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Laplace transform and Hyers–Ulam stability of linear differential equations

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ABSTRACT

In this paper, we prove the Hyers–Ulam stability of a linear differential equation of the nth order. More precisely, applying the Laplace transform method, we prove that the differential equation $y^{(n)}(t)+\sum_{k=0}^{n-1}\alpha_k y^{(k)}(t)=f(t)$ has Hyers–Ulam stability, where α_k is a scalar, y and f are n times continuously differentiable and of exponential order, respectively.

1. Introduction

In 1940, Ulam [22] posed a problem concerning the stability of functional equations: "Give conditions in order for a linear function near an approximately linear function to exist".

A year later, Hyers [5] gave an answer to the problem of Ulam for additive functions defined on Banach spaces: Let X_1 and X_2 be real Banach spaces and $\varepsilon > 0$. Then for every function $f : X \to Y$ satisfying

$$\|f(x+y)-f(x)-f(y)\| \le \varepsilon \quad (x, y \in X_1),$$

there exists a unique additive function $A : X_1 \to X_2$ with the property

$$\|f(x)-A(x)\| \le \varepsilon \quad (x \in X_1).$$

After Hyers's result, many mathematicians have extended Ulam's problem to other functional equations and generalized Hyers's result in various directions (see [3,6,10]). A generalization of Ulam's problem was recently proposed by replacing functional equations with differential equations: The differential equation $\varphi(f, y, y', \ldots, y^{(n)}) = 0$ has Hyers–Ulam stability if for given $\varepsilon > 0$ and a function y such that $|\varphi(f, y, y', \ldots, y^{(n)})| \le \varepsilon$, there exists a solution y_a of the differential equation such that $|y(t) - y_a(t)| \le K(\varepsilon)$ and $\lim_{\varepsilon\to 0} K(\varepsilon) = 0$. If the preceding statement is also true when we replace ε and $K(\varepsilon)$ by $\alpha(t)$ and $\beta(t)$, where α, β are appropriate functions not depending on y and y_a explicitly, then we say that the corresponding differential equation has the generalized Hyers–Ulam stability.

Obłoza seems to be the first author who has investigated the Hyers–Ulam stability of linear differential equations (see [14,15]). Thereafter, Alsina and Ger published their paper [1], which handles the Hyers–Ulam stability of the linear differential equation $y'(t) = y(t)$: If a differentiable function $y(t)$ is a solution of the inequality $|y'(t) - y(t)| \le \varepsilon$ for any $t \in (a, \infty)$, then there exists a constant c such that $|y(t) - ce^t| \le 3\varepsilon$ for all $t \in (a, \infty)$.

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Those previous results were extended to the Hyers–Ulam stability of linear differential equations of first order and higher order with constant coefficients in [12,20,21] and in [13], respectively. Furthermore, Jung has also proved the Hyers–Ulam stability of linear differential equations (see [7–9]). Rus investigated the Hyers–Ulam stability of differential and integral equations using the Gronwall lemma and the technique of weakly Picard operators (see [18,19]). Recently, the Hyers–Ulam stability problems of linear differential equations of first order and second order with constant coefficients were studied by using the method of integral factors (see [11,23]). The results given in [8,11,12] have been generalized by Cimpean and Popa [2] and by Popa and Raşa [16,17] for the linear differential equations of nth order with constant coefficients.

In this paper, we investigate the Hyers–Ulam stability of the linear differential equations by using the Laplace transform method.

2. Laplace transform and inverse transform

Throughout this paper, $\mathbb{F}$ will denote either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. A function $f : (0, \infty) \to \mathbb{F}$ is said to be of exponential order if there are constants $A, B \in \mathbb{R}$ such that

$$|f(t)| \leq Ae^{tB}$$

for all $t > 0$. For each function $f : (0, \infty) \to \mathbb{F}$ of exponential order, we define the Laplace transform of f by

$$F(s) = \int_0^\infty f(t)e^{-st}dt.$$

There exists a unique number $-\infty \leq \sigma < \infty$ such that this integral converges if $\Re(s) > \sigma$ and diverges if $\Re(s) < \sigma$. The number σ is called the abscissa of convergence and denoted by σ_f. It is well known that $|F(s)| \to 0$ as $\Re(s) \to \infty$. Furthermore, f is analytic on the open right half plane $\{s \in \mathbb{C} : \Re(s) > \sigma\}$ and we have

$$\frac{d}{ds}F(s) = -\int_0^\infty te^{-st}f(t)dt \quad (\Re(s) > \sigma).$$

The Laplace transform of f is sometimes denoted by $\mathcal{L}(f)$. It is well known that $\mathcal{L}$ is linear and one-to-one.

Conversely, let $f(t)$ be a continuous function whose Laplace transform $F(s)$ has the abscissa of convergence σ_f, then the formula for the inverse Laplace transforms yields

$$f(t) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\alpha-iT}^{\alpha+iT} F(s)e^{st}ds = \frac{1}{2\pi}\int_{-\infty}^\infty e^{(\alpha+iy)t}F(\alpha+iy)dy$$

for any real constant $\alpha > \sigma_f$, where the first integral is taken along the vertical line $\Re(s) = \alpha$ and converges as an improper Riemann integral and the second integral is used as an alternative notation for the first integral (see [4]). Hence, we have

$$\mathcal{L}(f)(s) = \int_0^\infty f(t)e^{-st}dt \quad (\Re(s) > \sigma_f)$$

$$\mathcal{L}^{-1}(F)(t) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{(\alpha+iy)t}F(\alpha+iy)dy \quad (\alpha > \sigma_f).$$

3. Hyers–Ulam stability of linear differential equations

Recall that the convolution of two integrable functions $f, g : (0, \infty) \to \mathbb{F}$ is defined by

$$(f * g)(t) = \int_0^t f(t-x)g(x)dx.$$

It is easy to check that

$$\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g).$$

Before stating the main theorem, we need the following two lemmas.

Lemma 3.1. *Let*

$$P(s) = \alpha_0 + \alpha_1 s + \alpha_2 s^2 + \cdots + \alpha_n s^n$$

and

$$Q(s) = \beta_0 + \beta_1 s + \beta_2 s^2 + \cdots + \beta_m s^m,$$

where m, n are nonnegative integers with $m < n$ and α_j, β_j are scalars. Then there exists an infinitely differentiable function $g : (0, \infty) \to \mathbb{F}$ such that

$$\mathcal{L}(g) = \frac{Q(s)}{P(s)} \quad (\Re(s) > \sigma_p)$$

and

$$g^{(i)}(0) = \begin{cases} 0 & (\textit{for } i = 0, 1, \ldots, n-m-2), \\ \beta_m/\alpha_n & (\textit{for } i = n-m-1), \end{cases}$$

where $\sigma_p = \max\{\Re(s) : P(s) = 0\}$.

Proof. Let $\ell = n - m$. Express $P(s)$ as a product of linear factors:

$$P(s) = \alpha_n (s - s_1)^{n_1} (s - s_2)^{n_2} \cdots (s - s_k)^{n_k}$$

for some complex numbers s_i and integers n_i, $i = 1, 2, \ldots, k$, with $n = n_1 + \cdots + n_k$. Applying the partial fraction decomposition of $Q(s)/P(s)$, we obtain

$$\frac{Q(s)}{P(s)} = \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{\lambda_{ij}}{(s - s_i)^j},$$

where λ_{ij} is a scalar for each $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, n_i$. Let

$$h_{ij}(t) = \frac{t^{j-1}}{(j-1)!} e^{s_i t}$$

for every integer $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, n_i$. We then define

$$g(t) = \sum_{i=1}^{k} \sum_{j=1}^{n_i} \lambda_{ij} h_{ij}(t)$$

and use the linearity of the Laplace transform to get

$$\mathcal{L}(g) = \mathcal{L}\left(\sum_{i=1}^{k} \sum_{j=1}^{n_i} \lambda_{ij} h_{ij}(t) \right) = \sum_{i=1}^{k} \sum_{j=1}^{n_i} \lambda_{ij} \mathcal{L}(h_{ij}) = \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{\lambda_{ij}}{(s - s_i)^j} = \frac{Q(s)}{P(s)}$$

for every s with $\Re(s) > \sigma_p$, where $\sigma_p = \max\{\Re(s_i) : i = 1, \ldots, k\}$. This completes the first part of our proof.

For the proof of the second part, applying the Maclaurin series expansion of g yields

$$g(t) = g(0) + g'(0)t + \cdots + \frac{g^{(n-1)}(0)}{(n-1)!} t^{n-1} + h(t),$$

where

$$h(t) = \sum_{i=n}^{\infty} \frac{g^{(i)}(0)}{i!} t^i.$$

Note that

$$\mathcal{L}(h) = \frac{H(s)}{s^{n+1}}$$

for some complex function H, and so

$$\mathcal{L}(g) = \frac{g(0)}{s} + \frac{g'(0)}{s^2} + \frac{g''(0)}{s^3} + \cdots + \frac{g^{(n-1)}(0)}{s^n} + \frac{H(s)}{s^{n+1}}.$$

Comparing the above relations, we get

$$\frac{g(0)}{s} + \frac{g'(0)}{s^2} + \frac{g''(0)}{s^3} + \cdots + \frac{g^{(n-1)}(0)}{s^n} + \frac{H(s)}{s^{n+1}} = \frac{\beta_0 + \beta_1 s + \cdots + \beta_m s^m}{\alpha_0 + \alpha_1 s + \cdots + \alpha_{m+\ell} s^{m+\ell}}.$$

If $\ell \geq 2$, multiply both sides of the above relation by s and then let $s \to \infty$ to get $g(0) = 0$. If $\ell > 2$, multiply both sides of the above equality by s^2 and let $s \to \infty$ to get $g'(0) = 0$. Continuing in this way, we get $g(0) = g'(0) = \cdots = g^{(\ell-2)}(0) = 0$. Finally, multiply both sides of the equality by s^ℓ and let $s \to \infty$ to get $g^{(\ell-1)}(0) = \beta_m/\alpha_n$. This completes the proof. □

In what follows, the functions $f, y : (0, \infty) \to \mathbb{F}$ are assumed to be of exponential order and continuous. If there is no danger of confusion, we write $f(0)$ and $y(0)$ instead of $f(0^+)$ and $y(0^+)$, respectively.

Lemma 3.2. *Given an integer $n > 1$, let $f : (0, \infty) \to \mathbb{F}$ be a continuous function and let $P(s)$ be a complex polynomial of degree n. Then there exists an n times continuously differentiable function $h : (0, \infty) \to \mathbb{F}$ such that*

$$\mathcal{L}(h) = \frac{\mathcal{L}(f)}{P(s)} \quad (\Re(s) > \max\{\sigma_p, \sigma_f\}),$$

where $\sigma_p = \max\{\Re(s) : P(s) = 0\}$ and σ_f is the abscissa of convergence for f. In particular, it holds that $h^{(i)}(0) = 0$ for every $i = 0, 1, \ldots, n-1$.

Proof. If we take $Q(s) = 1$ and $P(s) = \alpha_0 + \alpha_1 s + \cdots + \alpha_n s^n$ in Lemma 3.1, then there exists an infinitely differentiable function $g : (0, \infty) \to \mathbb{F}$ such that

$$\mathcal{L}(g) = \frac{1}{P(s)} \quad (\Re(s) > \sigma_p)$$

and $g^{(i)}(0) = 0$ if $i = 0, 1, \ldots, n-2$ and $g^{(n-1)}(0) = 1/\alpha_n$. If we define $h = g * f$, then

$$\mathcal{L}(h) = \mathcal{L}(g * f) = \mathcal{L}(g)\mathcal{L}(f) = \frac{\mathcal{L}(f)}{P(s)}.$$

By Leibniz's rule for differentiation under the integral sign, we have

$$h'(t) = g(0)f(t) + \int_0^t g'(t-x)f(x)dx = \int_0^t g'(t-x)f(x)dx,$$

and more generally

$$h^{(i)}(t) = g^{(i-1)}(0)f(t) + \int_0^t g^{(i)}(t-x)f(x)dx = \int_0^t g^{(i)}(t-x)f(x)dx$$

for every $i = 1, \ldots, n-1$. Hence, we get

$$h(0) = h'(0) = \cdots = h^{(n-1)}(0) = 0,$$

which completes the proof. $\square$

Theorem 3.3. *Let α be a scalar. If a function $y : (0, \infty) \to \mathbb{F}$ satisfies the inequality*

$$|y'(t) + \alpha y(t) - f(t)| \le \varepsilon \tag{3.1}$$

for all $t > 0$ and for some $\varepsilon > 0$, then there exists a solution $y_a : (0, \infty) \to \mathbb{F}$ of the differential equation

$$y'(t) + \alpha y(t) = f(t) \tag{3.2}$$

such that

$$|y_a(t) - y(t)| \le \begin{cases} \varepsilon t & (\text{for } \Re(\alpha) = 0), \\ \dfrac{(1 - e^{-\Re(\alpha)t})\varepsilon}{\Re(\alpha)} & (\text{for } \Re(\alpha) \neq 0) \end{cases}$$

for all $t > 0$.

Proof. If we define $z(t) = y'(t) + \alpha y(t) - f(t)$ for each $t > 0$, then we have

$$\mathcal{L}(z) = s\mathcal{L}(y) - y(0) + \alpha\mathcal{L}(y) - \mathcal{L}(f)$$

and so

$$\mathcal{L}(y) - \frac{y(0) + \mathcal{L}(f)}{s+\alpha} = \frac{\mathcal{L}(z)}{s+\alpha}. \tag{3.3}$$

If we set

$$y_a(t) = y(0)e^{-\alpha t} + (E_{-\alpha} * f)(t),$$

where $E_{-\alpha}(t) = e^{-\alpha t}$, then $y_a(0) = y(0)$ and

$$\mathcal{L}(y_a) = \frac{y(0) + \mathcal{L}(f)}{s+\alpha} = \frac{y_a(0) + \mathcal{L}(f)}{s+\alpha}. \tag{3.4}$$

Hence, we get

$$\mathcal{L}(y_a'(t) + \alpha y_a(t)) = s\mathcal{L}(y_a) - y_a(0) + \alpha\mathcal{L}(y_a) = \mathcal{L}(f).$$

Since $\mathcal{L}$ is one-to-one, it follows that $y_a'(t) + \alpha y_a(t) = f(t)$. Thus, y_a is a solution of (3.2). Applying (3.3) and (3.4) and considering $\mathcal{L}(E_{-\alpha} * z) = \mathcal{L}(z)/(s+\alpha)$, we obtain $\mathcal{L}(y) - \mathcal{L}(y_a) = \mathcal{L}(E_{-\alpha} * z)$ and consequently $y(t) - y_a(t) = (E_{-\alpha} * z)(t)$. In view of (3.1), it holds that $|z(t)| \le \varepsilon$ and it follows from the definition of the convolution that

$$|y(t) - y_a(t)| = |(E_{-\alpha} * z)(t)| \le \varepsilon e^{-\Re(\alpha)t} \int_0^t e^{\Re(\alpha)x} dx,$$

which completes the proof. □

The following theorem is our main theorem, in which we investigate the Hyers–Ulam stability problem of linear differential equations of nth order by using the Laplace transform method.

Theorem 3.4. *Let $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ be scalars, where n is an integer larger than 1. Then there exist a constant $M > 0$ such that for every function $y : (0, \infty) \to \mathbb{F}$ satisfying the inequality*

$$\left| y^{(n)}(t) + \sum_{k=0}^{n-1} \alpha_k y^{(k)}(t) - f(t) \right| \le \varepsilon \tag{3.5}$$

for all $t > 0$ and for some $\varepsilon > 0$, there exists a solution $y_a : (0, \infty) \to \mathbb{F}$ of the differential equation

$$y^{(n)}(t) + \sum_{k=0}^{n-1} \alpha_k y^{(k)}(t) = f(t) \tag{3.6}$$

such that

$$|y_a(t) - y(t)| \le \varepsilon M \frac{e^{\alpha t}}{\alpha}$$

for all $t > 0$ and for any $\alpha > \max\{0, \sigma_p, \sigma_f\}$, where σ_p is defined in (3.9).

Proof. Applying integration by parts repeatedly, we may derive

$$\mathcal{L}\big(y^{(n)}\big) = s^n \mathcal{L}(y) - \sum_{j=1}^{n} s^{n-j} y^{(j-1)}(0)$$

for any positive integer n. Let $\alpha_n = 1$. In view of the preceding formula, we see that a function y_0 is a solution of (3.6) if and only if

$$\begin{aligned} \mathcal{L}(f) &= \sum_{k=0}^{n} \alpha_k s^k \mathcal{L}(y_0) - \sum_{k=1}^{n} \alpha_k \sum_{j=1}^{k} s^{k-j} y_0^{(j-1)}(0) \\ &= \sum_{k=0}^{n} \alpha_k s^k \mathcal{L}(y_0) - \sum_{j=1}^{n} \sum_{k=j}^{n} \alpha_k s^{k-j} y_0^{(j-1)}(0) \\ &= P_{n,0}(s)\mathcal{L}(y_0) - \sum_{j=1}^{n} P_{n,j}(s) y_0^{(j-1)}(0), \end{aligned} \tag{3.7}$$

where the polynomials $P_{n,j}$ are determined by

$$P_{n,j}(s) = \alpha_j + \alpha_{j+1}s + \alpha_{j+2}s^2 + \cdots + \alpha_{n-1}s^{n-j-1} + \alpha_n s^{n-j} = \sum_{k=j}^{n} \alpha_k s^{k-j}$$

for $j = 0, 1, \ldots, n$.

Let us define

$$h(t) = y^{(n)}(t) + \sum_{k=0}^{n-1} \alpha_k y^{(k)}(t) - f(t)$$

for all $t > 0$. Then, similarly as in (3.6) and (3.7), we obtain

$$\mathcal{L}(h) = P_{n,0}(s)\mathcal{L}(y) - \sum_{j=1}^{n} P_{n,j}(s) y^{(j-1)}(0) - \mathcal{L}(f).$$

Hence, we get

$$\mathcal{L}(y) - \frac{1}{P_{n,0}(s)}\left(\sum_{j=1}^{n} P_{n,j}(s)y^{(j-1)}(0) + \mathcal{L}(f)\right) = \frac{\mathcal{L}(h)}{P_{n,0}(s)}. \tag{3.8}$$

Let σ_f be the abscissa of convergence for f, let $s_1, s_2, \ldots, s_n$ be the roots of the polynomial $P_{n,0}$, and let

$$\sigma_p = \max\{\Re(s_k) : k = 1, 2, \ldots, n\}. \tag{3.9}$$

For any s with $\Re(s) > \max\{\sigma_p, \sigma_f\}$, we set

$$G(s) = \frac{1}{P_{n,0}(s)}\left(\sum_{j=1}^{n} P_{n,j}(s)y^{(j-1)}(0) + \mathcal{L}(f)\right). \tag{3.10}$$

By Lemma 3.2, there exists an n times continuously differentiable function f_0 such that

$$\mathcal{L}(f_0) = \frac{\mathcal{L}(f)}{P_{n,0}(s)} \tag{3.11}$$

for all s with $\Re(s) > \max\{\sigma_p, \sigma_f\}$ and

$$f_0(0) = f_0'(0) = \cdots = f_0^{(n-1)}(0) = 0.$$

For $j = 1, 2, \ldots, n$, we note that

$$\frac{P_{n,j}(s)}{P_{n,0}(s)} = \frac{1}{s^j} - \frac{\alpha_0 + \alpha_1 s + \cdots + \alpha_{j-1}s^{j-1}}{s^j P_{n,0}(s)} \tag{3.12}$$

for every s with $\Re(s) > \max\{0, \sigma_p\}$. Applying Lemma 3.1 for the case of $Q(s) = \alpha_0 + \alpha_1 s + \cdots + \alpha_{j-1}s^{j-1}$ and $P(s) = s^j P_{n,0}(s)$, we can find an infinitely differentiable function g_j such that

$$\mathcal{L}(g_j) = \frac{\alpha_0 + \alpha_1 s + \cdots + \alpha_{j-1}s^{j-1}}{s^j P_{n,0}(s)} \tag{3.13}$$

and

$$g_j(0) = g_j'(0) = \cdots = g_j^{(n-1)}(0) = 0.$$

Let

$$f_j(t) = \frac{t^{j-1}}{(j-1)!} - g_j(t) \tag{3.14}$$

for $j = 1, 2, \ldots, n$. Then we have

$$f_j^{(i)}(0) = \begin{cases} 0 & (\text{for } i = 0, 1, \ldots, j-2, j, j+1, \ldots, n-1), \\ 1 & (\text{for } i = j-1). \end{cases}$$

If we define

$$y_a(t) = \sum_{j=1}^{n} y^{(j-1)}(0) f_j(t) + f_0(t),$$

then the conditions for f_j and f_0 imply that

$$y_a^{(i)}(0) = y^{(i)}(0)$$

for every $i = 0, 1, \ldots, n-1$. Moreover, it follows from (3.10) to (3.14) that $\mathcal{L}(y_a) = G(s)$ and hence

$$\mathcal{L}(y_a) = \frac{1}{P_{n,0}(s)}\left(\sum_{j=1}^{n} P_{n,j}(s)y_a^{(j-1)}(0) + \mathcal{L}(f)\right) \tag{3.15}$$

for each s with $\Re(s) > \max\{0, \sigma_p, \sigma_f\}$.

Now, (3.7) implies that y_a is a solution of (3.6). Moreover, by (3.8) and (3.15), we have

$$\mathcal{L}(y) - \mathcal{L}(y_a) = \frac{\mathcal{L}(h)}{P_{n,0}(s)}$$

and so

$$|y(t) - y_a(t)| = \left| \mathcal{L}^{-1}\left(\frac{\mathcal{L}(h)}{P_{n,0}(s)}\right) \right|$$

for $t > 0$. By the definition of h and (3.5), it holds that $|h(t)| \le \varepsilon$ for every $t > 0$ and so

$$|\mathcal{L}(h)| \le \int_0^\infty |e^{-st}||h(t)|dt \le \frac{\varepsilon}{\Re(s)} \tag{3.16}$$

for all s with $\Re(s) > 0$.

Finally, it follows from the formula for the inverse Laplace transform that

$$\begin{aligned} |y(t) - y_a(t)| &= \left| \mathcal{L}^{-1}\left(\frac{\mathcal{L}(h)}{P_{n,0}(s)}\right) \right| \\ &= \frac{1}{2\pi}\left| \int_{-\infty}^{\infty} e^{(\alpha+iy)t}\frac{\mathcal{L}(h)(\alpha+iy)}{P_{n,0}(\alpha+iy)}dy \right| \\ &\le \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\alpha t}\frac{\varepsilon}{\alpha}\frac{1}{|P_{n,0}(\alpha+iy)|}dy \quad \text{by (3.16)} \\ &\le \frac{\varepsilon}{2\pi\alpha}e^{\alpha t}\int_{-\infty}^{\infty}\frac{1}{|P_{n,0}(\alpha+iy)|}dy \\ &\le \varepsilon M\frac{e^{\alpha t}}{\alpha} \end{aligned}$$

for all $t > 0$ and any $\alpha > \max\{0, \sigma_p, \sigma_f\}$, where

$$M = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{|P_{n,0}(\alpha+iy)|}dy < \infty$$

because n is an integer larger than 1. □

Since $e^{\alpha t}/\alpha$ has its minimum at $\alpha = 1/t$, it holds that

$$\frac{e^{\alpha t}}{\alpha} \ge et$$

either for all $t > 0$ (if $\max\{0, \sigma_p, \sigma_f\} = 0$) or for all sufficiently small $t > 0$ with $1/t > \max\{0, \sigma_p, \sigma_f\}$. Thus, we have the following corollaries.

Corollary 3.5. *Assume that* $\max\{0, \sigma_p, \sigma_f\} = 0$. *Then there exists a constant* $M > 0$ *such that for every function* $y : (0, \infty) \to \mathbb{F}$ *satisfying the inequality* (3.5) *for all* $t > 0$ *and for some* $\varepsilon > 0$, *there exists a solution* $y_a : (0, \infty) \to \mathbb{F}$ *of the differential equation* (3.6) *such that*

$$|y_a(t) - y(t)| \le \varepsilon Met$$

for all $t > 0$, *where* σ_p *is defined in* (3.9).

Corollary 3.6. *Assume that* $\max\{0, \sigma_p, \sigma_f\} > 0$. *Then there exists a constant* $M > 0$ *such that for every function* $y : (0, \infty) \to \mathbb{F}$ *satisfying the inequality* (3.5) *for all* $t > 0$ *and for some* $\varepsilon > 0$, *there exists a solution* $y_a : (0, \infty) \to \mathbb{F}$ *of the differential equation* (3.6) *such that*

$$|y_a(t) - y(t)| \le \varepsilon Met$$

for all t *with* $0 < t < 1/\max\{0, \sigma_p, \sigma_f\}$.

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