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Sharp conditions for blowup of solutions of a chemotactical model for two species in $\mathbb{R}^2$ ☆

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ABSTRACT

We consider a model system of Keller–Segel type for the evolution of two species in the whole space $\mathbb{R}^2$ which are driven by chemotaxis and diffusion. It is well known that this problem admits global and blowup solutions. We show that there exists a sharp condition which allows to distinguish global and blowup solutions in the radially symmetric case. More precisely, let m_∞ and n_∞ be the total masses of the species. Then there exists a critical curve γ in the $m_\infty - n_\infty$ plane such that the solution blows up if and only if (m_∞, n_∞) is above γ. This gives an answer to a question raised by Conca et al. (2011) in [8]. We also study the asymptotic behaviour of global solutions in the subcritical case, showing that they are asymptotically self-similar.

1. Introduction

In this work we study the positive solutions $w(t,r) = (m(t,r), n(t,r))$ of the system

$$\begin{cases} \partial_t m - 4r\partial_{rr} m - \dfrac{\chi_1}{\pi}(m+n)\partial_r m = 0 & \text{in } (0,T)\times(0,\infty), \\ \partial_t n - 4r\partial_{rr} n - \dfrac{\chi_2}{\pi}(m+n)\partial_r n = 0 & \text{in } (0,T)\times(0,\infty), \\ w(t,\infty) = w_\infty & \text{in } (0,T), \\ w(0,r) = w_0(r) & \text{in } (0,\infty), \end{cases} \tag{1.1}$$

where $\chi_1 > 0$, $\chi_2 > 0$, $w_0 \in (C(0,\infty))^2$ is nonnegative, nondecreasing and satisfies $w_0(\infty) = w_\infty \in \mathbb{R}^2$. Here, $w(t,\infty) = \lim_{r\to\infty} w(t,r)$ and $w_0(\infty) = \lim_{r\to\infty} w_0(r)$. For radially symmetric solutions system (1.1) is derived from the following Keller–Segel elliptic–parabolic type problem for two species in $\mathbb{R}^2$

$$\begin{cases} u_t - \Delta u + \chi_1 \nabla\cdot(u\nabla c) = 0 & \text{in } (0,T)\times\mathbb{R}^2, \\ v_t - \Delta v + \chi_2 \nabla\cdot(v\nabla c) = 0 & \text{in } (0,T)\times\mathbb{R}^2, \\ -\Delta c = u + v & \text{in } (0,T)\times\mathbb{R}^2, \\ u(0) = u_0 & \text{in } \mathbb{R}^2, \\ v(0) = v_0 & \text{in } \mathbb{R}^2. \end{cases} \tag{1.2}$$

Here, u, v are the mass densities of the species, c is the chemical concentration and χ_1, χ_2 represent the intensities of chemotactical attraction of the species. A formal computation shows that when u_0 and v_0 are nonnegative integrable

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functions such that

$$\int_{\mathbb{R}^2} u_0(x)\,dx = m_\infty, \qquad \int_{\mathbb{R}^2} v_0(x)\,dx = n_\infty, \tag{1.3}$$

then the total mass is preserved, i.e.,

$$\int_{\mathbb{R}^2} u(t,x)\,dx = m_\infty, \qquad \int_{\mathbb{R}^2} v(t,x)\,dx = n_\infty \tag{1.4}$$

for $t > 0$. System (1.1) is obtained from (1.2) by assuming (1.4) and setting

$$m(t,r) = \int_{B(0,\sqrt{r})} u(t,x)\,dx, \qquad n(t,r) = \int_{B(0,\sqrt{r})} v(t,x)\,dx; \tag{1.5}$$

see [1,2] for the corresponding case of one organism.

Note that the diffusion coefficients in (1.1) degenerate at $r = 0$. As a consequence, no left hand boundary condition is imposed in (1.1). The right hand boundary condition, i.e., the condition at infinity is also nonstandard. To treat those difficulties we proceed as in [3], considering a regularized problem defined over a finite interval $(0, R)$ and replacing the diffusion coefficients by $4(r + \delta)$, where $\delta > 0$. Letting $\delta \to 0$ and $R \to \infty$ we prove the existence of a solution of (1.1) which is defined for all $t > 0$. A solution obtained by this regularization procedure will be called a r-solution. This construction yields the following result.

Theorem 1.1. *Suppose $w_0 \in (C(0,\infty))^2$ is a nonnegative nondecreasing function such that $w(0) = 0$ and $w_0(\infty) = w_\infty$. Then there exists a r-solution $w \in (C^{1,2}((0,\infty)\times(0,\infty)))^2$ of (1.1) such that*

$$w(t,r) \to w_0(r) \tag{1.6}$$

as $t \to 0$ uniformly in $[\rho,\infty)$ for all $\rho > 0$ and such that

$$w(t,r) \to w_\infty \tag{1.7}$$

as $r \to \infty$ uniformly in $t \in [0,T]$, for all $T > 0$. It holds that $w(t,r)$ is nonnegative, nondecreasing in r. Moreover, the following comparison principle takes place. Let $\tilde{w}_0 \geq w_0$ and suppose w is a r-solution such that $w(0) = w_0$. Then there exists a r-solution $\tilde{w} \geq w$ such that $\tilde{w}(0) = \tilde{w}_0$.

Assume further that there exists $C_0 > 0$ such that

$$w(t,r) \leq C_0 r. \tag{1.8}$$

Then there exist $T > 0$ and $C(T) > 0$ such that

$$w(t,r) \leq C(T) r \tag{1.9}$$

for all $t \leq T$, $r > 0$.

Finally, uniqueness holds in the class of solutions $w \in (C^{1,2}((0,T)\times(0,\infty)))^2$ satisfying (1.6), (1.7) and (1.9) for some $T > 0$.

As observed above r-solutions are defined for all $t > 0$. However, due to the degeneracy at the origin, if (1.8) holds (1.9) may break down in finite time. In fact, it may happen that $w(t,0) = \lim_{r\to 0} w(t,r)$ ceases to be equal to zero in finite time. This corresponds to the appearance of a Dirac mass at $r = 0$ in (1.2); see (1.4). We will say that a r-solution w is global if $w(0) = 0$ for a.a. $t > 0$, otherwise it is a blowup solution. In this article we give a sharp criterion to distinguish global from blowup r-solutions. Before presenting this characterization let us discuss some previous results.

In [4,5] the general (nonradial) case for one organism is considered. The authors use a sharp logarithmic Hardy–Sobolev inequality to prove that if $\chi m_\infty < 8\pi$ then a weak global solution $u(t) \in L^1(\mathbb{R}^2)$ may be defined. Here χ is the chemotactical intensity of the species. Assuming further that the second moment $\int |x|^2 u_0(x)\,dx$ is finite, they also show that in the supercritical case $\chi m_\infty > 8\pi$ all solutions blow up. Further results concerning this and analogous models may be found in [6]. The symmetric radial case is treated in [3], where the mass variable $m(t)$ is considered. The authors show that the solutions are global and asymptotically self-similar in the subcritical region $\chi m_\infty < 8\pi$. They also show that the critical case $\chi m_\infty = 8\pi$ corresponds to global solutions and provide an interesting analysis of the asymptotic behaviour of the solutions. Analogous results have been previously described in [1], where radial solutions in a disc of $\mathbb{R}^2$ were considered. The supercritical case $\chi m_\infty > 8\pi$ has been studied in [7], where it is shown that the corresponding radially symmetric solutions blow up.

The existence of global and of blowup solutions for two species is discussed in [8], where the authors study system (1.2). To present their main results let us define

$$T(w_\infty) = m_\infty + n_\infty - \frac{8\pi}{\max\{\chi_1,\chi_2\}}, \tag{1.10}$$

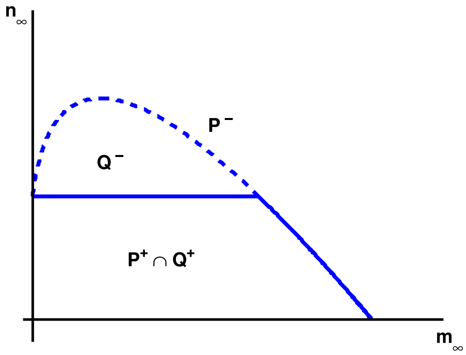

Fig. 1. The level curves $Q = 0$ and $P = 0$.

$$P(w_\infty) = 8\pi\left(\frac{m_\infty}{\chi_1} + \frac{n_\infty}{\chi_2}\right) - (m_\infty + n_\infty)^2, \tag{1.11}$$

$$Q(w_\infty) = 8\pi - \max\{\chi_1 m, \chi_2 n\}, \tag{1.12}$$

and denote

$$P^+ = \{w_\infty \in \mathbb{R}^+ \times \mathbb{R}^+, P(w_\infty) \geq 0\}, \tag{1.13}$$

$$P^- = \{w_\infty \in \mathbb{R}^+ \times \mathbb{R}^+, P(w_\infty) < 0\}, \tag{1.14}$$

$$\mathring{P}^+ = \{w_\infty \in \mathbb{R}^+ \times \mathbb{R}^+, P(w_\infty) > 0\}, \tag{1.15}$$

$$P_0 = \{w_\infty \in \mathbb{R}^+ \times \mathbb{R}^+, P(w_\infty) = 0\}; \tag{1.16}$$

see Fig. 1. Analogous notations stand for T and Q. Using as in [5] the logarithmic Hardy–Sobolev inequality they prove that solutions are global whenever $w_\infty \in T^-$ while blowup occurs when u_0 and v_0 have finite second moments and $w_\infty \in P^-$. Moreover, in the radially symmetric case they show that blowup also occurs when $w_\infty \in Q^-$. These results left open the determination of the nature of the solutions in the region $P^+ \cap Q^+$ in the general case, and $P^+ \cap Q^+ \cap T^+$ for radial solutions. In this work we give an answer to this question in the radial case showing that the two curves $Q(w_\infty) = 0$, $P(w_\infty) = 0$ are in fact critical. More precisely, we have the following.

Theorem 1.2. *Any r-solution w of (1.1) blows up when either*

$$w_\infty \in P^- \tag{1.17}$$

or

$$w_\infty \in Q^-. \tag{1.18}$$

Theorem 1.3. *Suppose (1.8) holds. Assume further that*

$$w_\infty \in P^+ \tag{1.19}$$

and

$$w_\infty \in Q^+. \tag{1.20}$$

Then any r-solution of (1.1) is global.

We next study the long time behaviour of global solutions for which $w_\infty \in \mathring{P}^+ \cap Q^+$. We first show there exists a unique self-similar solution w_s such that $w_s(t, 0) = 0$ and $w(t, \infty) = w_\infty$ for all $t > 0$. By self-similar we mean w_s satisfying $w_s(kt, kr) = w_s(t, r)$ for all $k > 0$. Then we prove that w_s describes the long-time behaviour of all solutions such that $w(t, \infty) = w_\infty$, as stated in Theorem 1.4 below. Analogous asymptotic behaviour holds for the case of one species; see [5] for the general case and [3] for radially symmetric solutions.

Theorem 1.4. *Let $w_\infty \in \mathring{P}^+ \cap Q^+$. Then there exists a unique self-similar solution w_s such that $w_s(t, 0) = 0$ and $w(t, \infty) = w_\infty$ for all $t > 0$.*

Given w_0 satisfying (1.8) and such that $w_0(\infty) = w_\infty$, there exists a unique global solution $w(t)$ of (1.1). It holds that

$$\lim_{t\to\infty} \|w(t) - w_s(t)\|_\infty = 0. \tag{1.21}$$

It is interesting to interpret the above results by looking at the stationary solutions of (1.1). They may be obtained using the connection between (1.1) and the so-called Liouville problem for systems. In fact, an appropriate change of variables transforms a solution of the Liouville system into a stationary solution of (1.1). Necessary and sufficient conditions for the existence of solutions of the Liouville problem were given in [9]. Translating those conditions to the present problem we see that stationary solutions lie in the critical region $P_0 \cap Q^+$. In the subcritical region solutions are trapped by stationary solutions and cannot blow up. A similar picture holds true in the case of a chemotactic system of multi-components in a bounded domain of $\mathbb{R}^2$; see [10] where a model for a general number of components and for the general non-radially symmetric case is considered. The author shows that the region $\mathring{P}^+ \cap \mathring{Q}^+$ corresponds to global solutions, which converge to some stationary solution. A similar picture also holds for the one species problem, where stationary solutions lie in the critical region $m_\infty = 8\pi/\chi$.

The proofs of Theorems 1.2 and 1.3 rely on a comparison principle and this is why they are restricted to r-solutions. Note however that when $w_\infty \in \mathring{P}^+ \cap Q^+$ the solution is unique; see Theorem 1.4. We finally remark that asymptotic self-similar behaviour takes places in the subcritical region $\mathring{P}^+ \cap Q^+$ and also in the part of the critical region $P^+ \cap \{w_\infty, R(w_\infty) = 0\}$. The stationary solutions lie in the remaining part of the critical region, $\mathring{Q}^+ \cap \{w_\infty, P(w_\infty) = 0\}$. Thus, we cannot expect the same asymptotically self-similar behaviour there.

In Section 2 we construct the r-solutions and prove Theorem 1.1. In Section 3 we study the stationary solutions of (1.1). In Section 4 we treat the supercritical case and show Theorem 1.2. The existence of global solutions in the critical and subcritical cases is studied in Section 4. Finally, in Section 5 we show the existence of self-similar solutions and describe the asymptotic self-similar behaviour of global solutions corresponding to the subcritical case and part of the critical case.

2. Existence and uniqueness

We will now discuss the existence and uniqueness of solutions of (1.1). To this end we introduce the following regularized boundary value problem. Given $\delta > 0$ and $R > 0$ find $w(t, r) = (m(t, r), n(t, r))$ such that

$$\begin{cases} \partial_t m - 4(r+\delta)\partial_{rr} m = \dfrac{\chi_1}{\pi}(m+n)\partial_r m, \\ \partial_t n - 4(r+\delta)\partial_{rr} n = \dfrac{\chi_2}{\pi}(m+n)\partial_r n, \\ w(0, r) = w_0(r), \\ w(t, 0) = 0, \\ w(t, R) = w_0(R). \end{cases} \tag{2.1}$$

We present a well-posedness result concerning (2.1).

Lemma 2.1. *Let $w_0 \in X := (C^0([0, R]))^2$ be nondecreasing, $w_0(0) = 0$. Then there exists a unique solution $w \in C([0, \infty); X) \cap (C^{1,2}((0, \infty) \times (0, R)))^2$ of (2.1) which is nondecreasing for all $t > 0$.*

Proof. The existence of a global solution $w(t)$ for $t > 0$ follows from a standard fixed-point argument and usual estimates. For two such solutions we also get

$$\sup_{t<T} \|w_1(t) - w_2(t)\|_\infty \le C(T)\|w_1(0) - w_2(0)\|_\infty, \tag{2.2}$$

from which uniqueness follows. We will now show that the solution preserves monotonicity. Suppose first that $w_0 \in (C^1([0, R]))^2$. Then $\mu = \partial_r m$ and $\nu = \partial_r n$ satisfy

$$\begin{cases} \partial_t \mu - 4\partial_r((r+\delta)\partial_r \mu) = \dfrac{\chi_1}{\pi}\partial_r((m+n)\mu), \\ \partial_t \nu - 4\partial_r((r+\delta)\partial_r \nu) = \dfrac{\chi_2}{\pi}\partial_r((m+n)\nu), \\ \partial_r \mu(t, 0) = 0, \\ \partial_r \nu(t, 0) = 0, \\ (R+\delta)\partial_r \mu(t, R) + \dfrac{\chi_1}{\pi}(m_0(R) + n_0(R))\mu(R) = 0, \\ (R+\delta)\partial_r \nu(t, R) + \dfrac{\chi_2}{\pi}(m_0(R) + n_0(R))\nu(R) = 0, \end{cases} \tag{2.3}$$

with $(\mu(t), \nu(t)) \to (m_0', n_0')$ uniformly as $t \to 0$. (This is a consequence of the fact that $w_0 = 0$ and of parabolic regularization.)

Multiplying the first equation of (2.3) by μ^- and integrating over $(0, R)$ we get

$$|\mu^-|^2 + 4\delta \int_0^R |\partial_r \mu^-|^2\, dr \le C \int_0^R |\mu^-||\partial_r \mu^-|\, dr.$$

Noting that $\mu^-(0) = 0$, it follows from Hölder, Young, Gronwall's inequalities that $\mu^-(t) = 0$ for $t > 0$. Thus, $m(t)$ is nondecreasing. We get analogously that $n(t)$ is nondecreasing. Consider now $w_0 \in C^0$ nondecreasing and let $\{w_0^n\}_{n\in\mathbb{N}}$ be a sequence of C^1 nondecreasing functions converging uniformly to w_0 and such that $w_0^n(0) = 0$, $w_0^n(R) = w_0(R)$. Then the corresponding solution $w^n(t)\, w_0^n$ is nondecreasing for each $t > 0$. Using (2.2), we conclude that $w(t)$ is nondecreasing. □

Lemma 2.2. *Let $w_0 \in X$ be nondecreasing and such that $w_0(0) = 0$. Given $T > 0$ there exists $C(R, T)$ independent of δ such that*

$$\int_0^T \int_0^R r|w(t)|^2\, dr\, dt + \int_0^T \|\partial_t w\|^2_{H^{-1}(0,R)}\, dt \le C(R, T). \tag{2.4}$$

Proof. We recall that $\tilde{m} = m - m_0(R)r/R$ satisfy

$$\partial_t \tilde{m} - 4(r+\delta)\partial_{rr}\tilde{m} - \frac{\chi_1}{\pi}(m+n)(\partial_r \tilde{m} + m_0(R)/R) = 0. \tag{2.5}$$

Hence,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^R |\tilde{m}|^2\, dr + 4\int_0^R (r+\delta)|\partial_r \tilde{m}|^2\, dr + \frac{\chi_1}{2\pi}\int_0^R (\partial_r m + \partial_r n)|\tilde{m}|^2\, dr \\ &\quad - \frac{\chi_1 m_0(R)}{R\pi}\int_0^R (m+n)\tilde{m}\, dr = 0. \end{aligned} \tag{2.6}$$

Note that $\partial_r m + \partial_r n \ge 0$, $m(t, r) \le m_0(R)$, $n(t, r) \le n_0(R)$ and $|\tilde{m}|_\infty \le 2m_0(R)$. Thus

$$\frac{1}{2}\frac{d}{dt}\int_0^R |\tilde{m}|^2\, dr + 4\int_0^R r|\partial_r \tilde{m}|^2\, dr \le 2m_0(R)^2(m_0(R) + n_0(R)). \tag{2.7}$$

We conclude from (2.7), and an analogous estimate for n, that

$$\int_0^T \int_0^R r(|\partial_r m|^2 + |\partial_r n|^2)\, dr\, dt \le C(R, T). \tag{2.8}$$

Next, multiplying (2.1) by $\varphi \in H_0^1(0, R)$ we have

$$\int_0^R \partial_t m\varphi\, dr + 4\int_0^R r\partial_r m\varphi'\, dr + 4\int_0^R \partial_r m\varphi\, dr = \frac{\chi_1}{\pi}\int_0^R (m+n)\partial_r m\varphi\, dr. \tag{2.9}$$

Using that $|\varphi(r)| \le \sqrt{r}\|\varphi'\|_2$ and Hölder's inequality we obtain $C(R) > 0$ such that

$$\int_0^R \left(r\partial_r m|\varphi'| + \partial_r m|\varphi| + (m+n)\partial_r m|\varphi|\right)\, dr \le C(R)\|\varphi'\|_2 \left(\int_0^R r|\partial_r m|^2\, dr\right)^{1/2}. \tag{2.10}$$

The bound for $\|\partial_t m\|_{H^{-1}}$ in (2.4) follows from (2.8)–(2.10). Analogous arguments give the desired estimate for $\|\partial_t n\|_{H^{-1}}$. □

The following comparison principle holds.

Lemma 2.3. *Let $w_0 \in X$ be nondecreasing, with $w_0 = 0$ and let $\overline{w} \in (C^{2,1}((0, \infty) \times (0, R)))^2 \cap (C([0, \infty) \times [0, R]))^2$ be a supersolution of (2.1), i.e., it holds that*

$$\begin{cases} \partial_t \overline{w} + A\overline{w} \ge F(\overline{w}), \\ \overline{w}(0, r) \ge w_0(r), \\ \overline{w}(t, 0) \ge 0, \\ \overline{w}(t, R) \ge w_0(R). \end{cases} \tag{2.11}$$

Then $\overline{w} \ge w$.

Proof. Let $\overline{w} = (m^1, n^1)$ and define $w^j = (m^j, n^j)$ recursively for $j \ge 2$ by

$$\begin{cases} \partial_t m^j - 4(r+\delta)\partial_{rr} m^j - \dfrac{\chi_1}{\pi}(m^{j-1} + n^{j-1})\partial_r m^j = 0, \\ \partial_t n^j - 4(r+\delta)\partial_{rr} n^j - \dfrac{\chi_2}{\pi}(m^{j-1} + n^{j-1})\partial_r n^j = 0, \\ w^j(0, r) = w_0(r), \\ w^j(t, 0) = 0, \\ w(t, R) = w_0(R). \end{cases} \tag{2.12}$$

It follows from the maximum principle for parabolic equations that w^j is a supersolution of (2.1), with $w^j(t)$ nondecreasing in r and nonincreasing in j for all $t > 0$. Thus $w^j \searrow w$ as $j \to \infty$. Since $\overline{w} \geq w^j$ for all j so that $\overline{w} \geq w$. □

We now prove the existence and uniqueness result concerning the r-solutions of (1.1).

Proof of Theorem 1.1. We first show the existence of a solution. Given $k \in \mathbb{N}$ call w_k the solution of (2.1) corresponding to $\delta = 1/k$ and $R = k$. It follows from Lemma 2.3 that $w_k \leq w_\infty$ for all k.

Extend $w_k(r) = w_k(R)$ for $r > R$. Using parabolic regularity and standard diagonal arguments, upon passing to a subsequence, we may write that $w_k \to w$ as $j \to \infty$ uniformly in $[t_1, t_2] \times [0, R]$, for all $0 < t_1 < t_2 < T$, $R > 0$, for some function w. Clearly, $w(t)$ is nondecreasing, $0 \leq w \leq w_\infty$ and w solves the partial differential equations of (1.1).

To study the behaviour of $w(t)$ for small times, we use (2.4) to get

$$\|w_k(t) - w_k(s)\|_{H^{-1}} \leq \int_s^t \|\partial_t w\|_{H^{-1}}\, dt \leq C(R, \tilde{T})(t-s)^{1/2}$$

for all $0 \leq s \leq t \leq \tilde{T} < T$ and all k. Hence, $\{w_k\}$ is equicontinuous and equibounded in $H^{-1}(0, R)$ for all $R > 0$, so that $w \in C([0, \tilde{T}], H^{-1}(0, R))$ with $w(0) = w_0$. Moreover, (2.4) also shows that $\{w_k\}$ is bounded in $L^2((0, \tilde{T}); H^1(\rho, R))$ for each $0 < \rho < R$, with $\{\partial_t w_k\}$ bounded in $L^2((0, \tilde{T}); H^{-1}(0, R))$. Thus $w \in C([0, \tilde{T}]; L^2(\rho, R))$ and since $w \leq \overline{w}$, we conclude that $w \in C([0, \tilde{T}]; L^2(0, R))$ for all $R > 0$. Finally, interior estimates for parabolic equations ensure that

$$w \in C([0, \tilde{T}] \times [\rho, R]) \tag{2.13}$$

for all $0 < \rho < R$.

We next prove that $w(t, r) \to w_\infty$ as $r \to \infty$ for all $t > 0$. Given $\varepsilon > 0$, define $\underline{m}(t, r) = (m_\infty - \varepsilon)\left(1 - C(t)(1 + r)^{-1}\right)$, where $C(t) = Ae^{4t}$, $A > 0$. A straightforward computation shows that for $\delta_k < 1$

$$\begin{aligned} \partial_t \underline{m} - 4\partial_r((r + \delta_k)\partial_r \underline{m}) &= -4C(t)(m_\infty - \varepsilon)(1 + r)^{-3}((1 + r)^2 - 2(r + \delta_k) + 1 + r) \\ &\leq 0 \leq \partial_t m_k - 4(r + \delta_k)\partial_{rr} m_k. \end{aligned} \tag{2.14}$$

Take now $\rho > 0$ such that $m_0(r) > m(\infty) - \varepsilon$ for $r > \rho$. Then $m_0(r) > \underline{m}(t, r)$ for $r > \rho$ and $t > 0$. We choose A large enough so that $m_0(r) > \underline{m}(t, r)$ for $r \leq \rho$. In this way, for all $r > 0$

$$m_0(r) > \underline{m}(0, r).$$

In particular for $t > 0$

$$0 > \underline{m}(0, 0) > \underline{m}(t, 0).$$

Moreover, for $R_k > \rho$,

$$m_k(t, R_k) = m_0(R_k) > m(\infty) - \delta > \underline{m}(t, R_k). \tag{2.15}$$

It follows from (2.14)–(2.15) that $m_k(t, r) \geq \underline{m}(t, r)$ for all $t > 0$, $r \in [0, R_k)$ and k large enough. Taking $k \to \infty$, we get that $m \geq \underline{m}$. Letting $\varepsilon \to 0$ and arguing analogously for $n(t)$ we conclude that given $T > 0$ there exists $C(T) > 0$ such that

$$m_\infty - m(t, r) + n_\infty - n(t, r) \leq C(T)(1 + r)^{-1} \tag{2.16}$$

for all $t \leq T$. Hence, (2.13) and (2.16) imply that $w(t) \to w_0$ as $t \to 0$ uniformly in $[\rho, \infty)$ for all $\rho > 0$ showing (1.6). Furthermore, (2.16) also yields (1.6).

The solutions constructed above will be called r-solutions. Since the approximate solutions are nonnegative, nondecreasing and admit a comparison principle the same is true for r-solutions.

Suppose now that (1.8) takes place. Set $\overline{w} = (\overline{m}, \overline{n})$ where $\overline{m}(t, r) = \overline{n}(t, r) = Ae^{kt} r$, $k > 0$ and $A > 0$ is such that $\overline{w}(0) \geq w_0$. It is immediate to see that k and T can be chosen so that $\overline{w}$ is a supersolution of (2.1) in some interval $(0, T)$, for all $\delta > 0$ and $R > 0$. In particular, (1.9) holds.

It remains to prove the uniqueness result. Suppose $w = (m, n)$ and $\tilde{w} = (\tilde{m}, \tilde{n})$ are two solutions satisfying (1.6)–(1.8) for some $T > 0$. Define $f = m - \tilde{m}$, $g = n - \tilde{n}$. We have that

$$\partial_t f - 4r\partial_{rr} f = \frac{\chi_1}{\pi}((m + n)\partial_r f + (f + g)\partial_r \tilde{m}).$$

Let $\varphi(r) = e^{-r}$. Using that $\varphi' = -\varphi$ we multiply the equation by φf and integrate to get

$$\begin{aligned} &\frac{d}{dt}\int_0^\infty \varphi |f|^2\, dr + 4\int_0^\infty r\varphi|\partial_r f|^2\, dr \\ &\quad = 4\int_0^\infty \varphi' |f|^2\, dr + \frac{\chi_1}{\pi}\int_0^\infty \left((m + n)\varphi f \partial_r f - (\partial_r f + \partial_r g)\tilde{m}\varphi f - (f + g)\tilde{m}(\varphi' f + \varphi \partial_r f)\right) dr. \end{aligned}$$

Applying Hölder and Young inequalities it follows from (1.8) that there exists $C > 0$ such that

$$\frac{d}{dt}\int_0^\infty \varphi|f|^2\,dr + 2\int_0^\infty r\varphi|\partial_r f|^2\,dr \le \int_0^\infty r\varphi|\partial_r g|^2\,dr + C\int_0^\infty \varphi(|f|^2+|g|^2)\,dr \tag{2.17}$$

we may write analogously that

$$\frac{d}{dt}\int_0^\infty \varphi|g|^2\,dr + 2\int_0^\infty r\varphi|\partial_r g|^2\,dr \le \int_0^\infty r\varphi|\partial_r f|^2\,dr + C\int_0^\infty \varphi(|f|^2+|g|^2)\,dr. \tag{2.18}$$

Since $f(0) = g(0) = 0$, from (2.17), (2.17) we conclude that $f(t) = g(t) = 0$ for $t < T$. $\square$

Remark 2.4. Let $\overline{w}$ be a supersolution of (2.1) and w be a r-solution of (1.1) such that $\overline{w}(0) \ge w(0)$. It follows from Lemma 2.3 and the construction of r-solutions that $\overline{w} \ge w$.

We observe that r-solutions are defined for all $t > 0$. Nevertheless, the definition below of a blowing up solution applies even to r-solutions.

Definition 2.5. Let w be a solution of (1.1) and set $w(t,0) = \lim_{r\to 0} w(t,r)$. Given $T > 0$ define $B(T) = \{t \in (0,T),\ w(t,0) > 0\}$. We say that w is a global solution if $\mu(B(T)) = 0$ for all $T > 0$, where μ is the Lebesgue measure in $\mathbb{R}$. Otherwise, we say that w blows up and define the blowup time $T_{\max}$ as $T_{\max} = \sup_{T>0}\{\mu(B(T)) = 0\}$.

From (1.5) we see that $T_{\max}$ corresponds to the time where at least one of the species collapses to a Dirac mass.

We next address the question of the uniqueness of solutions. We first show the following.

Proposition 2.6. *Let w_0 satisfy (1.8) and let $w(t)$ be a r-solution such that $w(0) = w_0$. Suppose (1.9) holds for some $T > 0$. Then $w(t) - w_0 \in (L^2(0,\infty))^2$ for all $t < T$.*

Proof. Let $\tilde{w}_0 = (\tilde{m}_0, \tilde{n}_0) \in (W^{2,\infty}(0,\infty))^2$ be such that $w_0 - \tilde{w}_0 \in (L^2(0,\infty))^2$. Given $k \in \mathbb{N}$ define $w_k = (m_k, n_k)$ as the solution of (2.1) corresponding to $\delta = k$ and $R = 1/k$; see the proof of Theorem 1.1. Let $\tilde{w}_k = (\tilde{m}_k, \tilde{n}_k) = w_k - \tilde{w}_0$. It follows from (2.1) that $\tilde{w}(0) = \tilde{w}(R) = 0$ and

$$\partial_t\tilde{m} - 4\partial_r((r+\delta)\partial_r(\tilde{m}+m_0) - (\tilde{m}+m_0)) = \frac{\chi_1}{\pi}(\tilde{m}+m_0+\tilde{n}+n_0)\partial_r(\tilde{m}+m_0).$$

Multiplying this by $\tilde{m}$ and integrating over $(0,R)$ yield

$$\frac{d}{dt}\int_0^R |\tilde{m}|^2\,dr + 4\int_0^R (r+\delta)|\partial_r\tilde{m}|^2\,dr + 4\int_0^R (r+\delta)\partial_r m_0\partial_r m\,dr$$
$$-4\int_0^R (\tilde{m}+m_0)\partial\tilde{m}\,dr = \frac{\chi_1}{\pi}\int_0^R (\tilde{m}+m_0+\tilde{n}+n_0)\partial_r(\tilde{m}+m_0)\tilde{m}\,dr. \quad \square$$

3. Stationary solutions

As noticed in [10], there exists a straight connection between the stationary solutions of (1.1) and the solutions of the following Liouville problem for systems.

Given $A = (a_{i,j})_{i,j=1,2}$ and $M_i > 0$, $i = 1, 2$ find z_1, z_2 such that

$$\begin{cases} -\Delta z_1 = e^{a_{1,1}z_1 + a_{1,2}z_2}, \\ -\Delta z_2 = e^{a_{2,1}z_1 + a_{2,2}z_2}, \\ \displaystyle\int_{\mathbb{R}^2} e^{a_{1,1}z_1+a_{1,2}z_2} = M_1, \\ \displaystyle\int_{\mathbb{R}^2} e^{a_{2,1}z_1+a_{2,2}z_2} = M_2. \end{cases} \tag{3.1}$$

In [9] the result below is proved. Suppose A is symmetric and $a_{i,j} \ge 0$, $i,j = 1,2$. Then (3.1) has a solution if and only if

$$8\pi(M_1+M_2) = \sum_{i=1,2}\sum_{j=1,2} a_{i,j}M_iM_j, \qquad M_1a_{1,1} \le 8\pi, \qquad M_2a_{2,2} \le 8\pi, \tag{3.2}$$

The connection between (1.2) and (3.1) goes as follows. Given $\chi_1, \chi_2, m_\infty, n_\infty$ positive numbers set $a_{1,1} = \chi_1^2$, $a_{2,2} = \chi_2^2$, $a_{1,2} = a_{2,1} = \chi_1\chi_2$, $M_1 = m_\infty\chi_1^{-1}$, $M_2 = n_\infty\chi_2^{-1}$, $w_\infty = (m_\infty, n_\infty)$. Then (3.2) is equivalent to

$$w_\infty \in P_0 \cap \mathring{Q}^+; \tag{3.3}$$

see (1.15) and (1.16). Suppose (3.3) holds and consider z_1, z_2 a solution of (3.1). Then $\tilde{u} = -\chi_1 \Delta z_1$, $\tilde{v} = -\chi_2 \Delta z_2$ and $z = \chi_1 z_1 + \chi_2 z_2$ satisfy (1.3) and

$$-\Delta z = \tilde{u} + \tilde{v}. \tag{3.4}$$

Moreover, it follows from (3.1) that $\log \tilde{u} = \log \chi_1 + \chi_1 z$ and $\log \tilde{v} = \log \chi_2 + \chi_2 z$, yielding $\nabla \tilde{u} = \chi_1 \tilde{u} \nabla z$, $\nabla \tilde{v} = \chi_2 \tilde{v} \nabla z$. Hence, $\Delta \tilde{u} = \chi_1 \nabla \cdot (\tilde{u} \nabla z)$, $\Delta \tilde{v} = \chi_2 \nabla \cdot (\tilde{v} \nabla z)$. This and (3.4) show that

$$(\tilde{u}, \tilde{v}) \in (L^1(\mathbb{R}^2))^2 \tag{3.5}$$

is a stationary solution of (1.2). Use $\tilde{u}, \tilde{v}$ in (1.5) to define $\tilde{m}$,$\tilde{n}$. It follows thus from the results of [9] that (3.3) ensures the existence of a stationary solution $\tilde{w} = (\tilde{m}, \tilde{n})$ of

$$\begin{cases} -4r\partial_{rr}\tilde{m} - \dfrac{\chi_1}{\pi}(\tilde{m} + \tilde{n})\partial_r \tilde{m} = 0, \\ -4r\partial_{rr}\tilde{n} - \dfrac{\chi_2}{\pi}(\tilde{m} + \tilde{n})\partial_r \tilde{n} = 0, \\ \tilde{w}(\infty) = w_\infty. \end{cases} \tag{3.6}$$

By (1.5) and (3.5) we get $\tilde{m}(0) = \tilde{n}(0) = 0$. Moreover, it is easy to see from (3.6) that $\tilde{m}$ and $\tilde{n}$ are increasing and concave. We now show that $\tilde{m}'(0)$, $\tilde{n}'(0)$ are finite. Indeed, setting

$$\mu = \frac{8\pi}{\chi_1}\tilde{m} + \frac{8\pi}{\chi_2}\tilde{n},$$
$$\eta = \tilde{m} + \tilde{n}$$

we get from (3.6) that $r\partial_{rr}\mu + \partial_r \eta^2 = 0$. We may assume that $\chi_1 \le \chi_2$. Since μ and η are concave we get

$$\frac{8\pi}{\chi_2} r\partial_{rr}\eta + \partial_r \eta^2 \ge 0.$$

Using that $r\partial_r \eta$ converges to 0 along a subsequence we have

$$\frac{8\pi}{\chi_2}(r\partial_r \eta - \eta) + \eta^2 \ge 0. \tag{3.7}$$

Choose $R > 0$ small enough so that $\chi_2 \eta < 8\pi$. Integrating (3.7) over (r, R) we obtain

$$\frac{8\pi\eta(r)}{r(8\pi - \chi_2\eta(r))} \le \frac{8\pi\eta(R)}{R(8\pi - \chi_2\eta(R))}.$$

Letting $r \to 0$ we conclude that $\eta'(0)$ is finite. Hence, $\tilde{m}'(0)$ and $\tilde{n}'(0)$ are finite.

4. Blowup solutions

In this section we show that r-solutions blow up in the supercritical case. It is useful to define

$$\mu = \frac{1}{\chi_1} m + \frac{1}{\chi_2} n, \quad \eta = m + n. \tag{4.1}$$

It follows from (1.1) that

$$\partial_t \mu - 4r\partial_{rr}\mu = \frac{1}{2\pi}\eta^2. \tag{4.2}$$

Proof of Theorem 1.2. Let $\tilde{w} = (\tilde{m}, \tilde{n})$ be a stationary solution of (1.1), so that $P(\tilde{w}_\infty) = 0$ and $R(\tilde{w}_\infty) > 0$. Given $k > 1$ consider $w_k = (m_k, n_k)$ a r-solution of (1.1) satisfying $m_k(0) = k\tilde{m}$, $n_k(0) = k\tilde{n}$. We have that w_k blows up. To see this, note that $kw_\infty \in P^-$, see (1.14), and that

$$-4r\partial_{rr} m_k(0, r) - \frac{\chi_1}{\pi}(m_k(0, r) + n_k(0, r))\partial_r m_k(0, r) = k(1 - k)(\tilde{m} + \tilde{n})\partial_r \tilde{m} \le 0,$$
$$-4r\partial_{rr} n_k(0, r) - \frac{\chi_1}{\pi}(m_k(0, r) + n_k(0, r))\partial_r n_k(0, r) = k(1 - k)(\tilde{m} + \tilde{n})\partial_r \tilde{n} \le 0.$$

Thus $w_k(t)$ is nondecreasing in t (this is true for the approximate regularized solutions constructed in Theorem 1.1 and *a fortiori* for w_k). Consider next μ_k and η_k as in (4.1) with m and n replaced by μ_k and n_k. Integrating (4.2) over (a, b) yields

$$\int_a^b \partial_t \mu_k(t, r)\, dr = 4r\partial_r \mu_k(t, r) + \frac{1}{2\pi}\eta_k(t, r)^2 - 4\mu_k(t, r)\,\big|_a^b.$$

Since $w_k(t)$ is nondecreasing in t we have from Fubini's lemma

$$\frac{d}{dt}\int_0^b \mu_k(t,r)\,dr \ge \int_0^b \partial_t\mu_k(t,r)\,dr \ge 4r\partial_r\mu_k(t,r) + \frac{1}{2\pi}\eta_k(t,r)^2 - 4\mu_k(t,r)\,|_a^b.$$

It follows from the boundedness of w_k that there exists a sequence $\{\rho_n\}_{n\in\mathbb{N}}$ such that $\rho_n\partial\mu_k(\rho_n) \to 0$ as $n \to \infty$. Writing (4.3) for $\rho = \rho_n$ and then letting $n \to \infty$ we obtain

$$\begin{aligned}\frac{d}{dt}\int_0^b \mu_k(t,r)\,dr &\ge 4b\partial_r\mu_k(t,b) + \frac{1}{2\pi}\eta_k^2(t,b) - 4\mu_k(t,b)\\ &\ge \frac{1}{2\pi}\eta_k^2(t,b) - 4\mu_k(t,b) = -\frac{1}{2\pi}P(w_k(t,b))\end{aligned}\tag{4.3}$$

for all t such that $w_k(t) = 0$. Choose b large enough so that $A = -P(w_k(0,b)) > 0$. Since $m_k(t), n_k(t)$ are nondecreasing $-P(w_k(t,R)) > A$ for all $t > 0$. Suppose that for some $T > 0$ $w_k(t) = 0$ a.e. in $(0,T)$. Integrating (4.3) we get

$$\left(\frac{\pi}{\chi_1}m_\infty + \frac{\pi}{\chi_2}n_\infty\right)b \ge \int_0^b \mu_k(t,r)\,dr \ge AT.$$

This shows that w_k blows up.

Consider now w a r-solution such that $w_\infty \in P^- \cap Q^+$. Then there exists $s < 1$ such that $sw_\infty \in P_0 \cap Q^+$. It follows that there exists a stationary solution $\tilde{w}$ such that $\tilde{w}(\infty) = sw_\infty$. Note from the scale invariance of the problem that $\tilde{w}^\lambda(r) = \tilde{w}(\lambda r)$ is also a stationary solution for all $\lambda > 0$. We may pick λ large enough and $1 < k < s^{-1}$ so that $w(0) \ge k\tilde{w}^\lambda$. Then there exists a r-solution $w_{k,\lambda}$ coming from $k\tilde{w}^\lambda$ which can be compared to w; see Theorem 1.1. As shown above, $w_{k,\lambda}$ blows up so that w also blows up.

We now treat the case $w_\infty \in Q^-$. Suppose $\chi_1 m_\infty > 8\pi$. We have that

$$\partial_t m - 4r\partial_{rr}m \ge \frac{\chi_1}{\pi}m\partial_r m.$$

Thus m is a supersolution of the corresponding scalar equation and we can argue as previously. The case $\chi_2 n_\infty > 8\pi$ is clearly analogous. □

5. Global solutions

We now show that solutions corresponding to the subcritical and critical regions are global. Inequality (5.1) below plays an essential role in the proof of Theorem 1.3 and is the system version of inequality (2.6) of [1]. The proof is included here for the reader's convenience.

Lemma 5.1. *Let $w = (m,n)$ be a r-solution of (1.1) and let μ, η be defined by (4.1) for $w = w_k$. Then given $T > 0$ there exists $C = C(T, w_\infty) > 0$ such that*

$$\int_0^T\int_0^1 \frac{P(w(t,r))}{r}\,dr\,dt \le C.\tag{5.1}$$

Proof. Let $w_k = (m_k, n_k)$ be a solution of (2.1) as constructed in the proof of Theorem 1.1 and let μ_k, η_k be defined by (4.1) for $w = w_k$. Then

$$\partial_t\mu_k - 4(r+1/k)\partial_{rr}\mu_k = \frac{1}{2\pi}\eta_k^2.\tag{5.2}$$

We have that

$$\begin{aligned}\int_0^1 (r+1/k)\log(r+1/k)\partial_{rr}\mu_k &= \int_0^1 \mu_k\partial_{rr}((r+1/k)\log(r+1/k))\\ &\quad - (r+1/k)\log(r+1/k)\partial_r\mu_k\,|_0^1 + (1+\log(r+1/k))\mu_k\,|_0^1\\ &= \int_0^1 \frac{1}{r}\mu_k - (r+1/k)\log(r+1/k)\partial_r\mu_k\,|_0^1 + (1+\log(1+1/k))\mu_k(t,1)\\ &\le \int_0^1 \frac{1}{r}\mu_k + (1+\log(1+1/k))\mu_k(t,1)\end{aligned}\tag{5.3}$$

and

$$\int_0^1 \log(r+1/k)\partial_r\eta_k^2 = -\int_0^1 \frac{1}{r}\eta_k^2 + \log(1+1/k)\eta_k^2(t,1).\tag{5.4}$$

Using (1.11) and (5.2)–(5.4) we get

$$\frac{d}{dt}\int_0^1 \mu_k \log(r+1/k) - \int_0^1 \frac{P(w_k)}{r} + (1+\log(1+1/k))\mu_k(t,1) \geq \log(1+1/k)\eta_k^2(t,1).$$

Integrating in time and letting $k \to \infty$ (5.1) follows easily. □

The following lemma will also be useful.

Lemma 5.2. *Let $T > 0$ and let $f : (0,T) \times (0,1) \to \mathbb{R}$ be a measurable nonnegative function such that*

$$\int_0^T \int_0^1 \frac{1}{r} f(t,r)\,dr\,dt < \infty. \tag{5.5}$$

Then $f(t,r) \to 0$ as $r \to 0$ for a.a. $t \in (0,T)$.

Proof. Let $A = \{t \in (0,T), \lim_{r\to 0} f(t,r) = 0\}$ and $B = (0,T) \subset A$. We argue by contradiction and assume that $\mu(B) > 0$, where μ is the usual Lebesgue measure. Given $n \in \mathbb{N}$, set

$$B_n = \{t \in B, \exists\, r_n(t) > 0 \text{ such that } f(t,r) > 1/n \text{ in } V_n(t) := (0, r_n(t))\}.$$

Then $B_n \subset B_{n+1}$ and

$$B = \bigcup_{n\in\mathbb{N}} B_n,$$

so that $\mu(B_m) > 0$ for some m. Thus

$$\int_{B_m} \frac{1}{m} \int_{V_{m(t)}} \frac{1}{r}\,dr\,dt \leq \int_0^T \int_0^1 \frac{1}{r} f(t,r)\,dr\,dt < \infty.$$

This is clearly a contradiction. □

Proof of Theorem 1.3. We recall that if $w_\infty \in P_0 \cap \mathring{Q}^+$ then there exists a stationary solution $\tilde{w}$ such that $\tilde{w}(\infty) = w_\infty$; see Section 2.

Suppose first that $w_\infty \in \mathring{P}^+ \cap \mathring{Q}^+$. Assuming without loss of generality that $\chi_1 \leq \chi_2$ set

$$f(m) = 8\pi \frac{m}{\chi_1} + 8\pi \frac{n_\infty}{\chi_2} - (m + n_\infty)^2.$$

Then $f(m_\infty) = P(w_\infty) > 0$. Furthermore,

$$f\left(\frac{8\pi}{\chi_1}\right) = 8\pi n_\infty \left(\frac{1}{\chi_2} - \frac{2}{\chi_1}\right) - n_\infty^2 < 0$$

so that there exists $m' \in (m_\infty, 8\pi\chi_1^{-1})$ such that $f(m') = 0$ so that $(m', n_\infty) \in P_0 \cap \mathring{Q}^+$. We can then clearly choose $\tilde{w}_\infty = (\tilde{m}_\infty, \tilde{n}_\infty) \in P_0 \cap \mathring{Q}^+$ such that $m_\infty < \tilde{m}_\infty < m'$ and $n_\infty < \tilde{n}_\infty$. Consider a stationary solution $\tilde{w} = (\tilde{m}, \tilde{n})$ satisfying

$$\tilde{w}(\infty) = \tilde{w}_\infty > w_\infty. \tag{5.6}$$

Recall that $\tilde{w}^\lambda(r) = \tilde{w}(\lambda r)$ is also a stationary solution for all $\lambda > 0$. Using (1.8) and (5.6) we may choose λ large enough so that $w_0 \leq \tilde{w}^\lambda$. It is easy to see that w^λ is a supersolution of (2.11). It follows then from Remark 2.4 that $w \leq \tilde{w}^\lambda$. Since $\tilde{w}^\lambda$ has finite derivative at zero, we conclude that w is a global solution.

Consider next the critical case $w_\infty \in P_0 \cap Q^+$. We have that $w(t,r) \leq w_\infty$ for all $t > 0, r > 0$ and so $w(t,r) \in P^+$. It follows from Lemmas 5.1 and 5.2 that $P(w(t,r)) \to 0$ as $r \to 0$ for a.a. $t > 0$. But then either $w(t,0) = 0$ or $w(t,0) = w_\infty$ a.e. in t. The second alternative cannot hold. Indeed, define $z(t,r)$ as the solution of

$$\begin{cases} \partial_t z - 4r\partial_{rr} z = \dfrac{\chi_1}{\pi} z + n_\infty \partial_r z & \text{in } (0,T+1)\times(1,2), \\ z(1,t) = z(2,t) = m_\infty & \text{in } (0,T+1), \\ z(0,r) = m_0(r) & \text{in } (1,2). \end{cases}$$

Then the strong maximum principle ensures that $z(T,r) < m_\infty$ for $r \in (1,2)$, so that $m_\infty(T,r) \leq z(t,r) < m_\infty$. We conclude that $w(t,0) = 0$ a.e., i.e., the solution is global. An analogous but simpler argument can be used to show that if the region $P^+ \cap Q_0$ solutions are global. □

6. Self-similar asymptotics

We now show that global solutions are asymptotically self-similar as described in Theorem 1.4. As observed in the proof of Theorem 1.3, when $w_\infty \in \mathring{P}^+ \cap Q^+$ there exists a stationary solution $\tilde{w}$ such that $\tilde{w} \geq w_0$. In this case uniqueness of solutions holds, see Theorem 1.1, so that the semigroup $S(t)$ associated to (1.1) is well defined. Consider also the dilation $d^\lambda w(r) = w(\lambda r)$ for $w \in (C[0,\infty))^2$ and $\lambda > 0$. Then the scale invariance of the problem means that

$$d^\lambda S(\lambda t) = S(t) d^\lambda \tag{6.1}$$

for all $\lambda > 0$. Let us first discuss the existence of a self-similar solution.

Proposition 6.1. *There exists a self-similar solution w of (1.1) such that $w(t,0) = 0$ for all $t > 0$.*

Proof. Let w_0 satisfy (1.8) and $w_\infty \in \mathring{P}^+ \cap Q^+$. Note that $d^\lambda w_0$ is nondecreasing in λ so that $S(t)d^\lambda w_0$ is also nondecreasing in λ. Define $\tilde{w}(t)$ such that $S(t)d^\lambda w_0 \nearrow \tilde{w}(t)$ as $\lambda \nearrow \infty$. Fix $\mu > 0$. Since $d^{\lambda\mu} = d^\lambda d^\mu$, from (6.1) we get

$$\begin{aligned}\tilde{w}(t) &= \lim_{\lambda\to\infty} S(t)d^\lambda w_0 = \lim_{\lambda\to\infty} d^\lambda S(\lambda t) w_0 = \lim_{\lambda\to\infty} d^{\lambda\mu} S(\lambda\mu t) w_0 \\ &= d^\mu \lim_{\lambda\to\infty} d^\lambda S(\lambda\mu t) w_0 = d^\mu \lim_{\lambda\to\infty} S(\mu t) d^\lambda w_0 = d^\mu \tilde{w}(\mu t).\end{aligned}$$

Thus $\tilde{w}$ is self-similar, i.e., $\tilde{w}(\mu t, \mu r) = \tilde{w}(t,r)$. Defining the profile h of $\tilde{w}$ by $h(y) = \tilde{w}(1,y)$, we have

$$\tilde{w}(t,r) = h(r/t). \tag{6.2}$$

Well-known arguments relying upon parabolic regularity ensure that $\tilde{w}$ solves (1.1). Moreover, since $\tilde{w} \geq d^1 w = w$ we see that $\tilde{w}(t,r) \to w_\infty$ as $r \to \infty$ for all $t > 0$. It remains to show that $\tilde{w}(t,0) = 0$ for $t > 0$. But (5.1) holds for $d^\lambda w$ with C independent of λ. Using Fatou's lemma we conclude that (5.1) also holds for $\tilde{w}$. Thus $\tilde{w}(t,0) = 0$ a.e. in t. Using (6.2) we see that in fact $\tilde{w}(t,0) = 0$ for all t. $\square$

We next prove the uniqueness of the self-similar solution.

Proposition 6.2. *The solution obtained in Proposition 6.1 is unique.*

Proof. Suppose $w(t,r) = h(y) = (f(y), g(y))$, where $y = r/t$, is a self-similar solution of (1.1) such that $h(0) = 0$. It follows from (1.1) that

$$\begin{cases} 4f'' + f' + \dfrac{\chi_1}{\pi}\left(\dfrac{f(y)}{y} + \dfrac{g(y)}{y}\right) f' = 0, \\ 4g'' + g' + \dfrac{\chi_2}{y\pi}\left(\dfrac{f(y)}{y} + \dfrac{g(y)}{y}\right) g' = 0. \end{cases} \tag{6.3}$$

We complete the proof in four steps.

Step 1—We show $f'(0)$ and $g'(0)$ are finite.

It is easy to see from (6.3) that f and g must be nondecreasing and concave. In particular, $f(y)/y$ and $g(y)/y$ are nonincreasing functions, so it holds that

$$f' + 4f'' + \frac{\chi_1}{\pi}\left(\frac{f(z)}{z} + \frac{g(z)}{z}\right) f' \geq 0$$

for $z < y$. Hence

$$\begin{aligned} 4f'(z) &\leq f(z) + 4f'(z) + \frac{\chi_1}{\pi}\left(\frac{f(z)}{z} + \frac{g(z)}{z}\right) \\ &\leq f(y) + 4f'(y) + \frac{\chi_1}{\pi}\left(\frac{f(z)}{z} + \frac{g(z)}{z}\right) f(y). \end{aligned} \tag{6.4}$$

Consider as in [3] the auxiliary function $v(y) = f(y) - 4yf'(y)$. Then (6.3) yields

$$v'(y) = f'(y)\left(-3 + y + \frac{4\chi_1}{\pi}(f(y) + g(y))\right).$$

Therefore, v is decreasing in some interval $[0, \tilde{y}]$. Moreover, there exists a sequence $y_n \to 0$ such that $y_n f'(y_n) \to 0$, otherwise f would be unbounded at zero. Thus $v(0) = 0$ and $v(y) < 0$ for $y < \tilde{y}$. Since an analogous argument holds for g, we may choose $\tilde{y}$ eventually smaller such that for $y < \tilde{y}$

$$\frac{f(y)}{y} \leq 4f'(y) \quad \text{and} \quad \frac{g(y)}{y} \leq 4g'(y).$$

Using this we get from (6.4) that

$$4f'(z) \le f(y) + 4f'(y) + \frac{4\chi_1}{\pi}(f'(z) + g'(z))f(y) \tag{6.5}$$

for $z < y < \tilde{y}$. Analogously we have

$$4g'(z) \le g(y) + 4g'(y) + \frac{4\chi_2}{\pi}(f'(z) + g'(z))g(y). \tag{6.6}$$

Choose y such that

$$\frac{4\chi_1}{\pi}f(y) + \frac{4\chi_2}{\pi}g(y) < 2.$$

It follows then from (6.5) and (6.6) that

$$2(f'(z) + g'(z)) \le f(y) + g(y) + 4(f'(y) + g'(y)),$$

showing that $f'(0)$ and $g'(0)$ are finite.

Step 2—We show that

$$\int (m_\infty - f(y))\,dy + (n_\infty - g(y))\,dy = \frac{1}{2}P(w_\infty). \tag{6.7}$$

Indeed, it follows from (6.3) that

$$\frac{4\pi}{\chi_1}yf'' + \frac{4\pi}{\chi_2}yg'' + \frac{\pi}{\chi_1}yf' + \frac{\pi}{\chi_2}yg' + \frac{1}{2}\left((f+g)^2\right)' = 0. \tag{6.8}$$

Integrating in $(0, y)$ yields

$$\begin{aligned} &\frac{4\pi}{\chi_1}(yf' - f) + \frac{4\pi}{\chi_2}(yg' - g) + \int_0^y \frac{\pi}{\chi_1}(f(y) - f(z)) + \frac{\pi}{\chi_2}(g(y) - g(z))\,dz + \frac{1}{2}(f+g)^2(y) \\ &\quad = \frac{4\pi}{\chi_1}(yf' - f) + \frac{4\pi}{\chi_2}(yg' - g) + \int_0^y \frac{\pi}{\chi_1}zf'(z) + \frac{\pi}{\chi_2}zg'(z)\,dz + \frac{1}{2}(f+g)^2(y) = 0. \end{aligned} \tag{6.9}$$

We now write (6.9) for $y = y_n$ where $\{y_n\}_{n\in\mathbb{N}}$ is a sequence such that $y_n \to \infty$, $y_nf'(y_n) \to 0$ and $y_ng'(y_n) \to 0$ as $n \to \infty$. Then (6.7) is a consequence of monotone convergence, upon letting $n \to \infty$.

Step 3—Consider

$$w_0(r) = \begin{cases} r & \text{if } r \le m_\infty, \\ w_\infty & \text{if } r > m_\infty \end{cases}$$

and use it to construct a self-similar solution $\overline{w}$ as in the proof of Proposition 6.1. Let w be another self-similar solution such that $w(0) = 0$. Then

$$\overline{w} \ge w. \tag{6.10}$$

Indeed, given $\tau > 0$ we know from Step 1 that $\partial_r w(\tau, 0)$ is finite. We may then choose $\lambda > 0$ large enough so that $d^\lambda w_0 \ge w(\tau)$. Therefore, $d^\lambda w(t) \ge w(\tau + t)$. Hence, $\overline{w}(t) \ge d^\lambda w(t) \ge w(\tau + t)$. Letting $\tau \to 0$, (6.10) follows.

Step 4—We now conclude. Let $\overline{h} = (\overline{f}, \overline{g})$, $h = (f, g)$ be the profiles of $\overline{w}$, w, respectively, where $\overline{w}$ and w are as in Step 3. From (6.10) we have $\overline{h} \ge h$ and from (6.7) we obtain $\overline{h} = h$. □

Proof of Theorem 1.4. Let w_0 satisfy (1.8) be such that $w_\infty \in \mathring{P}^+ \cap Q^+$. We have already observed in the beginning of this section that in this case there is a unique solution $w(t)$ starting at w_0. Moreover, for $\tau > 0\, S(\tau)d^\lambda w_0 \in C[0, \infty)$, $S(\tau)d^\lambda w_0(0) = 0$, $S(\tau)d^\lambda w_0(\infty) = w_\infty$ and $S(\tau)d^\lambda w_0 \nearrow w_s(t)$ as $\lambda \to 0$ pointwise in $[0, \infty)$, where w_s is self-similar. Under those conditions it is easy to see that in fact

$$\|S(\tau)d^\lambda w_0 - w_s(\tau)\|_\infty \to 0 \tag{6.11}$$

as $\lambda \to \infty$. Since $d^\lambda w_s(\lambda) = w_s(1)$, using (6.1) and letting $\lambda \to \infty$ we obtain from (6.11) for $\tau = 1$ that

$$\|S(t)w_0 - w_s(t)\|_\infty = \|d^t S(t)w_0 - d^t w_s(t)\|_\infty = \|S(1)d^t w_0 - w_s(1)\|_\infty \to 0.$$

This proves (1.21). □

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