



## Analytic fragmentation semigroups and continuous coagulation–fragmentation equations with unbounded rates

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### ABSTRACT

In this paper we show that continuous fragmentation operators are sectorial for a large range of physically relevant fragmentation rates and use this fact to prove classical solvability of the combined coagulation–fragmentation equation with unbounded coagulation kernels.

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### 1. Introduction

Coagulation and fragmentation models are abundant in the natural sciences and engineering, where they describe processes ranging from animals' groupings, the evolution of phytoplankton aggregates, blood agglutination, through planetesimals' formation and rock crushing, to polymerization and de-polymerization. One of the most efficient approaches to modelling the kinetics of such processes is through a rate equation which describes the evolution of the distribution of interacting clusters with respect to their size/mass. The first equation of this kind was derived by Smoluchowski [29] to describe pure coagulation in the discrete case, that is, if the ratio of the mass of the basic building block (monomer) to the mass of a typical cluster is positive and thus the size of a cluster is a finite multiple of the mass of the monomer. This equation was extended by Müller [27], to the continuous case, where it was assumed that the clusters can have arbitrary mass and hence that the mass of a single monomer is negligible. Since, typically, the clusters not only coalesce but also fragment into smaller clusters, the whole process must be described by a combined coagulation–fragmentation equation. With Müller's coagulation term, and with fragmentation modelled by terms introduced in [23] but written in the form proposed in [22], the full equation reads

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$$\begin{aligned} \partial_t u(x, t) = & -a(x)u(x, t) + \int_x^\infty a(y)b(x | y)u(y, t) dy - u(x, t) \int_0^\infty k(x, y)u(y, t) dy \\ & + \frac{1}{2} \int_0^x k(x - y, y)u(x - y, t)u(y, t) dy, \end{aligned} \tag{1}$$

where  $x \in \mathbb{R}_+ := (0, \infty)$  denotes the mass or size of a particle/cluster. Here  $u$  is the density of particles of mass/size  $x$ ,  $a$  is the fragmentation rate and  $b$  describes the distribution of masses  $x$  of particles spawned by fragmentation of a particle of mass  $y$ . Further,  $b \geq 0$  is assumed to be a measurable function of two variables satisfying  $b(x | y) = 0$  for  $x > y$ . The local law of mass conservation requires

$$\int_0^y xb(x | y) dx = y, \quad y \in \mathbb{R}_+, \tag{2}$$

and the expected number of particles resulting from a fragmentation of a size  $y$  parent,

$$n_0(y) := \int_0^y b(x | y) dx,$$

is assumed to satisfy

$$n_0(y) < +\infty \tag{3}$$

for any fixed  $y \in \mathbb{R}_+$ . Note also that

$$n_0 \geq 1, \tag{4}$$

since otherwise we would have

$$1 > \frac{y}{y} \int_0^y b(x | y) dx \geq \frac{1}{y} \int_0^y xb(x | y) dx,$$

contradicting (2).

In general, the fragmentation rate  $a$  is assumed to be a measurable nonnegative function. For the purpose of this paper, we have to impose some control on the growth of the fragmentation coefficients. Namely, we assume that there are  $j \in (0, \infty)$ ,  $l \in [0, \infty)$  and  $a_0, b_0 \in \mathbb{R}_+$  such that, for any  $x \in \mathbb{R}_+$ ,

$$a(x) \leq a_0(1 + x^j), \quad n_0(x) \leq b_0(1 + x^l). \tag{5}$$

We note that the reason for assuming  $j > 0$  is that for  $j = 0$  the fragmentation operator becomes bounded and the linear part of the theory becomes trivial.

The coagulation kernel  $k(x, y)$  represents the likelihood of a particle of size  $x$  attaching itself to a particle of size  $y$ . We assume that it is a measurable symmetric function such that for some  $K > 0$  and  $0 \leq \beta \leq \alpha < 1$

$$0 \leq k(x, y) \leq K((1 + a(x))^\alpha (1 + a(y))^\beta + (1 + a(x))^\beta (1 + a(y))^\alpha) \tag{6}$$

as  $x, y \rightarrow \infty$ . This will suffice to show local in time solvability of (1) whereas to show that the solutions are global in time we need to strengthen (6) to

$$0 \leq k(x, y) \leq K((1 + a(x))^\alpha + (1 + a(y))^\alpha) \tag{7}$$

for large  $x, y$  and some  $0 \leq \alpha < 1$ .

In fragmentation and coagulation problems, two spaces are most often used due to their physical relevance. In the space  $L_1(\mathbb{R}_+, x dx)$  the norm of a nonnegative element  $u$ , given by  $\int_0^\infty u(x)x dx$ , represents the total mass of the system, whereas the norm of a nonnegative element  $u$  in the space  $L_1(\mathbb{R}_+, dx)$ ,  $\int_0^\infty u(x) dx$ , gives the total number of particles in the system.

It is well known that the fragmentation equation, with a fragmentation rate  $a$  which is unbounded as  $x \rightarrow \infty$ , has good properties in  $L_1(\mathbb{R}_+, x dx)$  but is ill posed in  $L_1(\mathbb{R}_+, dx)$ , see [5]. On the other hand, the coagulation operator behaves well in  $L_1(\mathbb{R}_+, dx)$  and in  $L_1(\mathbb{R}_+, (1 + x) dx)$  but not in  $L_1(\mathbb{R}_+, x dx)$  alone.

Our approach is to follow [8] and use the scale of spaces with finite higher moments

$$X_m = L_1(\mathbb{R}_+, dx) \cap L_1(\mathbb{R}_+, x^m dx) = L_1(\mathbb{R}_+, (1 + x^m) dx), \tag{8}$$

where  $m \in \mathbb{M} := [1, \infty)$ . We extend this definition to  $X_0 = L_1(\mathbb{R}_+)$ . The natural norm in  $X_m$  is denoted by  $\|\cdot\|_m$ , and, to shorten notation, we define  $w_m(x) := 1 + x^m$ . We note that the continuous injection  $X_m \hookrightarrow X_1$ ,  $m > 1$ , means that any

solution in  $X_m$  is also a solution in the basic space  $X_1$ . Also, due to the nature of the problem, most of our analysis is carried out in  $X_{m,+}$  where, for any partially ordered space  $Z$ ,  $Z_+$  denotes the positive cone of  $Z$ .

There are two main strategies of approaching continuous coagulation-fragmentation problems (1). The first, introduced in [30] and later refined in e.g. [10,20] and recently used in [14,16], consists of considering a family of truncated problems, establishing weak compactness of their solutions and passing to the limit, establishing in this way existence of weak solutions to (1). Uniqueness, however, requires additional assumptions and other techniques. This approach has proved itself very effective in dealing with pure coagulation problems. However, for the full coagulation-fragmentation equation, the fragmentation part is required to be in some way or another subordinated to the coagulation kernel (see the discussion in [7]). This has meant that the truncation/compactness method has yielded so far results for a very restricted class of fragmentation rates, see e.g. [15] where in fact the fragmentation is required to be binary with linear growth at  $x = 0$  and  $x \rightarrow \infty$ . The second strategy, introduced in [1] and further developed in [6–8,24,25], treats (1) as a Lipschitz perturbation of the linear fragmentation problem. Application of substochastic semigroup theory [4], then enables a wide range of unbounded fragmentation kernels to be included at a cost, however, of making the coagulation process subordinate to fragmentation. This approach, apart from being able to include unbounded fragmentation rates, gave classical differentiable solutions to (1) but, so far, only for bounded coagulation kernels. Recently, [9], the discrete fragmentation semigroup has been proved to be analytic in spaces of sequences having sufficiently high moments for a large class of fragmentation processes including, in particular, physically relevant kernels  $b$  that are binary and homogeneous or decreasing with respect to the size of the daughter particles. This allowed local existence and uniqueness of classical solutions to be proved for a certain class of unbounded coagulation kernels.

Our aim in the present paper is to extend the results of [9] to continuous coagulation-fragmentation equations. In particular, we provide a simpler and more general proof of analyticity of a class of fragmentation semigroups in the scale of spaces  $X_m$  and show local existence and uniqueness of classical solutions in this scale for coagulation kernels satisfying (6). Furthermore, we show global solvability if  $k$  satisfies (7) and, in addition,  $\alpha_j \leq 1$ , see (5).

To illustrate what this means, we note that our approach establishes the global existence of classical solutions for all coagulation kernels mentioned in [14] such as the shear kernel, [2,28],

$$k(x, y) = k_0(x^{1/3} + y^{1/3})^{7/3},$$

or the modified Smoluchowski kernel, [19]

$$k(x, y) = k_0 \frac{(x^{1/3} + y^{1/3})^2}{x^{1/3}y^{1/3} + c}, \quad c > 0,$$

as long as  $a(x) \geq a_1 x^\delta$ ,  $a_1$  a constant, with  $\delta > 7/9$  in the first case and  $\delta > 2/3$  in the second case. Note that in the third example in [14],

$$k(x, y) = k_0 \frac{(x^{1/3} + y^{1/3})^q}{1 + \frac{(x^{1/3} + y^{1/3})^3}{8y^c}}, \quad 0 \leq q < 3,$$

is bounded at infinity and thus yields to the earlier classical solvability results, such as [7]. At the same time, we are able to handle a wide range of physically relevant daughter particle distribution functions  $b$ , including the homogeneous one, while  $b(x|y) = (\alpha + 2)x^\alpha/y^{\alpha+1}$ , given as an example of an application in [14], has a physical meaning only when  $\alpha = 0$ , that is, in the binary case, see [4, Subsection 8.2.1].

We would like to emphasize that local classical solvability does not place any restriction on the rate of growth of  $k$  as long as it is controlled by a small power of the fragmentation rate  $a$ . This shows that fast fragmentation of large clusters plays a stabilizing role in the process, which agrees with physical intuition and earlier weak solvability results such as in [13]. However, our assumptions (6) and (7) seem to display this relation in a more direct way than in [13].

The structure of the paper, and some of the techniques, are similar to the recent paper [9] in which the discrete coagulation and fragmentation equation was analysed and thus we have omitted some details. The main difference in the current paper is that to prove analyticity of the fragmentation semigroup, instead of the direct calculations in [9], we use the Arendt–Rhandi theorem:

**Theorem 1.1.** (See [3, Theorem 1.1].) *Assume that  $X$  is a Banach lattice,  $(A, D(A))$  is a resolvent positive operator which generates an analytic semigroup and  $(B, D(A))$  is a positive operator. If  $(\lambda_0 I - (A + B), D(A))$  has a nonnegative inverse for some  $\lambda_0$  larger than the spectral bound  $s(A)$  of  $A$ , then  $(A + B, D(A))$  generates a positive analytic semigroup.*

Thanks to this, we can avoid certain resolvent estimates which do not appear to be available in the continuous case. Also, we have been able to relax some assumptions on  $k$  and this allows a more general local solvability result to be obtained. On the other hand, the continuous case places an additional restriction on the order  $m$  of the space  $X_m$  in which the analyticity is available. This follows from the fact that in the continuous case we have to control the zeroth moment which is redundant in the discrete case. Furthermore, dealing with the relevant moment inequalities is technically more involved.

## 2. Formulation of main results

The main results of the paper are:

1. analyticity of the continuous fragmentation semigroups for a wide range of cases including, in particular, power law and homogeneous fragmentation in  $X_m$  for sufficiently large  $m$ ,
2. application of the analyticity of the fragmentation semigroup to show classical solvability of the full coagulation-fragmentation equation for a class of unbounded coagulation kernels in  $X_m$ .

To formulate these results, we have to introduce specific assumptions and notation. First we define

$$n_m(y) := \int_0^y b(x | y)x^m dx$$

for any  $m \in \mathbb{M}_0 := \{0\} \cup \mathbb{M}$  and  $y \in \mathbb{R}_+$ . Further, let

$$N_0(y) := n_0(y) - 1 \quad \text{and} \quad N_m(y) := y^m - n_m(y), \quad m \geq 1.$$

It follows from (3) and (4) that

$$n_m(y) \leq y^m \int_0^y b(x | y) dx = y^m n_0(y) < +\infty \quad \forall m \in \mathbb{M}_0$$

and  $N_0(y) = n_0(y) - 1 \geq 0$ . Moreover, by (2),

$$N_m(y) = y^m - \int_0^y b(x | y)x^m dx \geq y^m - y^{m-1} \int_0^y b(x | y)x dx = 0 \tag{9}$$

for  $m \geq 1$  and hence

$$N_m \geq 0, \quad m \in \mathbb{M}_0, \tag{10}$$

with  $N_1 = 0$ .

Next, for any  $m \in \mathbb{M}$ , let  $(A_m u)(x) := a(x)u(x)$  on

$$D(A_m) = \{u \in X_m : au \in X_m\}$$

and let  $B_m$  be the restriction to  $D(A_m)$  of the integral expression

$$[Bu](x) = \int_x^\infty a(y)b(x | y)u(y) dy.$$

**Theorem 2.1.** *Let  $a, b$  satisfy (2), (3) and (5), and let  $m$  be such that  $m \geq j + l$  if  $j + l > 1$  and  $m > 1$  if  $j + l \leq 1$ .*

- (a) *The closure  $(F_m, D(F_m)) = \overline{(-A_m + B_m, D(A_m))}$  generates a positive quasi-contractive semigroup, say  $(S_{F_m}(t))_{t \geq 0}$ , of type at most  $4a_0b_0$  on  $X_m$ . Furthermore, if  $u \in D(F_m)_+$ , then*

$$N_m(x)a(x)u(x) \in X_0, \quad m \in \mathbb{M}_0. \tag{11}$$

- (b) *If, moreover, for some  $m$  there is  $c_m > 0$  such that*

$$\liminf_{x \rightarrow \infty} \frac{N_m(x)}{x^m} = c_m, \tag{12}$$

*then  $F_m = -A_m + B_m$  and  $(S_{F_m}(t))_{t \geq 0}$  is an analytic semigroup on  $X_m$ .*

- (c) *If (12) holds for some  $m_0$ , then it holds for all  $m \geq m_0$ .*

We note that (12) cannot hold for  $m = 1$  as  $N_1 = 0$ .

**Example 1.** One of the forms of  $b(x | y)$  most often used in applications is

$$b(x | y) = \frac{1}{y} h\left(\frac{x}{y}\right) \tag{13}$$

which is referred to as the homogeneous fragmentation kernel, see e.g. [11]. In this case the distribution of the daughter particles does not depend directly on their relative sizes but on their ratio. In this case

$$n_m(y) = \frac{1}{y} \int_0^y h\left(\frac{x}{y}\right) x^m dx = y^m \int_0^1 h(z) z^m dz =: h_m y^m.$$

Since

$$y = n_1(y) = \frac{1}{y} \int_0^y h\left(\frac{x}{y}\right) x dx = y \int_0^1 h(z) z dz = h_1 y$$

we have  $h_1 = 1$  so that  $h_m < 1$  for any  $m > 1$  and  $N_m(y) = y^m(1 - h_m)$ . Hence, (12) holds.

On the other hand, fragmentation processes in which daughter particles tend to accumulate close both to 0 and to the parent's size may not satisfy (12). Examples of such distribution functions  $b$  are given in [9] (discrete case) and [8] (continuous case).

Next, we introduce a nonlinear operator  $C_m$  in  $X_m$  defined for  $u$  from a suitable subset of  $X_m$  by the formula

$$(C_m u)(x) := -u(x) \int_0^\infty k(x, y) u(y) dy + \frac{1}{2} \int_0^x k(x - y, y) u(x - y) u(y) dy$$

so that the initial value problem for (1) can be written as an abstract semilinear Cauchy problem in  $X_m$

$$u_t = -A_m u + B_m u + C_m u, \quad u(0) = \dot{u}, \quad (14)$$

where  $u_t$  denotes the strong  $X_m$  derivative of  $u$ . Note that we have used the same symbol  $u$  to denote the  $X_m$ -valued function of  $t$ ,  $t \rightarrow u(t)$ . However, as can be seen from the theorem below, this will not cause any misunderstanding. To formulate the next theorem we have to introduce a new class of spaces which, as we shall see later, is related to intermediate spaces associated with the fragmentation operator  $F_m$  and its fractional powers, [21]. We set

$$X_m^{(\alpha)} := \left\{ u \in X_m; \int_0^\infty |u(x)| (\omega + a(x))^\alpha (1 + x^m) dx < \infty \right\}, \quad (15)$$

where  $\omega$  is a sufficiently large constant. Then we have

**Theorem 2.2.** *Assume that  $a, b, k$  satisfy (2), (3), (5), (6) and (12) for some  $m_0 > 1$ , and let  $m \geq \max\{j + l, m_0\}$  hold. Then, for each  $\dot{u} \in X_{m,+}^{(\alpha)}$ , there is  $\tau > 0$  such that the initial value problem (14) has a unique nonnegative classical solution  $u \in C([0, \tau], X_m^{(\alpha)}) \cap C^1((0, \tau), X_m) \cap C((0, \tau), D(A_m))$ . Furthermore, there is a measurable representation of  $u$  which is absolutely continuous in  $t \in (0, \tau)$  for any  $x \in \mathbb{R}_+$  and which satisfies (1) almost everywhere on  $\mathbb{R}_+ \times (0, \tau)$ .*

Finally, for global in time solvability we need to restrict the growth rate of  $k$ . Namely, we have

**Theorem 2.3.** *Let the assumptions of Theorem 2.2 hold with  $\beta = 0$ , that is, let  $k$  satisfy (7). Furthermore, let the constant  $j$  from assumption (5) be such that  $\alpha j \leq 1$ . Then any local solution of Theorem 2.2 is global in time.*

### 3. Proof of Theorem 2.1

We shall fix  $m$  satisfying  $m \geq j + l$  if  $j + l > 1$  and  $m > 1$  otherwise; see (5).

The proof of part (a) depends on bringing together several results that are scattered in the literature and thus some standard calculations are omitted. First we show that  $B_m := \mathcal{B}|_{D(A_m)}$  is well defined. For this we establish that if  $0 \leq u \in D(A_m)$ , then

$$\|\mathcal{B}u\|_m = \int_0^\infty a(y)(n_m(y) + n_0(y))u(y) dy < +\infty. \quad (16)$$

Indeed, calculations as in [7,8] give

$$\int_0^\infty \left( \int_x^\infty a(y)b(x|y)u(y) dy \right) x^m dx \leq \int_0^\infty a(y)u(y)y^m dy < \infty \quad (17)$$

and, by (5),

$$\int_0^\infty \left( \int_x^\infty a(y)b(x|y)u(y) dy \right) dx \leq a_0b_0 \int_0^\infty (1+y^j)(1+y^l)u(y) dy \leq 4a_0b_0 \int_0^\infty w_m(y)u(y) dy < +\infty, \tag{18}$$

where we have used the fact that

$$(1+y^j)(1+y^l) \leq 4w_m(y) \tag{19}$$

if  $m \geq j+l$ . Hence, (16) follows by adding the above integrals.

Next, direct integration utilizing (17) and (18) gives

$$\int_0^\infty (-A_m + B_m)u(x)w_m(x) dx = -\phi_m(u) := \int_0^\infty (N_0(x) - N_m(x))a(x)u(x) dx, \quad u \in D(A_m). \tag{20}$$

If the term  $N_0(x) > 0$  had not been present, then (20) would have allowed a direct application of the substochastic semigroup theory, [4, Section 6.2]. Nevertheless, the inequalities in (5) allow us to proceed as in e.g. [7, Theorem 1.1]. Indeed, for  $u \in D(A_m)_+$  we have, by (9),

$$-\phi_m(u) \leq \int_0^\infty N_0(y)a(y)u(y) dy \leq 4a_0b_0 \int_0^\infty u(x)w_m(x) dx =: \eta \|u\|_m,$$

where we have used

$$0 \leq \frac{N_0(y)a(y)}{1+y^m} \leq \frac{n_0(y)a(y)}{1+y^m} \leq 4a_0b_0, \tag{21}$$

by (10), (5) and (19). Then we have  $\tilde{\phi}_m(u) := \phi_m(u) + \eta \int_0^\infty u(x)w_m(x) dx \geq 0$  for  $0 \leq u \in D(A_m)$  and the operator  $(\tilde{A}_m, D(A_m)) := (A_m + \eta I, D(A_m))$  satisfies

$$\int_0^\infty (-\tilde{A}_m + B_m)u(x)w_m(x) dx = -\tilde{\phi}_m(u) \leq 0.$$

Since  $(-\tilde{A}_m, D(A_m))$  also generates a positive semigroup of contractions, by [4, Corollary 5.17], an extension  $\tilde{F}_m$  of  $-\tilde{A}_m + B_m$  generates a substochastic semigroup  $(G_{\tilde{F}_m}(t))_{t \geq 0}$ . Arguing as in [4, Proposition 9.29], we see that there is an extension  $F_m$  of  $(-A_m + B_m, D(A_m))$  given by  $(F_m, D(F_m)) = (\tilde{F}_m + \eta I, D(\tilde{F}_m))$  generating a positive semigroup  $(G_{F_m}(t))_{t \geq 0} = (e^{\eta t} G_{\tilde{F}_m}(t))_{t \geq 0}$  on  $X_m$ .

Furthermore, by [4, Theorem 6.8],  $\tilde{\phi}_m$  extends to  $D(F_m)$  by monotone limits of elements of  $D(A_m)$ . Note that this does not require the assumption that  $\tilde{\phi}_m$  is an integral functional with a positive kernel, introduced in [4] (see also the discussion in [26, Remark 1.2]) as here  $\tilde{\phi}_m$  is a combination of positive and negative integral functionals and therefore the monotonic limit of each of them (finite or infinite) always exists. Thus, let  $u \in D(F_m)_+$  with  $D(A_m) \ni u_n \nearrow u$ . Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^\infty N_0(x)a(x)u_n(x) dx &= \int_0^\infty N_0(x)a(x)u(x) dx < \infty, \\ \lim_{n \rightarrow \infty} \int_0^\infty u_n(x)w_m(x) dx &= \int_0^\infty u(x)w_m(x) dx < \infty, \end{aligned}$$

where each right-hand side is well defined by (5),  $m \geq j+l$  and  $D(F_m) \subset X_m$ . But then the fact that  $\tilde{\phi}_m(u_n)$  tends to a finite limit shows that also the negative term of  $\tilde{\phi}_m(u_n)$  tends to a finite limit. Hence

$$\lim_{n \rightarrow \infty} \int_0^\infty N_m(x)a(x)u_n(x) dx = \int_0^\infty N_m(x)a(x)u(x) dx < +\infty.$$

Thus, by [4, Theorem 5.2], there exists an extension  $(F_m, D(F_m))$  of the operator  $(-A_m + B_m, D(A_m))$  which generates a positive quasi-contractive semigroup, say  $(G_{F_m}(t))_{t \geq 0}$ , with the growth rate (type) not exceeding  $\eta = 4a_0b_0$  and (11) is satisfied. That the extension is, in fact, the closure, can be proved by standard application of the Arlotti extensions, [4, Theorem 6.22], as in, e.g., [9, Theorem 2.1]. Since, however, this fact will not be used in the sequel, we shall skip the details of the calculations.

To prove part (b), we begin by observing that inequality (9) implies that  $0 \leq N_m(x) \leq x^m$ . This, together with the assumption, yields  $c_m x^m/2 \leq N_m(x) \leq x^m$  for large  $x$  which, by (11), establishes that if  $u \in D(F_m)$ , then  $au \in X_m$  or, in other words, that  $D(F_m) \subset D(A_m)$ . Since  $(F_m, D(F_m))$  is an extension of  $(-A_m + B_m, D(A_m))$ , we see that  $D(F_m) = D(A_m)$ .

It is clear that the semigroup generated by  $-A_m$  is bounded. Furthermore, if  $\lambda = r + is$  then  $|\lambda + a(x)|^2 = (r + a(x))^2 + s^2 \geq s^2$  and therefore

$$\|R(r + is, -A_m)f\|_m = \int_0^\infty \left| \frac{1}{r + is + a(x)} \right| |f(x)|(1 + x^m) dx \leq \frac{1}{|s|} \|f\|_m, \quad \forall r > 0.$$

The analyticity of the fragmentation semigroup then follows from Theorem 1.1.

The statement (c) that (12) holds for all  $m \geq m_0$  provided it holds for  $m_0$  follows as in [9, Theorem 2.1].  $\square$

#### 4. Intermediate spaces associated with $F_m$

Throughout this section, we shall assume that the size distribution function  $b$  is such that (12) is satisfied. It then follows from Theorem 2.1 that  $(F_m, D(F_m)) = (-A_m + B_m, D(A_m))$  is the infinitesimal generator of an analytic semigroup  $(S_{F_m}(t))_{t \geq 0}$  of type at most  $4a_0b_0$  on  $X_m$ , and consequently we can define intermediate spaces related to  $F_m$ , see [21]. As it is more convenient to deal with an invertible generator, we examine the operator  $F_{m,\omega}$  defined by

$$F_{m,\omega} := F_m - \omega I, \quad D(F_{m,\omega}) = D(F_m) = D(A_m), \tag{22}$$

where  $\omega > 4a_0b_0$  is a fixed constant. The abstract Cauchy problem associated with the fragmentation equation then takes the form

$$u_t = \omega u + F_{m,\omega}u = \omega u - A_{m,\omega}u + B_m u, \quad u(0) \in D(A_m),$$

where  $A_{m,\omega} := A_m + \omega I$  and  $D(A_{m,\omega}) = D(A_m)$ . The operators  $(F_{m,\omega}, D(A_m))$  and  $(-A_{m,\omega}, D(A_m))$  generate analytic semigroups  $(S_{F_{m,\omega}}(t))_{t \geq 0} = (e^{-\omega t} S_{F_m}(t))_{t \geq 0}$  and  $(S_{-A_{m,\omega}}(t))_{t \geq 0} = (e^{-\omega t} S_{-A_m}(t))_{t \geq 0}$  on  $X_m$ . Moreover, the fact that each operator is invertible, means that the norms  $\|u\|_{m,A} := \|A_{m,\omega}u\|_m$  and  $\|u\|_{m,F} := \|F_{m,\omega}u\|_m, u \in D(A_m)$  are equivalent to each other and also to the corresponding graph norms on  $D(A_m)$ , see [18, Remark 1.5, p. 191].

If  $(G, D(G))$  is the infinitesimal generator of an analytic semigroup  $(S_G(t))_{t \geq 0}$  on a Banach space  $X$ , one can construct a family of intermediate spaces,  $D_G(\alpha, r), 0 < \alpha < 1, 1 \leq r \leq \infty$  in the following way:

$$D_G(\alpha, r) := \{x \in X: t \rightarrow v(t) := \|t^{1-\alpha-1/r} G S_G(t)x\|_X \in L^r(J)\}, \tag{23}$$

$$\|x\|_{D_G(\alpha,r)} := \|x\|_X + \|v(t)\|_{L^r(J)}, \tag{24}$$

where  $J := (0, 1)$ ; see [21, p. 45]. From [21, Corollary 2.2.3], these spaces do not depend explicitly on  $G$ , but only on  $D(G)$  and its graph norm. If we apply this theory to the specific cases  $G = F_{m,\omega}$  and  $G = -A_{m,\omega}$  then, in view of the above discussion, we have (up to equivalence of the respective norms)

$$D_{F_{m,\omega}}(\alpha, r) = D_{-A_{m,\omega}}(\alpha, r). \tag{25}$$

We find it most convenient to use  $D_{-A_{m,\omega}}(\alpha, 1)$  which, by [21, Proposition 2.2.2], equals the real interpolation space  $(X_m, D(A_{m,\omega}))_{\alpha,1}$ . By [31, Section 1.18.5] we have

$$(X_m, D(A_{m,\omega}))_{\alpha,1} = X_m^{(\alpha)}, \tag{26}$$

see (15), which hereafter we equip with the norm

$$\|u\|_m^{(\alpha)} := \int_0^\infty |u(x)|(\omega + a(x))^\alpha (1 + x^m) dx. \tag{27}$$

In other words, there is a constant  $c_1 \geq 1$  such that

$$c_1^{-1} \|u\|_m^{(\alpha)} \leq \|u\|_{D_{F_{m,\omega}}(\alpha,1)} \leq c_1 \|u\|_m^{(\alpha)}, \quad \forall u \in D_{F_{m,\omega}}(\alpha, 1). \tag{28}$$

**Remark 1.** For the purpose of the forthcoming analysis, one could use any suitable intermediate space between  $X_m$  and  $D(A_{m,\omega})$ , see [21, Chapter 7], such as the domains of fractional powers of  $F_{m,\omega}$ , see [9]. Such a choice, however, seems to require some additional analysis which is missing in [9]. The choice of  $(X_m, D(A_{m,\omega}))_{\alpha,1}$  simplifies the calculations, also in the discrete case, without altering the final result.

**5. Local solvability – proof of Theorem 2.2**

In this section we assume that  $a$  and  $b$  satisfy the assumptions of Theorem 2.1 b) so that, in particular, (12) holds for some  $m \geq j + l$  or  $m > 1$  if  $j + l \leq 1$ . Furthermore, the coagulation kernel is such that (6) is satisfied. To shorten notation, for any constant  $c$  we define  $a_c(x) := c + a(x)$  and  $a_c^\alpha(x) := (c + a(x))^\alpha$  and fix a constant  $\omega > \max\{4a_0b_0, 1\}$ . Then we get easily

$$\frac{a_\omega^\alpha(x)}{\omega^\alpha} \leq a_1^\alpha(x) \leq a_\omega^\alpha(x). \tag{29}$$

We consider the following modified version of (1)

$$\begin{aligned} \partial_t u(x, t) = & -(a_\omega(x) + \gamma a_\omega^\alpha(x))u(x, t) + \int_x^\infty a(y)b(x|y)u(y, t) dy \\ & + (\gamma a_\omega^\alpha(x) + \omega)u(x, t) - u(x, t) \int_0^\infty k(x, y)u(y, t) dy + \frac{1}{2} \int_0^x k(x - y, y)u(x - y, t)u(y, t) dy, \end{aligned} \tag{30}$$

where  $\gamma$  is a constant to be determined and  $\alpha$  is the index appearing in (6).

If we define an operator  $A_\omega^\alpha$  by

$$(A_\omega^\alpha u)(x) := a_\omega^\alpha u(x), \quad D(A_\omega^\alpha) := \{u \in X_m : A_\omega^\alpha u \in X_m\},$$

then we clearly have  $D(A_\omega^\alpha) = X_m^{(\alpha)}$ . Therefore, from [21, Proposition 2.4.1],  $(F_\gamma, D(F_\gamma)) := (F_{m,\omega} - \gamma A_m^\alpha, D(A_m))$  generates an analytic semigroup, say  $(S_{F_\gamma}(t))_{t \geq 0}$ , on  $X_m$ . Since  $(S_{F_{m,\omega}}(t))_{t \geq 0}$  and  $(S_{-\gamma A_m^\alpha}(t))_{t \geq 0}$  are positive and contractive, we can use the Trotter product formula, [12, Corollary III.5.8] to deduce that  $(S_{F_\gamma}(t))_{t \geq 0}$  is also a positive contraction on  $X_m$ . Furthermore, since clearly  $S_{-\gamma A_m^\alpha}(t) \leq I$  for  $t \geq 0$ , using again the Trotter formula

$$S_{F_\gamma}(t)u \leq S_{F_{m,\omega}}(t)u, \quad u \in X_{m,+}. \tag{31}$$

Thus, since  $X_m^{(\alpha)}$  is a Banach lattice with order inherited from  $X_m$ , we obtain for  $u \in X_m^{(\alpha)}$

$$\begin{aligned} \|S_{F_\gamma}(t)u\|_m^{(\alpha)} & \leq \|S_{F_{m,\omega}}(t)u\|_m^{(\alpha)} \leq c_1 \left( \|S_{F_{m,\omega}}(t)u\|_m + \int_0^1 s^{1-\alpha} \|F_{m,\omega} S_{F_{m,\omega}}(s) S_{F_{m,\omega}}(t)u\|_m ds \right) \\ & \leq c_1 \left( \|u\|_m + \int_0^1 s^{1-\alpha} \|F_{m,\omega} S_{F_{m,\omega}}(s)u\|_m ds \right) = c_1 \|u\|_{D_{F_{m,\omega}}(\alpha,1)} \leq c_1^2 \|u\|_m^{(\alpha)}. \end{aligned} \tag{32}$$

Next consider the set

$$\mathcal{U} = \{u \in X_{m,+}^{(\alpha)} : \|u\|_m^{(\alpha)} \leq 1 + b\}, \tag{33}$$

for some arbitrary fixed  $b > 0$ . For  $u \in \mathcal{U}$  we obtain

$$\int_0^\infty k(x, y)u(y) dy \leq K \left( a_1^\alpha(x) \int_0^\infty a_1^\beta(y)u(y) dy + a_1^\beta(x) \int_0^\infty a_1^\alpha(y)u(y) dy \right) \leq 2Ka_1^\alpha(x) \|u\|_m^{(\alpha)} \leq 2Ka_\omega^\alpha(x)(1 + b),$$

for any  $x > 0$ . Thus, on setting

$$\gamma = 2K(b + 1), \tag{34}$$

we have

$$\begin{aligned} (C_\gamma u)(x) & := -u(x) \int_0^\infty k(x, y)u(y) dy + (\gamma (a_\omega^\alpha(x) + \omega))u(x) + \frac{1}{2} \int_0^x k(x - y, y)u(x - y)u(y) dy \\ & \geq (-2Ka_\omega^\alpha(x)(1 + b) + 2Ka_\omega^\alpha(x)(1 + b) + \omega)u(x) + \frac{1}{2} \int_0^x k(x - y, y)u(x - y)u(y) dy \geq 0 \end{aligned} \tag{35}$$

for all  $u \in \mathcal{U}$ . Note also that, for  $u, v \in X_m^{(\alpha)}$ , we have  $\|\gamma A_\omega^\alpha u + \omega u\|_m \leq (\omega + \gamma) \|u\|_m^{(\alpha)}$ ,

$$\int_0^\infty |u(x)| \left( \int_0^\infty k(x, y) |v(y)| dy \right) w_m(x) dx \leq K (\|u\|_m^{(\alpha)} \|v\|_0^{(\beta)} + \|u\|_m^{(\beta)} \|v\|_0^{(\alpha)}) \leq 2K \|u\|_m^{(\alpha)} \|u\|_m^{(\alpha)} \tag{36}$$

and, in a similar way,

$$\begin{aligned} \int_0^\infty \left( \int_0^x k(x-y, y) |u(y)| |v(x-y)| dy \right) w_m(x) dx &= \int_0^\infty \int_0^\infty k(x, y) |u(y)| |v(x)| w_m(x+y) dx dy \\ &\leq 2^{m+2} K \|u\|_m^{(\alpha)} \|v\|_m^{(\alpha)}, \end{aligned} \tag{37}$$

where we used  $\omega > 1$  and  $\beta \leq \alpha$ .

Therefore, using the definition of  $\gamma$ , on  $\mathcal{U}$  we have

$$\|C_\gamma u\|_m \leq (\omega + \gamma)(1 + b) + 2K(b + 1)^2 + 2^{m+1}K(b + 1)^2 \leq \omega \max\{1, 2K(1 + b)\}(b + 1)(2^m + 3). \tag{38}$$

If we now investigate the Lipschitz continuity of  $C_\gamma$ , then we observe that the linear component  $\gamma A_\omega^\alpha + \omega I$  satisfies

$$\|(\gamma A_m^\alpha + \omega I)u - (\gamma A_m^\alpha + \omega I)v\|_m \leq (\omega + \gamma)\|u - v\|_m^{(\alpha)}$$

and, by (36) and (37),

$$\|C_\gamma u - C_\gamma v\|_m \leq (\omega + \gamma + 2K(\|u\|_m^{(\alpha)} + \|v\|_m^{(\alpha)})(1 + 2^m))\|u - v\|_m^{(\alpha)}, \quad u, v \in \mathcal{U}. \tag{39}$$

Now, let us take  $\dot{u} \in X_{m,+}^{(\alpha)}$  satisfying  $\|\dot{u}\|_m^{(\alpha)} \leq c_1^{-2}b$ , for  $b$  of (33) and  $c_1$  from (32) and use the contraction mapping method for

$$(\mathcal{T}u)(t) = S_{F_\gamma}(t)\dot{u} + \int_0^t S_{F_\gamma}(t-s)C_\gamma u(s) ds$$

on  $Y = C([0, \tau], \mathcal{U})$ , with  $\mathcal{U}$  defined by (33) and the metric induced by the norm  $\|u(t)\|_Y := \sup_{0 \leq t \leq \tau} \|u(t)\|_m^{(\alpha)}$ . Having established the above estimates, the calculations that, for some  $\tau > 0$ ,  $\mathcal{T}$  is a contraction on  $Y$ , follow [21, Theorem 7.1.2] (with  $X_\alpha$  of [21] equal to  $X_m^{(\alpha)}$ ) and, for regularity, [21, Proposition 7.1.10 (iii)]. Therefore, for any  $\dot{u} \in X_{m,+}^{(\alpha)}$ , there is a unique mild solution  $u$  to (14) in  $X_{m,+}^{(\alpha)}$  which, moreover, satisfies  $u \in C^1((0, \tau), X_m) \cap C((0, \tau), D(A_m))$ .

To complete the proof, we note that  $X_m$  is a space of type  $L$ , see [17, pp. 69–71] or [4, pp. 38–41] and thus any  $u \in C^1((0, \tau), X_m)$  has a measurable representation  $\mathbb{R}_+ \times (0, \tau) \ni (x, t) \rightarrow \bar{u}(x, t)$  which, moreover, is absolutely continuous in  $t$  for any  $x$  and such that the partial derivative  $\partial_t \bar{u}$  exists almost everywhere on  $\mathbb{R}_+ \times (0, \tau)$  with  $\partial_t \bar{u}(\cdot, t) = u_t(t)$  for almost any  $t \in (0, \tau)$ . Since  $D(A_m) \subset D(\gamma A_m^\alpha) \subset X_m$ , we see that all terms in (30) are separately well defined and thus the solution  $u$  has a representation which satisfies (1) almost everywhere on  $\mathbb{R}_+ \times (0, \tau)$ .  $\square$

### 6. Global solvability – proof of Theorem 2.3

The local solution, constructed in the previous section, can be extended in a usual way to the maximal forward interval of existence  $[0, \tau_{\max}(\dot{u})]$ . By [21, Proposition 7.1.8] and the remark below it, if  $\tau_{\max}(\dot{u}) < +\infty$ , then  $\|u(t)\|_m^{(\alpha)}$  is unbounded as  $t \rightarrow \tau_{\max}(\dot{u})$ . Thus, to show that  $u$  is globally defined, we need to show that  $\|u(t)\|_m^{(\alpha)}$  is *a priori* bounded uniformly in time.

Noting that for any constant  $D > 1$  we have

$$\frac{D + y}{1 + y} \leq D, \quad y \geq 0$$

and, by elementary calculus, for any  $v, w > 0$

$$(1 + x^v)(1 + x^w) \leq 2(1 + x^{v+w}), \quad x \geq 0,$$

we obtain

$$\|u\|_m^{(\alpha)} \leq a_0^\alpha \int_0^\infty |u(x)| (1 + x^m)(1 + \omega a_0^{-1} + x^j)^\alpha dx \leq 2^{1+\alpha} (a_0 + \omega)^\alpha \int_0^\infty |u(x)| (1 + x^{m+j\alpha}) dx, \tag{40}$$

where  $a_0$  was defined in (5). For  $r \in \mathbb{M}_0$ , let us denote by  $M_r$  the  $r$ th moment of  $u$ ,

$$M_r(u) := \int_0^\infty x^r u(x) dx,$$

so that (40) can be written as

$$\|u\|_m^{(\alpha)} \leq L_\alpha (M_0(u) + M_{m+j\alpha}(u)),$$

where  $L_\alpha = 2^{1+\alpha}(a_0 + \omega)^\alpha$ . Thus, to prove the theorem it is enough to show that the moments  $M_0(u(\cdot))$  and  $M_{m+j\alpha}(u(\cdot))$  do not blow up in finite time. Though for a given  $m$ , Theorem 2.2 does not ensure the differentiability of  $M_{m+j\alpha}$ , it is valid in the scale of spaces  $X_r$  with  $r \geq m$  provided, of course,  $\hat{u} \in X_r^{(\alpha)}$ . Since the embedding  $X_r^{(\alpha)} \subset X_m^{(\alpha)}$  is continuous for  $r \geq m$ , the solutions emanating from the same initial value  $\hat{u} \in X_r^{(\alpha)} \subset X_m^{(\alpha)}$  in each space, by construction, must coincide. Hence, let  $\hat{u} \in X_{m+j\alpha}^{(\alpha)} \subset X_{m+j\alpha} \subset X_m^{(\alpha)}$ , where the last inclusion is due to (5) and (40), so that  $u \in C([0, \tau_{\max}(\hat{u})], X_{m+j\alpha}^{(\alpha)}) \cap C^1((0, \tau_{\max}(\hat{u})), X_{m+j\alpha}) \cap C((0, \tau_{\max}(\hat{u})), D(A_{m+j\alpha}))$ , with possibly different, but still nonzero,  $\tau_{\max}(\hat{u})$ . This, in particular, yields differentiability of  $\|u(\cdot)\|_0 = M_0(u(\cdot))$  and, consequently, of  $M_{m+j\alpha}(u(\cdot))$ . Since for  $1 \leq r \leq m + j\alpha$  we have

$$\int_0^\infty |u(x)|x^r dx \leq \int_0^1 |u(x)|x^r dx + \int_1^\infty |u(x)|x^r dx \leq \int_0^\infty |u(x)| dx + \int_0^\infty |u(x)|x^{m+j\alpha} dx = \|u\|_{m+j\alpha}, \tag{41}$$

we see that all lower order moments  $M_r(u(\cdot))$  are also differentiable on  $(0, \tau_{\max}(\hat{u}))$  and  $(M_r(u(t)))_t = M_r(u_t(t))$ . Further, all  $r$ th moments with  $1 \leq r \leq m + j\alpha$  of every term on the right-hand side of (1) exist and are continuous on  $(0, \tau_{\max}(\hat{u}))$ . To get the moment estimates we use the inequality

$$(x + y)^r - x^r - y^r \leq (2^r - 1)(x^{r-1}y + y^{r-1}x) =: G_r(x^{r-1}y + y^{r-1}x), \tag{42}$$

for  $r \geq 1$ ,  $x, y \in \mathbb{R}_+$ , established in [7]. Standard calculations, similar to (36) and (37) but for  $u \in X_{m+j\alpha,+}^{(\alpha)}$ ,  $1 < r \leq m + j\alpha$ , and (40) with  $\omega = 1$ , give

$$\begin{aligned} \int_0^\infty x^r (C_{m+j\alpha}u)(x) dx &= \int_0^\infty \int_0^\infty ((x + y)^r - x^r - y^r)k(x, y)u(x)u(y) dx dy \\ &\leq G_r K \int_0^\infty \int_0^\infty (x^{r-1}y + xy^{r-1})((1 + a(x))^\alpha + (1 + a(y))^\alpha)u(x)u(y) dx dy \\ &\leq \frac{G_r K L_\alpha}{2} \int_0^\infty \int_0^\infty (x^{r-1}y + xy^{r-1})(2 + x^{j\alpha} + y^{j\alpha})u(x)u(y) dx dy \\ &\leq G_r K L_\alpha (M_{r+j\alpha-1}M_1 + M_{r-1}M_{1+j\alpha} + 2M_{r-1}M_1). \end{aligned} \tag{43}$$

For the particular cases  $r = 0$  and  $r = 1$  we obtain

$$\begin{aligned} \int_0^\infty (C_{m+j\alpha}u)(x) dx &= -\frac{1}{2} \int_0^\infty k(x, y)u(x, t)u(y, t) dx dy \leq 0, \\ \int_0^\infty x(C_{m+j\alpha}u)(x) dx &= 0. \end{aligned}$$

Hence, using (20) and (21) for the linear part, we obtain on  $(0, \tau_{\max}(\hat{u}))$

$$\begin{aligned} M_{0,t} &\leq 4a_0b_0(M_1 + M_m), \\ M_{1,t} &= 0, \\ M_{m+j\alpha,t} &\leq G_{m+j\alpha} K L_\alpha (M_{m+2j\alpha-1}M_1 + M_{m+j\alpha-1}(M_{1+j\alpha} + 2M_1)). \end{aligned} \tag{44}$$

Arguing as in (41), we see that if  $1 \leq r \leq r'$ , then

$$M_r \leq M_1 + M_{r'} \tag{45}$$

as  $x^r \leq x$  on  $[0, 1]$  and  $x^r \leq x^{r'}$  on  $[1, \infty)$ . Thus, we see that in order for the moment system (44) to be closed, we must assume that  $j\alpha \leq 1$ . This allows us to re-write (44) as

$$\begin{aligned} M_{0,t} &\leq 4a_0b_0(2M_1 + M_{m+j\alpha}), \\ M_{m+j\alpha,t} &\leq G_{m+j\alpha} K L_\alpha ((M_{m+j\alpha} + M_1)M_1 + (M_{m+j\alpha} + M_1)(M_2 + 3M_1)), \end{aligned} \tag{46}$$

where  $M_1$  is constant and where we used  $j\alpha \leq 1$ . To find the behaviour of  $M_2$ , again we use (20) and (43), with an obvious simplification of (42), to get the estimate for  $M_2$  as

$$M_{2,t} \leq 4KL_\alpha(M_{1+j\alpha}M_1 + M_1^2) \leq 4KL_\alpha(M_2M_1 + 2M_1^2).$$

Hence,  $M_2$  is bounded on its interval of existence. Then, from the second inequality in (46), we see that  $M_{m+j\alpha}$  satisfies a linear inequality with bounded coefficients and thus it also is bounded on  $(0, \tau_{\max}(\hat{u}))$ . This in turn yields the boundedness of  $M_0$ . Hence, by (40),  $\|u(\cdot)\|_m^{(\alpha)}$  is bounded and hence  $u$  exists globally. To ascertain global existence of solutions emanating from any initial datum  $\hat{u} \in X_m^{(\alpha)}$  we observe that since  $X_{m+j\alpha}^{(\alpha)}$  is dense in  $X_m^{(\alpha)}$ , finite blow-up of such a solution would contradict the theorem on continuous dependence of solutions on the initial data, [21, Theorem 7.1.2].  $\square$

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## References

- [1] M. Aizenman, T.A. Bak, Convergence to equilibrium in a system of reacting polymers, *Comm. Math. Phys.* 65 (1979) 203–230.
- [2] D.J. Aldous, Deterministic and stochastic models for coalescence (aggregation and coagulation): A review of the mean-field theory for probabilists, *Bernoulli* 5 (1999) 3–48.
- [3] W. Arendt, A. Rhandi, Perturbation of positive semigroups, *Arch. Math.* 56 (1991) 107–119.
- [4] J. Banasiak, L. Arlotti, *Perturbations of Positive Semigroups with Applications*, Springer, London, 2006.
- [5] J. Banasiak, S.C. Oukouomi Noutchie, R. Rudnicki, Global solvability of a fragmentation-coagulation equation with growth and restricted coagulation, *J. Nonlinear Math. Phys.* 16 (Suppl. 1) (2009) 13–26.
- [6] J. Banasiak, W. Lamb, Coagulation, fragmentation and growth processes in a size structured population, *Discrete Contin. Dyn. Syst. Ser. B* 11 (3) (2009) 563–585.
- [7] J. Banasiak, W. Lamb, Global strict solutions to continuous coagulation–fragmentation equations with strong fragmentation, *Proc. Roy. Soc. Edinburgh Sect. A* 141 (2011) 465–480.
- [8] J. Banasiak, Transport processes with coagulation and strong fragmentation, *Discrete Contin. Dyn. Syst. Ser. B* 17 (2) (2012) 445–472.
- [9] J. Banasiak, Global classical solutions of coagulation–fragmentation equations with unbounded coagulation rates, *Nonlinear Anal. Real World Appl.* 13 (1) (2012) 91–105, doi:10.1016/j.nonrwa.2011.07.016.
- [10] P.B. Dubovskii, I.W. Stewart, Existence, uniqueness and mass conservation for the coagulation–fragmentation equation, *Math. Methods Appl. Sci.* 19 (1996) 571–591.
- [11] B.F. Edwards, M. Cai, H. Han, Rate equation and scaling for fragmentation with mass loss, *Phys. Rev. A* 41 (1990) 5755–5757.
- [12] K.-J. Engel, R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Grad. Texts in Math., Springer-Verlag, New York, 2000.
- [13] M. Escobedo, P. Laurençot, S. Mischler, B. Perthame, Gelation and mass conservation in coagulation–fragmentation models, *J. Differential Equations* 195 (2003) 143–174.
- [14] A.K. Giri, J. Kumar, G. Warnecke, The continuous coagulation equation with multiple fragmentation, *J. Math. Anal. Appl.* 374 (2011) 71–87.
- [15] A.K. Giri, G. Warnecke, Uniqueness for the coagulation–fragmentation equation with strong fragmentation, *Z. Angew. Math. Phys.* 62 (2011) 1047–1063, doi:10.1007/s00033-011-0129-0.
- [16] A.K. Giri, P. Laurençot, G. Warnecke, Weak solutions to the continuous coagulation equation with multiple fragmentation, arXiv:1101.4169v1 [math.AP].
- [17] E. Hille, R.S. Phillips, *Functional Analysis and Semi-Groups*, Colloquium Publications, vol. 31, American Mathematical Society, Providence, RI, 1957.
- [18] T. Kato, *Perturbation Theory for Linear Operators*, second ed., Springer-Verlag, Berlin, 1984.
- [19] J. Koch, W. Hackbusch, K. Sundmacher, H-matrix methods for linear and quasi-linear integral operators appearing in population balances, *Comput. Chem. Eng.* 31 (2007) 745–759.
- [20] P. Laurençot, On a class of continuous coagulation–fragmentation equations, *J. Differential Equations* 167 (2000) 245–274.
- [21] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Birkhäuser-Verlag, Basel, 1995.
- [22] E.D. McGrady, R.M. Ziff, “Shattering” transition in fragmentation, *Phys. Rev. Lett.* 58 (9) (1987) 892–895.
- [23] Z.A. Melzak, A scalar transport equation, *Trans. Amer. Math. Soc.* 85 (1957) 547–560.
- [24] D.J. McLaughlin, W. Lamb, A.C. McBride, A semigroup approach to fragmentation models, *SIAM J. Math. Anal.* 28 (1997) 1158–1172.
- [25] D.J. McLaughlin, W. Lamb, A.C. McBride, An existence and uniqueness result for a coagulation and multiple-fragmentation equation, *SIAM J. Math. Anal.* 28 (1997) 1173–1190.
- [26] M. Mokhtar-Kharroubi, J. Voigt, On honesty of perturbed substochastic  $C_0$ -semigroups in  $L^1$ -spaces, *J. Operator Theory* 64 (2010) 131–147.
- [27] H. Müller, Zur Allgemeinen Theorie der Raschen Koagulation, *Kolloideihefte* 27 (1928) 223–250.
- [28] D.J. Smit, M.J. Hounslow, W.R. Paterson, Aggregation and gelation-I. Analytical solutions for cst and batch operation, *Chem. Eng. Sci.* 49 (7) (1994) 1025–1035.
- [29] M. Smoluchowski, Drei vorträge über diffusion, brownische bewegung und koagulation von kolloidteilchen, *Phys. Z.* 17 (1916) 557–585.
- [30] I.W. Stewart, A global existence theorem for the global coagulation–fragmentation equation with unbounded kernels, *Math. Methods Appl. Sci.* 11 (1989) 627–648.
- [31] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, Amsterdam, 1978.