}_\xi u^k\right| \\
&\qquad - \int_0^{t_0}\int_{\Omega'_r} k\psi\operatorname{sign}\left(u^k - \mathcal{S}_\xi u^k\right)\left(F\left(u^k, v^k\right) - F\left(\mathcal{S}_\xi u^k, \mathcal{S}_\xi v^k\right)\right),
\end{aligned}$$

in which the local Lipschitz regularity of f implies that

$$\operatorname{sign}\left(u^k - \mathcal{S}_\xi u^k\right)\left(f\left(u^k\right) - f\left(\mathcal{S}_\xi u^k\right)\right) \leqslant \mathcal{L}_f\left|u^k - \mathcal{S}_\xi u^k\right|.$$

In the same way, we check that

$$\int_{\Omega_r'} \psi \left| v^k - \mathcal{S}_\xi v^k \right| (\cdot, t_0) - \int_{\Omega_r'} \psi \left| v^k - \mathcal{S}_\xi v^k \right| (\cdot, 0)$$

$$\leqslant d_2 \int_0^{t_0} \int_{\Omega_r'} \Delta\psi \left| v^k - \mathcal{S}_\xi v^k \right| + \mathcal{L}_g \int_0^{t_0} \int_{\Omega_r'} \psi \left| v^k - \mathcal{S}_\xi v^k \right|$$

$$- \int_0^{t_0} \int_{\Omega_r'} \alpha k \psi \operatorname{sign}\left(v^k - \mathcal{S}_\xi v^k\right)\left(F\left(u^k, v^k\right) - F\left(\mathcal{S}_\xi u^k, \mathcal{S}_\xi v^k\right)\right).$$

Combining the two previous inequalities, we get

$$\int_{\Omega_r'} \left(\left| u^k - \mathcal{S}_\xi u^k \right| + \frac{1}{\alpha} \left| v^k - \mathcal{S}_\xi v^k \right| \right)(\cdot, t_0)\psi$$

$$\leqslant \int_{\Omega_r'} \left(\left| u^k - \mathcal{S}_\xi u^k \right| + \frac{1}{\alpha} \left| v^k - \mathcal{S}_\xi v^k \right| \right)(\cdot, 0)\psi$$

$$+ d_1 \int_0^{t_0} \int_{\Omega_r'} \Delta\psi \left| u^k - \mathcal{S}_\xi u^k \right| + \frac{d_2}{\alpha} \int_0^{t_0} \int_{\Omega_r'} \Delta\psi \left| v^k - \mathcal{S}_\xi v^k \right|$$

$$+ \mathcal{L}_f \int_0^{t_0} \int_{\Omega_r'} \psi \left| u^k - \mathcal{S}_\xi u^k \right| + \frac{\mathcal{L}_g}{\alpha} \int_0^{t_0} \int_{\Omega_r'} \psi \left| v^k - \mathcal{S}_\xi v^k \right| - \int_0^{t_0} \int_{\Omega_r'} k \mathcal{E}\left(u^k, v^k\right)\psi,$$

where $\mathcal{E}(u^k, v^k)$ denotes the following quantity

$$\left(\operatorname{sign}\left(u^k - \mathcal{S}_\xi u^k\right) + \operatorname{sign}\left(v^k - \mathcal{S}_\xi v^k\right)\right)\left(F\left(u^k, v^k\right) - F\left(\mathcal{S}_\xi u^k, \mathcal{S}_\xi v^k\right)\right).$$

Next we show that

$$\mathcal{E}\left(u^k, v^k\right) \geqslant 0 \tag{8}$$

a.e. in $\Omega_r' \times (0, T)$. Indeed, we can check that $\mathcal{E}(u^k, v^k)$ is equal to

$$\underbrace{\left| F\left(u^k, v^k\right) - F\left(\mathcal{S}_\xi u^k, v^k\right) \right|}_{\mathcal{E}_1} + \underbrace{\left(F\left(u^k, v^k\right) - F\left(\mathcal{S}_\xi u^k, v^k\right)\right) \operatorname{sign}\left(v^k - \mathcal{S}_\xi v^k\right)}_{\mathcal{E}_2}$$

$$+ \underbrace{\left(F\left(\mathcal{S}_\xi u^k, v^k\right) - F\left(\mathcal{S}_\xi u^k, \mathcal{S}_\xi v^k\right)\right) \operatorname{sign}\left(u^k - \mathcal{S}_\xi u^k\right)}_{\mathcal{E}_3} + \underbrace{\left| F\left(\mathcal{S}_\xi u^k, v^k\right) - F\left(\mathcal{S}_\xi u^k, \mathcal{S}_\xi v^k\right) \right|}_{\mathcal{E}_4},$$

with $\mathcal{E}_1 + \mathcal{E}_2 \geqslant 0$ and $\mathcal{E}_3 + \mathcal{E}_4 \geqslant 0$.
From inequality (8) we deduce the inequality

$$\int_{\Omega_r'} \left(\left| u^k - \mathcal{S}_\xi u^k \right| + \frac{1}{\alpha} \left| v^k - \mathcal{S}_\xi v^k \right| \right)(\cdot, t_0)\psi \leqslant \int_{\Omega_r'} \left(\left| u^k - \mathcal{S}_\xi u^k \right| + \frac{1}{\alpha} \left| v^k - \mathcal{S}_\xi v^k \right| \right)(\cdot, 0)\psi$$

$$+ d_1 \int_0^{t_0} \int_{\Omega_r'} \Delta\psi \left| u^k - \mathcal{S}_\xi u^k \right| + \frac{d_2}{\alpha} \int_0^{t_0} \int_{\Omega_r'} \Delta\psi \left| v^k - \mathcal{S}_\xi v^k \right|$$

$$+ \max(\mathcal{L}_f, \mathcal{L}_g) \int_0^{t_0} \int_{\Omega_r'} \left(\left| u^k - \mathcal{S}_\xi u^k \right| + \frac{1}{\alpha} \left| v^k - \mathcal{S}_\xi v^k \right| \right)\psi.$$

Applying the Cauchy–Schwarz inequality yields

$$d_1\int_0^{t_0}\int_{\Omega_r'}\Delta\psi\left|u^k-\mathcal{S}_\xi u^k\right|\leqslant d_1\left(\int_0^T\int_{\Omega_r'}\left|u^k-\mathcal{S}_\xi u^k\right|^2\right)^{1/2}\left(\int_0^T\int_{\Omega_r'}|\Delta\psi|^2\right)^{1/2}$$
$$\leqslant\sqrt{c_1d_1}\left(\int_0^T\int_{\Omega_r'}|\Delta\psi|^2\right)^{1/2}|\xi|,$$
$$d_2\int_0^{t_0}\int_{\Omega_r'}\Delta\psi\left|v^k-\mathcal{S}_\xi v^k\right|\leqslant d_2\left(\int_0^T\int_{\Omega_r'}\left|v^k-\mathcal{S}_\xi v^k\right|^2\right)^{1/2}\left(\int_0^T\int_{\Omega_r'}|\Delta\psi|^2\right)^{1/2}$$
$$\leqslant\sqrt{c_2d_2}\left(\int_0^T\int_{\Omega_r'}|\Delta\psi|^2\right)^{1/2}|\xi|.$$

Thus we obtain

$$\int_{\Omega_r'}A_\xi(\cdot,t_0)\psi\leqslant\int_{\Omega_r'}A_\xi(\cdot,0)\psi+(\sqrt{c_1d_1}+\sqrt{c_2d_2})\left(\int_0^T\int_{\Omega_r'}|\Delta\psi|^2\right)^{1/2}|\xi|+\max(\mathcal{L}_f,\mathcal{L}_g)\int_0^{t_0}\int_{\Omega_r'}A_\xi\psi,$$

with

$$A_\xi:=\left|u^k-\mathcal{S}_\xi u^k\right|+\frac{1}{\alpha}\left|v^k-\mathcal{S}_\xi v^k\right|.$$

Applying Gronwall's lemma to the previous inequality, we finally get

$$\int_{\Omega_r'}A_\xi(\cdot,t_0)\psi\leqslant\left(\int_{\Omega_r'}A_\xi(\cdot,0)\psi+(\sqrt{c_1d_1}+\sqrt{c_2d_2})\sqrt{\int_0^T\int_{\Omega_r'}|\Delta\psi|^2|\xi|}\right)e^{\max(\mathcal{L}_f,\mathcal{L}_g)t_0}.$$

Now for $d_2\in(0,D^\star]$ (which may be assumed as we focus on the behaviour of the system for vanishing diffusion $d_2\to 0$), since $\Omega_r\subset\Omega_r'$ and $\psi=1$ on Ω_r, we get

$$\int_{\Omega_r}\left|u^k-\mathcal{S}_\xi u^k\right|+\frac{1}{\alpha}\left|v^k-\mathcal{S}_\xi v^k\right|\leqslant\left(\int_{\Omega_r'}A_\xi(\cdot,0)\psi+\left(\sqrt{c_1d_1}+\sqrt{c_2D^\star}\right)\sqrt{\int_0^T\int_{\Omega_r'}|\Delta\psi|^2|\xi|}\right)e^{\max(\mathcal{L}_f,\mathcal{L}_g)t_0}.$$

We have therefore completed the proof with $G(\xi)$ being equal to the right-hand side of this inequality.

∘ *Proof of* (ii). Let us introduce a cut-off function $\psi\in C_0^\infty(\Omega_r')$ such that $0\leqslant\psi\leqslant 1$ in Ω_r', $\psi\equiv 1$ in Ω_r. Then

$$\int_0^{T-\tau}\int_{\Omega_r'}\left|\mathcal{T}_\tau u^k-u^k\right|^2\psi=\int_0^{T-\tau}\int_{\Omega_r'}\left(\left[\mathcal{T}_\tau u^k-u^k\right](x,t)\int_t^{t+\tau}\partial_t u(x,s)\,\mathrm{d}s\right)\psi(x)\,\mathrm{d}x\,\mathrm{d}t$$
$$=\int_0^{T-\tau}\int_{\Omega_r'}\left(\left[\mathcal{T}_\tau u^k-u^k\right](x,t)\int_0^{\tau}\partial_t u(x,t+s)\,\mathrm{d}s\right)\psi(x)\,\mathrm{d}x\,\mathrm{d}t$$
$$=I_1+I_2+I_3,$$

with

$$I_1 := \int_0^\tau \int_0^{T-\tau} \int_{\Omega_r'} \big[\mathcal{T}_\tau u^k - u^k\big](x,t) d_1 \Delta u^k(x,t+s)\psi(x)\,dx\,dt\,ds,$$

$$I_2 := \int_0^\tau \int_0^{T-\tau} \int_{\Omega_r'} \big[\mathcal{T}_\tau u^k - u^k\big](x,t) f\big(u^k(x,t+s)\big)\psi(x)\,dx\,dt\,ds,$$

$$I_3 := -\int_0^\tau \int_0^{T-\tau} \int_{\Omega_r'} \big[\mathcal{T}_\tau u^k - u^k\big](x,t) k\big[F\big(u^k, v^k\big)\big](x,t+s)\psi(x)\,dx\,dt\,ds.$$

Since ψ vanishes on $\partial\Omega_r'$, one has

$$I_1 \leqslant -d_1 \int_0^\tau \int_0^{T-\tau} \int_{\Omega_r'} \nabla\big[\mathcal{T}_\tau u^k - u^k\big](x,t)\nabla u^k(x,t+s)\psi(x)\,dx\,dt\,ds$$
$$- d_1 \int_0^\tau \int_0^{T-\tau} \int_{\Omega_r'} \big[\mathcal{T}_\tau u^k - u^k\big](x,t)\nabla u^k(x,t+s)\cdot\nabla\psi(x)\,dx\,dt\,ds.$$

The right-hand side can be split into four terms:

$$I_1^{(1)} = -d_1 \int_0^\tau \int_0^{T-\tau} \int_{\Omega_r'} \nabla(\mathcal{T}_\tau u^k)(x,t)\nabla u^k(x,t+s)\psi(x)\,dx\,dt\,ds,$$

$$I_1^{(2)} = d_1 \int_0^\tau \int_0^{T-\tau} \int_{\Omega_r'} \nabla u^k(x,t)\nabla u^k(x,t+s)\psi(x)\,dx\,dt\,ds,$$

$$I_1^{(3)} = -d_1 \int_0^\tau \int_0^{T-\tau} \int_{\Omega_r'} (\mathcal{T}_\tau u^k)(x,t)\nabla u^k(x,t+s)\cdot\nabla\psi(x)\,dx\,dt\,ds,$$

$$I_1^{(4)} = d_1 \int_0^\tau \int_0^{T-\tau} \int_{\Omega_r'} u^k(x,t)\nabla u^k(x,t+s)\cdot\nabla\psi(x)\,dx\,dt\,ds.$$

Then, by the Cauchy–Schwarz inequality, using the property $\psi \leqslant 1$ and recalling that $\mathcal{T}_s f := f(\cdot,\cdot+s)$, we have

$$I_1^{(1)} \leqslant d_1 \int_0^\tau \left(\int_0^{T-\tau} \int_{\Omega_r'} \big|\nabla\big(\mathcal{T}_\tau u^k\big)\big|^2 \right)^{1/2} \left(\int_0^{T-\tau} \int_{\Omega_r'} \big|\nabla\big(\mathcal{T}_s u^k\big)\big|^2 \right)^{1/2} ds.$$

Simple computations allow us to control each term of the product:

$$\int_0^{T-\tau} \int_{\Omega_r'} \big|\nabla\big(\mathcal{T}_\tau u^k\big)\big|^2 \leqslant \int_0^{T-\tau} \int_{\Omega_r'} \mathcal{T}_\tau\big(\big|\nabla u^k\big|^2\big) \leqslant \int_0^T \int_{\Omega_r'} \big|\nabla u^k\big|^2,$$

and, for any $s \in (0,\tau)$,

$$\int_0^{T-\tau} \int_{\Omega_r'} \big|\nabla\big(\mathcal{T}_s u^k\big)\big|^2 \leqslant \int_0^{T-\tau} \int_{\Omega_r'} \mathcal{T}_s\big(\big|\nabla u^k\big|^2\big) \leqslant \int_0^T \int_{\Omega_r'} \big|\nabla u^k\big|^2,$$

so that we finally get

$$I_1^{(1)} \leqslant d_1 \tau \int_0^T \int_{\Omega_r'} \big|\nabla u^k(\cdot,\cdot)\big|^2.$$

With similar arguments, we have

$$I_1^{(2)} \leqslant d_1 \tau \int_0^T \int_{\Omega_r'} \left|\nabla u^k(\cdot,\cdot)\right|^2.$$

As $0 \leqslant u^k \leqslant 1$, we have also

$$\begin{aligned}
I_1^{(3)} &\leqslant d_1 \|\nabla\psi\|_{L^\infty(\Omega_r')} \int_0^\tau \left(\int_0^{T-\tau} \int_{\Omega_r'} \left|\nabla\left(\mathcal{T}_s u^k\right)\right| \right) \mathrm{d}s \\
&\leqslant d_1 \|\nabla\psi\|_{L^\infty(\Omega_r')} \int_0^\tau \left(\int_0^T \int_{\Omega_r'} \left|\nabla u^k\right| \right) \mathrm{d}s \\
&\leqslant d_1 \|\nabla\psi\|_{L^\infty(\Omega_r')} \tau \int_0^T \int_{\Omega_r'} \left|\nabla u^k\right|.
\end{aligned}$$

With similar arguments, we have

$$I_1^{(4)} \leqslant d_1 \|\nabla\psi\|_{L^\infty(\Omega_r')} \tau \int_0^T \int_{\Omega_r'} \left|\nabla u^k\right|.$$

The combination of the previous inequalities yields

$$\begin{aligned}
I_1 &\leqslant 2d_1 \tau \int_0^T \int_{\Omega_r'} \left|\nabla u^k\right|^2 + 2d_1 \|\nabla\psi\|_{L^\infty(\Omega_r')} \tau \int_0^T \int_{\Omega_r'} \left|\nabla u^k\right| \\
&\leqslant 2\left(1 + T \operatorname{meas}(\Omega) \|\nabla\psi\|_{L^\infty(\Omega_r')}\right) \tau d_1 \int_0^T \int_{\Omega_r'} \left|\nabla u^k\right|^2 \\
&\leqslant 2\left(1 + T \operatorname{meas}(\Omega) \|\nabla\psi\|_{L^\infty(\Omega_r')}\right) c_1 \tau.
\end{aligned}$$

The other terms are easier to handle: using the L^∞-bounds in the integral, we get

$$I_2 \leqslant \left(2\|f\|_{L^\infty(0,1)} \operatorname{meas}(\Omega) T\right) \tau,$$

and, using Lemma 1,

$$I_3 \leqslant 2c_0 \tau.$$

Thus, we have obtained the estimate on the left-hand side of (ii) with the constant:

$$c_3 := \left(1 + 2T \operatorname{meas}(\Omega) \|\nabla\psi\|_{L^\infty(\Omega_r')}\right) c_1 + 2\|f\|_{L^\infty(0,1)} \operatorname{meas}(\Omega) T + 2c_0.$$

An L^2-estimate for v^k can be obtained in a similar way (note that the boundary condition for u^k or v^k does not play any role in the proof, due to the use of a cut-off function). As a consequence, the desired L^1-estimate immediately follows. Note that the proof is even simpler for $d_2 = 0$.

∘ *Proof of* (iii). Let ε be arbitrary. Since u^k and v^k are bounded by 1, there exist $r_0 > 0$ and $\tau_0 > 0$ such that for $0 \leqslant r \leqslant r_0$ and $0 \leqslant \tau \leqslant \tau_0$,

$$\int_{T-\tau}^T \int_\Omega \left|u^k\right|^2 \leqslant \varepsilon, \qquad \int_0^T \int_{\Omega\setminus\Omega_r} \left|u^k\right|^2 \leqslant \varepsilon,$$

and similar inequalities hold for v^k in the L^1-norm. □

Interestingly, the previous estimates do not depend on k and d_2. This allows us to extend the definition of Problem $(\mathcal{P}^k)$ in the case $d_2 = 0$, corresponding to the PDE/ODE system. Thus, we have:

Lemma 4 *(Convergence results). Let $k > 0$ be fixed. There exists a pair $(u_\star^k, v_\star^k) \in (L^\infty(Q_T; [0,1]))^2$ such that, up to a subsequence,*

$$u^k \to u_\star^k \quad \text{in } L^2(Q_T),$$
$$v^k \to v_\star^k \quad \text{in } L^1(Q_T),$$
$$u^k - \bar{u} \rightharpoonup u_\star^k - \bar{u} \quad \text{in } L^2\big(0,T; H_0^1(\Omega)\big),$$

as $d_2 \to 0$.

Proof. We apply the Riesz–Fréchet–Kolmogoroff theorem[2]: we deduce from Proposition 3 that the sequence (u^k, v^k) is relatively compact in $L^2(Q_T) \times L^1(Q_T)$ which, together with the properties of (u^k, v^k), implies that there exist functions $(u_\star^k, v_\star^k) \in (L^\infty(Q_T; [0,1]))^2$ such that, up to a subsequence, (u^k, v^k) strongly converge to $(u_\star^k, v_\star^k)$ in $L^2(Q_T) \times L^1(Q_T)$. Since Q_T is a bounded domain, we easily check that v^k strongly converges to $v_\star^k$ in $L^2(Q_T)$. The weak convergence in $L^2(0,T; H_0^1(\Omega))$ follows from the estimate on ∇u^k (see Proposition 2). $\square$

The previous convergence result allows us to conclude this section with the well-posedness of the PDE/ODE system: existence is obtained by using simple convergence procedure, thanks to Lemma 4, whereas uniqueness of the solution has to be investigated in an independent way, as a consequence of a comparison principle.

Theorem 2. *Let $d_2 = 0$ and $k > 0$. Problem $(\mathcal{P}^k)$ admits a unique weak solution*

$$\big(u^k, v^k\big) \in W_2^{2,1}(Q_T) \times C^{0,1}\big([0,T]; L^\infty(\Omega)\big),$$

which means that it satisfies

$$\iint_{Q_T} u^k \partial_t \psi + \iint_{Q_T} \big\{ d_1 u^k \Delta\psi + \big(f\big(u^k\big) - kF\big(u^k, v^k\big)\big)\psi \big\} = -\int_\Omega u_0 \psi(\cdot, 0) + \int_0^T \int_{\partial\Omega} \bar{u}\partial_n \psi, \tag{9}$$

$$\iint_{Q_T} v^k \partial_t \psi + \iint_{Q_T} \big(g\big(v^k\big) - \alpha k F\big(u^k, v^k\big)\big)\psi = -\int_\Omega v_0 \psi(\cdot, 0) \tag{10}$$

for all $\psi \in \mathcal{F}_T := \{\psi \in C^{2,1}(\overline{Q}_T),\ \psi(\cdot, T) = 0 \text{ on } \Omega \text{ and } \psi = 0 \text{ on } \partial\Omega \times [0,T]\}$. Moreover, one has

$$0 \leqslant u^k, v^k \leqslant 1.$$

Proof. Existence of a solution follows from the convergence result stated in Lemma 4 applied to the formulations (1) and (2). In particular, strong convergence results in $L^2(Q_T)$ allow us to pass to the limit with respect to d_2 in the nonlinear terms. Uniqueness of the solution is a straightforward consequence of the following comparison principle (Lemma 5). $\square$

Lemma 5. *Let $d_2 = 0$ and $k > 0$. Let (u, v) and $(\tilde{u}, \tilde{v})$ be two solutions of $(\mathcal{P}^k)$ with different boundary and initial data. In particular, assume the following*:

(i) *$u, \tilde{u} \in W^{2,1}(Q_T)$ and $v, \tilde{v} \in C([0,T]; L^\infty(\Omega))$,*
(ii) *(u, v) and $(\tilde{u}, \tilde{v})$ satisfy*

---

[2] Let $\mathcal{F}$ be a bounded subset of $L^p(Q_T)$ with $1 \leqslant p < +\infty$. Suppose that

(i) for any $\varepsilon > 0$ and any subset $\omega \Subset Q_T$, there exists a positive constant $\delta < \operatorname{dist}(\omega, \partial Q_T)$ such that

$$\big\| f\big(\cdot + (\xi, 0)\big) - f(\cdot)\big\|_{L^p(\omega)} + \big\| f\big(\cdot + (0, \tau)\big) - f(\cdot)\big\|_{L^p(\omega)} < \varepsilon$$

for all ξ, τ and $f \in \mathcal{F}$ satisfying $|\xi| + |\tau| < \delta$,
(ii) for any $\varepsilon > 0$, there exists $\omega \Subset Q_T$ such that $\|f\|_{L^p(Q_T \setminus \omega)} < \varepsilon$ for all $f \in \mathcal{F}$.

Then $\mathcal{F}$ is precompact in $L^p(Q_T)$.

$$\begin{cases} \partial_t u = d_1 \Delta u + f(u) - kF(u,v), & \text{in } Q_T, \\ \partial_t \hat{u} = d_1 \Delta \hat{u} + f(\hat{u}) - kF(\hat{u},\hat{v}), & \text{in } Q_T, \\ \partial_t v = g(v) - \alpha k F(u,v), & \text{in } Q_T, \\ \partial_t \hat{v} = g(\hat{v}) - \alpha k F(\hat{u},\hat{v}), & \text{in } Q_T, \\ u(\cdot,0) \geqslant \hat{u}(\cdot,0), & \text{on } \Omega, \\ v(\cdot,0) \leqslant \hat{v}(\cdot,0), & \text{on } \Omega, \\ u \geqslant \hat{u}, & \text{on } \partial\Omega \times (0,T]. \end{cases}$$

Then $u \geqslant \tilde{u}$ and $v \leqslant \tilde{v}$ in Q_T.

Proof. We set $U = u - \hat{u}$ and $V = v - \hat{v}$. Then, we subtract the first two equations to obtain

$$0 = \partial_t U - d_1 \Delta U - f(u) + f(\hat{u}) + k\big(F(u,v) - F(\hat{u},v)\big) - k\big(F(\hat{u},\hat{v}) - F(\hat{u},v)\big),$$

which we multiply by $-U^-$ (with $U^- = \max(0,-U)$) and integrate over $\Omega \times (t_0,t)$. This yields

$$0 = \frac{1}{2}\int_\Omega |U^-|^2(\cdot,t) - \frac{1}{2}\int_\Omega |U^-|^2(\cdot,t_0) + d_1\int_{t_0}^t\int_\Omega |\nabla U^-|^2 - \int_{t_0}^t\int_\Omega U^-\big(f(\hat{u}) - f(u)\big)$$
$$+ \int_{t_0}^t\int_\Omega k\big(F(u,v) - F(\hat{u},v)\big)\big(-U^-\big) - \int_{t_0}^t\int_\Omega k\big(F(\hat{u},\hat{v}) - F(\hat{u},v)\big)\big(-U^-\big).$$

We obviously have that

$$\int_{t_0}^t\int_\Omega |\nabla U^-|^2 \geqslant 0.$$

Note that, since F is nondecreasing in u,

$$\int_{t_0}^t\int_\Omega k\big(F(u,v) - F(\hat{u},v)\big)\big(-U^-\big) \geqslant 0.$$

Also note that

$$-\int_{t_0}^t\int_\Omega U^-\big(f(\hat{u}) - f(u)\big) \geqslant -\mathcal{L}_f\int_{t_0}^t\int_\Omega |U^-|^2,$$

where $\mathcal{L}_f$ denotes the Lipschitz constant of f. Moreover, using the monotonicity of F with respect to its second argument as well as its Lipschitz continuity, one has

$$\int_{t_0}^t\int_\Omega k\big(F(\hat{u},\hat{v}) - F(\hat{u},v)\big)U^- \geqslant -\gamma\int_{t_0}^t\int_\Omega kV^+U^-,$$

so that we obtain

$$\frac{1}{2}\int_\Omega |U^-|^2(\cdot,t) - \frac{1}{2}\int_\Omega |U^-|^2(\cdot,t_0) - \int_{t_0}^t\int_\Omega \gamma kV^+U^- - \mathcal{L}_f\int_{t_0}^t\int_\Omega |U^-|^2 \leqslant 0.$$

Letting $t_0 \to 0$ and applying Cauchy–Schwarz inequality, we obtain

$$\frac{1}{2}\int_\Omega |U^-|^2(\cdot,t) \leqslant \int_0^t\int_\Omega \big((\gamma k + \mathcal{L}_f)|U^-|^2 + \gamma k|V^+|^2\big).$$

Using a similar procedure with respect to V, we get

$$\frac{1}{2}\int_\Omega |V^+|^2(\cdot,t) \leqslant \int_0^t\int_\Omega \big(\alpha\gamma k|U^-|^2 + (\alpha\gamma k + \mathcal{L}_g)|V^+|^2\big),$$

where $\mathcal{L}_g$ denotes the Lipschitz constant of g. Adding the two inequalities permits to conclude that there exists a positive constant $K := K(\alpha, \gamma, k, \mathcal{L}_f, \mathcal{L}_g)$ such that

$$\int_\Omega \left(\left|U^-\right|^2(\cdot,t) + \left|V^+\right|^2(\cdot,t)\right) \leqslant K \int_0^t \int_\Omega \left(\left|U^-\right|^2 + \left|V^+\right|^2\right).$$

Finally we deduce from Gronwall's lemma that $U^- = V^+ = 0$. □

Remark 1. The previous results show that Problem $(\mathcal{P}^k)$ for $d_2 = 0$ can be obtained as a limit of Problem $(\mathcal{P}^k)$ for $d_2 > 0$; although the functional frameworks are different (due to the loss of regularity when the diffusion vanishes), the corresponding solutions have similar properties. Some of the results which have been proved for $d_2 > 0$ may be extended to the PDE/ODE system: in particular Lemma 1, Eq. (4) in Proposition 2, and Proposition 3 still hold in the case $d_2 = 0$.

Remark 2. Eqs. (1)–(2) on the one hand and Eqs. (9)–(10) on the other hand show that the weak formulations of both problems are identical, which allows us to treat both cases in the same way. In particular, for convenience, we will denote (u^k, v^k) the unique solution of Problem $(\mathcal{P}^k)$, $d_2 \geqslant 0$.

3. Asymptotic analysis: The fast reaction limit

3.1. Derivation of the fast reaction problem

Now we focus on the behaviour of the unique solution (u^k, v^k) of Problem $(\mathcal{P}^k)$ (for $d_2 \geqslant 0$) in the sense of Theorems 1 and 2. As we noticed before, uniform estimates stated in Section 2 allow us to lead the asymptotic study with respect to k.

Lemma 6 *(Convergence results). Let $d_2 \geqslant 0$. There exists a pair $(u, v) \in (L^\infty(Q_T; [0,1]))^2$ such that, up to a subsequence,*

$$\begin{aligned} &u^k \to u \quad \text{in } L^2(Q_T), \\ &v^k \to v \quad \text{in } L^1(Q_T), \\ &u^k - \bar{u} \rightharpoonup u - \bar{u} \quad \text{in } L^2\big(0,T; H_0^1(\Omega)\big), \end{aligned}$$

as $k \to +\infty$.

Proof. We apply again the Riesz–Fréchet–Kolmogoroff theorem: in particular, Proposition 3 has been proved in the case $d_2 > 0$ but it can be easily extended to the case $d_2 = 0$ since the estimates are uniform with respect to d_2 (see Remark 1). We deduce from Proposition 3 that the sequence (u^k, v^k) is relatively compact in $L^2(Q_T) \times L^1(Q_T)$. Consequently, also in view of the properties of (u^k, v^k), there exist functions $(u, v) \in (L^\infty(Q_T; [0,1]))^2$ such that, up to a subsequence, (u^k, v^k) strongly converges to (u, v) in $L^2(Q_T) \times L^1(Q_T)$. Since Q_T is a bounded domain, we easily check that v^k strongly converges to v in $L^2(Q_T)$. The weak convergence follows from the estimate on ∇u^k (see Eq. (4) in Proposition 2). □

Next we prove that, in the limit $k \to +\infty$, the two biological populations are segregated or, in other words, that their habitats are disjoint.

Lemma 7 *(Disjoint habitats). Let $d_2 \geqslant 0$. One has:*

$$uv = 0, \quad \text{a.e. in } Q_T.$$

Proof. By Lemma 1 (which has been proved in the case $d_2 > 0$ but is easily extended to the case $d_2 = 0$ since the estimates are uniform in d_2) and by Lemma 6, we deduce that $F(u, v) = 0$ from the fact that F is nonnegative on $(0,1) \times (0,1)$. Furthermore, by Assumption 1, either $u = 0$ or $v = 0$, which concludes the proof. □

Lemma 7 shows the segregating effect of fast reaction: for fixed $k > 0$, we have in general a mixture of the two populations in the whole domain, whereas the habitats tend to spatially segregate as k becomes large. At the limit, the competition process concentrates on a free boundary. Now, let us focus on the behaviour of the two species at the boundary of the finite domain:

Proposition 8. *Let $d_2 > 0$ and let γ be the trace on the boundary $\partial\Omega \times (0,T)$; we have that*

$$\gamma\left(u^k - \frac{v^k}{\alpha}\right) \rightharpoonup \bar{u} \quad \text{in } L^2\big(\partial\Omega \times (0,T)\big),$$

as $k \to +\infty$.

Proof. It follows from Proposition 2 and the uniform L^∞-bounds on u^k and v^k that

$$\|u^k\|_{L^2(0,T;H^1(\Omega))} \leqslant c_5 := \left(d_1^{-1}c_1 + T\,\mathrm{meas}(\Omega)\right)^{1/2}, \tag{11}$$

$$\|v^k\|_{L^2(0,T;H^1(\Omega))} \leqslant c_6 := \left(d_2^{-1}c_2 + T\,\mathrm{meas}(\Omega)\right)^{1/2}, \tag{12}$$

where the constants c_5 and c_6 do not depend on k. Therefore,

$$u^k \rightharpoonup u, \qquad v^k \rightharpoonup v \quad \text{in } L^2\big(0,T;H^1(\Omega)\big),$$

as $k \to +\infty$. Thus, by linearity of the trace operator, we have that

$$\gamma\left(u^k - \frac{v^k}{\alpha}\right) \rightharpoonup \bar{u} - \frac{\gamma(v)}{\alpha} \quad \text{in } L^2\big(\partial\Omega \times (0,T)\big).$$

Moreover, since $\nabla(uv) = v\nabla u + u\nabla v \in L^2(Q_T)$, $uv \in L^2(0,T;H^1(\Omega))$ and, more precisely, we have that the trace of uv on $\partial\Omega \times (0,T)$ is well defined;

$$\gamma(uv) = \bar{u}\gamma(v) = 0.$$

Since $\bar{u}$ is a positive function, we conclude that $\gamma(v) = 0$. $\square$

Next we focus on the derivation of the limit problem. To this aim we take $u^k - v^k/\alpha$ as a new unknown function and state the following result:

Proposition 9. *For $d_2 \geqslant 0$, the function pair (u,v) defined in Lemma 6 (i.e. obtained as the fast reaction limit of (u^k,v^k)) satisfies the following weak formulation*

$$-\iint\limits_{Q_T}\left(u - \frac{v}{\alpha}\right)\partial_t\psi - \int\limits_{\Omega}\left(u_0 - \frac{v_0}{\alpha}\right)\psi(\cdot,0) = -\int\limits_0^T\int\limits_{\partial\Omega}\bar{u}\partial_n\psi + \iint\limits_{Q_T}\left\{\left(d_1u - d_2\frac{v}{\alpha}\right)\Delta\psi + \left(f(u) - \frac{g(v)}{\alpha}\right)\right\},$$

for all $\psi \in \mathcal{F}_T := \{\psi \in C^{2,1}(\overline{Q}_T),\ \psi(\cdot,T) = 0 \text{ on } \Omega,\ \psi = 0 \text{ on } \partial\Omega \times [0,T]\}$.

Proof. For $d_2 > 0$ (resp. $d_2 = 0$), divide Eq. (2) (resp. Eq. (10)) by α and subtract this equation from Eq. (1) (resp. Eq. (9)). In both cases, this yields

$$\begin{aligned} &-\iint\limits_{Q_T}\left(u^k - \frac{v^k}{\alpha}\right)\partial_t\psi - \int\limits_{\Omega}\left(u_0 - \frac{v_0}{\alpha}\right)\psi(\cdot,0) \\ &= \iint\limits_{Q_T}\left\{\left(d_1u^k - d_2\frac{v^k}{\alpha}\right)\Delta\psi + \left(f\big(u^k\big) - \frac{g(v^k)}{\alpha}\right)\right\} - \int\limits_0^T\int\limits_{\partial\Omega}\left(d_1\bar{u} - d_2\frac{v^k}{\alpha}\right)\partial_n\psi \end{aligned} \tag{13}$$

for all $\psi \in \mathcal{F}_T := \{\psi \in C^{2,1}(\overline{Q}_T),\ \psi(\cdot,T) = 0 \text{ on } \Omega,\ \psi = 0 \text{ on } \partial\Omega \times [0,T]\}$. Note that in Eq. (13) the boundary term should be read as

$$\int\limits_0^T\int\limits_{\partial\Omega}\left(d_1\bar{u} - d_2\frac{v^k}{\alpha}\right)\partial_n\psi, \quad \text{if } d_2 > 0,$$

$$\int\limits_0^T\int\limits_{\partial\Omega} d_1\bar{u}\partial_n\psi, \quad \text{if } d_2 = 0,$$

since the value of v^k may be undefined on the boundary if $d_2 = 0$. We let $k \to +\infty$ in Eq. (13). In particular, by Proposition 8, in both cases, the boundary term converges to

$$-\int\limits_0^T\int\limits_{\partial\Omega} d_1\bar{u}\partial_n\psi.$$

In view of the strong L^2 convergence result (see Lemma 6), the weak formulation is obtained by considering the corresponding limits in the linear and nonlinear integral terms. $\square$

The convergence result stated in Proposition 9 and the segregation principle lead us to work with the unknown functions:

$$w^k = u^k - \frac{1}{\alpha} v^k, \qquad w = u - \frac{1}{\alpha} v. \tag{14}$$

The key idea is that, because of the segregation property, function w completely characterizes the two unknown functions u and v. Indeed, we deduce from Lemmas 6 and 7 that there exists $w \in L^\infty(Q_T)$ such that the following strong convergence results hold:

$$u^k - \frac{v^k}{\alpha} \to w, \qquad u = w^+, \qquad v = \alpha w^-.$$

This suggests the definition of the following nonlinear diffusion operator and source terms:

Definition 4. We define

$$\mathcal{D}(s) := \begin{cases} d_1 s & \text{if } s \geqslant 0, \\ d_2 s & \text{if } s < 0, \end{cases} \qquad h(s) := \begin{cases} f(s) & \text{if } s \geqslant 0, \\ -\frac{g(-\alpha s)}{\alpha} & \text{if } s < 0. \end{cases}$$

It suggests the definition of the limit problem:

$$(\mathcal{P}^0) \quad \begin{cases} \partial_t w = \Delta \mathcal{D}(w) + h(w), & \text{in } \Omega \times (0, T], \\ \mathcal{D}(w) = \mathcal{D}(\bar{u}), & \text{on } \partial\Omega \times (0, T], \\ w(\cdot, 0) = w_0 := u_0 - v_0/\alpha, & \text{on } \Omega. \end{cases}$$

Remark 3. We remark that, since the function $\mathcal{D}$ is inversible, the boundary condition is a Dirichlet condition: indeed, $\mathcal{D}(\bar{u}) = d_1 \bar{u}$ so that $\mathcal{D}(w) = w^+$ and we get

$$w^+ = u = \bar{u}, \qquad \alpha w^- = v = 0, \quad \text{on } \partial\Omega \times (0, T),$$

so that the segregation principle is also valid on the boundary of Ω.

The way to analyze this problem relies on the following definition:

Definition 5. A function w is a weak solution of Problem $(\mathcal{P}^0)$ if it satisfies

(i) $w \in L^\infty(\Omega \times \mathbb{R}^+)$,
(ii) for all $T > 0$,

$$\iint_{Q_T} \big(w \partial_t \psi + \mathcal{D}(w) \Delta \psi + h(w) \psi \big) = \int_0^T \int_{\partial\Omega} \mathcal{D}(\bar{u}) \partial_n \psi - \int_\Omega w_0 \psi(\cdot, 0),$$

for all $\psi \in \mathcal{F}_T := \{\psi \in C^{2,1}(\overline{Q}_T),\ \psi(\cdot, T) = 0 \text{ on } \Omega,\ \psi = 0 \text{ on } \partial\Omega \times [0, T]\}$.

In the next subsection we will prove that function w defined by Eq. (14) is the unique weak solution of Problem $(\mathcal{P}^0)$; we will see below how Problem $(\mathcal{P}^0)$ can be expressed as a free boundary problem.

3.2. Well-posedness of the limiting free boundary problem $(\mathcal{P}^0)$

Theorem 3 *(Existence of a weak solution). Function w defined by Eq.* (14) *is a weak solution of Problem* $(\mathcal{P}^0)$.

Proof. This result is a straightforward consequence of Definition 5 and Proposition 9. □

Before proving the uniqueness result, we introduce the auxiliary problem:

$$(\mathcal{A}) \quad \begin{cases} \partial_t \psi + \sigma \Delta \psi = \eta, & \text{in } Q_T, \\ \psi = 0, & \text{on } \partial\Omega \times (0, T), \\ \psi(\cdot, T) = 0, & \text{on } \Omega, \end{cases}$$

and show the following preliminary result.

Proposition 10. *Let $T > 0$, $\eta \in C_0^\infty(Q_T)$ be such that $|\eta| \leqslant 1$ and $\sigma \in C^\infty(\overline{Q_T})$ be such that there exists a positive constant $\sigma_\star$ with $\sigma \geqslant \sigma_\star > 0$ in Q_T. Then there exists a unique solution $\psi \in C^{2,1}(\overline{Q}_T)$ of Problem $(\mathcal{A})$. It satisfies*

$$\iint_{Q_T} \sigma(\Delta\psi)^2 \leqslant 4T \iint_{Q_T} |\nabla\eta|^2. \tag{15}$$

$$|\psi| \leqslant T - t \quad \text{in } Q_T, \tag{16}$$

$$\iint_{Q_T} (\Delta\psi)^2 \leqslant \frac{T \operatorname{meas}(\Omega)}{\sigma_\star^2}. \tag{17}$$

Proof. Let us prove the existence and uniqueness result. Since σ is bounded away from zero, Problem $(\mathcal{A})$ is a uniformly parabolic problem in which the time variable is reversed, and since both σ and η are smooth functions, Problem $(\mathcal{A})$ has a unique solution $\psi \in \mathcal{F}_T$, with

$$\mathcal{F}_T := \bigl\{v \in C^{2,1}(\overline{Q_T}),\ v = 0 \text{ on } \partial\Omega \times [0, T],\ v(\cdot, T) = 0 \text{ on } \Omega\bigr\}.$$

In order to prove inequality (15), we multiply the main equation of Problem $(\mathcal{A})$ by $\Delta\psi$ and integrate by parts. This gives for all $t \in (0, T)$

$$\frac{1}{2}\int_\Omega \bigl|\nabla\psi(\cdot,0)\bigr|^2 - \frac{1}{2}\int_\Omega \bigl|\nabla\psi(\cdot,t)\bigr|^2 + \int_0^t\int_\Omega \sigma|\Delta\psi|^2 = -\int_0^t\int_\Omega \nabla\eta\,\nabla\psi, \tag{18}$$

which implies in particular that

$$\frac{1}{2}\int_\Omega \bigl|\nabla\psi(\cdot,0)\bigr|^2 + \int_0^T\int_\Omega \sigma|\Delta\psi|^2 = -\int_0^T\int_\Omega \nabla\eta\nabla\psi,$$

and that

$$\frac{1}{2}\int_{Q_T} |\nabla\psi|^2 \leqslant \frac{T}{2}\int_\Omega \bigl|\nabla\psi(\cdot,0)\bigr|^2 + T\int_{Q_T} \sigma|\Delta\psi|^2 + T\int_{Q_T} |\nabla\eta\,\nabla\psi|.$$

This implies that

$$\frac{1}{2}\int_{Q_T} |\nabla\psi|^2 \leqslant 2T\int_{Q_T} |\nabla\eta\,\nabla\psi|. \tag{19}$$

Next, we use the Cauchy–Schwarz inequality

$$\biggl(\int_{Q_T} |\nabla\eta\,\nabla\psi|\biggr)^2 \leqslant \int_{Q_T} |\nabla\eta|^2 \int_{Q_T} |\nabla\psi|^2,$$

in which we substitute Eq. (19) to obtain

$$\biggl(\int_{Q_T} |\nabla\eta\,\nabla\psi|\biggr)^2 \leqslant 4T\int_{Q_T} |\nabla\eta|^2 \int_{Q_T} |\nabla\eta\,\nabla\psi|.$$

Therefore,

$$\int_{Q_T} |\nabla\eta\,\nabla\psi| \leqslant 4T\int_{Q_T} |\nabla\eta|^2,$$

which together with Eq. (18) (with $t = T$) implies inequality (15). Inequalities (16)–(17) can be proved as in [4]. □

Lemma 11 *(Technical result). Assume that $d_2 \geqslant 0$. Let w_i, $i \in \{1, 2\}$, be two solutions of Problem $(\mathcal{P}^0)$ with initial conditions $w_0^{(i)}$. Then,*

$$\iint_{Q_T} |w_1 - w_2| \leqslant T\int_\Omega \bigl|w_0^{(1)} - w_0^{(2)}\bigr| + \iint_{Q_T} (T - t)\bigl|h(w_1) - h(w_2)\bigr|. \tag{20}$$

Proof. Set $\tilde{w} := w_1 - w_2$, $\tilde{w}_0 := w_0^{(1)} - w_0^{(2)}$, $z := h(w_1) - h(w_2)$ and define for all $(x,t) \in Q_T$

$$q(x,t) := \begin{cases} \frac{\mathcal{D}(w_1(x,t)) - \mathcal{D}(w_2(x,t))}{w_1(x,t) - w_2(x,t)} & \text{if } w_1(x,t) \neq w_2(x,t), \\ \min(d_1, d_2) & \text{otherwise.} \end{cases}$$

Note that

$$\min(d_1, d_2) \leqslant q(x,t) \leqslant \max(d_1, d_2) \quad \text{in } Q_T.$$

It follows from Definition 5 that for all $\psi \in \mathcal{F}_T$,

$$\iint\limits_{Q_T} \left\{ \tilde{w}(\partial_t \psi + q\Delta\psi) + z\psi \right\} = -\int\limits_{\Omega} \tilde{w}_0 \psi(\cdot, 0). \tag{21}$$

Now let $n \in \mathbb{N}$. Using mollifiers one can find a smooth function q_n such that

$$\|q_n - q\|_{L^2(Q_T)} \leqslant \frac{1}{n}, \qquad \min(d_1, d_2) \leqslant q_n(x,t) \leqslant \max(d_1, d_2) \quad \text{in } Q_T.$$

Additionally, we define $\tilde{q}_n = q_n + 1/n$. Then,

$$\min(d_1, d_2) + \frac{1}{n} \leqslant \tilde{q}_n \leqslant \max(d_1, d_2) + \frac{1}{n} \quad \text{in } Q_T,$$

so that

$$\iint\limits_{Q_T} \frac{(\tilde{q}_n - q)^2}{\tilde{q}_n} \leqslant 2\left(\iint\limits_{Q_T} \frac{(\tilde{q}_n - q_n)^2}{\tilde{q}_n} + \frac{(q_n - q)^2}{\tilde{q}_n} \right) \leqslant \frac{2}{n}\big(T \operatorname{meas}(\Omega) + 1\big).$$

Fix $\eta \in C_0^\infty(Q_T)$ with $|\eta| \leqslant 1$ and let ψ_n be the solution of Problem $(\mathcal{A})$ with the same function η and function σ replaced by $\tilde{q}_n$. Setting $\psi = \psi_n$ in (21) gives

$$\iint\limits_{Q_T} \left\{ \tilde{w}(\partial_t \psi_n + q\Delta\psi_n) + z\psi_n \right\} - \int\limits_{\Omega} \tilde{w}_0 \psi_n(\cdot, 0) = 0,$$

and hence, since

$$\partial_t \psi_n + \tilde{q}_n \Delta\psi_n = \eta,$$

we obtain

$$\iint\limits_{Q_T} \tilde{w}\eta = \iint\limits_{Q_T} \left\{ \tilde{w}(\tilde{q}_n - q)\Delta\psi_n - z\psi_n \right\} + \int\limits_{\Omega} \tilde{w}_0 \psi_n(\cdot, 0),$$

and consequently

$$\left| \iint\limits_{Q_T} \tilde{w}\eta \right| \leqslant \iint\limits_{Q_T} |z\psi_n| + \int\limits_{\Omega} \left| \tilde{w}_0 \psi_n(\cdot, 0) \right| + \iint\limits_{Q_T} \left| \tilde{w}(q - q_n)\Delta\psi_n \right|. \tag{22}$$

Next we analyze each term of the right-hand side of inequality (22) to obtain

- by Proposition 10 (see inequality (16)),

$$\iint\limits_{Q_T} |z\psi_n| \leqslant \iint\limits_{Q_T} (T - t)|z|;$$

- by Proposition 10 (see Eq. (16) in $t = 0$),

$$\int\limits_{\Omega} \left| \tilde{w}_0 \psi_n(\cdot, 0) \right| \leqslant T \int\limits_{\Omega} |\tilde{w}_0|;$$

- by the Cauchy–Schwarz inequality and Proposition 10 (see inequalities (15) and (17)),

$$\iint_{Q_T} \left|\tilde{w}(\tilde{q}_n - q)\Delta\psi_n\right| \leqslant \|\tilde{w}\|_{L^\infty(Q_T)} \sqrt{\iint_{Q_T} \frac{(\tilde{q}_n - q)^2}{\tilde{q}_n}} \sqrt{\iint_{Q_T} \tilde{q}_n(\Delta\psi)^2}$$
$$\leqslant 2\sqrt{\frac{8T}{n}\bigl(T\,\mathrm{meas}(\Omega)+1\bigr)\iint_{Q_T} |\nabla\eta|^2}.$$

Now letting $n \to +\infty$ in inequality (22) gives

$$\left|\iint_{Q_T} \tilde{w}\eta\right| \leqslant \iint_{Q_T} (T-t)|z| + T\int_\Omega |\tilde{w}_0|, \tag{23}$$

for each $\eta \in C_0^\infty(Q_T)$ with $|\eta| \leqslant 1$. Next we take as the functions η the elements of a subsequence $\{\eta_k\}_{k\in\mathbb{N}}$ such that $\{\eta_k\}$ converges to $\mathrm{sign}(\tilde{w})$ in $L^1(Q_T)$ as $k \to \infty$. Passing to the limit in (23) yields

$$\iint_{Q_T} |\tilde{w}| \leqslant \iint_{Q_T} (T-t)|z| + T\int_\Omega |\tilde{w}_0|,$$

which completes the proof. □

Theorem 4 *(Uniqueness of the weak solution). Assume that $d_2 \geqslant 0$. There exists at most one weak solution w of Problem $(\mathcal{P}^0)$ and the whole sequence (u^k, v^k) converges to $(u, v) = (w^+, \alpha w^-)$.*

Proof. Suppose that w_1 and w_2 are two weak solutions of Problem $(\mathcal{P}^0)$ with initial data $w_0^{(1)}$ and $w_0^{(2)}$. Since h is locally Lipschitz continuous on $\mathbb{R}$, there exists a constant L such that

$$\bigl|h(w_1) - h(w_2)\bigr| \leqslant L|w_1 - w_2| \quad \text{in } Q_T.$$

Applying (20) with Q_T replaced by $\Omega \times (t_0, t_0+\tau)$ gives

$$\int_{t_0}^{t_0+\tau}\int_\Omega |w_1 - w_2| \leqslant \tau\int_\Omega \bigl|w_1(\cdot,t_0) - w_2(\cdot,t_0)\bigr| + \int_{t_0}^{t_0+\tau}\int_\Omega (t_0+\tau-t)\bigl|h(w_1)-h(w_2)\bigr|$$
$$\leqslant \tau\int_\Omega \bigl|w_1(\cdot,t_0) - w_2(\cdot,t_0)\bigr| + \tau L\int_{t_0}^{t_0+\tau}\int_\Omega (t_0+\tau-t)|w_1 - w_2|,$$

from which it follows that, for all $\tau \leqslant (2L)^{-1}$,

$$\int_{t_0}^{t_0+\tau}\int_\Omega |w_1 - w_2| \leqslant 2\tau\int_\Omega \bigl|w_1(\cdot,t_0) - w_2(\cdot,t_0)\bigr|. \tag{24}$$

Let

$$\tilde{t} := \sup\bigl\{t \in [0,T],\ w_1(\cdot,s) = w_2(\cdot,s) \text{ for } 0 \leqslant s \leqslant t\bigr\}$$

and assume that $\tilde{t} < T$. Let

$$t_0 := \begin{cases} 0 & \text{if } \tilde{t} = 0, \\ \tilde{t} - \varepsilon & \text{if } \tilde{t} > 0 \text{ with } \varepsilon < \min(\tilde{t}, (2L)^{-1}). \end{cases}$$

Then $w_1(\cdot,t_0) = w_2(\cdot,t_0)$ so that by (24),

$$w_1 = w_2 \quad \text{on } \Omega \times (t_0, t_0+\tau)$$

with $\tau \in [0, \min\{(2L)^{-1}, T - t_0\}]$, which contradicts the definition of $\tilde{t}$. Therefore, Problem $(\mathcal{P}^0)$ has at most one weak solution w. To complete the proof, we remark that the functions $u = w^+$ and $v = \alpha w^-$ are uniquely defined as well. □

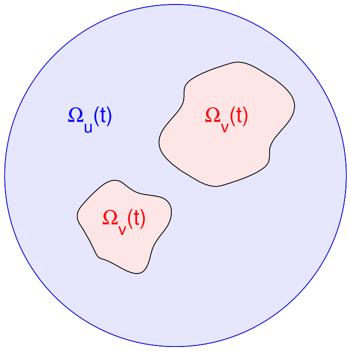

Fig. 1. Geometrical illustration (Ω is a ball) of the segregation principle.

The previous results highlight the fact that the same expression of the limit free boundary problem holds in both cases that $d_2 > 0$ and $d_2 = 0$. In the next section, we present a strong form of the limit free boundary problem, under a simple regularity assumption of the free boundary.

4. Behaviour of the free boundary

4.1. Strong formulation and interface jump conditions

Next we show that under suitable regularity assumptions $(\mathcal{P}^0)$ can be more explicitly written as a free boundary problem, where the free boundary is the level set where $w = 0$. This free boundary formulation unifies those either in the case $d_2 > 0$ or in the case that $d_2 = 0$.

Theorem 5 *(Free boundary problem under the regularity assumption). Let w be the unique solution of Problem $(\mathcal{P}^0)$. Suppose that $T^* > 0$ is such that for all $t \in [0, T^*]$, there exists a closed hypersurface $\Gamma(t)$, and two subdomains $\Omega_u(t)$, $\Omega_v(t)$ such that (see Fig. 1)*

$$\overline{\Omega} = \overline{\Omega_u(t)} \cup \overline{\Omega_v(t)}, \qquad \Gamma(t) = \overline{\Omega_u(t)} \cap \overline{\Omega_v(t)},$$

and

$$\begin{aligned} &w(\cdot,t) > 0, \quad \text{on } \Omega_u(t),\\ &w(\cdot,t) < 0, \quad \text{on } \Omega_v(t). \end{aligned}$$

Assume furthermore that $t \mapsto \Gamma(t)$ is smooth enough and that $(u, v) := (w^+, \alpha w^-)$ is smooth up to $\Gamma(t)$, then u and v satisfy

$$(\mathcal{P}^0)\quad \begin{cases} \partial_t u = d_1 \Delta u + f(u), & \text{in } Q_u := \displaystyle\bigcup_{t\in[0,T^*]} \{\Omega_u(t) \times \{t\}\},\\ \partial_t v = d_2 \Delta v + g(v), & \text{in } Q_v := \displaystyle\bigcup_{t\in[0,T^*]} \{\Omega_v(t) \times \{t\}\},\\ [u] = d_2[v] = 0, & \text{on } \Gamma := \displaystyle\bigcup_{t\in[0,T^*]} \{\Gamma(t) \times \{t\}\},\\ [v]V_n = \alpha\left[d_1 \partial_n u - \dfrac{d_2}{\alpha}\partial_n v\right], & \text{on } \Gamma := \displaystyle\bigcup_{t\in[0,T^*]} \{\Gamma(t) \times \{t\}\},\\ u = \bar{u}, & \text{on } \partial\Omega \times [0, T^*],\\ u(\cdot,0) = \left[u_0 - \dfrac{v_0}{\alpha}\right]^+, & \text{in } \Omega,\\ v(\cdot,0) = \alpha\left[u_0 - \dfrac{v_0}{\alpha}\right]^-, & \text{in } \Omega, \end{cases}$$

where $[\cdot]$ denotes the jump across $\Gamma(t)$ from $\Omega_u(t)$ to $\Omega_v(t)$, n denotes the outward normal unit vector from $\Omega_u(t)$ to $\Omega_v(t)$ (see Fig. 2) and V_n denotes the normal speed of propagation of the free boundary. We use here the convention that all the terms containing d_2 as a factor vanish in the case that $d_2 = 0$.

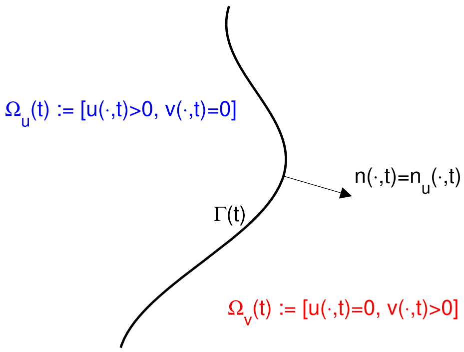

Fig. 2. Free boundary assumption: n denotes the outward normal unit vector from $\Omega_u(t)$ to $\Omega_v(t)$ (i.e. $n(\cdot,t) = n_u(\cdot,t)$).

Before proving Theorem 5, let us make some comments on the interface jump conditions at the free boundary:

Remark 4. We analyze the behaviour of (u,v) at the boundaries: for this, we will denote by Γ_u the points of Γ when they are reached as limits of points of Q_u and by Γ_v the points of Γ when they are reached as limits of points of Q_v, so that in particular $[A] = A_{|\Gamma_v} - A_{|\Gamma_u}$, where A is an arbitrary function. This can be rewritten as

$$\big[A(\cdot,t)\big] = \lim_{\varrho\to 0^+} A\big(\cdot + \varrho n_u(t), t\big) - \lim_{\varrho\to 0^-} A\big(\cdot + \varrho n_u(t), t\big) \quad \text{on } \Gamma(t).$$

■ First we analyze the jump condition. By the segregation principle, we have

$$[u] = \underbrace{u_{|\Gamma_v}}_{=0} - \underbrace{u_{|\Gamma_u}}_{\geqslant 0}, \qquad d_2[v] = \underbrace{d_2 v_{|\Gamma_v}}_{\geqslant 0} - \underbrace{d_2 v_{|\Gamma_u}}_{=0}$$

so that the jump condition reduces to

$$d_1 u_{|\Gamma_u} = d_2 v_{|\Gamma_v} = 0.$$

In particular, we have the following properties:
– if $d_2 \geqslant 0$, the function $u(\cdot,t)$ is continuous on Ω, i.e.

$$u_{|\Gamma_u} = u_{|\Gamma_v} = 0 \quad \text{on } \Gamma(t);$$

– if $d_2 > 0$, the function $v(\cdot,t)$ is continuous on Ω, i.e.

$$v_{|\Gamma_u} = v_{|\Gamma_v} = 0 \quad \text{on } \Gamma(t);$$

– if $d_2 = 0$, $v(\cdot,t)$ jumps across $\Gamma(t)$:

$$v_{|\Gamma_v} \neq v_{|\Gamma_u} = 0 \quad \text{on } \Gamma(t).$$

The loss of regularity is not surprising since the diffusion process has vanished. This is somehow similar to the loss of boundary conditions (in a classical sense) when passing from a parabolic problem to a hyperbolic problem by the vanishing viscosity method (see e.g. [2]).

■ Next we consider the Rankine–Hugoniot condition:

$$[v]V_n = \alpha\left[d_1\partial_n u - \frac{d_2}{\alpha}\partial_n v\right].$$

- If $d_2 > 0$, then the Rankine–Hugoniot reduces a jump condition on the normal derivatives

$$\left[d_1\partial_n u - \frac{d_2}{\alpha}\partial_n v\right] = 0.$$

- If $d_2 = 0$, then the speed of propagation of the free boundary is given by

$$[v]V_n = \alpha[d_1\partial_n u] \geqslant 0.$$

Proof. We recall that (u, v) satisfies:

$$-\iint_{Q_T}\left(u-\frac{v}{\alpha}\right)\partial_t\psi-\int_{\Omega}\left(u_0-\frac{v_0}{\alpha}\right)\psi(\cdot,0)$$

$$=-\int_0^T\int_{\partial\Omega} d_1\bar{u}\partial_n\psi+\iint_{Q_T}\left\{\left(d_1u-d_2\frac{v}{\alpha}\right)\Delta\psi+\left(f(u)-\frac{g(v)}{\alpha}\right)\right\}$$

for all $\psi\in\mathcal{F}_T:=\{\psi\in C^{2,1}(\overline{Q}_T),\ \psi(\cdot,T)=0 \text{ on } \Omega \text{ and } \psi=0 \text{ on } \partial\Omega\times[0,T]\}$. Next we consider the time derivative term and the diffusion term, namely

$$(\star)=-\iint_{Q_T}\left(u-\frac{v}{\alpha}\right)\partial_t\psi,$$

$$(\star\star)=\iint_{Q_T}\left(d_1u-d_2\frac{v}{\alpha}\right)\Delta\psi.$$

• *Analysis of the time derivative term.* Since the space domains depend on time, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_u(t)} u\psi=\int_{\Omega_u(t)}(\partial_t u\psi+u\partial_t\psi)+\int_{\Gamma(t)} u\psi V_n,$$

where V_n denotes the speed of propagation of the boundary $t\mapsto\Gamma(t)$. We apply the following convention: when $\Omega_u(t)$ increases, then V_n is nonnegative. Moreover, the term u in the boundary integral term should be understood in the following sense:

$$u:=\lim_{\varrho\to 0^-} u\big(\cdot+\varrho n(t),t\big)\quad\text{on } \Gamma(t).$$

In the same way, taking into account the property $n_v=-n_u=-n$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_v(t)} v\psi=\int_{\Omega_v(t)}(\partial_t v\psi+v\partial_t\psi)-\int_{\Gamma(t)} v\psi V_n$$

where the expression v in the boundary integral term should be understood in the following sense:

$$v:=\lim_{\varrho\to 0^+} u\big(\cdot+\varrho n(t),t\big)\quad\text{on } \Gamma(t).$$

Now, since the jump $[\cdot]$ is defined as

$$\big[w(\cdot,t)\big]=\lim_{\varrho\to 0^+} w\big(\cdot+\varrho n(t),t\big)-\lim_{\varrho\to 0^-} w\big(\cdot+\varrho n(t),t\big)\quad\text{on } \Gamma(t),$$

integrating in time gives

$$(\star)=-\iint_{Q_u} u\partial_t\psi+\frac{1}{\alpha}\iint_{Q_v} v\partial_t\psi$$

$$=\iint_{Q_u}\partial_t u\psi-\frac{1}{\alpha}\iint_{Q_v}\partial_t v\psi+\int_0^T\int_{\Gamma(t)}\left[-u+\frac{v}{\alpha}\right]\psi V_n.$$

• *Analysis of the diffusion term.* After two integrations by parts, we get

$$\iint_{Q_u} u\Delta\psi=\iint_{Q_u}\Delta u\psi-\int_0^T\int_{\Gamma(t)}\partial_n u\psi+\int_0^T\int_{\partial\Omega_u(t)} u\partial_n\psi,$$

$$\iint_{Q_v} v\Delta\psi=\iint_{Q_v}\Delta v\psi+\int_0^T\int_{\Gamma(t)}\partial_n v\psi-\int_0^T\int_{\Omega_v(t)} v\partial_n\psi,$$

where we have taken into account the property that $n_v = -n_u = -n$. Again, the values of u and v in the boundary terms have to be considered in the sense that has been explained before. Moreover, note that, due to the regularity assumption, one has

$$\Omega_u(t) = \partial\Omega \cup \Gamma(t), \qquad \Omega_v(t) = \Gamma(t).$$

Thus, integrating in time gives

$$\begin{aligned}(\star\star) &= \iint\limits_{Q_u} d_1 u \Delta\psi - \iint\limits_{Q_v} \frac{d_2}{\alpha} v \Delta\psi \\ &= \iint\limits_{Q_u} d_1 \Delta u \psi - \iint\limits_{Q_v} \frac{d_2}{\alpha} \Delta v \psi - \int_0^T \int\limits_{\partial\Omega} d_1 u \partial_n \psi + \int_0^T \int\limits_{\Gamma(t)} \left[-d_1 \partial_n u + \frac{d_2}{\alpha} \partial_n v\right]\psi + \int_0^T \int\limits_{\Gamma(t)} \left[-d_1 u + \frac{d_2}{\alpha} v\right]\partial_n \psi.\end{aligned}$$

• *Conclusion of the proof.* The computations yield

$$\begin{aligned}0 &= \iint\limits_{Q_u} \big(\partial_t u - d_1 \Delta u - f(u)\big)\psi - \frac{1}{\alpha} \iint\limits_{Q_v} \big(\partial_t v - d_2 \Delta v - g(v)\big)\psi \\ &\quad - \int_0^T \int\limits_{\partial\Omega} d_1 (u - \bar{u}) \partial_n \psi - \iint\limits_{\Gamma} \left[-d_1 u + \frac{d_2}{\alpha} v\right]\partial_n \psi + \iint\limits_{\Gamma} \left(\left[-u + \frac{v}{\alpha}\right] V_n + \left[-d_1 \partial_n u + \frac{d_2}{\alpha} \partial_n v\right]\right)\psi,\end{aligned}$$

for all $\psi \in \widetilde{\mathcal{F}}_T := \{\phi \in \mathcal{F}_T,\ \phi(\cdot,0) = 0 \text{ on } \Omega\}$. Now, by using suitable test-functions with suitable supports, namely $\psi \in C_0^\infty(Q_u)$ and $\psi \in C_0^\infty(Q_v)$, we obtain

$$\begin{aligned}&\partial_t u = d_1 \Delta u + f(u), \quad \text{in } Q_u, \quad \text{and} \\ &\partial_t v = d_2 \Delta v + g(v), \quad \text{in } Q_v.\end{aligned}$$

Besides, one has

$$\left[-d_1 u + \frac{d_2}{\alpha} v\right] = 0, \quad \text{on } \Gamma,$$

which follows from either the continuity in space of u and v if $d_2 > 0$, or the continuity in space of u combined with the fact that the term $d_2 v$ does not exist if $d_2 = 0$. By the segregation principle and nonnegativity of u and v, we get

$$[d_1 u] = \left[\frac{d_2}{\alpha} v\right] = 0, \quad \text{i.e.} \quad [u] = d_2[v] = 0.$$

The remaining term in the initial integral equality allows us to conclude that

$$\iint\limits_{\Gamma} \left(\left[-u + \frac{v}{\alpha}\right] V_n + \left[-d_1 \partial_n u + \frac{d_2}{\alpha} \partial_n v\right]\right)\psi = 0, \quad \forall \psi \in \widetilde{\mathcal{F}}_T,$$

which gives

$$\left[-u + \frac{v}{\alpha}\right] V_n + \left[-d_1 \partial_n u + \frac{d_2}{\alpha} \partial_n v\right] = 0, \quad \text{on } \Gamma.$$

Because of the jump condition, this equality reduces to

$$[v] V_n = \alpha \left[d_1 \partial_n u - \frac{d_2}{\alpha} \partial_n v\right] = 0, \quad \text{on } \Gamma.$$

The initial condition is obtained by restarting all the previous computations with a slightly modified space of test-functions: indeed, considering test-functions in $\mathcal{F}_T$, we obtain

$$\begin{aligned}0 &= \iint\limits_{Q_u} \big(\partial_t u - d_1 \Delta u - f(u)\big)\psi - \frac{1}{\alpha} \iint\limits_{Q_v} \big(\partial_t v - d_2 \Delta v - g(v)\big)\psi \\ &\quad - \int\limits_{\Omega} \left(u(\cdot,0) - \frac{v(\cdot,0)}{\alpha}\right)\psi(\cdot,0) + \int\limits_{\Omega} \left(u_0 - \frac{v_0}{\alpha}\right)\psi(\cdot,0)\end{aligned}$$

$$- \int_0^T \int_{\partial\Omega} d_1(u - \bar{u})\partial_n\psi - \iint_\Gamma \left[-d_1 u + \frac{d_2}{\alpha} v \right] \partial_n\psi$$
$$+ \iint_\Gamma \left(\left[-u + \frac{v}{\alpha} \right] V_n + \left[-d_1 \partial_n u + \frac{d_2}{\alpha} \partial_n v \right] \right) \psi,$$

for all $\psi \in \mathcal{F}_T$. Thanks to the previous computations, we obtain in a straightforward way

$$\int_\Omega \left\{ \left(u(\cdot, 0) - \frac{v(\cdot, 0)}{\alpha} \right) - \left(u_0 - \frac{v_0}{\alpha} \right) \right\} \psi(\cdot, 0) = 0 \quad \text{for all } \psi \in \mathcal{F}_T.$$

As a consequence,

$$u(\cdot, 0) - \frac{v(\cdot, 0)}{\alpha} = u_0 - \frac{v_0}{\alpha},$$

which implies that

$$u(\cdot, 0) = \left[u_0 - \frac{v_0}{\alpha} \right]^+, \qquad v(\cdot, 0) = \alpha \left[u_0 - \frac{v_0}{\alpha} \right]^-. \qquad \square$$

4.2. Concentration effect of the interspecific reaction term

In the previous subsection we have described the behaviour of the species at the free boundary. On each side of this moving free boundary, intraspecific reaction–diffusion only involves one species, whereas the interspecific reaction terms concentrate on the free boundary, where also the Rankine–Hugoniot type condition is satisfied. In order to describe this concentration effect, we focus on the singular limit as k tends to infinity of the interspecific reaction term. Previous estimates (see Lemma 1) ensure that, up to a subsequence,

$$kF\big(u^k, v^k\big) \rightharpoonup \mu \quad \text{in the sense of measures.}$$

It remains to identify μ.

Theorem 6 *(Singular limit of the interspecific reaction term). Under the assumptions of Theorem* 5*, there exists a measure μ such that*

$$kF\big(u^k, v^k\big) \rightharpoonup \mu, \quad \text{in the sense of measures as } k \to \infty.$$

The measure μ is localized on Γ and is given by

$$\mu(x, t) = \frac{1}{1+\alpha} \big([d_1 \partial_n u + d_2 \partial_n v] + [v] V_n \big) \delta\big(x - \xi(t)\big),$$

which we rewrite as

$$\mu(x, t) = \begin{cases} \frac{1}{1+\alpha} [(d_1 \partial_n u + d_2 \partial_n v)(\xi(t), t)] \delta(x - \xi(t)), & \text{if } d_2 > 0, \\ \frac{1}{\alpha} [v(\xi(t), t)] V_n \delta(x - \xi(t)), & \text{if } d_2 = 0, \end{cases}$$

where $(x, t) \in Q_T^* = \Omega \times (0, T^*)$ and the function $t \mapsto \xi(t)$ is a parametrization of the free boundary Γ.

Proof. Defining $\mu^k = kF(u^k, v^k)$ and taking $\psi \in C_0^\infty(Q_T)$, we have

$$\iint_{Q_T} \mu^k \psi = \iint_{Q_T} \big(u^k \partial_t \psi + d_1 u^k \Delta\psi + f\big(u^k\big)\psi \big)$$
$$= \frac{1}{\alpha} \iint_{Q_T} \big(v^k \partial_t \psi + d_2 v^k \Delta\psi + g\big(v^k\big)\psi \big).$$

Therefore, letting $k \to +\infty$ gives

$$\iint_{Q_T} \mu\psi = \iint_{Q_T} \big(u \partial_t \psi + d_1 u \Delta\psi + f(u)\psi \big)$$
$$= \frac{1}{\alpha} \iint_{Q_T} \big(v \partial_t \psi + d_2 v \Delta\psi + g(v)\psi \big)$$

which we integrate by parts to obtain

$$\iint_{Q_T} \mu\psi = \int_0^T \int_{\Omega_u(t)} \underbrace{\left(-\partial_t u + d_1 \Delta u + f(u)\right)}_{=0} \psi + \int_0^T \int_{\Gamma(t)} \underbrace{[u]}_{=0} (V_n \psi - d_1 \partial_n \psi) + [d_1 \partial_n u]\psi,$$

$$\alpha \iint_{Q_T} \mu\psi = \int_0^T \int_{\Omega_v(t)} \underbrace{\left(-\partial_t v + d_2 \Delta v + g(v)\right)}_{=0} \psi + \int_0^T \int_{\Gamma(t)} [v] V_n \psi - \underbrace{d_2 [v]}_{=0} \partial_n \psi + [d_2 \partial_n v]\psi.$$

This yields

$$\iint_{Q_T} \mu\psi = \frac{1}{1+\alpha} \int_0^T \int_{\Gamma(t)} \left([d_1 \partial_n u + d_2 \partial_n v] + [v] V_n\right)\psi,$$

which concludes the proof. □

This result highlights the particular behaviour of the two species in the following sense: the fast reaction limit enforces the segregation of the two populations so that the interspecific competition effects focus on the free boundary. Thus, the interspecific reaction is governed by this localized measure whereas each subdomain rules the behaviour of each intraspecific (diffusion–)reaction process.

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